INVERSE FORMULA FOR THE MELLIN–MAZUR TRANSFORM

by

Ha Huy Khoai

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

Institute of Mathematics
P.O. Box 631 Bo Ho
10 000 Hanoi
Vietnam

MPI/90-31

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§ 1. Introduction and Notation.

Let p be a fixed prime number, $\mathbb{Q}_p$ the field of p–adic numbers and $\mathbb{C}_p$ the p–adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{C}_p$ is normalized so that $|p| = p^{-1}$. We use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p. Let Δ_0 be an integer to p and let

$$q = \begin{cases} 4 \text{ if } p = 2 \\ p \text{ otherwise} . \end{cases}$$

We set $\Delta_0 q = \Delta$ and denote

$$\mathbb{Z}_\Delta^* = \varprojlim (\mathbb{Z}/\Delta p^m \mathbb{Z})^* .$$

The group of p–adic characters is the group of continuous holomorphisms of $\mathbb{Z}_\Delta^*$ into $\mathbb{C}_p^*$:

$$X(\mathbb{Z}_\Delta^*) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_\Delta^*, \mathbb{C}_p^*) .$$

Each Dirichlet character χ of conductor Δp^n is an element of the group $\mathrm{Hom}((\mathbb{Z}/\Delta p^m\mathbb{Z})^*,\mathbb{C}_p)$ for each $m \geq n$, and is prolonged to an unique element of the group $X(\mathbb{Z}_\Delta^*)$ which is again denoted by χ.

We set

$$U = 1 + q\mathbb{Z}_p = \{z \in \mathbb{Z}_p, v(z-1) \geq v(q)\} .$$

Then, for every $g \in U$ such that $v(g-1) = v(q)$, the map $z \longmapsto g^z$ is an isomorphism of $\mathbb{Z}_p$ onto U. We call g a topological generator of the group U.

For each generator g of the group U the map

$$X(U) = \mathrm{Hom}_{\mathrm{cont}}(U,\mathbb{C}_p^*) \longrightarrow \mathbb{C}_p^*$$

transforming a continuous character χ of the group U into a point $\chi(g)-1$ defines an isomorphism between $X(U)$ and the unit disk of $\mathbb{C}_p$:

$$D = \{z \in \mathbb{C}_p, |z|_p < 1\} .$$

Also we have isomorphisms:

$$\mathbb{Z}_\Delta^* \cong (\mathbb{Z}/\Delta_0\mathbb{Z})^* \times \mathbb{Z}_p^* ; \mathbb{Z}_p^* \cong (\mathbb{Z}/q\mathbb{Z})^* \times U . \qquad (1.1)$$

From isomorphisms (1.1) it follows that $X(\mathbb{Z}_\Delta^*)$ is a product of a finite group and $X(U)$, while the last is isomorphic to D. Since D is an open disk of $\mathbb{C}_p$, this isomorphism makes $X(\mathbb{Z}_\Delta^*)$ into an analytic manifold. For this analytic structure

$X(\mathbb{Z}_\Delta^*)$ is an analytic group. A function $f(\chi)$ is called an analytic function on the analytic group $X(\mathbb{Z}_\Delta^*)$ if its restriction on each component isomorphic to D is an analytic function. This means that for each character χ_0 of modulo Δ there exists a convergent series $\sum_{n=0}^{\infty} a_n(\chi_0)z^n$ in D such that for every character $\chi = \chi_0 \cdot \chi_1$, where $\chi_1 \in X(U)$, we have

$$f(\chi) = \sum_{n=0}^{\infty} a_n(\chi_0)\,[\chi_1(g) - 1]^n .$$

Now let μ be a measure on $\mathbb{Z}_\Delta^*$, i.e. μ is a continuous linear functional with values in $\mathbb{C}_p$ on the space of continuous functions in $\mathbb{Z}_\Delta^*$. Then the restriction of μ on the analytic group $X(\mathbb{Z}_\Delta^*)$ gives an analytic function

$$L(\mu,\chi) = \int_{\mathbb{Z}_\Delta^*} \chi d\mu . \qquad (1.2)$$

The function $L(\mu,\chi)$ is called the p–adic Mellin–Mazur transform of the measure μ.

In this paper we give an inverse formula for the transform (1.2) and some applications to the study of p–adic functions. In particular we obtain the Mellin–Mazur transform for Morita's p–adic Γ–function .

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§ 2. The inverse formula.

In [2] and [7] D. Barsky and M. Vishik prove that each Krasner analytic function in $\mathbb{C}_p - \mathbb{Z}_p^*$ subject to some growth conditions is the Cauchy transform of a measure on $\mathbb{Z}_p^*$ and they give an inverse formula for the p–adic Cauchy transform. Here we consider a similar problem for a large class of Mellin–Mazur transform.

Definition 2.1. Let f and g be two analytic functions in the unit disk D . We say f is of class o(g) if

$$\sup_{|z|\leq r} |f(z)| = o(\sup_{|z|\leq r} |g(z)|)$$

when $r \longrightarrow 1-0$.

If f , g are analytic functions on the group X(U) , we consider them as functions on D in view of the isomorphism (1.1).

Definition 2.2. A h–addmissible measure on $\mathbb{Z}_\Delta^*$ is a linear functional on the space of functions on $\mathbb{Z}_\Delta^*$ which are locally polynomials of degrees less than h and satisfy the following condition:

$$|\mu\{(z-a)^k \psi_{a,m}\}| = o(p^{(h-k)m}), \quad k = 0,1, \dots ,h-1 ,$$

where $\psi_{a,m}$ is the characteristic function of the set $a + U_m$.

It is proved in [7] that a h–addmissible measure is prolonged to a continuous linear functional on the space of $(h-1)$ – differentiable functions whose derivates of order $h-1$ satisfy the Lipschitz condition. The restriction of such a functional on the group $X(U)$ is an analytic function of class $o(\log^h)$ and is called the Mellin–Mazur transform of the measure μ. The class of such measures contains, for example, the measures associated to modular forms (see [5], [7]).

We will show that every analytic function on D of class $o(\log^h)$ is a Mellin–Mazur transform of some h–addmissible measure on U.

Theorem 1. Let $F(\chi)$ be a function of class $o(\log^h)$ on $X(U)$. Then the following formula defines a h–addmissible measure:

$$\mu(z^k\psi_{a,m}) = \frac{1}{p^m - p^{m-1}} \sum_{\chi} \chi^{-1}(a)F(z^k\chi)\ , \quad k = 0,1,\dots,h-1\ , \qquad (2.1)$$

where $\psi_{a,m}$ is the characteristic function of the set $a + U_m$ and χ runs on the set of Dirichlet characters of modulo p^m. Furthermore we have

$$F(\chi) = \int_U \chi d\mu\ . \qquad (2.2)$$

Proof. We first show formula (2.1) correctly defines a linear functional μ on the space of functions on U that are locally polynomials of degrees less than h. In fact, we have

$$\sum_{k=0}^{p-1} \mu\left\{ z^k\psi_{a+kp^m,m+1} \right\} = \sum_{k=0}^{p-1} \frac{1}{p^{m+1}-p^m} \sum_{\chi} \chi^{-1}(a + kp^m)F(z^k\chi)$$

$$= \frac{1}{p^{m+1} - p^m} \sum_{\chi} F(z^k\chi) \sum_{k=0}^{p-1} \chi^{-1}(a + kp^m),$$

$k = 0, \ldots, h-1$, where χ runs on the set of Dirichlet characters of modulo p^{m+1}. We note first

$$\sum_{k=0}^{p-1} \chi^{-1}(a + kp^m) = \begin{cases} p\chi^{-1}(a) & \text{if } \chi \bmod p^m \\ 0 & \text{if } \chi \text{ is of conductor } p^{m+1} \end{cases}.$$

From this we obtain

$$\sum_{k=0}^{p-1} \mu\left\{ z^k \psi_{a+kp^m, m+1} \right\} = \frac{1}{p^m - p^{m+1}} \sum_{\chi \bmod p^m} \chi^{-1}(a) F(z^k\chi) =$$

$$= \mu\{z^k \psi_{a,m}\}, \quad k = 0, 1, \ldots, h-1.$$

Now we prove μ satisfies the conditions of h–addmissibility. In regard to isomorphism (1.1) we may consider $F(\chi)$ as a function on the unit disk D : $F(\chi) = F\{\chi(g) - 1\} = F(z)$, where g is a topological generator of the group U .

For each analytic function f(z) on D and each $t_0 > 0$ we set

$$\|f\|_{t_0} = \sup_{v(z)=t_0} |f(z)|.$$

Then we obtain:

$$\|\log^h(1+z)\|_{t_m} = p^{mh} ,$$

where $t_m = 1/p^m - p^{m-1}$, $m = 1,2, \dots$ ($|\log^h(1+z)|$ is calculated by the Newton polygon (see [4]). From the hypothesis we have

$$\|F(z)\|_{t_m} = o(p^{mh}) \ (m \longrightarrow \infty) .$$

Let u be the sequence $\{g^i\xi - 1\}$, $i = 0,1, \dots ,h-1$, where $\{\xi\}$ is the sequence of primitive roots of unity of order $p^m (m = 1,2, \dots)$. Since the function $F(z)$ is of class $o(\log^h)$, one infers u is an interpolating sequence of $F(z)$ (see [3], [4]). We denote $\{S_m(z)\}$ the sequence of Lagrange's interpolation polynomials for the function $F(z)$ and the sequence u . Then $S_m(z)$ is defined by the following conditions:

$$\deg S_m(z) \leq hp^m - 1$$

$$S_m(g^i\xi - 1) = F(g^i\xi - 1) , \ i = 0, \dots ,h-1 .$$

By a Lazard's lemma ([6]) we may represent $F(z)$ in the form

$$F(z) = \varphi(z) \prod_{\substack{\gamma \in E_m \\ i=0, \dots ,h-1}} (1 - z/g^i\xi - 1) + Q_m(z) , \qquad (2.3)$$

where E_m is the set of primitive roots of unity of order p^m , $Q_m(z)$ are polynomials of order hp^m satisfying the condition:

$$\|Q_m\|_{t_m} \le \|F\|_{t_m} .$$

Since the representation (2.3) is unique, we have $S_m(z) \equiv Q_m(z)$, and hence that

$$\|S_m\|_{t_m} \le \|F\|_{t_m} .$$

From this it follows

$$\|S_m\|_{t_m} = o(p^{mh}) . \tag{2.4}$$

Supposing $S_m(z)$ is written in the form

$$S_m(z) = \sum_{\ell=0}^{h_p^m-1} b_\ell^{(m)} z^\ell$$

we have then

$$\|S_m\|_{t_m} = \max_{0\le \ell \le h_p^m-1} \{|b_\ell^{(m)} z^\ell|_{t_m}\} =$$

$$\max_\ell \{|b_\ell^{(m)}| p^{-1/(p^m-p^{m-1})}\} > p^{-hp/(p-1)} \max_\ell \{|b_\ell^{(m)}|\} .$$

Thus we have $\max|b_\ell^{(m)}| = o(p^{mh})$ $(m \longrightarrow \infty)$. Note that if we write

$$S_m(z-1) = \sum_{\ell=0}^{hp^m-1} a_\ell^{(m)} z^\ell$$

then we obtain also $\max_\ell |a_\ell^{(m)}| = o(p^{mh})$. By definition of the measure μ we have:

$$\mu\{(z-a)^k \psi_{a,m}\} = \sum_{j=0}^{k} (-a)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} \frac{1}{p^m - p^{m-1}} \sum_\chi \chi^{-1}(a) F(z^j \chi) =$$

$$= \sum_{j=0}^{k} (-a)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} \frac{1}{p^m - p^{m-1}} \sum_\chi \chi^{-1}(a) F(g^j \chi(g) - 1) =$$

$$= \sum_{j=0}^{k} (-a)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} \frac{1}{p^m - p^{m-1}} \sum_\chi \chi^{-1}(a) S_m(g^j \chi(g) - 1) =$$

$$= \sum_{j=0}^{k} (-a)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} \frac{1}{p^m - p^{m-1}} \sum_\chi \chi^{-1}(a) \sum_{\ell=0}^{hp^m-1} a_\ell^{(m)} g^{j\ell} \chi^\ell(g) =$$

$$= \sum_{j=0}^{k} (-a)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} \sum_{\ell=0}^{hp^m-1} a_\ell^{(m)} g^{j\ell} \frac{1}{p^m - p^{m-1}} \sum_\chi \chi(a^{-1} g^\ell) =$$

$$= \sum_{j=0}^{k} (-a)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} \sum_{\substack{\ell=0 \\ g^\ell \equiv a \mod qp^m}}^{hp^m-1} a_\ell^{(m)} g^{j\ell} =$$

$$= \sum_{\substack{\ell=0 \\ g^{\ell}\equiv a \mod qp^m}}^{hp^m-1} a_\ell^{(m)}(g^\ell - a)^k .$$

Thus we obtain

$$\sup_a |\mu\{(z-a)^k \psi_{a,m}\}| = \sup_a |\sum_{\substack{\ell=0 \\ g^{\ell}\equiv a \mod qp^m}}^{hp^m-1} a_\ell^{(m)}(g^\ell - a)^k| =$$

$$= o(p^{m(h-k)}) , \ k = 0, \dots ,h-1 ,$$

because $\max_\ell |a_\ell^{(m)}| = o(p^{mh})$, $|g^\ell - a| \le p^{-m}$. It remains to prove that the given function $F(\chi)$ is the Mellin–Mazur transform of the measure μ . For this we note that the Mellin–Mazur transform of the measure μ is an analytic function of class $o(\log^h)$ (see [7]). Consequently, the function $f(\chi) = F(\chi) - \int_U \chi d\mu$ is of class $o(\log^h)$. Hence, the sequence $\{g^k\xi - 1\}$, $k = 0, \dots ,h-1$, $\xi \in E_m$, $m = 1,2, \dots$ is an interpolating sequence of the function $f(\chi)$ (see [5]). It suffices to show for this sequence the function $f(\chi)$ vanishes identically. In fact, for every Dirichlet character of modulo p^m we have

$$\int_U z^k \chi d\mu = \sum_{a \mod p^m} \chi(a)\mu\{z^k \psi_{a,m}\} =$$

$$= \sum_{a \bmod p^m} \chi(a) \sum_{\overline{\chi}} \frac{1}{p^m - p^{m-1}} \chi^{-1}(a) F(z^k \overline{\chi}) = F(z^k \chi) .$$

Theorem 1 is proved.

§ 3. Integral representation of p–adic analytic functions

We now use the inverse formula for the Mellin–Mazur transform (Theorem 1) to find the integral representation of p–adic analytic functions of class $o(\log^h)$ on D before applying these results to Morita's p–adic Γ–function.

Let $f(z)$ be an analytic function of class $o(\log^h)$ on D. We then regard $f(z)$ as function on the analytic group $X(U)$ and obtain

$$f(z) = f(\chi_z(g) - 1) = F(\chi_z) , \tag{3.1}$$

where χ_z is the character of the group $X(U)$ defined by the condition $\chi_z(1 + p) = 1 + z$.

For each $x \in U$ we have $x = (1 + p)^{\log x/\log(1+p)}$ and hence that

$$\chi_z(x) = (1 + z)^{\log x/\log(1+p)} .$$

By Theorem 1 we obtain the integral representation of the function $F(\chi_z)$:

$$F(\chi_z) = \int_U \chi_z d\mu , \tag{3.2}$$

where the measure μ is defined by the formula:

$$\mu\{x^k \psi_{a,m}\} = \frac{1}{p^m - p^{m-1}} \sum_{\chi \bmod p^m} \chi^{-1}(a) F(x^k \chi) , \tag{3.3}$$

where χ runs on the set of Dirichlet characters of modulo p^m. We note that by isomorphism (1.1) a character χ corresponds to a point $\xi - 1$ where ξ is a root of order p^m of unity. Hence formula (3.3) takes the form:

$$\mu\{x^k \psi_{a,m}\} = \frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m}=1, \xi \neq 1} \xi^{-a} f\{(1+p)^k \xi - 1\} , \qquad k = 0,1, \dots ,h-1 . \tag{3.4}$$

Thus, we obtain an integral representation of the function $f(z)$ in the form:

$$f(z) = \int_U (1+z)^{\log x / \log(1+p)} d\mu , \tag{3.5}$$

where the measure μ is defined by formula (3.4). By the isomorphism $U \longrightarrow \mathbb{Z}_p : (1+p)^x \longmapsto x$ we may write formula (3.5) in the form:

$$f(z) = \int_{\mathbb{Z}_p} (1+z)^x d\mu(x) . \tag{3.6}$$

We now turn to consider p–adic bounded analytic functions on D. By using an argument similar to that showing Theorem 1 we obtain the following inverse formula for the bounded Mellin–Mazur transform.

Theorem 2. Let $f(x)$ be a bounded analytic function on $X(U)$. Then the following formula defines a bounded measure on $X(U)$:

$$\mu\{a + U_m\} = \frac{1}{p^m - p^{m-1}} \sum_{\chi} \chi^{-1}(a) F(\chi) \tag{3.7}$$

where χ runs on the set of Dirichlet characters of modulo p^m, $m = 1,2, \dots$. Furthermore we have

$$F(\chi) = \int_U \chi \, d\mu .$$

Using Theorem 2 we may obtain integral representations of bounded analytic functions on D as follows. Let $f(z)$ be a bounded analytic function. As in the case concerning functions of class $o(\log^h)$ we have

$$f(z) = \int_{\mathbb{Z}_p} (1 + z)^x d\mu(x)$$

where the measure μ is defined by the formula

$$\mu(a + U_m) = \frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m}=1,\xi\neq 1} \xi^{-a} f(\xi - 1) .$$

We now apply this representation to Morita's p–adic Γ–function . In [1] it is proved that we may consider the function $\Gamma_p(x)$ as the restriction on $\mathbb{Z}_p$ of a locally analytic function $\Gamma_p(z)$ on D of local analyticity ratio $p = p^{-1/p - 1/p-1}$. This means for each point $x \in \mathbb{Z}_p$ there exists ρ_x such that on $D(x,\rho_x) \cap \mathbb{Z}_p$ the function $\Gamma_p(x)$ is the restriction of $f(z) = \sum_{n\geq 0} a_n(z - x)^n$ which is analytic on $D(x,\rho_x)$. The local analyticity ratio, by definition, is the number

$$\rho = \inf_{x\in\mathbb{Z}_p} \rho_x > 0 .$$

Thus, on the disk $D(0,p^{-1/p - 1/p-1})$ the function $\Gamma_p(z)$ is represented by a convergent power series. We set

$$f(z) = \Gamma_p(p^{-1/p - 1/p-1}z) ,$$

then $f(z)$ is a bounded analytic function on the unit disk D . We have an integral represention of the function $f(z)$:

$$f(z) = \int_{\mathbb{Z}_p} (1 + z)^x d\mu \qquad (3.8)$$

where the measure μ is defined by the following formula:

$$\mu\{a + U_m\} = \frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m}=1,\xi\neq 0} \xi^{-a} f(\xi - 1) . \qquad (3.9)$$

Hence, for Morita's p–adic Γ–function we have the following integral representation

$$\Gamma_p(p^{1/p - 1/p-1} z) = \int_{\mathbb{Z}_p} (1 + z)^x d\mu ,$$

where the measure μ is defined by the formula:

$$\mu\{a + U_m\} = \frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m}=1,\xi\neq 1} \xi^{-a} \Gamma_p(p^{1/p - 1/p-1}(\xi - 1)) .$$

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