Reduction of the Hurwitz space of metacyclic covers

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Abstract

We compute the stable reduction of some Galois covers of the projective line branched at three points. These covers are constructed using Hurwitz spaces parameterizing metacyclic covers. The reduction is determined by a certain hypergeometric differential equation. This generalizes the result of Deligne–Rapoport on the reduction of the modular curve $X(p)$.

Introduction

Recently, much progress has been made in understanding the reduction of Galois covers of the projective line from characteristic zero to characteristic $p > 0$. The starting point is a series of papers by Raynaud, [19], [20], [21] in which the theory of the stable reduction of a Galois cover is developed. In [21], one finds a criterion for good reduction for Galois covers of the projective line branched at three points, in case the order of the Galois group is strictly divisible by p. Raynaud's Criterion implies, for example, that many covers which have been constructed with the method of rigidity have good reduction to characteristic p.

Raynaud's results have been extended in [29] and [30]. Let $f : Y \to \mathbb{P}^1$ be a G-Galois cover branched at three points x_1, x_2, x_3, defined over a field K of characteristic zero. Suppose that $p > 2$ is a prime which strictly divides the order of G. It is proved that the stable reduction of f at a prime of K above p is of a very easy kind. This theorem implies that to describe the stable reduction $\bar{f}$ of f, it suffices to compute the associated deformation datum (Definition 1.1) for which there are only finitely many possibilities. However, in general, it is not known how to compute $\bar{f}$.

In the literature, the stable reduction of a Galois cover with bad reduction has been computed in very few cases. Most attention has been focused on cyclic covers of the projective line. Coleman and McCallum [8] computed the reduction of cyclic covers of the projective line branched at three points. In [10], [22], [15], [16] the reduction of p-cyclic covers of $\mathbb{P}^1$ branched at $r \geq 3$ points is studied. But even in this case the results are far from complete.

Other known results use the reduction of modular curves. For example, let $X(N)$ be the modular curve parameterizing elliptic curves with full level-N structure. Then $f : X(2p) \to X(2)$ is a Galois cover with Galois group $\mathrm{PSL}_2(p)$ branched at three points of order p. Its stable reduction can be deduced from [9] and [14].

In this paper, we compute the stable reduction of Galois covers of the projective line branched at three points with Galois group $\mathrm{PSL}_2(p)$, $\mathrm{SL}_2(p)$ or $\mathrm{PGL}_2(p)$. The construction of these covers goes back to Völklein, [27]. Essentially, the covers we consider arise as Hurwitz spaces $H(\mathbf{a})$ parameterizing covers of the projective line branched at four points. More precisely, choose $m|(p-1)$ and let N be an extension of $\mathbb{Z}/m$ by $\mathbb{Z}/p$. Then $H(\mathbf{a})$ parameterizes N-Galois covers of $\mathbb{P}^1$ branched at $0, 1, \infty, \lambda$ with ramification of order prime-to-p described by the type $\mathbf{a} = (a_1, a_2, a_3, a_4)$, Definition 2.1. Write $\pi(\mathbf{a}) : H(\mathbf{a}) \to \mathbb{P}^1_\lambda$ for the cover which sends g to the branch point λ. The cover we consider is the Galois closure $\varpi(\mathbf{a})$ of $\pi(\mathbf{a})$. It is expected that all covers of the projective line which are branched at three points and whose Galois group is $\mathrm{PSL}_2(p)$ or $\mathrm{SL}_2(p)$ arise via a construction like this, Section 8.

Theorem 4.2. *The cover $\varpi(\mathbf{a})$ has good reduction if and only if $a_1 + a_2 + a_3 + a_4 \neq 2m$.*

In Section 8 we illustrate that this theorem does not follow from Raynaud's criterion for good reduction. For $m = 2$ the cover $\varpi(\mathbf{a})$ is just the cover $X(2p) \to X(2)$.

In case the cover $\varpi(\mathbf{a})$ has bad reduction, we compute its stable reduction in Section 6. We do not give a modular interpretation for the stable model, therefore our approach differs from the approach of [9] and [14]. However, also in our approach, the key ingredient for computing the reduction of $\varpi(\mathbf{a})$ is the study of the reduction of the metacyclic covers which the Hurwitz space parameterizes. It turns out that the stable reduction is determined by a so-called $\mathbf{a}$-Hasse invariant $\Phi_{\mathbf{a}}$. For $m = 2$ this is just the classical Hasse invariant. An important property of $\Phi_{\mathbf{a}}$ is that it is the solution to a hypergeometric differential equation, Section 5. This hypergeometric differential equation determines all combinatorial invariants which are associated to the stable reduction, see Section 6 for a precise statement.

The organization of this paper is as follows. In Section 1 we define the stable reduction and collect some results on its structure. In Section 2 we introduce the metacyclic covers and study their reduction. In Section 3 we define the Hurwitz space $H(\mathbf{a})$ in characteristic zero and describe the map $\pi(\mathbf{a}) : H(\mathbf{a}) \to \mathbb{P}^1_\lambda$ and its Galois closure. In Section 4 we prove the good reduction theorem. Sections 5 and 6 study the stable reduction of the covers with bad reduction. First we define the $\mathbf{a}$-Hasse invariant, Section 5. Then we show how this gives the stable reduction, Section 6. The rest of the paper contains complements and examples.

1 The stable model

In this section we recall the definition of the stable model. We refer to [30] for proofs and more details. Suppose that G is a finite group whose order is strictly divisible by p. Let R be a complete discrete valuation ring with fraction field K of characteristic zero and residue field $k = \bar{k}$ of characteristic $p > 2$. Let $f : Y \to X = \mathbb{P}^1_K$ be a G-Galois cover branched at $\mathbf{x} = (x_1, \ldots, x_r)$, where the x_i are distinct K-rational points. Write $\mathbb{B}_0 = \{1, \ldots, r\}$ for the set indexing the branch points. We assume that $r \geq 3$. We assume moreover that $(X; \mathbf{x})$ has *good reduction*, i.e. there exists a model $X_{0,R} = \mathbb{P}^1_R$ of X over R such that the x_i extend to pairwise disjoint sections $\mathrm{Spec}(R) \to X_{0,R}$.

Denote the ramification points of f by $\mathbf{y} = (y_1, \ldots y_s)$. Let $(Y_R; \mathbf{y})$ be the unique extension of $(Y; \mathbf{y})$ to a stably pointed curve, which exists after replacing R by a finite extension, if necessary. Recall that $(Y_R; \mathbf{y})$ is called *stably pointed* if Y_R is a semistable curve such that the usual "three point condition" is fulfilled for the union of the marked points and the singular points of $\bar{Y} := Y_R \otimes_R k$. The action of G extends to Y_R; denote the quotient of Y_R by G by X_R. Then $f_R : Y_R \to X_R$ is called the *stable model* of f. The special fiber $\bar{f} : \bar{Y} \to \bar{X}$ of f_R we call the *stable reduction* of f. We say that f has *good reduction* if and only if the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$ is a separable morphism. This is equivalent to X_R being smooth. If f does not have good reduction, we say it has *bad reduction*.

Suppose that f has bad reduction. Let $\bar{f} : \bar{Y} \to \bar{X}$ be the stable reduction of f. We denote the dual graph of $\bar{X}$ by T. The vertices $\mathbb{V}$ of T correspond to the irreducible components of $\bar{X}$ and the (oriented) edges $\mathbb{E}$ to the singular points. The source and target of an edge e are denoted by $s(e)$ and $t(e)$.

Since $(X; \mathbf{x})$ has good reduction, there exists a natural map $X_R \to X_{0,R}$. This map restricts to the identity on a unique irreducible component of $\bar{X}$ and contracts all other components of the special fiber to a point. Therefore we may (and will) consider $\bar{X}_0$ as an irreducible component of $\bar{X}$, which we call the *original component*. The dual graph T of $\bar{X}$ is a tree. We denote the vertex of T corresponding to the original component $\bar{X}_0$ by v_0 and consider T to be oriented from v_0.

An edge e is called *positive* if its orientation coincides with the orientation of T. For every $v \in \mathbb{V}$, we denote the corresponding component of $\bar{X}$ by X_v. We choose a component Y_v of $\bar{Y}$ above X_v. For every $v \in \mathbb{V}$ we denote by ∞_v the unique point of X_v which is singular in $\bar{X}$ and such that the corresponding edge e with $s(e) = v$ is negative. If e is an edge with $s(e) = v$, we write x_e for the singular point of X_v corresponding to e.

An irreducible component X_v of $\bar{X}$ is called a *tail* if it is different from the original component and meets the rest of $\bar{X}$ in just one point. We denote by $\mathbb{B}$ the subset of $\mathbb{V}$ corresponding to the tails of $\bar{X}$. A tail is called *primitive* if one of the branch points specializes to this component; otherwise the tail is called *new*. Let $\mathbb{B}_{\mathrm{prim}}$ (resp. $\mathbb{B}_{\mathrm{new}}$) be the subset of $\mathbb{B}$ corresponding to the primitive (resp. new) tails. We identify $\mathbb{B}_{\mathrm{prim}}$ with a subset of $\mathbb{B}_0 = \{1, \dots, r\}$: we write X_i for the primitive tail to which the branch point x_i specializes. The complement $\mathbb{B}_{\mathrm{wild}} := \mathbb{B}_0 - \mathbb{B}_{\mathrm{prim}}$ corresponds to the branch points whose ramification index is divisible by p. We use the notation x_i to denote both the point on X and its specialization to $\bar{X}$.

For every $b \in \mathbb{B}$, the map $Y_b \to X_b$ is separable. The inertia group of a point above ∞_b has order pm_b, for some m_b prime-to-p. Denote the corresponding conductor by h_b. We define the *ramification invariant* to be $\sigma_b = h_b/m_b$. (This is the jump in the filtration of the higher ramification groups in the upper numbering.) If $b \in \mathbb{B}_{\mathrm{new}}$, the cover $Y_b \to X_b$ is unbranched outside ∞_b. If $i \in \mathbb{B}_{\mathrm{prim}}$, then $Y_i \to X_i$ is tamely branched at x_i and unbranched outside x_i and ∞_i. If $v \in \mathbb{V} - \mathbb{B}$, then the restriction of $\bar{f}$ to X_v is inseparable.

Choose an irreducible component $\bar{Y}_0$ of $\bar{Y}$ above the original component $\bar{X}_0$. Since f has bad reduction, the inertia group I_0 if $\bar{Y}_0$ has order p. The decomposition group D_0 of $\bar{Y}_0$ is the extension of a group H_0 of order prime-to-p by I. The map $\bar{Y}_0 \to \bar{X}_0$ is the composition of a purely inseparable map $\bar{Y}_0 \to \bar{Z}_0$ and an H_0-Galois cover $\bar{Z}_0 \to \bar{X}_0$. Write $\bar{Y}_0'$ (resp. $\bar{Z}_0'$) for the quotient of $\bar{Y}_0$ (resp. $\bar{Z}_0$) by the prime-to-p part of the center of D_0. Then $\bar{Z}_0' \to \bar{X}_0$ is an m-cyclic cover, for some m dividing $p-1$. Let $\chi : \mathbb{Z}/m \to \mathbb{F}_p^\times$ be an injective character. We choose generators ϕ and ψ of the decomposition group D_0' of $\bar{Y}_0'$ such that $\phi^p = \psi^m = 1$ and $\psi^i \phi \psi^{-i} = \phi^{\chi(i)}$.

The restriction of $\bar{Y}_0 \to \bar{Z}_0$ to some open U of $\bar{Z}_0$ is an $\boldsymbol{\alpha}_p$ or $\boldsymbol{\mu}_p$-torsor. We say that f has *multiplicative (bad) reduction* if $\bar{Y}_0 \to \bar{Z}_0$ restricts to a $\boldsymbol{\mu}_p$-torsor. If $\bar{Y}_0 \to \bar{Z}_0$ restricts to an $\boldsymbol{\alpha}_p$-torsor, we say that f has *additive (bad) reduction*.

To an $\boldsymbol{\alpha}_p$ or $\boldsymbol{\mu}_p$-torsor, we can associate a differential ω, [17, Proposition 4.14]. Let W be the inverse image of U in $\bar{Y}_0$. There exists an open covering $\{U_j\}_j$ of U and elements $f_j \in \Gamma(U_j, \mathcal{O})$ such that the torsor restricted to U_j is given by an equation $v^p = f_j$.

Suppose first that $W \to U$ is a $\boldsymbol{\mu}_p$-torsor, then $f_j \in \Gamma(U_j, \mathcal{O})^*$. To the torsor we associate the differential

$$\omega = \frac{\mathrm{d}f_j}{f_j} \in H^0(U, \Omega^1).$$

Here we assume that we fixed an isomorphism $\mathbb{Z}/p \simeq \boldsymbol{\mu}_p(K)$. Note that ω does not depend on j or the choice of f_j. In case $W \to U$ is an $\boldsymbol{\alpha}_p$-torsor, we associate to it the differential

$$\omega = \mathrm{d}f_j \in H^0(U, \Omega^1).$$

Moreover, $\psi^* \omega = \chi(1)\omega$. See [30] for more details.

Definition 1.1 We call the tuple $(\bar{Z}_0', \omega)$ the *deformation datum* corresponding to f.

Lemma 1.2 *(a) The differential ω is regular outside x_b for $b \in \mathbb{B}_{\mathrm{wild}}$. For $b \in \mathbb{B}_{\mathrm{wild}}$, we have $\mathrm{ord}_{z_b}(\omega) = -1$. Here $z_b \in \bar{Z}_0$ is a point above x_b.*

(b) The differential ω has a zero at $z \in \bar{Z}_0$ if and only if the image of z in $\bar{X}_0$ is a singular point of $\bar{X}$.

Proof: First suppose that $w \in \bar{Z}_0$ is a point whose image in $\bar{X}_0$ is x_b for $b \in \mathbb{B}_{\mathrm{wild}}$. Let t be a local parameter at w. Then $\bar{Y}_0 \to \bar{Z}_0$ is given by $y^p = t^n \bmod t^{n+1}$, locally around w, for some $n > 0$ which is prime-to-p. It follows that ω is a logarithmic differential which has a simple pole at w.

Let $w \in \bar{Z}_0$ be a point which does not map to x_b for $b \in \mathbb{B}_{\rm wild}$. Let t be a local parameter at w. Then the natural map $\bar{Y}_0 \to \bar{Z}_0$ is given by $y^p = 1 + t^n \bmod t^{n+1}$, locally around w, [19, p. 192]. It follows from the definition of ω that ω does not have a pole at w and that ω has a zero at w if and only if w is singular in $\bar{Z}$. $\square$

Definition 1.3 Let $\mathbb{E}_0 = \{e \in \mathbb{E} \mid s(e) = v_0\}$. Let $z_e \in \bar{Z}_0$ be a point above x_e. For $e \in \mathbb{E}_0$, the *conductor* of e is defined as

$$h_e = \mathrm{ord}_{z_e}(\omega) + 1.$$

Suppose that $e \in \mathbb{E}_0$. Let m_e be the ramification index of z_e in $\bar{Z}_0 \to \bar{X}_0$. Define

$$\nu_e := \left[\frac{h_e}{m_e}\right], \quad a_e := m\left(\frac{h_e}{m_e} - \nu_e\right).$$

Lemma 1.4 *Let $e \in \mathbb{E}_0$.*

(a) If the subtree of T with root $t(e)$ contains a new tail, then $\nu_e \geq 1$.

(b)

$$\sum_{e \in \mathbb{E}_0} \nu_e = |\mathbb{E}_0| - 2 - \frac{1}{m} \sum_{b \in \mathbb{B}} a_b.$$

Proof: The lemma is proved in [30]. Part (a) is a reformulation of [21, Proposition 3.3.5]. Part (b) follows from the vanishing cycle formula [21, Section 3.4.2]. $\square$

Theorem 1.5 *Let G be a group whose order is strictly divisible by p. Let $f : Y \to X = \mathbb{P}^1_K$ be a G-Galois cover over K branched only at $0, 1, \infty$. Suppose that f has bad reduction.*

(a) The cover f has multiplicative reduction.

(b) Every irreducible component of $\bar{X}$ other than $\bar{X}_0$ is a tail.

(c) Let $e \in \mathbb{E}_0$ with $t(e) = b \in \mathbb{B}$. Choose a point z_e of $\bar{Z}'_0$ above x_e. Then z_e is a ramification point of $\bar{Z}'_0 \to \bar{X}_0$. Write m_e for its ramification index. The ramification invariant σ_b of the tail X_b is equal to h_e/m_e.

Proof: [30]. $\square$

Remark 1.6 We use the notation of Theorem 1.5. Lemma 1.4 and Theorem 1.5 imply that $\nu_e = 1$ if $t(e) \in \mathbb{B}_{\rm new}$ and $\nu_e = 0$ if $t(e) \in \mathbb{B}_{\rm prim}$. Moreover, we have that $\sum_{b \in \mathbb{B}} a_b = m$. One easily checks from the proof of Lemma 1.2 that ψ^{a_b} is the canonical generator of the inertia group of z_e in $\bar{Z}'_0 \to \bar{X}_0$ (with respect to the character χ). In the terminology of Definition 2.1, we say that the *type* of $\bar{Z}'_0 \to \bar{X}_0$ is (x_e, a_b), where $e \in \mathbb{E}_0$ and $b = t(e)$.

2 The reduction of metacyclic covers

In this section, we study metacyclic covers of the projective line branched at four points. We start by defining the class of metacyclic covers we are interested in in this paper. We determine the different types of reduction that can occur. In Definition 2.8, we distinguish the three different cases that may occur. These play an important role in the rest of the paper.

We fix the following data.

- Let $p \neq 2$ be a prime number and $m > 1$ an integer such that $p \equiv 1 \bmod m$.
- Let R be a complete discrete valuation ring whose fraction field K has characteristic zero and whose residue field k is algebraically closed of characteristic p.
- Fix a character $\chi : \mathbb{Z}/m \to \mathbb{F}_p^\times$ of order m. Put $\zeta_m = \chi(1)$. We choose a lift $\chi_K : \mathbb{Z}/m \to K^\times$ of χ. We denote χ_K also by χ.
- Let N be an extension of $\mathbb{Z}/m$ by $\mathbb{Z}/p$. We fix once and for all generators ϕ and ψ of N. We suppose that $\phi^p = \psi^m = 1$ and $\psi^i \phi \psi^{-i} = \phi^{\chi(i)}$.

Definition 2.1 Let $\mathbf{x} = (x_1, \ldots x_r)$ be r distinct points of $X = \mathbb{P}^1_{\bar{K}}$. Let $\mathbf{a} = (a_1, \ldots, a_r)$ be an r-tuple of integers with $0 < a_i < m$ and $\sum a_i \equiv 0 \bmod m$. Let $f : Y \to X$ be an N-Galois cover. We say that f is *of type* $(\mathbf{x}; \mathbf{a})$ (with respect to χ) if the following holds.

(a) The cover f is branched only at $x_1, \ldots, x_r$.

(b) The element ψ^{a_i} is the canonical generator with respect to χ of some point y_i of Y above x_i. In other words, if u_i is a local parameter of y_i then
$$(\psi^{a_i})^* u_i \equiv \chi(1)^{(a_i, m)} u_i \pmod{u_i^2}.$$

In concrete terms this means the following. Let $f : Y \to X$ be a metacyclic cover of type $(\mathbf{x}, \mathbf{a})$ and suppose that none of the x_i is equal to ∞, for simplicity. Let Z be the quotient of Y by the normal subgroup of N of order p. Then $Y \to Z$ is étale and Z is the complete nonsingular curve given by the equation
$$z^m = \prod_i (x - x_i)^{a_i},$$
where $\psi^* z = \chi(1) z$. We also use the terminology 'of type $(\mathbf{x}; \mathbf{a})$' for the (unique) m-cyclic cover $Z \to X$.

Definition 2.2 Let $g : Z \to X$ be the m-cyclic cover of type $(\mathbf{x}; \mathbf{a})$. Write $V := \mathrm{Hom}(\pi_1(Z), \mathbb{Z}/p)$. Then $\mathbb{Z}/m$ acts on V and we may consider $V_\chi := \{\xi \in V \mid \psi\xi = \chi(1)\xi\}$. Suppose $f : Y \to X$ is a metacyclic cover of type $(\mathbf{x}; \mathbf{a})$. Then f factors through g. We define $\xi_f \in V$ as the element corresponding to the exact sequence
$$1 \to \pi_1(Y) \to \pi_1(Z) \to \mathrm{Gal}(Y, Z) = \mathbb{Z}/p \to 1.$$
One checks that $\psi \xi_f = \chi(1) \xi_f$ and therefore $\xi_f \in V_\chi$.

Lemma 2.3 *With notations as above:*
$$\dim_{\mathbb{F}_p} V_\chi = r - 2.$$

Proof: This follows from the main result of [7]. See also [27, Section 2.4]. □

Two metacyclic covers $f_i : Y_i \to X$ of type $(\mathbf{x}; \mathbf{a})$ are isomorphic if there exists an N-equivariant isomorphism $h : Y_1 \to Y_2$ such that $f_1 = f_2 \circ h$. Note that such h, if it exists, is unique, since the center of N is trivial. Two metacyclic covers f_1, f_2 of type $(\mathbf{x}; \mathbf{a})$ are isomorphic if and only if the corresponding elements $\xi_i \in V_\chi$ are in the same orbit under the action of $\mathbb{Z}/m$ on V_χ. Therefore Lemma 2.3 implies that, up to isomorphism, there are $(p^{r-2} - 1)/m$ metacyclic covers of X of type $(\mathbf{x}; \mathbf{a})$.

Since we suppose that $(X; \mathbf{x})$ has *good reduction*, the m-cyclic cover $Z \to X$ extends to an m-cyclic cover $Z_{0,R} \to X_{0,R}$ of smooth curves branched at the x_i. Let $\bar{Z}_0$ and $\bar{X}_0$ be the special fibers of $Z_{0,R}$ and $X_{0,R}$. Let $\bar{V} = \mathrm{Hom}(\pi_1(\bar{Z}_0), \mathbb{Z}/p)$ and let $\bar{V}_\chi$ be the χ-isotypical part of $\bar{V}$. Since every étale p-cyclic cover of $\bar{Z}_0$ can be uniquely lifted to an étale p-cyclic cover of Z, we obtain a canonical injection $\bar{V}_\chi \hookrightarrow V_\chi$. This proves the following criterion for good reduction.

Lemma 2.4 *Let $f : Y \to X$ be a metacyclic cover of type $(\mathbf{x}; \mathbf{a})$ and let $\xi \in V_\chi$ be the corresponding element. Then f has potentially good reduction if and only if $\xi \in \bar{V}_\chi$.*

Artin–Schreier Theory implies that there exists a canonical isomorphism $\bar{V} \simeq H^1(\bar{Z}, \mathcal{O}_{\bar{Z}})^F$, where F denotes the absolute Frobenius. Write

$$H^1(\bar{Z}_0, \mathcal{O}) = \oplus_{i=1}^{m-1} L_{\chi^i},$$

where $L_{\chi^i} = \{\xi | \psi\xi = \chi^i(1)\xi\}$. Then $\bar{V}_\chi = L_\chi^F$.

Lemma 2.5 *The dimension of L_χ as a k-vector space is $(a_1 + \cdots + a_r - m)/m$.*

Proof: [4, Lemma 4.3]. □

Note that $0 \leq \dim_k L_\chi \leq r - 2$. In fact, $\dim_k L_\chi + \dim_k L_{\chi^{-1}} = r - 2$. This gives an upper bound on the number of metacyclic covers of type $(\mathbf{x}; \mathbf{a})$ in characteristic p. If $\dim L_\chi < r-2$ there are strictly less N-covers of type $(\mathbf{x}; \mathbf{a})$ in characteristic p than in characteristic zero. In principle one can compute $\dim_{\mathbb{F}_p} \bar{V}_\chi$ by using the explicit basis for L_χ and the matrix of F with respect to this basis as computed in [4, Section 5].

Suppose that $f : Y \to X$ is a metacyclic cover of type $(\mathbf{x}; \mathbf{a})$ with bad reduction. Lemma 1.2 implies that ω is a regular differential on $\bar{Z}_0$, i.e. an element of $H^0(\bar{Z}_0, \Omega^1)$. The vector space $H^0(\bar{Z}_0, \Omega^1)$ is dual to $H^1(\bar{Z}_0, \mathcal{O}_{\bar{Z}_0})$. The transpose of the absolute Frobenius $F : H^1(\bar{Z}_0, \mathcal{O}_{\bar{Z}_0}) \to H^1(\bar{Z}_0, \mathcal{O}_{\bar{Z}_0})$ is called the Cartier operator $\mathcal{C} : H^0(\bar{Z}_0, \Omega^1) \to H^0(\bar{Z}_0, \Omega^1)$, [23]. An element $\omega \in H^0(\bar{Z}_0, \Omega^1)$ defines a $\boldsymbol{\mu}_p$-torsor if $\mathcal{C}\omega = \omega$ and an $\boldsymbol{\alpha}_p$-torsor if $\mathcal{C}\omega = 0$, [17, Proposition 4.14]. Let $\bar{V} = H^1(\bar{Z}_0, \mathcal{O}_{\bar{Z}_0})$ and $\bar{V}_\chi = \{\xi \in \bar{V} | \psi\xi = \chi(1)\xi\}$. Note that this definition coincides with the definition we gave in Section 2.

Define $\bar{M} = H^0(\bar{Z}_0, \Omega^1)^{\mathcal{C}}$ and let $\bar{M}_\chi = \{\omega \in \bar{M} \,|\, \psi\omega = \chi(1)\omega\}$. Note that $\bar{M}_\chi$ is an $\mathbb{F}_p[\mathbb{Z}/m]$-module. Since $\mathcal{C} : H^0(\bar{Z}_0, \Omega^1) \to H^0(\bar{Z}_0, \Omega^1)$ is the transpose of $F : H^1(\bar{Z}_0, \mathcal{O}_{\bar{Z}_0}) \to H^1(\bar{Z}_0, \mathcal{O}_{\bar{Z}_0})$, we conclude that

$$\bar{M}_\chi = \bar{V}_{\chi^{-1}}^{\mathrm{dual}},$$

where 'dual' means duality of $\mathbb{F}_p[\mathbb{Z}/m]$-modules.

Proposition 2.6 *There exists a surjective morphism of $\mathbb{F}_p[\mathbb{Z}/m]$-modules*

$$V_\chi \twoheadrightarrow \bar{M}_\chi. \tag{1}$$

Proof: Since $\bar{M}_\chi = \bar{V}_{\chi^{-1}}^{\mathrm{dual}}$, there is an inclusion $\bar{M}_\chi^{\mathrm{dual}} \hookrightarrow V_{\chi^{-1}}$. The Weil pairing defines a map

$$V_\chi \times V_{\chi^{-1}} \to \boldsymbol{\mu}_p.$$

The choice of an isomorphism $\mathbb{Z}/p \simeq \boldsymbol{\mu}_p(K)$ identifies therefore V_χ and $V_{\chi^{-1}}^{\mathrm{dual}}$. This defines (1). □

In more concrete terms we can describe the surjection of Proposition 2.6 as follows. Let $f : Y \to X$ be a metacyclic cover of type $(\mathbf{x}; \mathbf{a})$ and let ξ be the corresponding element of V_χ. If f has multiplicative reduction, then ξ maps to ω_0 in $\bar{M}_\chi$. Otherwise, ξ maps to zero. In case $H^0(\bar{Z}, \Omega_1)^{\mathcal{C}=0} = 0$ there are no metacyclic covers with additive reduction and the following sequence is exact:

$$0 \to \bar{V}_\chi \to V_\chi \to \bar{M}_\chi \to 0.$$

This happens, for example, when $\bar{Z}_0$ is ordinary. Namely, in that case $F : L_{\chi^i} \to L_{\chi^i}$ is a bijection for every i, in particular for $i = 1, -1$. Therefore $\dim_{\mathbb{F}_p} \bar{V}_\chi = \dim_k L_\chi$ and $\dim_{\mathbb{F}_p} \bar{M}_\chi = \dim_k L_{\chi^{-1}}$. Therefore $H^0(\bar{Z}, \Omega_\chi)^{\mathcal{C}=0} = 0$.

Suppose that $r = 4$. Define $a := \dim_{\mathbb{F}_p} \bar{V}_\chi = \dim_{\mathbb{F}_p} L_\chi^F$ and $b := \dim_{\mathbb{F}_p} \bar{M}_\chi = \dim_{\mathbb{F}_p} L_{\chi^{-1}}^F$. Note that $a, b \leq 2$. Lemma 2.5 implies that $\dim_k L_{\chi^i} = r - 2 = 2$, therefore $a, b \leq 2$. The following proposition lists the possibilities for (a, b).

Proposition 2.7 *Suppose $r = 4$. Then $(a, b) \in \{(2,0),(0,2),(1,1),(0,0)\}$.*

Proof: The $\mathbb{F}_p[\mathbb{Z}/m]$-module $\bar{V}$ is equal to $\mathrm{Hom}(\boldsymbol{\mu}_p, J(\bar{Z}_0)[p])$, where 'Hom' should be considered in the category of finite flat group schemes. The p-divisible group $J(\bar{Z}_0)[p^\infty]$ decomposes into eigenspaces of the automorphism ψ, since $\mathbb{Z}_p$ contains the mth roots of unity. Write

$$J(\bar{Z}_0)[p^\infty] = \oplus_i J(\bar{Z}_0)[p^\infty]_{\chi^i}$$

for this decomposition. We conclude that the finite flat group scheme $J(\bar{Z}_0)[p]_\chi = J(\bar{Z}_0)[p^\infty]_\chi/p$ is a direct summand of the group scheme $J(\bar{Z}_0)[p]$. Recall that there is a direct sum decomposition

$$J(\bar{Z}_0)[p] = (\mathbb{Z}/p)^\sigma \times (\boldsymbol{\mu}_p)^\sigma \times \Lambda,$$

where σ is the p-rank of $\bar{Z}_0$ and Λ is some local-local finite flat group scheme. Since $J(\bar{Z}_0)$ is an abelian variety, $J(\bar{Z})[p]$ does not have any direct summand $\boldsymbol{\alpha}_p$.

Since the dimension of $J(\bar{Z}_0)[p]_\chi$ is two, we conclude that it is isomorphic to either $(\mathbb{Z}/p)^2$, $(\boldsymbol{\mu}_p)^2$, $\mathbb{Z}/p \oplus \boldsymbol{\mu}_p$ or it is a local-local group scheme. The definition of a and b implies that

$$J(\bar{Z}_0)[p]_\chi \simeq (\mathbb{Z}/p)^b \times (\boldsymbol{\mu}_p)^a \times \Lambda_\chi,$$

where Λ_χ is local-local. This proves the proposition. $\square$

It is easy to check that all possibilities listed in Proposition 2.7 occur; we give explicit conditions for each possibility to occur further on.

Definition 2.8 Distinguish three cases.

- **The multiplicative case**: $a_1 + a_2 + a_3 + a_4 = m$.
- **the mixed case**: $a_1 + a_2 + a_3 + a_4 = 2m$.
- **the étale case**: $a_1 + a_2 + a_3 + a_4 = 3m$.

Proposition 2.9 *Suppose $a_1+a_2+a_3+a_4 = m$ and $\lambda \not\equiv 0, 1, \infty \bmod p$. Then all metacyclic covers of type $(\mathbf{x}; \mathbf{a})$ have multiplicative bad reduction. Therefore $(a, b) = (0, 2)$, for all $\lambda \in \mathbb{P}^1_k - \{0, 1, \infty\}$.*

Proof: The proposition is proved in [29, Proposition 1.3] for every $r \geq 3$. $\square$

Proposition 2.10 *Suppose $a_1 + a_2 + a_3 + a_4 = 2m$. Then $F : L_\chi \to L_\chi$ is an isomorphism, for sufficiently general $\bar{\lambda} \in k$.*

Proof: In fact, a similar statement holds without assumption on r and the type. In [4, lemma 4.6] it is shown that the locus U where $F : L_\chi \to L_\chi$ is an isomorphism is open. The fact that U is non-empty follows from [4, Proposition 7.4]. $\square$

Let $U \subset \mathbb{P}^1_k - \{0, 1, \infty\}$ be the locus where $F : L_\chi \to L_\chi$ is an isomorphism. In case $m = 2$, this U is the subset of $\lambda \in \mathbb{P}^1_k - \{0, 1, \infty\}$ such that the elliptic curve with Weierstrass equation $y^2 = x(x-1)(x-\lambda)$ is ordinary. Therefore, in the mixed case, U is in general strictly smaller than $\mathbb{P}^1_k - \{0, 1, \infty\}$, in contrast to what happens in the étale case (see below).

For $\lambda \in U$ we have $(a, b) = (1, 1)$. Lemma 2.4 implies that $(p-1)/m$ metacyclic cover of type $(\mathbf{x}; \mathbf{a})$ have good reduction and the other $p(p-1)/m$ have bad reduction. There are finitely many $\lambda \in \mathbb{P}^1_k$ for which $F : L_\chi \to L_\chi$ is the zero morphism. For these λ we have $(a, b) = (0, 0)$ and all metacyclic covers of type $(\mathbf{x}; \mathbf{a})$ have additive bad reduction.

Proposition 2.11 *Let $f : Y \to X$ be a metacyclic cover of type $(\mathbf{x}; \mathbf{a})$. Suppose that $a_1 + \cdots + a_4 = 3m$ and $\lambda \not\equiv 0, 1, \infty \bmod p$. Then f has good reduction and $(a, b) = (2, 0)$.*

Proof: Let f as in the statement of the proposition. The assumption $a_1 + \cdots + a_4 = 3m$ implies that $\dim_k L_\chi = 2$. Since $\dim_k L_\chi + \dim_k L_{\chi^{-1}} = r-2$, we conclude that $\dim_k L_{\chi^{-1}} = 0$. Therefore Proposition 2.9 implies that $V_{\chi^{-1}} = \bar{M}_{\chi^{-1}}$. By duality, we conclude that $V_\chi = \bar{V}_\chi$. Therefore f has good reduction. $\square$

3 Description of the Hurwitz space in characteristic zero

In this section, we define the Hurwitz space parameterizing metacyclic covers in characteristic zero.

Let G be a group whose order is strictly divisible by p. Let $R_0 = W(\bar{\mathbb{F}}_p)$ and K_0 its quotient field. Let $N \simeq \mathbb{Z}/p \rtimes \mathbb{Z}/m$ be as in Section 2. Recall that we have fixed generators ϕ and ψ of N which satisfy

$$\phi^p = \psi^m = 1, \qquad \psi\phi\psi^{-1} = \phi^{\chi(1)},$$

where $\chi : \mathbb{Z}/m \to \mathbb{F}_p^\times$ is an injective character. We represent the element $\phi^i\psi^j$ of N as $[i, \chi(1)^j]$, with $i \in \mathbb{F}_p$ and $\chi(1)^j \in \mathrm{Im}(\chi) \subset \mathbb{F}_p^\times$.

Fix a type $\mathbf{a}$ with $\gcd(m, a_1, \ldots, a_4) = 1$. The inverse image of $b \in \mathbb{Z}/m$ is a conjugacy class of N which we denote by $C(b)$. Let $\mathbf{C} = (C(a_1), C(a_2), C(a_3), C(a_4))$. Put $\zeta_i := \chi(1)^{a_i} \in \mathbb{F}_p^\times$. Let $\mathbb{Q}_p(\mathbf{a})$ be the smallest extension of $\mathbb{Q}_p$ over which $\mathbf{C}$ is rational, [24, Section 7.1].

Definition 3.1 Define $H(\mathbf{a}) = H_4^{\mathrm{in}}(\mathbf{a})/\mathbb{Q}_p(\mathbf{a})$ as the (inner) Hurwitz space parameterizing N-Galois covers $Y \to X = \mathbb{P}^1$ of type $(\mathbf{x}; \mathbf{a})$. Write $\pi(\mathbf{a}) : H(\mathbf{a}) \to \mathbb{P}^1_\lambda - \{0, 1, \infty\}$ for the cover defined by $[f] \mapsto \lambda$. We denote th Galois closure of $\pi(\mathbf{a})$ by $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1_\lambda$.

If $\mathbf{a}$ is understood, we sometimes drop the index $\mathbf{a}$ from the notation. For generalities on Hurwitz spaces, we refer to [31]. This Hurwitz space has also been considered in [3]. Berger considers types of the form $\mathbf{a} = (1, 1, 1, m-3)$. He calls the Hurwitz space $H(\mathbf{a})$ a "fake modular curve" and shows that it is a quotient of the complex upper half-plane by a non-congruence subgroup.

Lemma 3.2 *The degree of* $\pi \otimes \bar{\mathbb{Q}}_p$ *is* $(p^2-1)/m$.

Proof: Let $f : Y \to X$ be an N-cover of type $(\mathbf{x}; \mathbf{a})$ and $Z \to X$ the intermediate m-cyclic cover. The étale p-cyclic cover $Y \to Z$ corresponds to a nonzero element ξ in $V_\chi \simeq (\mathbb{Z}/p)^2$, Lemma 2.3. Two elements ξ_1 and ξ_2 define the same cover if $\psi^i(\xi_1) = \xi_2$, for some i. $\square$

One can describe describe $\bar{H} \otimes \bar{\mathbb{Q}}_p$ using Nielsen tuples, see for example [28, Section 10.1.7]. Define

$$\mathcal{E}(\mathbf{a}) = \{\mathbf{g} = (g_1, g_2, g_3, g_4) |\, g_i \in C(a_i),\ N = \langle g_i \rangle,\ \prod_i g_i = 1\}.$$

The Nielsen class is equal to

$$\mathrm{Ni}_4^{\mathrm{in}}(\mathbf{a}) = \mathcal{E}(\mathbf{a})/N,$$

where the group N acts via uniform conjugation. The elements of this set are called the *Nielsen tuples*. Choose a presentation of the fundamental group

$$\pi_1(\mathbb{P}^1 - \mathbf{x}, *) = \langle \gamma_1, \ldots, \gamma_4 |\, \prod_i \gamma_i = 1 \rangle,$$

where γ_i corresponds to a "loop" around the point x_i, as usual. One can identify the Nielsen tuples with isomorphism classes of surjective homomorphisms

$$\pi_1(\mathbb{P}^1 - \mathbf{x}, *) \to N, \qquad \mathrm{Im}(\gamma_i) \in C(a_i).$$

We also consider $\mathcal{E}^*(\mathbf{a}) := \mathcal{E}(\mathbf{a})/\mathbb{F}_p$, where $\mathbb{F}_p \subset N$ acts via uniform conjugacy.

Choose a base point $\lambda_0 \in \mathbb{P}^1 - \{0,1,\infty\}$. The cover $\pi : H \to \mathbb{P}^1_\lambda$ defines an action of $\Pi := \pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \lambda_0)$ on $\pi^{-1}(\lambda_0)$. Let t be a transcendental and $K = \bar{\mathbb{Q}}_p(t)$. Choose a K-model $Z \to \mathbb{P}^1_K$ of the m-cyclic cover of type $(\mathbf{x};\mathbf{a})$, where $x_4 = t$. Define $W := J_Z[p]_\chi(\bar{K}) \simeq (\mathbb{F}_p)^2$. This defines the *monodromy representation*

$$\rho : \mathrm{Gal}(\bar{K}, K) \to \mathrm{GL}(W) \simeq \mathrm{GL}_2(p). \tag{2}$$

Let Γ be the image of the monodromy representation. The first goal of this section is to describe Γ. These results are due to Völklein [26], [27]. We repeat the proofs as our assumptions are somewhat different.

The action of Π on $\pi(\mathbf{a})^{-1}(\lambda_0)$ can be described via the action of the *Artin braid group* $\mathcal{B}_4$. Recall that $\mathcal{B}_4$ is generated by the standard generators Q_1, Q_2, Q_3 which act on Nielsen tuples via:

$$\mathbf{g}Q_i = (g_1, \dots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \dots, g_r).$$

We follow here the convention of [27]. In some other texts the inverses of the Q_i are used; this does not make much difference, [28, p. 167].

The *pure Artin braid group* $\mathcal{B}^{(4)}$ is the kernel of the map $\mathcal{B}_4 \to S_4$, sending Q_i to the transposition $(i, i+1)$. The pure Artin braid group $\mathcal{B}^{(4)}$ acts naturally on the set $\mathcal{E}^*(\mathbf{a})$. The fundamental group Π can be embedded into the Hurwitz braid group which is a quotient of the Artin braid group. The action of Π on the Nielsen class $\mathrm{Ni}_4^{\mathrm{in}}(\mathbf{a})$ is induced from the action of $\mathcal{B}^{(4)}$ on $\mathcal{E}^*(\mathbf{a})$. We refer to [27] for details on braid groups.

The pure Artin braid group is generated by the braids

$$b_1 = Q_3 Q_2 Q_1^2 Q_2^{-1} Q_3^{-1}, \quad b_2 = Q_3 Q_2^2 Q_3^{-1}, \quad b_3 = Q_3^2. \tag{3}$$

Write $g^h = h^{-1} g h$ and $[g,h] = g^{-1} h^{-1} g h$. One checks that

$$(g_1, g_2, g_3, g_4) b_i = \begin{cases} (g_1^{g_4}, g_2^{[g_1,g_4]}, g_3^{[g_1,g_4]}, g_4^{g_1 g_4}) & \text{if } i = 1, \\ (g_1, g_2^{g_4}, g_3^{[g_2,g_4]}, g_4^{g_2 g_4}) & \text{if } i = 2, \\ (g_1, g_2, g_3^{g_4}, g_4^{g_3 g_4}) & \text{if } i = 3. \end{cases} \tag{4}$$

We start by studying the action of $\mathcal{B}^{(4)}$ on the set $\mathcal{E}^*(\mathbf{a})$. The elements of $\mathcal{E}^*(\mathbf{a})$ may be represented as

$$([0,\zeta_1], [v_1,\zeta_2], [v_2,\zeta_3], [v_3,\zeta_4]), \quad v_i \in \mathbb{F}_p, \tag{5}$$

where $v_3 = -(\zeta_2\zeta_3)^{-1} v_1 - \zeta_3^{-1} v_2$. Recall that $\zeta_i = \chi(1)^{a_i}$ and therefore $\zeta_1\zeta_2\zeta_3\zeta_4 = 1$. We may identify a tuple (5) with a vector $\mathbf{v} = (v_1, v_2) \in W := J(Z)[p]_\chi \simeq (\mathbb{F}_p)^2$. Here $Z = Z_\lambda$ is the m-cyclic cover of type $(\mathbf{x};\mathbf{a})$. Note that $\mathbf{v} \in W$ corresponds to a tuple in $\mathcal{E}^*(\mathbf{a})$ if and only if $\mathbf{v} \neq 0$. The action of $\mathcal{B}^{(4)}$ on $\mathcal{E}^*(\mathbf{a})$ induces an action on W.

Proposition 3.3 *The image in* $\mathrm{GL}(W)$ *of the action of* $\mathcal{B}^{(4)}$ *on* W *is equal to* Γ.

Proof: [27, Theorem A] □

This proposition gives us a concrete way to compute the image of the monodromy action.

For $i = 1,2,3$, define $B_i \in \mathrm{GL}_2(p)$ as the matrix corresponding to the braid b_i. As in [26, Lemma 3], one computes that

$$B_1 = \begin{pmatrix} \zeta_2 & -1+\zeta_3 \\ \zeta_2(-1+\zeta_2) & 1-\zeta_2+\zeta_2\zeta_3 \end{pmatrix}, \tag{6}$$

$$B_2 = \begin{pmatrix} \zeta_1(1-\zeta_2+\zeta_2\zeta_3) & \zeta_2^{-1}(-1+\zeta_3)(-1+\zeta_1-\zeta_1\zeta_2+\zeta_1\zeta_2\zeta_3) \\ \zeta_1\zeta_2(1-\zeta_2) & 1-\zeta_1+\zeta_1\zeta_2+\zeta_1\zeta_3-\zeta_1\zeta_2\zeta_3 \end{pmatrix}, \tag{7}$$

$$B_3 = \begin{pmatrix} 1 & \zeta_1(1-\zeta_3) \\ 0 & \zeta_1\zeta_2 \end{pmatrix}. \tag{8}$$

Let $\bar{\Gamma}$ be the image of Γ under the projection $\mathrm{GL}_2(p) \to \mathrm{PGL}_2(p)$. We show that $\bar{\Gamma}$ is either a group of order prime-to-p or $\mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$.

Definition 3.4 If $\bar{\Gamma} \neq \mathrm{PSL}_2(p), \mathrm{PGL}_2(p)$, we say that $\mathbf{a}$ is *exceptional.*

Proposition 3.5 *Suppose that* $p > 5$.

(a) *Suppose that m is even and, after permuting the branch points, we have $\mathbf{a} = (a, a, -a + m/2, -a + m/2)$. Then Γ is a dihedral group of order prime-to-p.*

(b) *Suppose that $\mathbf{a}$ is exceptional, but not as in (a). The order of $\zeta_i\zeta_j$ in $\mathbb{F}_p^\times$ is either 2,3,4,5, for all $i \neq j$. If $\mathbf{a}$ is non-exceptional, Γ contains $\mathrm{SL}_2(p)$.*

(c) *Suppose $\mathbf{a}$ is non-exceptional. The image of $\det : \Gamma \to \mathbb{F}_p^\times$ is generated by $\zeta_1\zeta_2, \zeta_1\zeta_3, \zeta_2\zeta_3$.*

(d) *Suppose $\mathbf{a}$ is non-exceptional. The Hurwitz space H is connected.*

Proof: Part (a) follows easily from the expression for the matrices B_i, cf. [26, Section 1.7].

Suppose that no permutation of $\mathbf{a}$ is equal to $(a, a, -a+m/2, -a+m/2)$. Recall that a subgroup of $\mathrm{GL}_2(W)$ is called *primitive* if it is irreducible and does not permute the summands in any non-trivial direct sum decomposition of W. It is proved in [26, Section 1.7] that Γ acts primitively on W. We may regard Γ as a subgroup of $\mathrm{PSL}_2(p^2)$. Dickson's classification of the subgroups of $\mathrm{PSL}_2(p^2)$, [11, Theorem 8.27], implies that $\bar{\Gamma}$ is isomorphic to either $A_4, S_4, A_5, \mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$. Therefore, if $\mathbf{a}$ is non-exceptional, Γ contains $\mathrm{SL}_2(p)$.

Suppose that $\mathbf{a}$ is exceptional. Then the matrices B_1, B_2, B_3 have order less than or equal to 5. One checks that there exists vectors $w_i, w_i' \in W$ which are eigenvectors of B_i with eigenvalue 1 and $(\zeta_i\zeta_4)^{-1}$. If $\zeta_i\zeta_j = 1$ for some $i \neq j$, then $\bar{\Gamma}$ contains an element of order $p > 5$ which is impossible. This proves (b).

Suppose that $\mathbf{a}$ is non-exceptional. To prove (c), note that the index of $\mathrm{SL}_2(p)$ in Γ is equal to the order of the image of $\det : \Gamma \to \mathbb{F}_p^\times$. The subgroup $\det(\Gamma)$ of $\mathbb{F}_p^\times$ is generated by the determinant of the matrices B_i for $i = 1, 2, 3$. We have

$$\det(B_i) = \begin{cases} \zeta_2\zeta_3 & \text{if } i = 1, \\ \zeta_1\zeta_3 & \text{if } i = 2, \\ \zeta_1\zeta_2 & \text{if } i = 3. \end{cases}$$

Recall that we have identified $\mathcal{E}^*(\mathbf{a})$ with the set of $\mathbf{v} \in W - \{0\}$. Since Γ contains $\mathrm{SL}_2(p)$, it follows that Γ acts transitively on $\mathcal{E}^*(\mathbf{a})$. But this implies that G acts transitively on $\mathrm{Ni}_4^{\mathrm{in}}(\mathbf{C}) = \mathcal{E}^*(\mathbf{a})/(\mathbb{Z}/m)$ and therefore that the Hurwitz space is connected. $\square$

It is not clear for which types the group $\bar{\Gamma}$ is either A_4, S_4 or A_5. The necessary condition of Proposition 3.5 is certainly not sufficient. However, for us this is not important since in the exceptional case p does not divide the order of the Galois group G, since $p > 5$. Therefore, the Galois closure of $H(\mathbf{a}) \to \mathbb{P}^1_\lambda$ has good reduction to characteristic p. If $\sum a_i = 2m$ we show in Section 4 that the Galois closure of $H(\mathbf{a}) \to \mathbb{P}^1_\lambda$ has bad reduction and we are not in the exceptional case.

Write $M := \langle \chi(1)I \rangle$ for the m-cyclic subgroup of the center of $\mathrm{GL}_2(p)$. If $\mathbf{a}$ is non-exceptional, then $\mathrm{SL}_2(p) \subset \Gamma \subset \mathrm{GL}_2(p)$. Therefore the index of $\mathrm{SL}_2(p)$ in Γ is equal to the order of $\det(\Gamma)$. It follows from Proposition 3.5.(c) that $\det(\Gamma)$ is a subgroup of $\mathrm{Im}(\chi) \simeq \mathbb{Z}/m \subset \mathbb{F}_p^\times$.

Corollary 3.6 *Suppose that $\mathbf{a}$ is non-exceptional. Let G be the Galois group of the Galois closure of $\pi(\mathbf{a})$. Then*

$$G \simeq \begin{cases} \mathrm{SL}_2(p) & \text{if } m \text{ is odd,} \\ \mathrm{PSL}_2(p) & \text{if } m \text{ if even and } \det(\Gamma) \subset \det(M), \\ \mathrm{PGL}_2(p) & \text{if } \det(\Gamma) \not\subset \det(M). \end{cases}$$

Proof: Note that the action of $\psi^i \in \mathbb{Z}/m$ on $\mathcal{E}^*(\mathbf{a})$ corresponds to multiplication by the matrix

$$\begin{pmatrix} \chi(1)^i & 0 \\ 0 & \chi(1)^i \end{pmatrix}$$

on W. Therefore G is the image of Γ under the projection $\Gamma M \to \Gamma M/M$.

It follows from Proposition 3.5(b) that $\det(\Gamma) \subset \mathrm{Im}(\chi) \simeq \mathbb{Z}/m \subset \mathbb{F}_p^\times$. We first suppose that m is odd. Then $\mathbb{Z}/m$ does not contain an element of order 2 and $-I \notin M$. This implies that $\det : M \to \mathbb{F}_p^\times$ is injective and $\det(M) \simeq \mathbb{Z}/m$. Therefore $\det(\Gamma) \subset \det(M)$. We conclude that the image G of Γ in $\Gamma M \to \Gamma M/M$ is equal to the image of $\mathrm{SL}_2(p)$ in $\Gamma M \to \Gamma M/M$. Since $-I \notin \Gamma$, we have $G \simeq \mathrm{SL}_2(p)$.

Suppose that m is even. Then $-I \in M$. If $\det(\Gamma) \subset \det(M)$, then G is isomorphic to the image of $\mathrm{SL}_2(p)$ in $\Gamma M/M$. Since $-I \in M$, we conclude that $G \simeq \mathrm{PSL}_2(p)$.

Suppose that $\det(\Gamma)$ is not contained in $\det(M)$. Then M contains $-I$. The image of Γ in $\Gamma M/M$ is strictly larger than the image of $\mathrm{SL}_2(p)$ in $\Gamma M/M$. We conclude that $G \simeq \mathrm{PGL}_2(p)$. □

Remark 3.7 We denote the Galois closure of $\pi(\mathbf{a})$ by $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1$. Following [27], we can give a modular interpretation for $\mathbb{H} = \mathbb{H}(\mathbf{a})$. We explain the definition and refer to [27] for details.

Let $\mathcal{N} := W \rtimes \mathbb{Z}/m$, where $\mathbb{Z}/m$ acts on W via χ. Let $\mathcal{H}$ be the Hurwitz space over $\bar{\mathbb{Q}}_p$ parameterizing tuples (f, h, η), up to isomorphism. Here $f : Y \to \mathbb{P}^1$ is an $\mathcal{N}$-Galois cover of type $(\mathbf{x}; \mathbf{a})$ with $x_1 = 0$, $x_2 = 1$, $x_3 = \infty$, $x_4 = \lambda$ and λ different from the base point λ_0 and $h : \mathrm{Aut}(Y, \mathbb{P}^1) \to \mathcal{N}$ is an isomorphism. Moreover, η is an point of Y above λ_0. We have a natural map $\mathcal{H} \to \mathbb{P}^1_\lambda$, which sends (f, h, η) to λ.

There is a natural action of W on $\mathcal{H}$. Therefore we can define $\mathcal{H}^{(W)}$ as the quotient of $\mathcal{H}$ by W. Let $\mathbb{H}^{(W)}$ be a connected component of $\mathcal{H}^{(W)}$. It follows from the results of [27] that $\varpi^{(W)} : \mathbb{H}^{(W)} \to \mathbb{P}^1_\lambda$ is a Γ-Galois cover. The Galois closure ϖ of π is the quotient of $\varpi^{(W)}$ by $M \cap \Gamma$. We do not need this in the rest of the paper.

Our next goal is to describe the ramification of $\pi(\mathbf{a}) : H(\mathbf{a}) \to \mathbb{P}^1_\lambda$. Write $x_1 = 0, x_2 = 1, x_3 = \infty \in \mathbb{P}^1_\lambda$. Let $i \in \{1, 2, 3\}$. The points $\pi(\mathbf{a})^{-1}(\{x_i\})$ are called the *cusps* of $\pi(\mathbf{a})$. By [6, Section 4.2.2], there is a 1-1 correspondence between cusps above x_i for $i \in \{1, 2, 3\}$ and elements of

$$\mathrm{Ni}_4^{\mathrm{in}}(\mathbf{a})/\langle b_i \rangle,$$

where the braids b_i defined in (3) act as in (4). Define $d_i = m/\gcd(a_i + a_4, m)$ if $a_i + a_4 \not\equiv 0 \bmod m$ and $d_i = p$ otherwise.

Proposition 3.8 *The signature of the Galois closure $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1$ of $\pi(\mathbf{a})$ is (d_1, d_2, d_3).*

Proof: Choose a cusp c of $H(\mathbf{a})$ above x_i corresponding to $\mathbf{g} = (g_1, g_2, g_3, g_4)$. For simplicity we suppose that $i = 3$, i.e. $x_i = \infty$. The ramification index of the cusp c in $\pi(\mathbf{a}) : H(\mathbf{a}) \to \mathbb{P}^1$ is equal to the length of the orbit of $\mathbf{g}$ under b_3. It follows from the definition of the action of b_i that the ramification index of c divides the order of $g_3 g_4$. Let $\mathbf{g}' = (\psi^{a_1}, \psi^{a_2}, \phi\psi^{a_3}, g_4')$, where g_4' is chosen such that the product of the g_i' is one.

We distinguish two cases. First suppose that $a_3 + a_4 = m$. Then for every Nielsen tuple $\mathbf{g}$, the order of $g_3 g_4$ divides p. One easily checks that the orbit of $\mathbf{g}'$ has length p. This implies that the ramification index of ∞ in $\varpi(\mathbf{a})$ is p.

Now suppose that $a_3 + a_4 < m$. Then the order of $g_3 g_4$ divides $d_3 m/\gcd(a_3 + a_4, m)$. Since the orbit of $\mathbf{g}'$ has length d_3, we conclude that ∞ is ramified in $\varpi(\mathbf{a})$ of order d_3. □

4 A good reduction theorem

Let $\mathbf{a} = (a_1, a_2, a_3, a_4)$ be a type. To $\mathbf{a}$ we associated in Section 3 a Galois cover $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1$. In this section we determine when $\varpi(\mathbf{a})$ has good reduction.

Let $K = \bar{\mathbb{Q}}_p(t)$, where t is a transcendental element. Denote by G_K the absolute Galois group of K. Recall from Proposition 3.3 that the monodromy representation $\rho : G_K \to \mathrm{GL}_2(p)$ has image Γ. Let L'/K be the corresponding Γ-Galois extension. Recall that the Galois group G of $\varpi(\mathbf{a})$ is the quotient of Γ by the (central) subgroup $\Gamma \cap M$ of Γ. Let L be the sub field of L of invariants under $\Gamma \cap M$. Denote the Gauß-valuation of K by ν_0; this valuation corresponds to the original component of the stable model of $\varpi(\mathbf{a})$.

Lemma 4.1 *The cover $\varpi(\mathbf{a})$ has good reduction if and only if ν_0 is unramified in L/K.*

Proof: The forward implication is obvious. Suppose that ν_0 is unramified in L/K. Then the restriction of the stable model of $\varpi(\mathbf{a})$ to the original component is separable. It is well known that this implies that $\varpi(\mathbf{a})$ has good reduction, [6, Proposition 1.1.4]. □

Theorem 4.2 *The cover $\varpi(\mathbf{a})$ has good reduction if and only if $a_1 + a_2 + a_3 + a_4 \neq 2m$.*

Proof: Let R be the ring of integers of the completion of K at ν_0. Let k be the residue field of R. Denote by $Z_{0,R}$ the m-cyclic cover of $\mathbb{P}^1_R$ of type $\mathbf{a}$ branched at 0, 1, ∞, t. Let $\bar{Z}_0$ be the special fiber of $Z_{0,R}$. There is an exact sequence of group schemes

$$1 \to J_{\bar{Z}_0}[p]_\chi^{\mathrm{loc}} \to J_{\bar{Z}_0}[p]_\chi \to J_{\bar{Z}_0}[p]_\chi^{\mathrm{et}} \to 1. \tag{9}$$

Suppose that $\sum_i a_i = m$, i.e. we are in the multiplicative case of Definition 2.8. It follows from Proposition 2.9 that $J_{Z_{0,R}}[p]_\chi$ is étale. This implies that the action of Γ on $J_{Z_{0,R}}[p]_\chi$ is unramified, since R is a Henselian local ring. Lemma 4.1 implies therefore that $\varpi(\mathbf{a})$ has good reduction.

Suppose that $\sum_i a_i = 3m$, i.e. we are in the étale case. The group scheme $J_{Z_{0,R}}[p]_\chi$ is dual to $J_{Z_{0,R}}[p]_{\chi^{-1}}$. But $J_{Z_{0,R}}[p]_{\chi^{-1}} \simeq J_{Z^*_{0,R}}[p]_\chi$, where $Z^*_{0,R}$ is the m-cyclic cover of type $\mathbf{a}^* = (m - a_1, m - a_2, m - a_3, m - a_4)$ branched at 0, 1, ∞, t. Since $\sum_i (m - a_i) = m$, we may apply the theorem in the multiplicative case to $\mathbf{a}^*$. The theorem in the étale case follows by duality.

Suppose that $\sum_i a_i = 2m$, i.e. we are in the mixed case. Since the fourth branch point $x_4 = t$ is generic, it follows from Proposition 2.10 that both $J_{\bar{Z}_0}[p]_\chi^{\mathrm{loc}}$ and $J_{\bar{Z}_0}[p]_\chi^{\mathrm{et}}$ have rank p. Choose a valuation ν' of L' above ν_0 and write $\Gamma_{\nu'}$ for its decomposition group. Suppose that $\varpi(\mathbf{a})$ has good reduction. The Γ-Galois cover $\varpi'(\mathbf{a})$ corresponding to L'/K has good reduction also. (Here we use that the index of G in Γ is prime-to-p.) We conclude that Γ is equal to $\Gamma_{\nu'}$ and that Γ acts faithfully on the stable reduction of $\varpi'(\mathbf{a})$. In particular, Γ acts transitively on the subgroups of $J_{\bar{Z}_0}[p]_\chi$. But the action of Γ on $J_{\bar{Z}_0}[p]_\chi$ respects the splitting (9) of $J_{\bar{Z}_0}[p]_\chi$ into its local and étale parts. This gives a contradiction. □

5 The a-Hasse invariant

In Sections 5 and 6, we suppose that $\sum_i a_i = 2m$, i.e. $\varpi(\mathbf{a})$ has bad reduction. Our goal is to compute the stable reduction of $\varpi(\mathbf{a})$. In this section we define the $\mathbf{a}$-Hasse invariant $\Phi_{\mathbf{a}}$. This is a generalization if the classical Hasse invariant for elliptic curves, cf. [14, Section 12.4]. The $\mathbf{a}$-Hasse invariant satisfies a hypergeometric differential equation, just as the classical Hasse invariant. In Section 6 we show that $\Phi_{\mathbf{a}}$ essentially determines the stable reduction.

Let $\mathbf{a}$ be a type such that $a_1 + a_2 + a_3 + a_4 = 2m$. Let $\bar{Z}_0$ be the m-cyclic cover of $\mathbb{P}^1_k$ branched at 0, 1, ∞, λ of type $\mathbf{a}$. Since we are in the mixed case, Lemma 2.5 implies that the χ-isotypical part $H^0(\bar{Z}_0, \Omega)_\chi$ has k-dimension one. Recall that $\mathcal{C} : H^0(\bar{Z}_0, \Omega)_\chi \to H^0(\bar{Z}_0, \Omega)_\chi$ is a bijection for all but finitely many λ, Proposition 2.10.

Definition 5.1 The $\bar{\lambda}$ for which $\mathcal{C}: H^0(\bar{Z}_0, \Omega)_\chi \to H^0(\bar{Z}_0, \Omega)_\chi$ is a bijection are called **a**-*ordinary*. If $\bar{\lambda}$ is not **a**-ordinary it is called **a**-*supersingular*. Write $\Lambda(\mathbf{a})$ for the set of **a**-supersingular λ-values.

For $m = 2$, the only possible type is $(1,1,1,1)$. In this case $\bar{Z}_0$ is an elliptic curve and λ is **a**-supersingular if and only if $\bar{Z}_0$ is supersingular.

Now suppose that $\lambda = t$ corresponds to the generic point of $\mathbb{P}^1_k$. Then $\bar{Z}_0$ is given by an equation $z^m = x^{a_1}(x-1)^{a_2}(x-\lambda)^{a_4}$. Define

$$\omega = \frac{z\,\mathrm{d}x}{x(x-1)(x-\lambda)} \in H^0(\bar{Z}_0, \Omega^1)_\chi.$$

Lemma 2.5 implies that $\dim_k H^0(\bar{Z}_0, \Omega^1)_\chi = \dim_k H^1(\bar{Z}_0, \mathcal{O}_{\bar{Z}_0})_{\chi^{-1}} = 1$. Therefore ω is a basis of $H^0(\bar{Z}_0, \Omega^1)_\chi$.

Definition 5.2 Suppose that $a_1 + a_2 + a_3 + a_4 = 2m$. Define the **a**-Hasse invariant as

$$\mathcal{C}\omega = \Phi_{\mathbf{a}}(\lambda)^{(1/p)}\omega.$$

Lemma 5.3 *Write $a_i^* = m - a_i$ and $\alpha = (p-1)/m$. Then*

$$\Phi_{\mathbf{a}}(\lambda) = (-1)^{\alpha a_3} \sum_{i+j=\alpha a_3} \binom{\alpha a_2^*}{i} \binom{\alpha a_4^*}{j} \lambda^j. \tag{10}$$

Proof: Write

$$\omega = \frac{z^p}{x^p(x-1)^p(x-\lambda)^p} x^{p-\alpha a_1}(x-1)^{p-1-\alpha a_2}(x-\lambda)^{p-1-\alpha a_4}\frac{\mathrm{d}x}{x}$$

and

$$e := x^{p-\alpha a_1}(x-1)^{p-1-\alpha a_2}(x-\lambda)^{p-1-\alpha a_4} = x^{1+\alpha a_1^*}(x-1)^{\alpha a_2^*}(x-\lambda)^{\alpha a_4^*} = \sum_i e_i x^i.$$

It follows from standard properties of the Cartier operator that

$$\mathcal{C}\omega = \frac{z}{x(x-1)(x-\lambda)}\left(\sum_i e_{pi}^{1/p} x^i\right)\frac{\mathrm{d}x}{x},$$

[23]. Note that $\deg e = 1 + \alpha(a_1^* + a_2^* + a_4^*) < 1 + \alpha(a_1^* + a_2^* + a_3^* + a_4^*) = 1 + 2\alpha m = 2p-1$ and $\deg(e) \geq 1 + \alpha(a_1^* + a_2^* + a_3^* + a_4^*) - \alpha m = 1 + \alpha m = p$. Therefore

$$\mathcal{C}\omega = \frac{z(e_p^{1/p}x)}{x(x-1)(x-\lambda)}\frac{\mathrm{d}x}{x} = e_p^{1/p}\omega.$$

One easily checks that e_p is equal to the right hand side of (10). $\square$

We find back the classical Hasse invariant Φ for $m = 2$ and $\mathbf{a} = (1,1,1,1)$. It is well known that Φ is the solution to a hypergeometric differential equation. The same holds for the **a**-Hasse invariant.

Proposition 5.4 *The polynomial $\Phi_{\mathbf{a}}(\lambda)$ is a solution to the hypergeometric differential equation*

$$\lambda(1-\lambda)u'' + [C - (A+B+1)\lambda]u' - ABu = 0, \tag{11}$$

where $A \equiv -\alpha a_3 \bmod p$ and $B \equiv \alpha(a_4 - m) \bmod p$ and $C \equiv -\alpha(a_2 + a_3) \bmod p$.

Proof: Write $\Phi_{\mathbf{a}}(\lambda) = \sum_j \varphi_j \lambda^j$. To prove the proposition, we have to show that the coefficients φ_j satisfy the recursion relation $\varphi_{j+1}/\varphi_j = (A+j)(B+j)/(C+j)(1+j)$, [32, Section III.2]. Here j and $j+1$ are in the range where the coefficients are not congruent to zero. Obviously,

$$\frac{\varphi_{j+1}}{\varphi_j} = \frac{(-\alpha a_3 + j)(-\alpha a_4^* + j)}{(-\alpha(a_2+a_3)+j)(j+1)} \equiv \frac{(A+j)(B+j)}{(C+j)(1+j)} \pmod{p},$$

with A, B, C as in the statement of the proposition. □

Corollary 5.5 *(a) The zeros of $\Phi_{\mathbf{a}}(\lambda)$ different from 0 and 1 are simple.*

(b) At $\lambda = 0$, the polynomial $\Phi_{\mathbf{a}}(\lambda)$ has a zero of order $\max(\alpha(a_2+a_3-m), 0)$.

(c) At $\lambda = 1$, the polynomial $\Phi_{\mathbf{a}}(\lambda)$ has a zero of order $\max(\alpha(a_1+a_3-m), 0)$.

Proof: Part (a) follows immediately from Proposition 5.4 as the hypergeometric differential equation (11) has essential singularities only at $0, 1, \infty$. Part (b) is obvious. Part (c) follows from Part(b) by interchanging the branch points x_1 and x_2. □

Corollary 5.6 *The number of* $\mathbf{a}$*-supersingular* λ*-values is*

$$\min(\alpha a_4^*, \alpha a_3) - \max(\alpha(a_2+a_3-m), 0) - \max(\alpha(a_1+a_3-m), 0).$$

Proof: The degree of $\Phi_{\mathbf{a}}$ is equal to $\min(\alpha a_4^*, \alpha a_3)$. Therefore the corollary follows from Corollary 5.5. One easily checks from the formula that the number of $\mathbf{a}$-supersingular λ-values is non-zero. □

6 Computation of the deformation datum

The goal of this section is to describe the stable reduction of $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1$ By Theorem 1.5, this amounts to computing the deformation datum associated to $\varpi(\mathbf{a})$.

The notation is as in Section 4. In particular, we suppose that $\sum_i a_i = 2m$. Let k be an algebraically closed field of characteristic p. We denote by P the Sylow p-subgroup of Γ consisting of upper triangular matrices with ones on the diagonal. Choose a valuation ν of L' above ν_0 such that the inertia group I_ν of ν is P. This is possible as $\varpi(\mathbf{a})$ has bad reduction by Theorem 4.2. Let Γ_ν be the decomposition group of ν. By our assumption on ν, the group Γ_ν is a subgroup of the Borel group $\mathcal{B}$ of $\mathrm{GL}_2(p)$ consisting of upper triangular matrices. Define

$$\Gamma_\nu^{\mathrm{res}} = \Gamma_\nu / I_\nu.$$

The monodromy representation (2) induces a representation

$$\rho^{\mathrm{res}} : \Gamma_\nu^{\mathrm{res}} \to \mathcal{B}/P = \mathbb{F}_p^\times \times \mathbb{F}_p^\times.$$

We may identify ρ^{res} with two characters $\xi_i : \Gamma_\nu^{\mathrm{res}} \to \mathbb{F}_p^\times$, where ξ_i is the composition of ρ^{res} with the ith projection.

Proposition 6.1 *Denote by $G_{k(t)}$ the absolute Galois group of $k(t)$. Under the natural identification*

$$\begin{array}{rcl} k(t)^\times/(k(t)^\times)^{p-1} & \simeq & \mathrm{Hom}(G_{k(t)}, \boldsymbol{\mu}_{p-1}) \\ u & \mapsto & \left[\sigma \mapsto \frac{\sigma(\sqrt[p-1]{u})}{\sqrt[p-1]{u}}\right] \end{array}$$

the character ξ_2 (resp. ξ_1) corresponds to $\Phi_{\mathbf{a}}(t)$ (resp. $\Phi_{\mathbf{a}^*}(t)^{-1}$). Here $\Phi_{\mathbf{a}}$ is the $\mathbf{a}$*-Hasse invariant and $\mathbf{a}^* := (m-a_1, m-a_2, m-a_3, m-a_4)$ is the dual type.*

Proof: Let $\mathbf{x} = (0, 1, \infty, t)$ and write $\bar{Z}_0$ for the m-cyclic cover of type $(\mathbf{x}; \mathbf{a})$ over $k(t)$. It follows from (9) that the character ξ_2 (resp. ξ_1) corresponds to the action of $G_{k(t)}$ on $J_{\bar{Z}_0}[p]^{\rm et}_\chi$ (resp. $J_{\bar{Z}_0}[p]^{\rm loc}_\chi$).

We start by computing ξ_2. For this we use the canonical identification

$$J_{\bar{Z}_0}[p]^{\rm et}_\chi \simeq H^0(\bar{Z}_0, \Omega)^{\mathcal{C}}_\chi,$$

as in Section 2. The differential

$$\omega := \frac{z \,\mathrm{d}x}{x(x-1)(x-t)}$$

is a basis of $H^0(\bar{Z}_0, \Omega)_\chi$, cf. Section 5. It follows from Definition 5.2 that $\mathcal{C}\omega = \Phi_{\mathbf{a}}(t)^{1/p}\omega$. Since $\mathcal{C}c\omega = c^{1/p}\mathcal{C}\omega$, we see that

$$\mathcal{C}\Phi_{\mathbf{a}}(t)^{\frac{1}{p-1}}\omega = \Phi_{\mathbf{a}}(t)^{\frac{1}{p-1}}\omega.$$

We conclude that ξ_2 corresponds to the $(p-1)$-cyclic extension of $k(t)$ obtained by adjoining a $(p-1)$th root of the $\mathbf{a}$-Hasse invariant $\Phi_{\mathbf{a}}(t)$.

To prove the statement of the proposition for ξ_1, we remark that $J_Z[p]^{\rm loc}_\chi$ is dual to $J_Z[p]^{\rm et}_{\chi^{-1}}$. Therefore the above argument applied to the character χ^{-1} shows that ξ_1 corresponds to the inverse of the Hasse invariant $\Phi_{\mathbf{a}^*}(t)$ corresponding to the dual type $\mathbf{a}^* = (m-a_1, m-a_2, m-a_3, m-a_4)$. $\Box$

Let $\bar{W}_0$ be the original component of the stable reduction of the G-Galois cover $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1_\lambda$. We choose a component $\bar{U}_0$ of $\bar{\mathbb{H}}(\mathbf{a})$ above $\bar{W}_0$. The map $\bar{U}_0 \to \bar{W}_0$ factors through a curve $\bar{V}_0$ such that $\bar{U}_0 \to \bar{V}_0$ is a $\boldsymbol{\mu}_p$-torsor and $\bar{V}_0 \to \bar{W}_0$ is a tame cover, cf. Section 1. The following theorem describes the cover $\bar{V}_0 \to \bar{W}_0$.

Theorem 6.2 *Suppose that* $\mathbf{a}$ *is non-exceptional, Definition 3.4. Write* $d := \gcd(m, a_1 + a_3, a_2 + a_3)$. *If* $m \neq 2$, *we assume that* $d \neq m$. *This is no restriction.*

(a) Suppose that $m = 2$. *Then* $G = \mathrm{PSL}_2(p)$ *and the cover* $\bar{V}_0 \to \bar{W}_0$ *is given by*

$$\theta^{(p-1)/2} = \Phi_{\mathbf{a}}.$$

(b) Suppose that $G = \mathrm{SL}_2(p)$. *Write* $m/d = 1 + 2j$. *The cover* $\bar{V}_0 \to \bar{W}_0$ *is given by*

$$\theta^{p-1} = \Phi_{\mathbf{a}}^{1+j}\Phi_{\mathbf{a}^*}^{-j}.$$

(c) Suppose $G = \mathrm{PSL}_2(p)$ *or* $G = \mathrm{PGL}_2(p)$ *and* d *is even. Then there exists a function* $g \in k(t)$ *such that*

$$\Phi_{\mathbf{a}}^{1+m/d}(t)\Phi_{\mathbf{a}^*}^{1-m/d}(t) = g^2.$$

The cover $\bar{V}_0 \to \bar{W}_0$ *is given by*

$$\theta^{(p-1)/2} = g.$$

(d) Suppose that $G = \mathrm{PGL}_2(p)$ *and* d *is odd. The cover* $\bar{V}_0 \to \bar{W}_0$ *is given by*

$$\theta^{p-1} = \Phi_{\mathbf{a}}^{1+m/(2d)}\Phi_{\mathbf{a}^*}^{1-m/(2d)}.$$

Proof: We first note that

$$\Phi_{\mathbf{a}^*}(t)=(-1)^{\alpha a_3^*}\sum_{i+j=\alpha a_3^*}\binom{\alpha a_2}{i}\binom{\alpha a_4}{j}\lambda^j.$$

The polynomial $\Phi_{\mathbf{a}^*}(t)$ has a zero at $t=0$ of order $\max(\alpha(m-a_2-a_3),0)$ and a zero at $t=1$ of order $\max(\alpha(m-a_1-a_3),0)$.

Since the group scheme $J_{\bar{Z}_0}[p]_\chi$ is dual to $J_{\bar{Z}_0}[p]_{\chi^{-1}}$, it is obvious that $J_{\bar{Z}_0}[p]_\chi$ is local-local if and only if $J_{\bar{Z}_0}[p]_{\chi^{-1}}$ is local-local. This implies that $t\neq 0,1$ is a zero of $\Phi_{\mathbf{a}}(t)$ if and only if it is a zero of $\Phi_{\mathbf{a}^*}(t)$. Recall from Corollary 5.5 that all zeros of $\Phi_{\mathbf{a}}$ and $\Phi_{\mathbf{a}^*}$, except 0 and 1, are simple. Therefore

$$\frac{\Phi_{\mathbf{a}}(t)}{\Phi_{\mathbf{a}^*}(t)}=t^{\pm\alpha(a_2+a_3-m)}(t-1)^{\pm\alpha(a_1+a_3-m)}.$$

Case (a). Suppose that $m=2$. The only possibility for the type is $\mathbf{a}=(1,1,1,1)$. Since $a_i+a_j=m$, for all i and j, we conclude that $\Phi_{\mathbf{a}}=\Phi_{\mathbf{a}}^*$. This implies the theorem in this case. This is well known, see for example [14].

Suppose now that $m\neq 2$. Then the exist $i\neq j$ such that $a_i+a_j\neq m$. Therefore, after permuting the branch points if necessary, we may assume that $d\neq m$. Let L_2 (resp. L_1) be the extension of $k(t)$ corresponding to adjoining a $(p-1)$th root θ_i of $\Phi_{\mathbf{a}}(t)$ (resp. $1/\Phi_{\mathbf{a}^*}(t)$). Let $\tilde{L}=L_1\otimes_{k(t)}L_2$. Choose a primitive $(p-1)$th root of unity $\xi\in k$. Choose generators σ_i for the Galois group of $\tilde{F}$ over $k(t)$ such that

$$\sigma_1\theta_1=\xi\theta_1,\qquad \sigma_1\theta_2=\theta_2,\qquad \sigma_2\theta_1=\theta_1,\qquad \sigma_2\theta_2=\xi\theta_2.$$

For $\ell|(p-1)$, denote by $L_i^{(\ell)}$ the (unique) subfield of L_i containing $k(t)$ whose degree over $k(t)$ is ℓ. Note that $L_1^{(\ell)}\simeq L_2^{(\ell)}$ if and only if $\Phi_{\mathbf{a}}\Phi_{\mathbf{a}^*}^i=\delta^\ell$ for some $\delta\in k(t)$ and some i prime to $(p-1)$. Let d be as in the statement of the theorem and $\alpha=(p-1)/m$. Then $[L_1\cap L_2:k(t)]=\alpha d$. Therefore the algebra $\tilde{L}$ is the product of αd fields $L(\eta)$ with

$$L(\eta)=k(t)[\theta_1,\theta_2|\,\theta_2^{p-1}=\Phi_{\mathbf{a}},\quad \theta_1^{p-1}=1/\Phi_{\mathbf{a}^*},\quad (\theta_1\theta_2)^{m/d}=\eta\delta],$$

where η is a αdth root of unity and $\delta^{\alpha d}=\Phi_{\mathbf{a}}/\Phi_{\mathbf{a}^*}$. The group $G(\eta)=\mathrm{Gal}(L(\eta),k(t))$ is generated by $\sigma_1\sigma_2^{-1}$ and $\sigma_1^{\alpha d}$.

Recall that M is the subgroup of $\mathrm{GL}_2(p)$ consisting of the scalar matrices xI with $x\in\mathbb{Z}/m\subset\mathbb{F}_p^\times$. To find the subfield of $L(\eta)$ which corresponds to $\bar{V}_0$, we have to take invariants under the restriction of $M\cap\Gamma$ to $L(\eta)$, cf. Section 3. From now on, we suppose $\eta=1$. This is no restriction.

Define J as the intersection of $M\cap\Gamma$ with $G(\eta)$. The group $M\cap\Gamma$ is generated by $(\sigma_1\sigma_2)^\alpha$. Therefore I is generated by $(\sigma_1\sigma_2)^{\alpha d/2}$ if d is even and by $(\sigma_1\sigma_2)^{\alpha d}$ if d is odd.

Case (b) Suppose that $G=\mathrm{SL}_2(p)$. Corollary 3.6 implies that m is odd. Write $m/d=1+2j$. The degree of $L(\eta)^J$ over $k(t)$ is $p-1$. Define $\theta:=\theta_1^j\theta_2^{1+j}$. We claim that θ generates $L(\eta)^J$ over $k(t)$. It is clear that $\theta\in L(\eta)^J$. Since we assumed that $m\neq d$, the integer j does not divide $p-1$. Therefore $\theta^i\notin k(t)$, for every $0<i<p-1$ and θ generates $L(\eta)^J$ over $k(t)$. We have

$$\theta^{p-1}=\Phi_{\mathbf{a}}^{1+j}\Phi_{\mathbf{a}^*}^{-j}=x^{b_1}(x-1)^{b_2}\prod_{\lambda\in\Lambda(\mathbf{a})}(x-\lambda).\tag{12}$$

The b_i are expressions in terms of the a_j which we leave to the reader to compute.

Case (c). Suppose that $G=\mathrm{PSL}_2(p)$. Corollary 3.6 implies that m is even and that $\det(\Gamma)$ is a subgroup of $\det(M)$. Therefore the order of $\det(\Gamma)$ divides $m/2$. This implies that $\gcd(m,a_1+$

$a_3, a_1 + a_3, a_3 + a_4)$ is even and so d is even. We conclude that J is generated by $(\sigma_1\sigma_2)^{\alpha d/2}$ and that the degree of $L(\eta)^J$ over $k(t)$ is $(p-1)/2$.

One checks that $L(\eta)^J$ is generated over $k(t)$ by $\theta := \theta_1^{-1-m/d}\theta_2^{+1-m/d}$ which satisfies

$$\theta^{p-1} = \Phi_{\mathbf{a}}^{1+m/d}\Phi_{\mathbf{a}^*}^{1-m/d} = x^{b_1}(x-1)^{b_2}\prod_{\lambda\in\Lambda(\mathbf{a})}(x-\lambda)^2.$$

Then

$$b_i = \begin{cases} \alpha(a_i + a_4 - m)(1-\frac{m}{d}) & \text{if } a_i + a_4 > m, \\ \alpha(m - a_i - a_4)(1+\frac{m}{d}) & \text{if } a_i + a_4 < m. \end{cases}$$

In case $a_i + a_4 = m$ the branch point x_i is unramified in $\bar{V}_0 \to \bar{W}_0$.

Since d is even, also b_1 and b_2 are even. We conclude that the $(p-1)/2$-cyclic cover $\bar{V}_0 \to \bar{W}_0$ is given by the equation

$$(\theta)^{(p-1)/2} = x^{b_0/2}(x-1)^{b_1/2}\prod_{\lambda}(x-\lambda). \tag{13}$$

We leave it for the reader to verify that if $G = \mathrm{PGL}_2(p)$ and d is even the cover $\bar{V}_0 \to \bar{W}_0$ is given by the same equation.

Case (d) Suppose that $G = \mathrm{PGL}_2(p)$ and that d is odd. It follows from the proof of Corollary 3.6 that in this case m is even. Define $j = m/(2d)$. Since d is odd, we know that J is generated by $(\sigma_1\sigma_2)^{\alpha d}$ and that the degree of $L(\eta)^J$ over $k(t)$ is $p-1$. One checks that $L(\eta)^J/k(t)$ is generated by $\theta := \theta_1^{-1+j}\theta_2^{1+j}$ which satisfies

$$\theta^{p-1} = \Phi_{\mathbf{a}}^{1+j}\Phi_{\mathbf{a}^*}^{1-j} = x^{b_1}(x-1)^{b_2}\prod_{\lambda\in\Lambda(\mathbf{a})}(x-\lambda)^2. \tag{14}$$

$\square$

Recall the following from Section 1. The stable reduction $\bar{\varpi}(\mathbf{a}) : \bar{\mathbb{H}}(\mathbf{a}) \to \bar{\mathbb{P}}$ of $\varpi(\mathbf{a})$ has a very simple structure. The curve $\bar{\mathbb{P}}$ is a comb. It consist of tails W_b for $b \in \mathbb{B}$ which intersect the original component in one point which we denote by τ_b. There are two types of tails: new tails and primitive tails. A tail W_b is primitive if W_b contains the specialization of one of the branch points of $\varpi(\mathbf{a})$ whose ramification index is prime-to-p. Therefore x_i for $i \in \{1,2,3\}$ specializes to a primitive tail if $a_i + a_4 \neq m$, Proposition 3.8. All other tails are called the new tails. They intersect the original component $\bar{W}_0$ in an $\mathbf{a}$-supersingular $\lambda \in \Lambda(\mathbf{a})$. We may regard the set $\mathbb{B}$ indexing the tails as a subset of $\{1,2,3\} \cup \Lambda(\mathbf{a})$.

To a tail W_b of $\bar{\mathbb{P}}$ we associated a ramification invariant $\sigma_b = h_b/m_b$ which describes the ramification above the intersection point of W_b with $\bar{W}_0$. As a consequence of Theorem 6.2, we can now describe the ramification invariants of the tails of $\bar{\mathbb{P}}$.

Recall that we have chosen an irreducible component $\bar{U}_0$ of $\bar{\mathbb{H}}(\mathbf{a})$ such that the inertia group of $\bar{U}_0$ is the Sylow p-subgroup P of G. The map $\bar{U}_0 \to \bar{W}_0$ factors through $\bar{V}_0$ and $\bar{U}_0 \to \bar{V}_0$ is a $\boldsymbol{\mu}_p$-torsor. Write $\bar{U}_0'$ for the quotient of $\bar{U}_0$ by the prime-to-p part of the center of its decomposition group and write $\bar{V}_0'$ for the quotient of $\bar{U}_0'$ by $\boldsymbol{\mu}_p$. Then $\bar{V}_0' \to \bar{W}_0$ is a cyclic cover, which is branched at $0, 1, \infty, \lambda$ for $\lambda \in \Lambda(\mathbf{a})$. Let n be the order of this cover.

Remark 1.6 implies that $0 < \sigma_i < 1$ for $i \in \{1,2,3\}$ and $1 < \sigma_\lambda < 2$ for $\lambda \in \Lambda(\mathbf{a})$. Moreover, the ramification invariants are related to the type of the cover $g_0' : \bar{V}_0' \to \bar{W}_0$, [30]. The relation is as follows. Write $\sigma_b = h_b/n_b$, where n_b is the ramification index of τ_b in g_0'. Let (β_b, τ_b) be the type of g_0', as defined in Definition 2.1. Then $h_b n/n_b \equiv \beta_b \bmod n$. The invariants β_b are easily computed from the explicit equation for g_0'. This determines the ramification invariants.

We will make this explicit in case $\lambda \in \Lambda(\mathbf{a})$. Suppose first that we are in Case (b) of Theorem 6.2, i.e. $G = \mathrm{SL}_2(p)$. Then $\bar{V}_0 \to \bar{W}_0$ has order $p-1$. The normalizer $N_G(P)$ of the Sylow p-subgroup P in $\mathrm{SL}_2(p)$ has order $p(p-1)$, therefore the decomposition group of $\bar{U}_0$ is the full normalizer $N_G(P)$. The order of the prime-to-p centralizer of $N_G(P)$ is two. Therefore the degree of $\bar{V}_0' \to \bar{W}_0$ is $(p-1)/2$. It follows from (12) that the integer $\beta_\lambda = 1$, for every $\lambda \in \Lambda(\mathbf{a})$. The ramification invariant is now determined by the fact that $(p-1)/2 = n_\lambda < h_\lambda < 2n_\lambda = (p-1)$ and $2h_\lambda \equiv 1 \bmod (p-1)$. We conclude that $\sigma_\lambda = (p+1)/(p-1)$. One checks that the same holds in the other cases.

We note that the fact that all new tails have ramification invariant $(p+1)/(p-1)$ is a consequence of the fact that the $\mathbf{a}$-Hasse invariant is a solution of a hypergeometric differential equation. The decomposition group of a new tail is either $\mathrm{PSL}_2(p)$ or $\mathrm{SL}_2(p)$. One can show the following. Let f be a Galois cover of $\mathbb{P}^1_k$ with Galois group either $\mathrm{PSL}_2(p)$ or $\mathrm{SL}_2(p)$. Suppose that f is branched only at ∞ and that the ramification invariant σ of f satisfies $1 < \sigma < 2$. Then $\sigma = (p+1)/(p-1)$. The proof uses ideas from [5, Section 3.3].

Corollary 6.3 *The differential ω is given by*

$$\omega = u\frac{\theta\,\mathrm{d}x}{x(x-1)}, \tag{15}$$

where θ and x are as in Theorem 6.2 and $u \in k^\times$.

Proof: This is immediate. □

An important property of ω is that $\mathcal{C}\omega = \omega$. This determines the constant u in (15), up to an element of $\mathbb{F}_p^\times$. (Actually, $u \in \mathbb{F}_p^\times$.) One can also check directly that the differential defined by (15) with $u \in \mathbb{F}_p^\times$ is fixed by the Cartier operator, for example by using the method of [12, Section 4.3]. It turns out that $\mathcal{C}\omega = \omega$ is equivalent to the fact that $\Phi_{\mathbf{a}}$ satisfies the hypergeometric differential equation of Proposition 5.4.

7 Reduction of the Hurwitz spaces

In the previous sections, we computed the stable reduction of $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1_\lambda$ as Galois cover. We did not give an interpretation of the stable model $\bar{\mathbb{H}}(\mathbf{a})$ as moduli space; it is not clear whether the model $\bar{\mathbb{H}}(\mathbf{a})$ has a "reasonable" interpretation as a moduli space. We merely used the interpretation of $\mathbb{H}(\mathbf{a})$ (or rather of its quotient $H(\mathbf{a})$) as Hurwitz space to determine the stable reduction of the cover $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1_\lambda$.

Let $H(\mathbf{a})/K$ be the Hurwitz space parameterizing metacyclic covers of type $(\mathbf{x};\mathbf{a})$, as defined in Section 3. Here K is some finite extension of $\mathbb{Q}_p$. Write $\pi(\mathbf{a}) : H(\mathbf{a}) \to \mathbb{P}^1_\lambda$ for the projection to the λ-line. In [6], we find a definition of a compactification $\bar{H}(\mathbf{a})$ of $H(\mathbf{a})$ over the ring of integers R of K. Roughly speaking, $\bar{H}(\mathbf{a}) \otimes_R \mathbb{F}_p$ parameterizes the stable reduction of the metacyclic covers of type $(\mathbf{x};\mathbf{a})$. In this section, we want to describe the structure of $\bar{H}(\mathbf{a}) \otimes_R \mathbb{F}_p$ without proofs. To make the proofs rigorous one needs to carefully analyze the deformation theory. Details will appear elsewhere.

The étale case. In this case all metacyclic covers of type $(\mathbf{x};\mathbf{a})$ have good reduction, Proposition 2.11. Therefore the complete model $\bar{H}(\mathbf{a})$ parameterizes so called admissible covers. It is well known that the moduli space of admissible covers is smooth, [31]. We conclude that $\pi(\mathbf{a}) : H(\mathbf{a}) \to \mathbb{P}^1_\lambda$ has good reduction in this case, confirming Theorem 4.2.

The mixed case. Let $\lambda \in \mathbb{P}^1_K$ be such that its reduction $\bar{\lambda} \in \mathbb{P}^1_k$ is $\mathbf{a}$-ordinary and different from $0, 1, \infty$. (See Definition 5.1 for the definition of $\mathbf{a}$-ordinary and $\mathbf{a}$-supersingular.) Let $x_1 = 0, x_2 =$

$1, x_3 = \infty, x_4 = \lambda$. It follows from Proposition 2.10 that there are $(p-1)/m$ metacyclic covers of type $(\mathbf{x};\mathbf{a})$ with good reduction. The other $p(p-1)/m$ such covers have multiplicative bad reduction. If $\bar\lambda$ is $\mathbf{a}$-supersingular, then all covers have additive bad reduction.

Suppose $f: Y \to X$ is a metacyclic cover of type $(\mathbf{x};\mathbf{a})$ with bad reduction. Lemma 1.4 implies that $\bar X$ has 5 irreducible components: the original component $\bar X_0$ and the four primitive tails X_1, X_2, X_3, X_4. Lemma 1.4.(b) implies that the ramification invariant σ_i corresponding to the tail X_i is smaller than 1. Therefore it follows from [18, Lemma 2.3.3] that there are only finitely many possibilities for $\bar f|_{X_i}$. This implies that $\bar\pi(\mathbf{a}) : \bar H(\mathbf{a}) \to \mathbb{P}^1_R$ is finite, outside the cusps.

It follows from the description of the monodromy given in Section 4 that $\bar H(\mathbf{a}) \otimes_R \mathbb{F}_p$ has two irreducible components H^{good} and H^{bad}. Over the $\mathbf{a}$-ordinary locus, H^{good} parameterizes metacyclic covers and H^{bad} parameterizes the stable reduction of the covers with multiplicative bad reduction. The two components intersect above $\lambda \in \Lambda(\mathbf{a})$ (the $\mathbf{a}$-supersingular λ's).

For $m = 2$, the Hurwitz space $H(\mathbf{a})$ is a version of the modular curve $X_1(p)$ with ordered branch points. (In other words, one adds a full level 2-structure, cf. [5, Section 2.2].) In this case, the reduction of H is of course well known.

The multiplicative case. All metacyclic covers of type $(\mathbf{x};\mathbf{a})$ have multiplicative bad reduction, Proposition 2.9. To describe the reduction of $\pi(\mathbf{a}) : H(\mathbf{a}) \to \mathbb{P}^1_\lambda$, we first need some results on the stable reduction of metacyclic covers.

Let $\mathbf{a} = (a_1, a_2, a_3, a_4)$ be a type such that $a_1+a_2+a_3+a_4 = m$. Let $f: Y \to X$ be a metacyclic cover of type $\mathbf{a}$, branched at $x_1 = 0, x_2 = 1, x_3 = \infty, x_4 = \lambda$ and suppose that $\lambda \not\equiv 0, 1, \infty \bmod p$. Write $(\bar Z_0, \omega)$ for the deformation datum of f, as defined in Definition 1.1. As in Remark 1.6, one checks that for $z_i \in \bar Z_0$ above x_i we have

$$\gcd(a_i, m)\mathrm{ord}_{z_i}(\omega) \equiv a_i \pmod{m}.$$

Let $e \in \mathbb{E}$ be an edge with source v_0. Let x_e be the corresponding point of $\bar X_0$. Choose a point $z_e \in \bar Z_0$ above x_e and write m_e for the ramification index of this point in $\bar Z_0 \to \bar X_0$. Analogous to the notation introduced in Section 1, we define $\nu_e = [h_e/m_e]$ and $a_e = m(h_e/m_e - \nu_e)$.

Suppose that the subtree of T with root e contains the primitive tail X_i. As in Proposition 1.6, one checks that $a_e = a_i$. The analog of Remark 1.6 becomes in this case $\sum_{s(e)=v_0} \nu_e = 1$. It follows that there are two cases.

- The first possibility is that $\nu_i = 0$, for all $i \in \mathbb{B}_{\mathrm{prim}}$. In this case, there is one new tail X_5 which intersects the original component.
- The second possibility is that $\nu_i = 1$ for a unique $i \in \mathbb{B}_{\mathrm{prim}}$.

Definition 7.1 For $\lambda \in k - \{0,1\}$, let $\bar Z_0 \to \mathbb{P}^1_k$ be the m-cyclic cover of type $(\mathbf{x};\mathbf{a})$ branched at $x_1 = 0, x_2 = 1, x_3 = \infty, x_4 = \lambda$. We say that $\lambda \in k$ is $\mathbf{a}$-*special* if there exists an $\omega \in H^0(\bar Z_0, \Omega^1)_\chi$ and $1 \le i \le 4$ such that $\nu_i(\omega) = 1$. If λ is not $\mathbf{a}$-special, we call it is $\mathbf{a}$-*general*.

This terminology is inspired by [29], but the notation of a special cover in that paper is more restrictive than the way we use this concept here.

Lemma 7.2 *Suppose that $a_1 + \cdots + a_4 = m$. There are at most finitely many* $\mathbf{a}$*-special λ's.*

Proof: Similar to [29, Section 3.5] □

Suppose that $\bar\lambda \in \mathbb{P}^1_k - \{0, 1, \infty\}$ is $\mathbf{a}$-general. One checks that $\bar X$ has 6 irreducible components: the original component $\bar X_0$, the primitive tails X_1, X_2, X_3, X_4 and the new tail X_5.

We have $\sigma_i < 1$ if $i \le 4$ and $\sigma_5 = 2$. We claim that this implies that, for given $\bar\lambda$, that there are only finitely many possibilities for $\bar f$. It follows from [18, Lemma 2.3.3] that there are only finitely

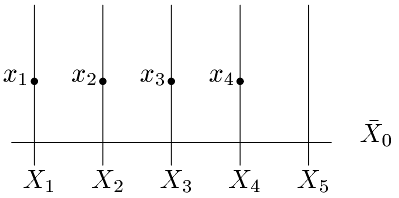

Figure 1: Reduction in the multiplicative case for general λ

many possibilities for $\bar{f}|_{X_i}$. The intersection point τ of X_5 with $\bar{X}_0$ is the image of a zero of ω. Therefore it is obvious that there are at most finitely many possibilities for τ.

Suppose now that $\bar{\lambda} \in \mathbb{P}^1_k$ is **a**-special. For simplicity, we suppose that $\nu_1 = 1$. There are two possibilities for the stable reduction $\bar{f}$.

(a) The curve $\bar{X}$ consists of 5 irreducible components: the original component $\bar{X}_0$ and the primitive tails X_1, X_2, X_3, X_4, see Figure 2. The ramification invariants are: $\sigma_1 = (m+a_1)/m$ and $\sigma_i = a_i/m$ for $i > 1$.

(b) There is one new tail which does not intersect the original component, see Figure 3.

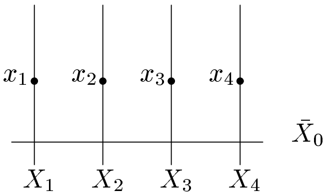

Figure 2: Reduction in the multiplicative case for **a**-special λ (Possibility a)

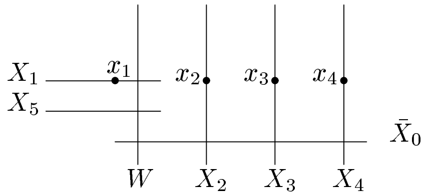

Figure 3: Reduction in the multiplicative case for **a**-special λ (Possibility b)

□

One can show using a patching argument as in [10] that both possibilities occur. It follows from [18, Proposition 2.2.6] that there is a one dimensional family of covers of type (a), which all occur as the reduction of a metacyclic cover. Therefore the map $\bar{\pi} : \bar{H} \to \mathbb{P}^1_R$ is not finite: there are *vertical components* whose image under $\bar{\pi}$ is a point. The occurrence of vertical components in the stable reduction is not unexpected. The same phenomenon occurs for the reduction of the Hurwitz space of 2-cyclic covers branched at four points to characteristic 2, [1, Section 4].

Theorem 4.2 implies that there exists a smooth model of H over R, but the model $\bar{H}$ which has a modular interpretation is not smooth.

8 Examples

Suppose that $p > 5$ and choose $2 \neq d|(p-1)/2$. Let $f : Y \to \mathbb{P}^1_{\mathbb{C}}$ be a cover with Galois group $\mathrm{PSL}_2(p)$ branched at three points of order p, p, d. In this section we show that there exists an $m|(p-1)$ and a type $\mathbf{a}$ such that f is isomorphic to the quotient of $\varpi(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1$ by the center of its Galois group. A similar result is shown in [5, Section 2.2] for $\mathrm{PSL}_2(p)$-covers of $\mathbb{P}^1_{\mathbb{C}}$ branched at three points of order p. In that case one may take $m = 2$ and $\mathbf{a} = (1,1,1,1)$. The cover $\varpi(\mathbf{a})$ is in this case isomorphic to $X(2p) \to X(2)$, where $X(N)$ is the modular curve with full level-N structure.

Recall that $\mathrm{SL}_2(p)$ has two conjugacy classes of elements of order p, which we denote by pA and pB, [27, Lemma 3.27]. We make the convention that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in pA.$$

Choose a primitive $(p-1)$th root of unity $\xi \in \mathbb{F}_p^\times$. For $0 < i < (p-1)/2$, let $\mathcal{C}(i) = \{A \in \mathrm{SL}_2(p) |\ \mathrm{Tr}(A) = \xi^i + \xi^{-i}\}$. Then $\mathcal{C}(i)$ is a conjugacy class of $\mathrm{SL}_2(p)$ of elements whose order is equal to the order of ξ^i in $\mathbb{F}_p^\times$. Moreover, every element of $\mathrm{SL}_2(p)$ of order n dividing $p-1$ with $n > 2$ is contained in some $\mathcal{C}(i)$. For a triple of conjugacy classes $\mathbf{C} = (C_1, C_2, C_3)$ of $\mathrm{SL}_2(p)$, we write $\mathrm{Ni}_3^{\mathrm{in}}(\mathbf{C})$ for the set of isomorphism classes of $\mathrm{SL}_2(p)$-cover $Y \to \mathbb{P}^1_{\mathbb{C}}$ with class vector $\mathbf{C}$. (This means that the canonical generator of some point of Y above x_i is contained in the conjugacy class C_i, with respect to some fixed system of roots of unity.)

Lemma 8.1 *Let $0 < i < (p-1)/2$. Then either $|\mathrm{Ni}_3^{\mathrm{in}}(pA, pA, \mathcal{C}(i))| = 0$ and $|\mathrm{Ni}_3^{\mathrm{in}}(pA, pB, \mathcal{C}(i))| = 1$, or $|\mathrm{Ni}_3^{\mathrm{in}}(pA, pA, \mathcal{C}(i))| = 1$ and $|\mathrm{Ni}_3^{\mathrm{in}}(pA, pB, \mathcal{C}(i))| = 0$.*

Proof: The proof is similar to [28, Lemma 3.29]. □

The lemma says that the triples $(pA, pA, \mathcal{C}(i))$ and $(pA, pB, \mathcal{C}(i))$ are *rigid*. Since there is an outer automorphism of $\mathrm{SL}_2(p)$ which interchanges the conjugacy classes pA and pB and fixes $\mathcal{C}(i)$, the lemma may be rephrased as follows. For every $0 < i < (p-1)/2$, there exists a unique $\mathrm{SL}_2(p)$-cover of $\mathbb{P}^1_{\mathbb{C}}$ with class vector $(p*, p**, \mathcal{C}(i))$ *up to isomorphism*.

We will now construct types $\mathbf{a}$ such that $\varpi(\mathbf{a})$ is ramified of order p, p, d, for some $d|(p-1)$. The cover $\varpi(\mathbf{a})$ is branched at 0 and 1 of order p if and only if $a_1 + a_4 = a_2 + a_4 = m$, by Proposition 3.8. Therefore the only types we need to consider are types of the form $(a, a, m-a, m-a)$, for some $m|(p-1)$ with $\gcd(a, m) = 1$.

Lemma 8.2 *Suppose $m \neq 2$ divides $p-1$. Choose $0 < a < m$ such that $\gcd(m, a) = 1$ and write $\mathbf{a} = (a, a, m-a, m-a)$. Then the Galois group of $\varpi(\mathbf{a})$ is $\mathrm{SL}_2(p)$ if m is odd and $\mathrm{PSL}_2(p)$ if m is even.*

Proof: We first note that p divides the order of the Galois group of $\varpi(\mathbf{a})$, since the ramification index at 0 and 1 is p. Therefore we are not in the exceptional case, Definition 3.4.

We denote by B_1, B_2, B_3 the matrices defined in (6)–(8). Then $\det(B_1) = \det(B_2) = 1$ and $\det(B_3) = (\xi^{\alpha a})^2$. Corollary 3.6 implies that the Galois group of $\varpi(\mathbf{a})$ is $\mathrm{SL}_2(p)$ if m is odd and $\mathrm{PSL}_2(p)$ if m is even. □

For every type $\mathbf{a}$, we can define an $\mathrm{SL}_2(p)$-cover $\varpi'(\mathbf{a})$. If m is odd, this is just $\varpi(\mathbf{a})$. If m is even, we choose a lift of $\varpi(\mathbf{a})$ to an $\mathrm{SL}_2(p)$-cover of $\mathbb{P}^1$ branched at three points. We may suppose that $\varpi'(\mathbf{a})$ is branched of order p at 0 and 1. This uniquely characterizes the lift.

The following proposition states that there exists a type $\mathbf{a}$ such that $\varpi'(\mathbf{a})$ has class vector $(C_1, C_2, \mathcal{C}(i))$, for some choice of C_1, C_2. In other words, every $\mathrm{SL}_2(p)$-cover with class vector $(p*, p*, \mathcal{C}(i))$ is isomorphic to one of the covers $\varpi'(\mathbf{a})$.

Proposition 8.3 *Let $2 \neq m|(p-1)$ and $\mathbf{a} = (a, a, m-a, m-a)$. Write $\alpha = (p-1)/m$. Suppose $a < m/2$ and $\gcd(m, a) = 1$. Then the class vector of $\varpi'(\mathbf{a}) : \mathbb{H}(\mathbf{a}) \to \mathbb{P}^1$ is $(C_1, C_2, \mathcal{C}(a\alpha))$, for some $C_1, C_2 \in \{pA, pB\}$.*

Proof: The Nielsen tuple of $\varpi'(\mathbf{a})$ is (B_1, B_2, B_3') with $B_3' = B_3/\xi^{\alpha a} \in \mathrm{SL}_2(p)$. Therefore $\mathrm{Tr}(B_3') = \xi^{\alpha a} + \xi^{-\alpha a}$ and $B_3' \in \mathcal{C}(\alpha a)$. □

Corollary 8.4 *Suppose $d \neq 2$ divides $(p-1)/2$. Let $f : Y \to \mathbb{P}^1_{\mathbb{C}}$ be a cover with Galois group $\mathrm{PSL}_2(p)$ which is branched only at $0, 1, \infty$ with ramification index p, p, d, respectively. Then there exists a type $\mathbf{a}$ such that f is isomorphic to the quotient of $\varpi(\mathbf{a}) \otimes_K \mathbb{C}$ by the center of the Galois group G of $\varpi(\mathbf{a})$.*

Proof: This follows from Proposition 8.3. □

One may expect Corollary 8.4 to hold more generally. Let $G = \mathrm{SL}_2(p)$ and suppose that $\mathbf{C} = (C_1, C_2, C_3)$ is a triple of conjugacy classes of G. In general, $\mathbf{C}$ is not rigid, but it is still *linearly rigid*, [25, Example 2.4]. This means the following. Let (M_1, M_2, M_3) be a triple of matrices in $\mathrm{GL}_2(p)$ such that $M_i \in C_i$ and $M_1 M_2 M_3 = 1$. Then if (M_1', M_2', M_3') is another such triple, there exists an $N \in \mathrm{GL}_2(p)$ such that $N M_i' N^{-1} = M_i$ for all i. This was essentially already known to Riemann, see [13, Introduction].

It follows from Proposition 3.8 that the ramification indices of $\varpi(\mathbf{a})$ are either p or divide $p-1$. Therefore there exist covers of $\mathbb{P}^1$ branched at three points with Galois group $\mathrm{PSL}_2(p)$, $\mathrm{SL}_2(p)$ or $\mathrm{PGL}_2(p)$ which are not isomorphic to some $\varpi(\mathbf{a})$. To obtain covers with ramification indices dividing $p+1$ one should omit the condition $p \equiv 1 \bmod m$.

Comparison to Raynaud's criterion for good reduction. In the rest of this paper, we want to compare Theorem 4.2 to Raynaud's criterion for good reduction, [21]. We illustrate that our result does not follow from Raynaud's Criterion.

Suppose that $m = 5$ and $p \equiv 1 \bmod 5$. Choose a primitive 5th root of unity $\zeta_5 \in \mathbb{F}_p^\times$. For $1 \leq i \leq 4$, we write $\mathbf{a}_i = (i, i, i, -3i)$. One checks that $\mathbf{a}_i$ is non-exceptional. Therefore $\varpi_i := \varpi(\mathbf{a}_i)$ is a Galois cover with Galois group $\mathrm{SL}_2(p)$, Corollary 3.6. Proposition 3.8 implies that ϖ_i is branched at three points of order 5.

Let $5A$ (resp. $5B$) be the conjugacy class of $\mathrm{SL}_2(p)$ of matrices with trace $\zeta_5 + \zeta_5^4$ (resp. $\zeta_5^2 + \zeta_5^3$). Write $\mathbf{C}_A = (5A, 5A, 5A)$ and $\mathbf{C}_B = (5B, 5B, 5B)$. As before, we denote by $\mathrm{Ni}_3^{\mathrm{in}}(\mathbf{C})$ the Nielsen class.

Lemma 8.5 *(a) We have $\varpi_1, \varpi_4 \in \mathrm{Ni}_3^{\mathrm{in}}(\mathbf{C}_B)$ and $\varpi_2, \varpi_3 \in \mathrm{Ni}_3^{\mathrm{in}}(\mathbf{C}_A)$.*

(b) The sets $\mathrm{Ni}_3^{\mathrm{in}}(\mathbf{C}_A)$ and $\mathrm{Ni}_3^{\mathrm{in}}(\mathbf{C}_B)$ both contain exactly two elements.

Proof: For each type, we defined matrices B_1, B_2, B_3 in (6)–(8). We denote these matrices corresponding to the type $\mathbf{a}_i$ by $B_1(i), B_2(i), B_3(i)$.

The class vector of ϖ_i is described by matrices $B_1'(i), B_2'(i), B_3'(i)$ which are characterized by the following two properties:

- $B_j'(i) = B_j(i) \cdot \gamma_j(i) I$, for some scalar matrix $\gamma_j(i) I \in M$, such that $\det(B_j'(i)) = 1$,

- the order of $B'_j(i)$ is 5.

It follows that $\gamma_j(i) = \zeta_5^{-i}$, for every j. Part (a) follows from this. The triples $\mathbf{C}_A$ and $\mathbf{C}_B$ are linearly rigid, [25, Example 2.4]. The outer automorphism group of $\mathrm{SL}_2(p)$ has order two and fixes the conjugacy classes of order five. Therefore, the Nielsen class has either zero or two elements and (b) follows from (a). $\Box$

In $\mathrm{SL}_2(p)$ there are two conjugacy classes of elements of order 5. Therefore the triples $\mathbf{C}_A$ and $\mathbf{C}_B$ are rational over $\mathbb{Q}(\sqrt{5})$, [24, Section 7.1]. Lemma 8.5.(b) implies that the $\mathrm{SL}_2(p)$-covers ϖ_i are defined over some extension K_i of $\mathbb{Q}(\sqrt{5})$ of degree at most two. Let $\wp$ be a prime of K_i above p and write e_i for the ramification index of $\wp$. Since the degree of K_i over $\mathbb{Q}(\sqrt{5})$ is at most two, we have $e_i \leq 2$. Raynaud's Criterion for good reduction states in this case that if ϖ_i has bad reduction then e_i is greater than or equal to the number of Sylow p-subgroups in $\mathrm{SL}_2(p)$, i.e. $e_i \geq 2$. Therefore we cannot conclude directly whether ϖ_i has good reduction or not, unless we have more information on the field of definition K_i of ϖ_i. But such information is hard to obtain for triples which are not rigid. In what follows we take an alternative approach. We use Theorem 4.2 to decide whether ϖ_i has good reduction and deduce from this information on the field of definition K_i.

The following proposition is essentially the statement of Raynaud's criterion for good reduction, in our special case.

Proposition 8.6 *The cover ϖ_i has good reduction at $\wp$ if and only if $e_i = 1$.*

Proof: If ϖ_i has good reduction at $\wp$ it follows from Beckmann's Theorem that $e_i = 1$, [2]. Suppose that ϖ_i has bad reduction. Raynaud's Criterion [21] implies that $e_i \geq 2$. In [21] the assumption is made that the Galois group G has trivial center, but this is not essential. $\Box$

Proposition 8.7 *(a) Let $f : Y \to \mathbb{P}^1$ be an $\mathrm{SL}_2(p)$-cover with class vector $\mathbf{C}_A$. Then f has bad reduction.*

(b) Let $f : Y \to \mathbb{P}^1$ be an $\mathrm{SL}_2(p)$-cover with class vector $\mathbf{C}_B$. Then f has good reduction.

Proof: Lemma 8.5 implies that, over $\mathbb{C}$, there are two covers with class vector $\mathbf{C}_A$ and two with class vector $\mathbf{C}_B$. Write $\mathrm{Ni}_4^{\mathrm{in}}(\mathbf{C}_B) = \{f_1, f_4\}$ and $\mathrm{Ni}_4^{\mathrm{in}}(\mathbf{C}_A) = \{f_2, f_3\}$. Let $K/\mathbb{Q}$ be a field over which all f_i are defined. The definition of the type $\mathbf{a}_i$ depends on a choice of primitive 5th root of unity $\xi_5 \in K$. Recall that we have also fixed a primitive 5th root of unity $\zeta_5 \in \mathbb{F}_p^\times$. Choose a prime ideal $\wp$ of K above p such that $\xi_5 \equiv \zeta_5 \bmod \wp$.

We may assume that $f_1 = \varpi_1$ and $f_2 = \varpi_2$. Theorem 4.2 implies that f_1 has good reduction and f_2 has bad reduction at $\wp$. It follows from Proposition 8.6 that f_2 is defined over a field K_2 such that $\mathbb{Q}(\sqrt{5}) \subset K_2 \subset K$ in which the ramification index of $\wp \cap K_2$ is two. Let σ_2 be a generator of $\mathrm{Gal}(K_2, \mathbb{Q}(\sqrt{5}))$. Then $f_3 = f_2^\sigma$. we conclude that f_3 has bad reduction at $\wp$ also.

The Galois group of $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$ permutes the class vectors $\mathbf{C}_A$ and $\mathbf{C}_B$, therefore the covers f_1, f_2, f_3, f_4 are conjugated over K by an automorphism of order four. It follows that we may take K to be the Galois closure of $K_2/\mathbb{Q}$. The Galois group of $K/\mathbb{Q}$ is a transitive group on four letters which contains an element of order four and has a quotient of order two, i.e. $\mathrm{Gal}(K, \mathbb{Q})$ is either cyclic of order four or a dihedral group of order eight.

Suppose $\mathrm{Gal}(K, \mathbb{Q}) \simeq \mathbb{Z}/4$. Then $K_2/\mathbb{Q}$ is Galois and f_1 is defined over K_2. Moreover, $f_1^\sigma = f_4$. This implies that K_2 is a minimal field of definition for f_1. Since $\wp$ is ramified in K_2, Proposition 8.6 implies that f_1 has bad reduction at $\wp$ contradicting Theorem 4.2. We conclude that $\mathrm{Gal}(K, \mathbb{Q})$ is a dihedral group of order eight. We represent the elements of $\mathrm{Gal}(K, \mathbb{Q})$ as permutations on $\{1, 2, 3, 4\}$. It follows that K_2 is the subfield of K of invariants under $(1\,4)$.

The covers f_1 and f_4 may be defined over the subfield K_1 of K of invariants of $(2\,3)$. The permutation $(1\,4)(2\,3)$ restricts to an automorphism of K_1 which permutes f_1 and f_4, therefore K_1

is a minimal field of definition of f_1. Since f_1 has good reduction at $\wp$, we conclude that $\wp \cap K_1$ is unramified in K_1. Therefore f_4 has good reduction at $\wp$ also.

We remark that the choice of $\wp$ depends on the choice of the 5th root of unity $\xi_5 \in K$. The permutation $(1\,2\,4\,3) \in \mathrm{Gal}(K, \mathbb{Q})$ sends $\wp$ to a different prime $\wp'$ and also changes the class vector of a cover f_i. If f_i has good reduction at $\wp$ then f_i has bad reduction at $\wp'$, and conversely. $\square$

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