Automorphisms and derivations of free Poisson algebras in two variables

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Abstract

Let P be a free Poisson algebra in two variables over a field of characteristic zero. We prove that the automorphisms of P are tame and that the locally nilpotent derivations of P are triangulable.

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1 Introduction

It is well known [6, 9, 10, 11] that the automorphisms of polynomial algebras and free associative algebras in two variables are tame. It was recently proved [17, 18] that polynomial algebras and free associative algebras in three variables in the case of characteristic zero have wild automorphisms. P. Cohn [4] proved that the automorphisms of a free Lie algebra with a finite set of generators are tame.

There are many other results, some of them quite deep, known about the structure of polynomial algebras, free associative algebras, and free Lie algebras. Though free Poisson

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algebras are very closely connected with these algebras, only few results are known about them up to now. Say, one of the fundamental results about free associative algebras is the Bergman Centralizer Theorem (see [3]) which says that the centralizer of any nonconstant element is a polynomial algebra on a single variable. An analogue of this theorem for free Poisson algebras in the case of characteristic zero was proved in [12].

The question on the tameness of automorphisms of free Poisson algebras in two variables was open and was formulated in [12, Problem 5]. Note that the Nagata automorphism [13, 17] gives an example of a wild automorphism of a free Poisson algebra in three variables.

In [14] R. Rentschler proved that the locally nilpotent derivations of polynomial algebras in two variables over a field of characteristic 0 are triangulable. Using this result he gave a new proof of Jung's Theorem [9] on the tameness of automorphisms of these algebras.

In this paper we study automorphisms and locally nilpotent derivations of free Poisson algebras over a field of characteristic zero. In Section 2 we introduce several gradings of free Poisson algebras and describe some properties of homogeneous derivations of these algebras. In Section 3 we prove that the locally nilpotent derivations of two generated free Poisson algebras are triangulable and the automorphisms of these algebras are tame. These results are analogues of Rentschler's Theorem [14] and Jung's Theorem [9], respectively.

2 Homogeneous derivations

A vector space B over a field k endowed with two bilinear operations $x \cdot y$ (a multiplication) and $\{x, y\}$ (a Poisson bracket) is called *a Poisson algebra* if B is a commutative associative algebra under $x \cdot y$, B is a Lie algebra under $\{x, y\}$, and B satisfies the following identity (the Leibniz identity):

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}.$$

Of course, the Leibniz identity just says that for every $x \in B$ the map

$$ad_x : B \longrightarrow B, \quad (y \mapsto \{x, y\}),$$

is a derivation of B as an associative algebra.

The map ad_x also satisfies another similar identity:

$$ad_x\{y, z\} = \{ad_x(y), z\} + \{y, ad_x(z)\}.$$

It is just the Jacobi identity for B as a Lie algebra.

Let us call a linear homomorphism D of B to B a derivation of B as a Poisson algebra if it satisfies both the Leibniz and Jacobi identities. In other words, D is simultaneously a derivation of B as an associative algebra and as a Lie algebra.

There are two important classes of Poisson algebras.

1) Symplectic algebras S_n. For each n algebra S_n is a polynomial algebra $k[x_1, y_1, \ldots, x_n, y_n]$ endowed with the Poisson bracket defined by

$$\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0,$$

where δ_{ij} is the Kronecker symbol and $1 \leq i, j \leq n$.

2) Algebras of Lie type. Let g be a Lie algebra with a linear basis $e_1, e_2, \ldots, e_k, \ldots$. The symmetric algebra $S(g)$ of g (i. e. the usual polynomial algebra $k[e_1, e_2, \ldots, e_k, \ldots]$) endowed with the Poisson bracket defined by

$$\{e_i, e_j\} = [e_i, e_j]$$

for all i, j, where $[x, y]$ is the multiplication of the Lie algebra g is the Poisson algebra of type g.

From now on let g be a free Lie algebra with free (Lie) generators $x_1, x_2, \ldots, x_n$. It is well known (see, for example [15]) that in this case $S(g)$ is a free Poisson algebra on the same set of generators. We denote this algebra by $P = P\langle x_1, x_2, \ldots, x_n\rangle$.

By deg we denote the standard degree function of the homogeneous algebra P, i.e. $\deg(x_i) = 1$, where $1 \leq i \leq n$. Note that

$$\deg\{f, g\} = \deg f + \deg g$$

if f and g are homogeneous and $\{f, g\} \neq 0$. By $\deg_{x_i}$ we denote the degree function on P with respect to x_i. We have $\deg_{x_i}(x_j) = \delta_{ij}$, where $1 \leq i, j \leq n$. The homogeneous elements of P with respect to $\deg_{x_i}$ can be defined in the ordinary way.

If f is homogeneous with respect to each $\deg_{x_i}$, where $1 \leq i \leq n$, then f is called multihomogeneous. For every multihomogeneous element $f \in P$ we put

$$mdeg(f) = (m_1, m_2, \ldots, m_n),$$

where $\deg_{x_i} f = m_i$ for all i and $1 \leq i \leq n$.

Let us choose a multihomogeneous linear basis

$$x_1, x_2, \ldots, x_n,\ [x_1, x_2], \ldots, [x_1, x_n], \ldots, [x_{n-1}, x_n],\ [[x_1, x_2], x_3], \ldots$$

of the free Lie algebra g and denote the elements of this basis by

$$e_1, e_2, \ldots, e_m, \ldots. \tag{1}$$

Note that

$$mdeg\{e_i, e_j\} = mdeg(e_i) + mdeg(e_j)$$

if $i \neq j$. So if $i < j$ then $\{e_i, e_j\}$ is a linear combination of e_m where all $m > j$.

The algebra $P = P\langle x_1, x_2, \ldots, x_n\rangle$ coincides with the polynomial algebra on the elements (1). Consequently, the words

$$u = e_{i_1}e_{i_2}\ldots e_{i_k}, \quad i_1 \leq i_2 \leq \ldots \leq i_k \tag{2}$$

form a linear basis of P. The basis (2) is multihomogeneous since so is (1).

Consider the Lie algebra $Der(P)$ of all derivations of the Poisson algebra P. For every system of elements $f_1, f_2, \ldots, f_n$ of P denote by

$$D = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \tag{3}$$

a unique derivation of P such that $D(x_i) = f_i$ where $1 \leq i \leq n$. Then the derivations

$$v = u\frac{\partial}{\partial x_i}, \tag{4}$$

where $1 \leq i \leq n$ and u is an element of (2), constitute a linear basis of $Der(P)$. For every element v of the form (4) we put

$$mdeg(v) = mdeg(u) - \epsilon_i,$$

where $\epsilon_i \in Z^n$ is the standard basis vector with 1 in the ith position and with zeroes everywhere else. Now one can define the multihomogeneous derivations of the algebra P and every element of $Der(P)$ can be uniquely represented as the sum of multihomogeneous derivations of different multidegrees.

To each nonzero vector $w \in Z^n$ we associate the so called w-degree (or weight degree) function $wdeg$ on P and $Der(P)$. Put

$$wdeg(u) =< mdeg(u), w >, \quad wdeg(v) =< mdeg(v), w >,$$

where u and v are elements of the form (2) and (4) respectively, and $<,>$ is the standard inner product in R^n. Let P_m and $Der_m P$ be the subsets of all w-homogeneous elements of degree m of P and $Der(P)$, respectively. It is clear that the decompositions

$$P = \oplus_{m\in Z} P_m, \quad Der(P) = \oplus_{m\in Z} Der_m P$$

are gradings of the corresponding algebras. Moreover, for every element $d \in Der_m P$ we have

$$d(P_k) \subseteq P_{m+k}.$$

There is another natural degree function on P, just the total degree on P as a polynomial ring, where the degree is one for all elements of the homogeneous basis (1). Denote it by $pdeg$ and observe that

$$pdeg[a, b] = pdeg a + pdeg b - 1$$

for any p-homogeneous a, $b \in P$ if $[a, b] \neq 0$.

If v is an element of the form (4) then we put

$$pdeg v = pdeg u - 1.$$

Let P^*_m and $Der^*_m P$ be the subsets of all p-homogeneous elements of degree m of P and $Der(P)$, respectively. It is again clear that the decompositions

$$P = \oplus_{m \in Z} P^*_m, \quad Der(P) = \oplus_{m \in Z} Der^*_m P$$

are gradings of the corresponding algebras and that for every element $d \in Der^*_m P$ we have

$$d(P^*_k) \subseteq P^*_{m+k}.$$

Recall that a derivation D of an algebra R is called locally nilpotent if for every $a \in R$ there exists a natural number $m = m(a)$ such that $D^m(a) = 0$. The statement of the next proposition is well known (see, for example [8, Proposition 5.1.15]).

Proposition 1 *Let $R = \oplus_{m \in Z} R_m$ be a graded algebra and suppose D be a locally nilpotent derivation of R such that*

$$D = D_p + D_{p+1} + \ldots + D_q, \quad D_i(R_m) \subseteq R_{i+m}, \quad p \leq i \leq q, \quad D_q \neq 0.$$

Then D_q is locally nilpotent.

Proof. If

$$f = f_r + f_{r+1} + \ldots + f_s \in R,$$

where $f_i \in R_i$, $r \leq i \leq s$, and $f_s \neq 0$, then we put $\widehat{f} = f_s$.

Let $a \in R_m$ and assume that $D^i_q(a) \neq 0$ for any i. It can be easily proved by induction on i that

$$\widehat{D^i(a)} = D^i_q(a).$$

Consequently, $D^i(a) \neq 0$ for any i and this gives a contradiction. □

Let f be an arbitrary element of P and D be an arbitrary derivation of P of the form (3). We put

$$fD = \sum_{i=1}^{n} (f f_i) \frac{\partial}{\partial x_i}.$$

Put also

$$S(f) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$$

if $f \in k[S(f)]$ and $f \notin k[S(f_i) \setminus \{e_{i_j}\}]$, where $1 \leq j \leq k$. For D we put

$$S(D) = S(f_1) \cup S(f_2) \cup \ldots \cup S(f_n).$$

If $x = e_i$ then we denote by $pdeg_x$ the polynomial degree function with respect to x on P. Elements $f \in P$ and $D \in Der\, P$ can be uniquely written as

$$f = f_0 + xf_1 + \ldots + x^m f_m, \quad x \notin S(f_i), \quad 0 \leq i \leq m,$$

and

$$D = D_0 + xD_1 + \ldots + x^m D_m, \quad x \notin S(D_i), \quad 0 \leq i \leq m,$$

respectively. If $f_m \neq 0$ then $pdeg_x(f) = m$ and we put $l_x(f) = f_m$. Put also $pdeg_x D = m$ and $l_x(D) = D_m$ if $D_m \neq 0$.

Put $e_i < e_j$ if $i < j$.

Proposition 2 *Let D be a derivation of P and x be the minimal element of $S(D)$. Then*

$$pdeg_x D(f) \leq pdeg_x D + pdeg_x f.$$

This inequality becomes an equality iff $l_x(D)(l_x(f)) \neq 0$ and in this case

$$l_x(D(f)) = l_x(D)(l_x(f)).$$

Proof. Without loss of generality we may assume that f is an element of the basis (2) and D is an element of the basis (4).

If $f = uv$ then $D(f) = D(u)v + uD(v)$. So if the Proposition is true for u and v it is also true for f. Because of that we can assume that the polynomial degree of f is one. Let us prove that in this case $pdeg_x D(f) \leq pdeg_x D$.

If $f = \{u, v\}$ then $D(f) = \{D(u), v\} + \{u, D(v)\}$. Denote by $L(x)$ the set of all elements e_i such that $e_i > x$. If, say $D(u) = x^d u_1$ where $S(u_1) \subset L(x)$ then $\{D(u), v\} = \{x^d u_1, v\} = x^d\{u_1, v\} + dx^{d-1}u_1\{x, v\}$. As we remarked if $i < j$ then $\{e_i, e_j\}$ is a linear combination of e_m where all $m > j$. So both $S(\{u_1, v\})$ and $S(\{x, v\})$ are subsets of $L(x)$ and we can conclude that $pdeg_x D(f) \leq pdeg_x D$ if it is true for u and v. It remains to check that $pdeg_x D(f) \leq pdeg_x D$ for f with $deg(f) = 1$. Since we can assume that $D = x^d u \frac{\partial}{\partial x_i}$ where $u \in L(x)$ and $1 \leq i \leq n$ we have $D(x_j) = 0$ when $j \neq i$ and $D(x_i) = x^d u$.

So we proved that $pdeg_x D(f) \leq pdeg_x D + pdeg_x f$. To prove that $l_x(D(f)) = l_x(D)(l_x(f))$ in the case of equality take $f = x^n f_n$ and $D = x^m u \frac{\partial}{\partial x_i}$ where $pdeg_x(f_n) = 0$ and $u \in L(x)$. Since $D(f) = x^n D(f_n) + nx^{n-1} f_n D(x)$ only $x^n D(f_n)$ can contain x^{n+m} and we should show that $l_x(D(f_n)) = x^m D_m(f_n)$ where $D_m = u \frac{\partial}{\partial x_i}$. It can be done exactly as above by reduction first to the case when $pdeg(f_n) = 1$ and then to the case when $deg(f_n) = 1$. □

Lemma 1 *Let D be a derivation of P and x be the minimal element of $S(D)$. If D is locally nilpotent then so is $l_x(D)$.*

Proof. If $l_x(D)$ is not locally nilpotent then there exists x_i such that $l_x(D)^k(x_i) \neq 0$ for all $k \geq 0$. Put $a = l_x(D)(x_i)$. Note that $x \notin S(a)$ and $l_x(a) = a$. Using this and Proposition 2, we get

$$l_x(D^k(a)) = l_x(D)(l_x(D^{k-1}(a))) = \ldots = l_x(D)^{k-1}(l_x(D)(a)) = l_x(D)^k(a) \neq 0.$$

Consequently, D is not locally nilpotent. □

Proposition 3 *Let D be a derivation of P of the form*

$$D = D_0 + xD_1 + \ldots + x^{m-1}D_{m-1} + x^m \frac{\partial}{\partial x_1}, \quad x \notin S(D_i), \quad 0 \leq i \leq m-1,$$

where x is the minimal element of $S(D)$. Let f be an element of P such that $x_1 \notin S(f)$. Then

$$pdeg_x D(f) \leq m - 1 + pdeg_x f.$$

This inequality becomes an equality iff $D'(l_x(f)) \neq 0$, where $D' = D_{m-1} + mx\frac{\partial}{\partial x_1}$, and in this case

$$l_x(D(f)) = D'(l_x(f)).$$

Proof. The same considerations as in the proof of Proposition 2 show that

$$pdeg_x(x^m \frac{\partial}{\partial x_1}(f)) \leq m - 1 + pdeg_x f$$

and if $\frac{\partial}{\partial x_1}(l_x(f)) \neq 0$ then

$$l_x(x^m \frac{\partial}{\partial x_1}(f)) = ml_x(x\frac{\partial}{\partial x_1}(f)).$$

Note that $D = D^* + x^m \frac{\partial}{\partial x_1}$ and $pdeg_x(D^*) \leq m - 1$. So applying Proposition 2, we can complete the proof of Proposition 3. □

Lemma 2 *Let D be a locally nilpotent derivation of P of the form*

$$D = D_0 + xD_1 + \ldots + x^{m-1}D_{m-1} + x^m \frac{\partial}{\partial x_1}, \quad x \notin S(D_i), \quad 0 \leq i \leq m-1,$$

where x is the minimal element of $S(D)$. If $x \neq x_1$ then $D_{m-1} + mx\frac{\partial}{\partial x_1}$ is also locally nilpotent.

Proof. Assume that $D' = D_{m-1} + mx\frac{\partial}{\partial x_1}$ is not locally nilpotent. Then there exists x_i such that $D'^k(x_i) \neq 0$ for all $k \geq 0$. We put $a = D'^2(x_i)$. It is not difficult to show that $x_1, x \notin S(a)$. So $l_x(a) = a$. Using this and Proposition 3, we get

$$l_x(D^k(a)) = D'(l_x(D^{k-1}(a))) = \ldots = D'^k(a) \neq 0.$$

Consequently, D is not locally nilpotent. □

Lemma 3 *Let D be a multihomogeneous derivation of $P = P\langle x_1, x_2\rangle$ and $mdeg(D) = (m_1, m_2)$. If $m_i \geq 0$ for $i = 1, 2$ then D is not locally nilpotent.*

Proof. Let D be a counterexample to the lemma with the minimal $\deg(D)$. By Proposition 1, we can also assume that D is p-homogeneous. Let x be the minimal element of $S(D)$. By Lemma 1, it follows that $l_x(D)$ is also locally nilpotent. Put $mdeg(l_x(D)) = (n_1, n_2)$. We can assume that $n_1 = -1$ since $\deg(l_x(D)) < \deg(D)$. Then $l_x(D) = \alpha x_2^{n_2} \frac{\partial}{\partial x_1}$.

If $x = x_1$ then D contains a summand $l_x(D) = \alpha x_1^{m_1+1} x_2^r \frac{\partial}{\partial x_1}$. In this case D induces a nonzero locally nilpotent derivation of the polynomial algebra $k[x_1, x_2]$ with the same multidegree. It is impossible (see, for example [8], p. 91).

So $x \neq x_1$. If $x = x_2$ then $m_1 = -1$. So $x > x_2$ and D can be written as in Lemma 2. By Lemma 2, it follows that $D' = D_{m-1} + mx\frac{\partial}{\partial x_1}$ is a nonzero locally nilpotent derivation. Note that $pdeg(D') = 0$ and D' is p-homogeneous. Therefore D' is a derivation of the free Lie algebra g generated by x_1, x_2. Obviously, $exp(D')$ gives a nonlinear automorphism of g. But all automorphisms of g are linear [4]. $\square$

3 The main results

Recall that a derivation of the free Poisson algebra $P\langle x_1, x_2, \ldots, x_n\rangle$ of the form (3) is called triangular if $f_i \in P\langle x_{i+1}, x_{i+2}, \ldots, x_n\rangle$ for any i. It is clear that every triangular derivation is locally nilpotent. A derivation D of $P\langle x_1, x_2, \ldots, x_n\rangle$ is called triangulable if there exists an automorphism φ such that $\varphi^{-1} D\varphi$ is triangular. R. Rentschler proved [14] that the locally nilpotent derivations of polynomial algebras in two variables over a field of characteristic 0 are triangulable. H. Bass gave [1] an example of a nontriangulable derivation of polynomial algebras in three variables.

Theorem 1 *Let D be a locally nilpotent derivation of $P = P\langle x_1, x_2\rangle$. Then there exist a tame automorphism φ of P and $f(x_2) \in k[x_2]$ such that $\varphi^{-1} D\varphi = f(x_2)\frac{\partial}{\partial x_1}$.*

Proof. Denote by I the ideal of P generated by $\{x_1, x_2\}$. Then $P/I \cong k[x_1, x_2]$ and D induces a locally nilpotent derivation D' of $k[x_1, x_2]$. By Rentschler's theorem [14], there exists a tame automorphism ψ of $k[x_1, x_2]$ and $f(x_2) \in k[x_2]$ such that $\psi^{-1} D'\psi = f(x_2)\frac{\partial}{\partial x_1}$. Denote by φ the extension of ψ to P such that $\varphi|_{k[x_1,x_2]} = \psi$. Replacing D by $\varphi^{-1} D\varphi$ we can assume that $D' = f(x_2)\frac{\partial}{\partial x_1}$. Then

$$D = (f(x_2) + a)\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2},$$

where $a, b \in I$.

We would like to show that $a = b = 0$. Assume it is not the case. Consider $\deg_{x_1}$ and the corresponding highest homogeneous derivation R which is locally nilpotent by Proposition 1. But $R = c\frac{\partial}{\partial x_1} + d\frac{\partial}{\partial x_2}$ where $c, d \in I$ and either c or d is not zero. So R cannot be locally nilpotent by Lemma 3. $\square$

Corollary 1 *Let D be a locally nilpotent derivation of $P = P\langle x_1, x_2\rangle$. Then $D\{x_1, x_2\} = 0$.*

Proof. If D is triangular then $D\{x_1, x_2\} = 0$. Note that $\varphi\{x_1, x_2\} = \alpha\{x_1, x_2\}$ for every tame automorphism since it is true for every elementary automorphism. □

Theorem 2 *Automorphisms of free Poisson algebras in two variables over a field of characteristic zero are tame.*

Proof. Let θ be an arbitrary automorphism of $P = P\langle x_1, x_2\rangle$. Then θ induces an automorphism ψ of $k[x_1, x_2]$. Denote by φ the extension of ψ to P such that $\varphi|_{k[x_1,x_2]} = \psi$. By Jung's theorem [9], ψ and φ are tame. Changing θ to $\theta\varphi^{-1}$ we can assume that θ induces the identical automorphism of $k[x_1, x_2]$. Then,

$$\theta(x_1) = x_1 + a, \quad \theta(x_2) = x_2 + b, \quad a, b \in I,$$

where I is the ideal of P generated by $\{x_1, x_2\}$.

For every $h \in k[x]$ denote by D_h a derivation of P defined by $D_h(x_1 + a) = h(x_2 + b)$, $D_h(x_2 + b) = 0$. This derivation is locally nilpotent.

Now,

$$D_h = (h(x_2) + (h(x_2 + b) - h(x_2)) - D_h(a))\frac{\partial}{\partial x_1} - D_h(b)\frac{\partial}{\partial x_2}$$

since $D_h(x_1) = h(x_2) + (h(x_2 + b) - h(x_2)) - D_h(a)$ and $D_h(x_2) = -D_h(b)$.

The ideal I is invariant under every derivation. Hence $h(x_2+b)-h(x_2))-D_h(a), D(b) \in I$. Since D_h is locally nilpotent it is possible only if $D_h(b) = h(x_2+b)-h(x_2))-D_h(a) = 0$ (see the proof of Theorem 1). Therefore $D_h(x_1) = h(x_2)$, $D_h(x_2) = 0$ and $D_h(a) = h(x_2 + b) - h(x_2)$.

Put $h = x$. Then $D_x(a) = b$. Note that $\deg D_x(a) \leq \deg a$ since $D_x(x_1) = x_2$ and $D_x(x_2) = 0$. So $\deg b \leq \deg a$. We can exchange x_1 and x_2 in the definition of D_h, so $\deg a \leq \deg b$ and $\deg a = \deg b$. Of course, $\deg a = \deg b \geq 2$ since $a, b \in I$.

We now put $h = x^2$. Then $D_h(a) = 2x_2 b + b^2$. Note that in this case $\deg D_h(a) \leq \deg a + 1$ since $D_h(x_1) = x_2^2$ and $D_h(x_2) = 0$. Consequently, $\deg a + 1 \geq 2 \deg b = 2 \deg a$, and $\deg a \leq 1$. This contradiction gives $a = 0$ and $b = 0$. □

Corollary 2 *Let φ be an arbitrary automorphism of $P = P\langle x_1, x_2\rangle$. Then $\varphi\{x_1, x_2\} = \alpha\{x_1, x_2\}$, where $0 \neq \alpha \in k$.*

So every automorphism of $P\langle x_1, x_2\rangle$ preserves $\{x_1, x_2\}$ up to the proportionality. An analogue of this result for free associative algebras is also true, i.e., every automorphism of the free associative algebra $k < x_1, x_2 >$ in the variables x_1, x_2 preserves the commutator $[x_1, x_2]$ up to the proportionality. Moreover, the so called commutator test theorem [7] says that any endomorphism of $k < x_1, x_2 >$ which preserves $[x_1, x_2]$ is an automorphism.

Problem 1 *Is any endomorphism of the free Poisson algebra $P\langle x_1, x_2\rangle$ over a field of characteristic* 0 *which preserves* $\{x_1, x_2\}$ *an automorphism?*

Note that the positive answer to Problem 1 implies the Jacobian Conjecture for $k[x_1, x_2]$ [8].

It is well known [6, 11] that $Aut\, k[x_1, x_2] \cong Aut\, k < x_1, x_2 >$, where $k < x_1, x_2 >$ is the free associative algebra generated by x_1, x_2.

Corollary 3 *Let* k *be a field of characteristic zero. Then,*

$$Aut\, k[x_1, x_2] \cong Aut\, k < x_1, x_2 > \cong Aut\, P\langle x_1, x_2\rangle.$$

This isomorphism is also interesting in the context of paper [2] since $k < x_1, x_2 >$ is a deformation quantization of $P\langle x_1, x_2\rangle$ and because it shows that the group $Aut\, P\langle x_1, x_2\rangle$ has a nice representation as a free amalgamated product of its subgroups (see, for example [5]).

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