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TENSOR RANK: MATCHING POLYNOMIALS AND SCHUR RINGS

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Abstract. We study the polynomial equations vanishing on tensors of a given rank. By means of polarization we reduce them to elements A of the group algebra $\mathbb{Q}[S_n \times S_n]$ and describe explicitely linear equations on the coefficients of A to vanish on tensors of a given rank. Further, we reduce the study to the Schur ring over the group $S_n \times S_n$ that arises from the diagonal conjugacy action of S_n. More closely, we consider elements of $\mathbb{Q}[S_n \times S_n]$ vanishing on tensor of rank $n-1$ and describe them in terms of triples of Young diagrams, their irreducible characters and nonvanishing of their Kronecker coefficients. Also, we construct a family of elements in $\mathbb{Q}[S_n \times S_n]$ vanishing on tensors of rank $n-1$ and illustrate our approach by a sharp lower bound on the border rank of an explicitly produced tensor. Finally, we apply this construction to prove a lower bound $5n^2/4$ on the border rank of the matrix multiplication tensor (being, of course, weaker than the best known one $3n^2/2+n/2-1$, due to T.Lickteig).

1. Introduction

In this paper we propose an approach to producing equations for tensors of a given rank with the goal of obtaining lower bounds on the rank. We recall (see e.g. [20, 23, 4, 11, 12]) that the rank $\mathsf{rk}(A)$ of a tensor $A \in U \otimes V \otimes W$ is defined to be the minimal positive integer r such that there exist vectors $u^{(i)} \in U$, $v^{(i)} \in V$, $w^{(i)} \in W$, $i = 1, \dots, r$, for which

$$A = \sum_{1 \le i \le r} u^{(i)} \otimes v^{(i)} \otimes w^{(i)}.$$

(Throughout this paper we assume that the vector spaces U, V and W are defined over an algebraically closed field of characteristic zero.) Clearly, the concept of the rank of a tensor generalizes the one of the matrix rank. But unlike the matrix rank the tensor rank is not semicontinuous. That is why one studies the border rank $\underline{\mathsf{rk}}(A)$ being the maximal semicontinuous function for which $\underline{\mathsf{rk}}(A) \le \mathsf{rk}(A)$.

The tensor rank equals the multiplicative complexity of computing a family of bilinear forms with noncommuting variables [20]. One of the main inspiring problems in this context is to estimate the multiplicative complexity of $n \times n$ matrix multiplication, that is equal to the rank $\mathsf{rk}(M_n)$ of the structure tensor M_n of the algebra of $n \times n$ matrices. The best known bounds are

$$5n^2 - 3n \le \mathsf{rk}(M_n) \le O(n^{2.38}),$$

we refer to [23, 4, 11, 12] for the development and the history of the upper bound, and to [2] for the lower bound. In [18] it was established a lower bound on the border rank $\underline{\mathsf{rk}}(M_n) \ge 3n^2/2 + n/2 - 1$. In [16] the best known current bound $\underline{\mathsf{rk}}(M_n) \ge (2-\epsilon) \cdot n^2$ for any $\epsilon > 0$ was proved.

Thus, the gap between the upper and lower bounds is big. One of its reasons is the lack of explicit equations on the variety $T_r \subset U \otimes V \otimes W$ of the tensors with the border rank less or equal to r. There are several approaches in this direction.

Strassen in [21] has constructed explicit equations on the variety T_r for certain r's in the case $\dim(U) = 3$ and $\dim(V) = \dim(W) = n$. This result was extended in [14] to more general tensors of order more than 3. Another approach is based on the general idea of "embedding" tensors into appropriate matrices (called flattenings) and estimating the rank of these matrices [15, 13, 6, 16]. A study of the closures of the $\mathsf{GL}(U) \otimes \mathsf{GL}(V) \otimes \mathsf{GL}(W)$-orbits of tensors was proposed first in [22] and then developed further in [5].

A similar problem of estimating the rank for the symmetric product (rather than the tensor product) was studied e.g. in [17] (see also the numerous references in the latter paper); earlier a method to obtain lower bound for the rank in this sitution was suggested in [7]. We mention also a topological approach that was proposed in [8] for a related problem on lower bounds for the complexity of polynomials.

Let us briefly discuss the contents of our paper. In Section 2 we establish a reduction from general polynomials on tensors $A = (A_{ijk})$ to the *matching polynomials* which are homogeneous, and polylinear in a strong sense: the indices of variables X_{ijk} occuring in any given monomial form a 3-dimensional matching

$$\{(i^f, i^g, i^h) : \ i = 1, \ldots, n\} \tag{1}$$

where n is the degree of the matching polynomial, and $f, g, h \in S_n$ are permuatations depending on the monomial. One can treat such a matching polynomial on $n \times n \times n$ tensors, which vanishes on the rank $n-1$ tensors, as a 3-dimensional analogue of the customary determinant (or more generally, a 3-dimensional subdeterminant vanishing on tensors of fixed rank $r < n$). Our reduction is a special polarization which preserves the property "to vanish on T_r" . Subsequently, having a 3-dimensional determinant D one can pass to a polynomial vanishing on T_r for tensors of a smaller size $n_1 \times n_2 \times n_3$ $(n_1, n_2, n_3 \le n)$ by means of depolarization just identifying suitable variables of D. Since the polarization and the depolarization are transformation being inverse to each other, one may reduce the study of equations on T_r to matching polynomials.

In its turn the 3-dimensional matchings (1) are in 1-1 correspondence with the elements $(f^{-1}g, f^{-1}h)$ of the group $S_n \times S_n$. This enables us to identify a matching polynomial with an element of the group algebra $\mathbb{Q}[S_n \times S_n]$. In Section 3 we describe explicitly (linear) equations on the coefficients of an element of this algebra that corresponds to a 3-dimensional (sub)determinant vanishing on tensors of rank at most r. Since these equations, and thereby, their space of solutions

$$V_{n,n-r} \subset \mathbb{Q}[S_n \times S_n],$$

are invariant under the (diagonal) conjugacy action of S_n, the space $V_{n,n-r}$ is generated as a right ideal in $\mathbb{Q}[S_n \times S_n]$ by the intersection $V_{n,n-r} \cap \mathcal{A}$ where $\mathcal{A} \subset \mathbb{Q}[S_n \times S_n]$ is the Schur ring of this action (see Section 4). Moreover,

$$V_{n,n-r} \cap \mathcal{A} = \bigoplus_{\pi} V_{n,n-r} \cap \mathcal{A}_\pi$$

where the direct sum ranges over the irreducible representations of $\mathcal{A}$. Furthermore, in Section 4 we describe the latter representations in terms of triples of Young diagrams and nonvanishing of their Kronecker coefficients. Finally, in this section we provide the conditions (in terms of the Young diagrams) when the depolarization does not vanish identically.

In Section 5 we study more closely the case of the rank of $n \times n \times n$ tensors equal $n-1$, and give an explicit criterion for a matching polynomial to be a 3-dimensional determinant; this criterion is expressed in terms of the triples of Young diagrams. Also in this section we show that, unfortunately, if a depolarization of 3-dimensional determinant does not vanish on $n_1 \times n_2 \times n_3$ tensors, then $n_i > n/3$ for some $i \in \{1, 2, 3\}$. This implies that in order to obtain nonlinear lower bound on the tensor rank one should consider elements of $V_{n,n-r}$ with $r < n-1$ (perhaps, with the rank r significantly less than $n-1$).

Finally, in Section 5 we construct a particular family of elements in $V_{n,n-1}$ which we apply in Section 6 to yield a $(2m+1) \times (2m+1) \times (2m+1)$ tensor A such that $\mathsf{rk}(A) = \underline{\mathsf{rk}}(A) = 3m$. Also as an illustration of our approach we apply in Section 6 the latter construction to get a bound $\underline{\mathsf{rk}}(M_n) \geq 5n^2/4$ (being, of course, weaker than the best known bound from [18].

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Notations.

For positive integers $m \leq n$ we set $[m, n] = \{m, m+1, \ldots, n\}$ and $[n] = [1, n]$.

The set of all (resp. r-class) partitions of $[n]$ is denoted by $\Lambda(n)$ (resp. $\Lambda(n, r)$). The Young diagram of $\lambda \in \Lambda(n)$ is denoted by $[\lambda]$.

The set of all Young diagrams with n nodes is denoted by $\mathbf{\Lambda}(n)$; the subset of $\mathbf{\Lambda}(n)$ that consists of diagrams with r rows is denoted by $\mathbf{\Lambda}(n, r)$.

The Young subgroup of a partition $\lambda \in \Lambda(n)$ is denoted by S_λ.

The identity of a group G is denoted by e_G.

Given a group G and a set $H \subset G$ the sum $\sum_{h \in H} h$ in the group ring $\mathbb{Q}G = \mathbb{Q}[G]$ is denoted by $\underline{H}$.

For a set S the algebra of all rational $S \times S$-matrices is denoted by $\mathsf{Mat}_S(\mathbb{Q})$, or $\mathsf{Mat}_n(\mathbb{Q})$ if $S = [n]$.

2. Tensor rank, polarization and matching polynomials

For the sake of simplifying notations in what follows we will study cubic tensors (i.e. tensors of order 3), but actually one could study parallelepiped tensors as well.

We observe that the variety T_r (see Section 1) is defined over the field $\mathbb{Q}$. Therefore it suffices to look for polynomials which vanish on T_r with rational coefficients. Thus throughout the paper we assume that all the polynomials have rational coefficients.

Let $\mathcal{P}(n)$ be the set of all homogeneous polynomials $P(X)$ on $n \times n \times n$ tensors; here $X = \{X_{ijk}\}_{i,j,k \in [n]}$ denotes the set of variables. Our goal is to find those polynomials $P(X)$ for which $P(A) = 0$ for all $n \times n \times n$ tensors A of rank at most r. The linear space of all these polynomials is denoted by $\mathcal{P}(n, r)$. The key point of our approach to look for elements of this space is the concept of a matching poynomial introduced below.

We say that $P(X) \in \mathcal{P}(n)$ is a (3-dimensional) *matching polynomial* if it is multilinear and any of its monomial is given by two permutations of the symmetric group S_n; more precisely, $P(X)$ is a linear combination of the monomials

$$M_g(X) = \prod_{i=1}^{n} X_{i\ i^{g_1}\ i^{g_2}}, \qquad g \in S_n^2 \tag{2}$$

where $S_n^2 = S_n \times S_n$, and g_1 and g_2 are the coordinates of g. The linear space of matching polynomials is denoted by $\mathcal{M}(n)$.

Polarization. Given any polynomial $P(X) \in \mathcal{P}(n)$ one can construct a matching polynomial belonging to $\mathcal{M}(N)$ for some $N \geq n$ by means of a series of "local" polarizations of one of the following types:

$$\cdots \prod_{j=1}^{k} X_{ia_jb_j} \cdots \mapsto \cdots \frac{1}{k!}\left(\sum_{\pi\in S_k}\prod_{j=1}^{k} X_{i_{j^\pi}\,a_jb_j}\right)\cdots, \tag{3}$$

$$\cdots \prod_{j=1}^{k} X_{a_jib_j} \cdots \mapsto \cdots \frac{1}{k!}\left(\sum_{\pi\in S_k}\prod_{j=1}^{k} X_{a_ji_{j^\pi}\,b_j}\right)\cdots,$$

$$\cdots \prod_{j=1}^{k} X_{a_jb_ji} \cdots \mapsto \cdots \frac{1}{k!}\left(\sum_{\pi\in S_k}\prod_{j=1}^{k} X_{a_jb_ji_{j^\pi}}\right)\cdots,$$

where the monomial in the left-hand side of (3) contains k variables of the form X_{iab} for a fixed i (similar for other two polarizations), and $i_1, \ldots, i_k$ are new indices (each such a transformation increases the number of variables).

Given a monomial $M(X) \in \mathcal{P}(n)$ denote by u, v and w the nonnegative integer n-vectors such that u_s, v_s and w_s are respectively the numbers (taken with multiplicities) of variables X_{sjk}, X_{isk} and X_{ijs} that occur in $M(X)$ when the indices i, j, k run over $[n]$. The triple $(u, v, w) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^n$ will be called the *multidegree* of $M(X)$. The sum of all monomials of a polynomial $P(X) \in \mathcal{P}(n)$ that have the same multidegree is called a *multihomogeneous* component of $P(X)$. The polynomial $P(X) \in \mathcal{P}(n)$ is multihomogeneous if it has exactly one multihomogeneous component; if, in additiion, $\sum_i u_i = \sum_i v_i = \sum_i w_i =: N$, the polynomial $P(X)$ is called *N-uniform.*

Lemma 2.1. *Let $r \in [n]$ and $P(X) \in \mathcal{P}(n, r)$. Then*

(1) *each multihomogeneous component of $P(X)$ belongs to $\mathcal{P}(n, r)$,*
(2) *if $P(X)$ is N-uniform, then the polarization of $P(X)$ belongs to $\mathcal{P}(N, r)$.*

Proof. To prove statement (1) take $3n$ new variables $\varepsilon_s^{(u)}$ with $s \in [n]$ and $u \in [3]$. Let us consider

$$P^\varepsilon(X) = P(\ldots, X_{ijk} := X_{i,j,k}\,\varepsilon_i^{(1)}\varepsilon_j^{(2)}\varepsilon_k^{(3)}, \ldots).$$

as a polynomial in these variables. The coefficient of this polynomial at the monomial $M = \cdots(\varepsilon_s^{(u)})^{d(s,u)}\cdots$ coincides with the multihomogeneous component of $P(X)$, the multidegree of which is $(\ldots, d(s, u), \ldots)$ where $d(s, u)$ is the degree of the variable $\varepsilon_s^{(u)}$ in M. Now, since $P(X) \in \mathcal{P}(n, r)$, the above coefficient at M is equal to zero. This implies that the multihomogeneous component of $P(X)$ that contains M, vanishes at any $n \times n \times n$ tensor of rank $\le r$. Thus each multihomogeneous component of a polynomial $P(X)$ belongs to $\mathcal{P}(n, r)$.

To prove statement (2) for the sake of simplifying notation suppose that the polarization $Q(Y)$ of $P(X)$ is obtained by a single "local" polarization of the form (3). Thus $P(X)$ is defined on $n_1 \times n_2 \times n_3$-tensors for some positive integers n_1, n_2, n_3, while $Q(Y)$ is defined on $(n_1 + k - 1) \times n_2 \times n_3$-tensors.

Take any $(n_1+k-1)\times n_2\times n_3$-tensor A of rank $\le r$. Then there exist (n_1+k-1)-dimensional vector $X^{(u)}$, n_2-dimensional vector $Y^{(u)}$ and n_3-dimensional vector $Z^{(u)}$, $u = 1, \dots, r$, such that

$$A_{\alpha,\beta,\gamma} = \sum_{u=1}^{r} X^{(u)}_{\alpha} Y^{(u)}_{\beta} Z^{(u)}_{\gamma} \tag{4}$$

for all α, β, γ. For each index u and a nonempty set $I \subset [k]$ denote by $X^{(u)}(I)$ the n_1-dimensional vector which is obtained from $X^{(u)}$ be removing its $i_1, \dots, i_k$-entries and taking $\sum_{j\in I} X^{(u)}_{i_j}$ as the i-entry, Then it is straightforward to check that

$$Q(A) = \frac{1}{k!} \sum_{\emptyset \ne I \subset [k]} (-1)^{k-|I|} \cdot P(A(I)), \tag{5}$$

where

$$A(I) = \sum_{u=1}^{r} X^{(u)}(I) \cdot Y^{(u)}(I) \cdot Z^{(u)}(I)$$

is the $n_1 \times n_2 \times n_3$-tensor (cf. (4)). Since obviously $\mathsf{rk}(A(I)) \le r$ for all I, and $P(X) \in \mathcal{P}(n, r)$, the right-hand side in (5) is equal to 0, and we are done. ∎

Depolarization. Given a matching polynomial $P(X) \in \mathcal{M}(n)$ and partitions $\lambda_1, \lambda_2, \lambda_3 \in \Lambda(n, m)$ we define the $(\lambda_1, \lambda_2, \lambda_3)$-*contraction* of $P(X)$ to be the polynomial

$$P^{\lambda_1,\lambda_2,\lambda_3}(Y) = P(\dots, X_{ijk} := Y_{c_1(i)c_2(j)c_3(k)}, \dots) \tag{6}$$

where $Y = \{Y_{ijk}\}_{i,j,k\in[m]}$ and $c_a(b)$ is the number of the class of the partition λ_a that contains b. Thus, $P^{\lambda_1,\lambda_2,\lambda_3}(Y) \in \mathcal{P}(m)$.

Lemma 2.2. *In the above notations $P \in \mathcal{P}(n, r) \;\Rightarrow\; P^{\lambda_1,\lambda_2,\lambda_3} \in \mathcal{P}(m, r)$.*

Proof. Given an $m \times m \times m$ tensor A set

$$(A^{\lambda_1,\lambda_2,\lambda_3})_{i,j,k} = A_{c_1(i),c_2(j),c_3(k)}, \quad i, j, k \in [n].$$

Then $A^{\lambda_1,\lambda_2,\lambda_3}$ is an $n\times n\times n$ tensor, and it is easily seen that $\mathsf{rk}(A) = \mathsf{rk}(A^{\lambda_1,\lambda_2,\lambda_3})$. On the other hand from (6) it immediately follows that

$$P^{\lambda_1,\lambda_2,\lambda_3}(A) = P(A^{\lambda_1,\lambda_2,\lambda_3})$$

and we are done. ∎

Since for any multihomogeneous polynomial $P(X)$ the result of its polarization with a subsequent appropriate depolarization coincides with $P(X)$, any multihomogeneous polynomial from $\mathcal{P}(n, r)$ can be obtained as the depolarization of a certain matching polynomial from $\mathcal{P}(N, r)$ where N is as statement (2) of Lemma 2.1.

3. A reduction to the group algebra of $S_n \times S_n$

Thoughout the section we fix a number $d \in [n]$. Under a *defect d cubic determinant* we mean any matching polynomial belonging the set $\mathcal{P}(n, n-d)$. This name is justified since such a polynomial vanishes at any tensor of the rank less or equal to $n-d$ (generalizing polynomials in matrices). A transition from $\mathcal{M}(n)$ to the group algebra $\mathbb{Q}S_n^2$ is provided by the linear space isomorphism

$$F_n : \mathcal{M}(n) \to \mathbb{Q}S_n^2, \quad \sum_{g \in S_n^2} a(g) M_g \mapsto \sum_{g \in S_n^2} a(g) g.$$

Lemma 3.1. *Let $\chi \in \mathbb{Q}S_n^2$ and $P = F_n^{-1}(\chi)$. Then P is a cubic determinant of defect d if and only if χ is a solution of the following system of linear equations:*

$$\underline{S}_\lambda^2 \cdot \chi = 0, \quad \lambda \in \Lambda(n, r) \tag{7}$$

where $S_\lambda^2 = S_\lambda \times S_\lambda$ and $r = n - d$.

Proof. To prove the sufficiency suppose that χ is a solution of system (7). Take an $n \times n \times n$ tensor A of rank $\le r$. Then there exist n-vectors $X^{(u)}$, $Y^{(u)}$ and $Z^{(u)}$, $u = 1, \ldots, r$, such that

$$A_{ijk} = \sum_{u=1}^{r} X_i^{(u)} Y_j^{(u)} Z_k^{(u)}, \qquad i, j, k \in [n].$$

Without loss of generality we can assume that the elements of the vectors $X^{(u)}$, $Y^{(u)}$ and $Z^{(u)}$ are independent pairwise commuting variables. Then $P(A)$ is a polynomial in these variables, and for any monomial $M_g(X)$, $g \in S_n^2$, of the polynomial $P(X)$ we have (see (2)):

$$M_g(A) = \prod_{v=1}^{n} \Big(\sum_{u=1}^{r} X_v^{(u)} Y_{v^{g_1}}^{(u)} Z_{v^{g_2}}^{(u)} \Big).$$

This implies that any monomial of the polynomial $M_g(A)$, and hence any monomial of the polynomial $P(A)$, is uniquely determined by a map $\theta : [n] \to [r]$, in terms of which this monomial can be written as follows:

$$M(g, \theta) = \prod_{v=1}^{n} X_v^{(\theta(v))} Y_{v^{g_1}}^{(\theta(v))} Z_{v^{g_2}}^{(\theta(v))}.$$

Though not all of these monomials are distinct, we have

$$P(A) = (F_n^{-1}(\chi))(A) = \sum_{g \in S_n^2} \sum_{\theta : [n] \to [r]} \chi(g) M(g, \theta) \tag{8}$$

where $\chi(g)$ is the coefficient of χ at g. In fact, two monomials $M(g, \theta)$ and $M(g', \theta')$ can be equal only if $\theta = \theta'$. Denote by λ the partition of $[n]$ with r' non-empty classes $\theta^{-1}(i)$, $i \in [r]$. Then $r' \le r$, $\lambda \in \Lambda(n, r')$, and $M(hg, \theta) = M(g, \theta)$ does not depend on $h \in S_\lambda^2$. This shows that

$$M(g, \theta) = M(g', \theta) \iff g' \in S_\lambda^2 g.$$

Thus given a right S_λ^2-coset C in the group S_n^2, the monomial $M(g, \theta)$ does not depend on $g \in C$; we denote it by $M(C, \lambda)$. Then by (8) we obtain that

$$P(A) = \sum_{\lambda \in \Lambda_r} \sum_{C \in C_\lambda} \sum_{g \in C} \chi(g) M(C, \lambda) = \sum_{\lambda \in \Lambda_r} \sum_{C \in C_\lambda} \left(\sum_{g \in C} \chi(g) \right) M(C, \lambda)$$

where C_λ is the set of all S^2_λ-cosets in S^2_n, and Λ_r is the set of all partitions of $[n]$ into at most r classes. Thus, to prove that P is of defect d it suffices to verify that given $\lambda \in \Lambda_r$ and $C \in C_\lambda$ we have

$$\sum_{g\in C} \chi(g) = 0. \tag{9}$$

First, we note that for any partition $\lambda \in \Lambda_r$ one can find a refinement $\lambda' \in \Lambda(n,r)$ of it. By the hypothesis of the lemma we have

$$0 = \underline{S}^2_{\lambda'} \cdot \chi = \underline{S}^2_{\lambda'} \cdot \sum_{g\in S} \chi(g) g = \sum_{C'\in C_{\lambda'}} \left(\sum_{h\in C'} \chi(h) \right) \underline{C}'.$$

Therefore

$$\sum_{g\in C'} \chi(g) = 0 \quad \text{for all } C' \in C_{\lambda'}.$$

Since $S^2_{\lambda'}$ is a subgroup of S^2_λ, each S^2_λ-coset C is a union of $S^2_{\lambda'}$-cosets $C' \in C_{\lambda'}$. This implies (9), and hence the sufficiency. The necessity can be proved in a similar way. ■

Lemma 3.1 shows that all cubic determinants of defect d are in 1-1 correspondence with the solutions of the linear system defined by (7). The set of all of them is denoted by $V = V_{n,d}$. Clearly, this set is a right ideal of the algebra $\mathbb{Q}S$ where $S = S^2_n$. Lemma 3.2 below enables us to reduce the system (7) to a single linear equation by means of the element

$$\zeta = \zeta_{n,d} = \sum_{\lambda\in\Lambda(n,n-d)} \underline{S}^2_\lambda. \tag{10}$$

Obviously, given $g \in S$ and $u \in S_n$ we have $g \in S^2_\lambda$ if and only if $g^u \in S^2_{\lambda^u}$ where $g^u = (g^u_1, g^u_2)$. Therefore the coefficients of ζ are constant on the orbits of the coordinatewise conjugacy action of S_n on S.

To compute the element $\zeta_{n,1}$, we note that every partition $\lambda \in \Lambda(n, n-1)$ has exactly one class of size 2 and $n-2$ singleton classes. Therefore S^2_λ is a Klein group whose non-identity elements are (t,t), $(1_n, t)$ and $(t, 1_n)$ for some transposition $t \in S_n$ where 1_n is the identity permutation in S_n. It follows that

$$\zeta_{n,1} = \binom{n}{2} \underline{1}_n + \underline{C}_1 + \underline{C}_2 + \underline{C}_3 \tag{11}$$

where $C_1 = \mathrm{Diag}(T \times T)$, $C_2 = \{1_n\} \times T$ and $C_3 = T \times \{1_n\}$ with $T = T_n$ being the set of transpositions in S_n.

Lemma 3.2. *In the above notations* $V = \{\chi \in \mathbb{Q}S : \ \zeta \cdot \chi = 0\}$.

Proof. Clearly, $\zeta \cdot \chi = 0$ for all $\chi \in V$. Conversely, let $\chi \in \mathbb{Q}S$ be such that $\zeta \cdot \chi = 0$. We have to verify that $\underline{S}^2_\lambda \cdot \chi = 0$ for all $\lambda \in \Lambda(n,r)$ where $r = n - d$. However,

$$\zeta \cdot \chi = 0 \ \Rightarrow \ \langle \zeta \cdot \chi, \chi \rangle = 0 \ \Rightarrow \ \sum_{\lambda\in\Lambda(n,r)} \langle \underline{S}^2_\lambda \cdot \chi, \chi \rangle = 0 \tag{12}$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{Q}S$. Since S^2_λ is a subgroup of S, the quadratic form $\langle \underline{S}^2_\lambda \cdot \chi, \chi \rangle$ is positive semidefinite. Hence the right-hand side equality in (12) implies that $\langle \underline{S}^2_\lambda \cdot \chi, \chi \rangle = 0$, and hence $\underline{S}^2_\lambda \cdot \chi = 0$ for all $\lambda \in \Lambda(n,r)$. ■

We complete the section by giving a necessary and sufficient condition for a matching polynomial to have a nonzero contraction in terms of the Young subgroups of the corresponding partitions. Below we will use the group monomorphisms from S_n into S defined by

$$\Delta_1 : g \mapsto (g, g), \qquad \Delta_2 : g \mapsto (g, 1_n), \qquad \Delta_3 : g \mapsto (1_n, g). \tag{13}$$

Each of them is linearly extended to the "coordinate" embedding of the algebra $\mathbb{Q}S_n$ into $\mathbb{Q}S$. In what follows we will denote the corresponding algebra monomorphisms also by Δ_1, Δ_2 and Δ_3. For an element $\chi \in \mathbb{Q}S$ set

$$\chi^{\lambda_1,\lambda_2,\lambda_3} = \Delta_1(\underline{S}_{\lambda_1}) \cdot \chi \cdot \Delta_2(\underline{S}_{\lambda_2}) \cdot \Delta_3(\underline{S}_{\lambda_3}).$$

This element will be called the $(\lambda_1, \lambda_2, \lambda_3)$-*contraction* of χ.

Lemma 3.3. *Let $P \in \mathcal{M}(n)$, $\chi = F_n(P)$ and $\lambda_1, \lambda_2, \lambda_3 \in \Lambda_n$. Then*

$$P^{\lambda_1,\lambda_2,\lambda_3} \neq 0 \iff \chi^{\lambda_1,\lambda_2,\lambda_3} \neq 0.$$

Proof. By the definition of the mapping F_n we have

$$P(X) = \sum_{g \in S} \chi(g) M_g(X).$$

Let us define the equivalence relation $\sim$ on the group S by setting $g \sim h$ if and only if $g \in C_h := \mathrm{Diag}(S_{\lambda_1,\lambda_1})\, h\, S_{\lambda_2,\lambda_3}$. It is easily seen that the monomial $(M^{\lambda_1,\lambda_2,\lambda_3})_g(Y)$ does not depend on the choice of g in an equivalence class C of this equivalence relation. Denote this monomial by $M_C(Y)$. Then

$$P^{\lambda_1,\lambda_2,\lambda_3}(Y) = \sum_C \left(\sum_{g \in C} \chi(g) \right) M_C(Y)$$

where C runs over the equivalence classes of $\sim$. Therefore $P^{\lambda_1,\lambda_2,\lambda_3} \neq 0$ if and only if $\sum_{g \in C} \chi(g) \neq 0$ for some equivalence class C. However, obviously,

$$\sum_{h \in S} \left(\sum_{g \in C_h} \chi(g) \right) h = \Delta_1(\underline{S}_{\lambda_1}) \cdot \chi \cdot \Delta_2(\underline{S}_{\lambda_2}) \cdot \Delta_3(\underline{S}_{\lambda_3}).$$

This implies the required statement because $\{C_h : h \in S\}$ is the set of classes of the equivalence relation $\sim$. ∎

4. A decomposition of the cubic determinant space via a Schur ring

Throughout this section we fix positive integers n and $d \in [n]$, and set $V = V_{n,d}$. The group $G = G_n = S_n^3$ has two subgroups $G_0 \cong S_n$ and $S_0 \cong S_n^2$, defined by

$$G_0 = \mathrm{Diag}(S_n^3) \quad \text{and} \quad S_0 = \{1_n\} \times S_n^2.$$

Since $G = G_0 \cdot S_0$ and the right G_0-cosets are in the natural 1-1 correspondence with the elements of S_0, the action of G on the right G_0-cosets induces a permutation group on $S = S_n^2$. In this action we have

$$(G_0 \cdot (1_n, x, y))^{(1_n, g, h)} = G_0 \cdot (1_n, xg, yh) \quad \text{and} \quad (G_0 \cdot (1_n, x, y))^{(g,g,g)} = G_0 \cdot (1_n, x^g, y^g)$$

for all $x, y, g, h \in S_n$. Therefore after identifying in a natural way S_0 with S the group G becomes a permutation group on S, in which the subgroup S acts on itself by the right multiplications, and the subgroup G_0 acts by coordinatewise conjugation. In particular, G is transitive, and the stabilizer of the point $e_S = (1_n, 1_n)$ in G is equal to G_0. The action of G on S defines a permutation representation of G defined by

$$R : G \to \mathsf{Mat}_S(\mathbb{Q}),\ g \mapsto P_g \tag{14}$$

where P_g is the permutation matrix of g: $(P_g)_{x,y} = \delta_{x^g,y}$ for all $x, y \in S$. In particular, $(\sum_x a_x x)P_g = \sum_x a_x x^g$ for all $a_x \in \mathbb{Q}$.

Lemma 4.1. *In the above notations the linear space V is a G-invariant. In particular, it is the direct sum of G-irreducible subspaces.*

Proof. The set V is S-invariant because S acts on itself by the right multiplications, and V is a right ideal of the algebra $\mathbb{Q}S$. Next, it was mentioned just after the definition (10) that the coefficients of the element ζ are constant on the orbits of the coordinatewise conjugacy action of S_n on S. So by Lemma 3.2 the set V is G_0-invariant. Thus V is a G-invariant and we are done. ∎

Denote by $\mathcal{A} = \mathcal{A}_n$ the linear subspace of the algebra $\mathbb{Q}S$ spanned by the set S/G_0 of all orbits of the permutation group $G_0 \leq \mathsf{Sym}(S)$,

$$\mathcal{A} = \mathsf{Span}_{\mathbb{Q}}\{\underline{X} :\ X \in S/G_0\}.$$

Then since S is a regular subgroup of G, the subspace V is also a subalgebra of the algebra $\mathbb{Q}S$, see [24, Chapter IV]. In terms of the latter book, $\mathcal{A}$ is the *transitivity module* of the group S_0, and hence *the Schur ring* over the group S, that corresponds to the permutation group G. In particular, $\{\underline{X} :\ X \in S/G_0\}$ is a linear base of $\mathcal{A}$ that contains 1_S and is closed with respect to taking inverses, $\underline{X} \to \underline{X}^{-1}$.

Denote by $\mathsf{Irr}(\mathcal{A})$ the set of all $\mathbb{Q}$-irreducible characters of $\mathcal{A}$. As we will see below (Lemma 4.3) these characters are absolutely irreducible. By the Wedderburn theorem this implies that

$$\mathcal{A} = \bigoplus_{\pi \in \mathsf{Irr}(\mathcal{A})} \mathcal{A}_\pi \tag{15}$$

where $\mathcal{A}_\pi$ is a simple algebra isomorphic to $\mathsf{Mat}_{n_\pi}(\mathbb{Q})$ with $n_\pi = \pi(1_S)$. The following statement shows how the space of cubic determinants of defect d can be constructed from spaces $V \cap \mathcal{A}_\pi$ where $V = V_{n,d}$.

Lemma 4.2. *The linear space V is generated (as a right ideal of the algebra $\mathbb{Q}S$) by the linear space $V \cap \mathcal{A}$. Moreover,*

$$V \cap \mathcal{A} = \bigoplus_{\pi \in \mathsf{Irr}(\mathcal{A})} V \cap \mathcal{A}_\pi. \tag{16}$$

Proof. By Lemma 4.1 the linear space V is the direct sum of G-irreducible subspaces. Let U be one of them. Then

$$U = \mathsf{Span}_{\mathbb{Q}}\{uP_g :\ g \in G\}$$

for all nonzero vectors $u \in U$. Therefore to prove the first statement it suffices to verify that $U \cap \mathcal{A} \neq \{0\}$. For this purpose we note that $U = \mathbb{Q}SE$ for some (not necessarily central) idempotent matrix E such that $EP_g = P_gE$ for all $g \in G$. Then for $u = 1_SE$ and $g \in G_0$ we have

$$uP_g = (1_SE)P_g = (1_SP_g)E = 1_SE = u.$$

This implies that $u \in \mathcal{A}$. Besides, $u \neq 0$, because otherwise by the transitivity of G we have $0 = uP_g = (1_S E)P_g = (1_S P_g)E = gE$ for all $g \in S$ which is impossible. Thus $0 \neq u \in U \cap \mathcal{A}$. To prove the second statement we note that $V \cap \mathcal{A}$ is a right $\mathcal{A}$-ideal. Therefore (16) immediately follows from (15). ∎

Given a character $\pi \in \mathsf{Irr}(\mathcal{A})$ denote by e_π the central primitive idempotent of the algebra $\mathcal{A}$ that corresponds to π. Then $\mathcal{A}_\pi = e_\pi \cdot \mathcal{A}$. Therefore to study the linear space $V \cap \mathcal{A}_\pi$ we find the explicit expression for the idempotent e_π in terms of the irreducible characters of the group S_n. For this purpose let us fix some notations. First, given $\rho \in \mathsf{Irr}(S_n)$ we denote by e_ρ the central primitive idempotent of the algebra $\mathbb{Q}S_n$ that corresponds to ρ.[1] The rationality of ρ implies that

$$e_\rho = \frac{\rho(1_n)}{n!} \cdot \sum_{g \in S_n} \rho(g) g. \tag{17}$$

Second, given any two irreducible characters $\mu, \nu \in \mathsf{Irr}(S_n)$ we have $\mu \cdot \nu = \sum_\rho k^\rho_{\mu,\nu} \rho$ where ρ runs over the set $\mathsf{Irr}(S_n)$ and $k^\rho_{\mu,\nu}$ is a nonnegative integer called the *Kronecker coefficient* [3].

Lemma 4.3. *Given a character $\pi \in \mathsf{Irr}(\mathcal{A})$ there exist uniquely determined characters $\rho, \mu, \nu \in \mathsf{Irr}(S_n)$ such that $k^\rho_{\mu,\nu} \neq 0$ and $e_\pi = e_{\rho,\mu,\nu}$ where*

$$e_{\rho,\mu,\nu} = \Delta_1(e_\rho) \cdot \Delta_2(e_\mu) \cdot \Delta_3(e_\nu). \tag{18}$$

Proof. Denote by $Z(G,S)$ the centralizer algebra of the group $G \leq \mathsf{Sym}(S)$; this algebra consists of all matrices $A \in \mathsf{Mat}_S(\mathbb{Q})$ such that $AP_g = P_g A$ for all $g \in G$. Then by [24, Theorem 28.8] the linear mapping $A \to 1_S A$ is an algebra isomorphism from $Z(G,S)$ onto $\mathcal{A}$. Therefore given a character $\pi \in \mathsf{Irr}(\mathcal{A})$ there exists an irreducible character π' of the algebra $Z(G,S)$ such that

$$e_\pi = 1_S e_{\pi'}. \tag{19}$$

Next, the algebras $Z(G,S)$ and $R(\mathbb{Q}G)$ where the representation R is defined by (14), have the same central primitive idempotents. So $e_{\pi'}$ is also a central primitive idempotent of the algebra $R(\mathbb{Q}G)$. Therefore there exists a character $\pi_0 \in \mathsf{Irr}(G)$ such that $e_{\pi'} = R(e_{\pi_0})$ where e_{π_0} is the central primitive idempotent of $\mathbb{Q}S$ that corresponds to π_0. Since $G = S_n^3$, by [9, Theorem 4.21] there are uniquely determined characters $\rho, \mu, \nu \in \mathsf{Irr}(S_n)$ such that $\pi_0(g) = \rho(g)\mu(g)\nu(g)$ for all $g \in S_n$. Due to (17) we obtain that

$$e_{\pi'} = R(e_{\pi_0}) = \frac{\rho(1_n)\mu(1_n)\nu(1_n)}{(n!)^3} \cdot \sum_{u,v,w \in S_n} \rho(u)\mu(v)\nu(w) P_{g(u,v,w)}$$

where $g(u,v,w)$ is the permutation of S that is the image in G of the element $(u,v,w) \in S_n^3$. However, $g(u,v,w) = g_u \cdot g_v \cdot g_w$ where

$$g_u = g(u, 1_n, 1_n), \quad g_v = g(1_n, v, 1_n), \quad g_w = g(1_n, 1_n, w).$$

Therefore

$$\sum_{u,v,w \in S_n} \rho(u)\mu(v)\nu(w) P_{g(u,v,w)} = \sum_{u \in S_n} \rho(u) P_{g_u} \cdot \sum_{v \in S_n} \mu(v) P_{g_v} \cdot \sum_{w \in S_n} \nu(w) P_{g_w} =$$

[1] Here and below we are working in the algebra $\mathbb{Q}S_n$, because the irreducible representations of S_n over $\mathbb{Q}$ are absolutely irreducible.

$$\left(\sum_C \rho(C) \cdot \sum_{u \in C} P_{g_u}\right) \cdot \left(\sum_C \mu(C) \cdot \sum_{v \in C} P_{g_v}\right) \cdot \left(\sum_C \nu(C) \cdot \sum_{w \in C} P_{g_w}\right)$$

where C runs over the conjugacy classes of S_n and $\rho(C)$, $\mu(C)$ and $\nu(C)$ are respectively the values of ρ, μ and ν on an element of C. Besides, from the definition of g_u, g_v and g_w it follows that $1_S P_{g_u} = (1_S)^{g_u} = (u^{-1}, u^{-1})$, and similarly, $1_S P_{g_v} = (v, 1_n)$ and $1_S P_{g_w} = (1_n, w)$. Therefore

$$1_S \left(\sum_{u \in C} P_{g_u}\right) = \Delta_1(\underline{C}), \quad 1_S \left(\sum_{v \in C} P_{g_v}\right) = \Delta_2(\underline{C}), \quad 1_S \left(\sum_{w \in C} P_{g_w}\right) = \Delta_3(\underline{C})$$

(here in the first equality we used the fact that $C^{-1} = C$). Together with (19) these equalities imply that

$$e_\pi = \left(\frac{\rho(1_n)}{n!} \sum_C \rho(C)\Delta_1(\underline{C})\right) \cdot \left(\frac{\mu(1_n)}{n!} \sum_C \mu(C)\Delta_2(\underline{C})\right) \cdot \left(\frac{\nu(1_n)}{n!} \sum_C \nu(C)\Delta_3(\underline{C})\right) =$$

$$= \Delta_1(e_\rho) \cdot \Delta_2(e_\mu) \cdot \Delta_3(e_\nu) = e_{\rho,\mu,\nu}$$

as required. To complete the proof we have to verify that $k^\rho_{\mu,\nu} \neq 0$. For this purpose let us compute the trace $\mathsf{Tr}(e_\pi)$ of the linear map $u \mapsto e_\pi u$, $u \in \mathbb{Q}S$. The matrix of this linear map coincides with $e_{\pi'}$ and so

$$\mathsf{Tr}(e_\pi) = \mathsf{Tr}(e_{\pi'}) = |S| \cdot (e_\pi)_1 = (n!)^2 \cdot (e_\pi)_1 \tag{20}$$

where $(e_\pi)_1$ is the coefficient of e_π at 1_S. Using (17) for $\rho = \rho, \mu, \nu$, (18) and (13) we obtain that

$$\frac{(n!)^3}{\rho(1_n)\mu(1_n)\nu(1_n)} \cdot (e_\pi)_1 = \sum_{x \in S} \rho(x)\mu(x^{-1})\nu(x^{-1}) = n! \cdot \langle \rho, \mu\nu \rangle = n! \cdot k^\rho_{\mu,\nu}. \tag{21}$$

Now from (20) and (21) it follows that $\mathsf{Tr}(e_p) = \rho(1_n)\mu(1_n)\nu(1_n)k^\rho_{\mu,\nu}$. On the other hand $\dim(\mathcal{A}_\pi) = \mathsf{Tr}(e_\pi)$ because e_π is idempotent. Thus $k^\rho_{\mu,\nu} \neq 0$. ■

Let us consider the contraction of a cubic determinants (see Section 2 and Lemma 3.3). For this purpose denote by $\boldsymbol{\lambda}_\rho$ the Young diagram corresponding a character $\rho \in \mathsf{Irr}(S_n)$, and for a partition $\lambda \in \Lambda(n)$ by π_λ the permutation character of the group S_n acting on the right cosets of S_λ by multiplications. Then by [10, Corollary 2.2.22] we have

$$\langle \pi_\lambda, \rho \rangle \neq 0 \quad \Leftrightarrow \quad \boldsymbol{\lambda}_\rho \trianglerighteq [\lambda] \tag{22}$$

where $\trianglerighteq$ denotes the partial order on $\boldsymbol{\Lambda}(n)$ in which $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ ($\boldsymbol{\lambda}$ *dominates* $\boldsymbol{\mu}$) if and only if for all i the inequality $\sum_{j=1}^i \boldsymbol{\lambda}_j \geq \sum_{j=1}^i \boldsymbol{\mu}_j$ holds.

Lemma 4.4. *Given $\rho, \lambda, \mu \in \mathsf{Irr}(S_n)$ and $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3 \in \boldsymbol{\Lambda}(n)$ set $e = e_{\rho,\mu,\nu}$ and set $\Lambda_i = \{\lambda_i \in \Lambda(n) : \ [\lambda_i] = \boldsymbol{\lambda}_i\}$, $i = 1, 2, 3$. Then*

(1) *if $\boldsymbol{\lambda}_1 \not\trianglerighteq \boldsymbol{\lambda}_\rho$ or $\boldsymbol{\lambda}_2 \not\trianglerighteq \boldsymbol{\lambda}_\mu$ or $\boldsymbol{\lambda}_3 \not\trianglerighteq \boldsymbol{\lambda}_\nu$, then $\chi^{\lambda_1,\lambda_2,\lambda_3} = 0$ for all $\chi \in e\mathbb{Q}S$ and all $\lambda_i \in \Lambda_i$,*

(2) *if $\boldsymbol{\lambda}_1 \trianglerighteq \boldsymbol{\lambda}_\rho$ and $\boldsymbol{\lambda}_2 \trianglerighteq \boldsymbol{\lambda}_\mu$ and $\boldsymbol{\lambda}_3 \trianglerighteq \boldsymbol{\lambda}_\nu$ and $k^\rho_{\mu,\nu} \neq 0$, then $e^{\lambda_1,\lambda_2,\lambda_3} \neq 0$ for some $\lambda_i \in \Lambda_i$.*

Proof. To prove statement (1) suppose that $\boldsymbol{\lambda}_1 \not\trianglerighteq \boldsymbol{\lambda}_\rho$ or $\boldsymbol{\lambda}_2 \not\trianglerighteq \boldsymbol{\lambda}_\mu$ or $\boldsymbol{\lambda}_3 \not\trianglerighteq \boldsymbol{\lambda}_\nu$. Then by (22) for any partitions $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$ and $\lambda_3 \in \Lambda_3$ we have

$$\langle \pi_{\lambda_1}, \rho \rangle = 0 \quad \text{or} \quad \langle \pi_{\lambda_2}, \mu \rangle = 0 \quad \text{or} \quad \langle \pi_{\lambda_3}, \nu \rangle = 0.$$

Notice that an irreducible representation τ of a group Y appears in a decomposition of a permutational representation 1_X^Y of a group $X \le Y$ if and only if

$$\sum_{x\in X}\tau(x)=0 \iff \underline{X}\cdot e_\tau=0.$$

Thus one of the elements $\underline{S}_{\lambda_1}\cdot e_\rho$, $\underline{S}_{\lambda_2}\cdot e_\mu$, $\underline{S}_{\lambda_3}\cdot e_\nu$ is equal to 0. Together with (18) we obtain that for any $g=(g_1,g_2)\in S$

$$(e\cdot g)^{\lambda_1,\lambda_2,\lambda_3}=\Delta_1(\underline{S_{\lambda_1}}e_\rho)\cdot\Delta_2(\underline{\{g_1\}}\cdot\underline{S_{\lambda_2}}e_\mu)\cdot\Delta_3(\underline{\{g_2\}}\cdot\underline{S_{\lambda_3}}e_\nu)=0$$

(here we used the facts that the elements $\Delta_2(X)$ and $\Delta_3(Y)$ commute each to other for all $X,Y\in\mathbb{Q}S_n$, and that the elements e_ρ, e_μ and e_ν belong to the center of the algebra $\mathbb{Q}S_n$). Now, statement (1) follows from Lemma 3.3.

To prove statement (2) suppose that $\boldsymbol{\lambda}_1\trianglerighteq\boldsymbol{\lambda}_\rho$ and $\boldsymbol{\lambda}_2\trianglerighteq\boldsymbol{\lambda}_\mu$ and $\boldsymbol{\lambda}_3\trianglerighteq\boldsymbol{\lambda}_\nu$. Then again as above from (22) it follows that

$$\sum_{\lambda_1\in\Lambda_1}\underline{S_{\lambda_1}}\cdot e_\rho=a_1\cdot e_\rho,\qquad \sum_{\lambda_2\in\Lambda_2}\underline{S_{\lambda_2}}\cdot e_\mu=a_2\cdot e_\mu,\qquad \sum_{\lambda_3\in\Lambda_3}\underline{S_{\lambda_3}}\cdot e_\nu=a_3\cdot e_\nu,$$

for some non-zero rational numbers a_1,a_2,a_3. Since $k_{\rho,\mu}^\nu\ne0$ by Lemma 4.3 we have

$$0\ne a_1a_2a_3\cdot e^{\lambda_1,\lambda_2,\lambda_3}=\sum_{\lambda_1}\Delta_1(\underline{S_{\lambda_1}}\cdot e_\rho)\cdot\sum_{\lambda_2}\Delta_2(\underline{S_{\lambda_2}}\cdot e_\mu)\cdot\sum_{\lambda_3}\Delta_3(\underline{S_{\lambda_3}}\cdot e_\nu).$$

Since any element of the form $\Delta_1(\underline{S_{\lambda_1}}\cdot e_\rho)\cdot\Delta_2(\underline{S_{\lambda_2}}\cdot e_\mu)\cdot\Delta_3(\underline{S_{\lambda_3}}\cdot e_\nu)$ is an idempotent in $\mathbb{Q}S$, at least one of them is not zero, and we are done. ∎

It is easily seen that any $(\lambda_1,\lambda_2,\lambda_3)$-contraction of the element $e_{\rho,\mu,\nu}$ is a nonnegative multiple of an idempotent of the algebra $\mathbb{Q}S$. Therefore under the condition of statement (2) of Lemma 4.4 the sum of all $(\lambda_1,\lambda_2,\lambda_3)$-contractions of $e_{\rho,\mu,\nu}$ is not zero. Thus, summing up the results from Lemmas 4.2, 4.3 and 4.4 we come to the following statement.

Theorem 4.5. *The linear space $V_{n,d}$ is generated as a right ideal of the algebra $\mathbb{Q}S_n^2$ by the sets*

$$V_{\rho,\mu,\nu}=\{\chi\in\mathcal{A}_{\rho,\mu,\nu}:\ \zeta_{n,d}\cdot e_{\rho,\mu,\nu}\cdot\chi=0\}$$

where ρ,μ,ν run over the set $\mathsf{Irr}(S_n)$ with $k_{\mu,\nu}^\rho\ne0$. In particular, $V_{\rho,\mu,\nu}=\mathcal{A}_{\rho,\mu,\nu}$ if and only if $e_{\rho,\mu,\nu}\in V_{n,d}$. In the latter case the sum of all $(\lambda_1,\lambda_2,\lambda_3)$-contractions of the element $e_{\rho,\mu,\nu}$ is not zero whenever the partitions λ_1, λ_2 and λ_3 have the same Young diagram and this diagram dominates the diagrams $\boldsymbol{\lambda}_\rho$, $\boldsymbol{\lambda}_\mu$ and $\boldsymbol{\lambda}_\mu$. ∎

Given a Young diagram $\boldsymbol{\lambda}$ denote by $r(\boldsymbol{\lambda})$ the number of rows of $\boldsymbol{\lambda}$. Then it is easily seen that $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$ only if $r(\boldsymbol{\lambda})\le r(\boldsymbol{\mu})$. Therefore the following statement immediately follows from statement (1) of Lemma 4.4.

Corollary 4.6. *Let $\rho,\mu,\nu\in\mathsf{Irr}(S_n)$ and $\lambda_1,\lambda_2,\lambda_3\in\Lambda(n)$. Then any cubic determinant in $\mathcal{M}(n)$ that corresponds to an element of $V_{\rho,\mu,\nu}$ has a nonzero $(\lambda_1,\lambda_2,\lambda_3)$-contraction only if $r(\lambda_1)\le r(\boldsymbol{\lambda}_\rho)$, $r(\lambda_2)\le r(\boldsymbol{\lambda}_\mu)$ and $r(\lambda_3)\le r(\boldsymbol{\lambda}_\nu)$.* ∎

5. Cubic determinants of defect 1

In this section we refine Theorem 4.5 for the case $d = 1$. To compute the elements $\zeta \cdot e_\pi$ with $\zeta = \zeta_{n,1}$ and $\pi \in \mathsf{Irr}(\mathcal{A})$ we need the following auxiliary lemma.

Lemma 5.1. *Let C and ρ be a conjugacy class and an irreducible character of S_n, respectively. Then*

(1) $\underline{C} \cdot e_\rho = \frac{|C|\rho(g)}{\rho(1_n)} e_\rho$ *for any* $g \in C$,

(2) $\Delta_i(\underline{C}) \cdot \Delta_j(e_\rho) = \Delta_j(e_\rho) \cdot \Delta_i(\underline{C})$ *for* $i, j = 1, 2, 3$.

Proof. To prove statement (1)[2] we note that the element $\underline{C}$ belongs to the center of the group algebra $\mathbb{Q}S_n$. Therefore $\chi := \underline{C} \cdot e_\rho = a \cdot e_\rho$ for some $a \in \mathbb{Q}$. After comparing the coefficients in both sides of the latter equality at 1_n, we get that $a = \chi(1_n)/e_\rho(1_n)$. However, by (17) we have

$$\chi(1_n) = |C|\rho(g)\frac{\rho(1_n)}{n!} \quad \text{and} \quad e_\rho(1_n) = \frac{(\rho(1_n))^2}{n!}$$

where $g \in C$. Thus $a = |C|\rho(g)/\rho(1_n)$ and we are done.

Statement (2) is obvious whenever $\{i, j\} \subset \{2, 3\}$ or $i = j = 1$. Suppose that $i = 1$ and $j = 2$ (the remaining three cases are proved in a similar way). Then

$$\frac{n!}{\rho(1_n)}\Delta_1(\underline{C}) \cdot \Delta_2(e_\rho) = \sum_{g\in C}\sum_{h\in S_n} \rho(h)(g, gh) = \sum_{g\in C}\sum_{h'\in S_n} \rho(g^{-1}h'g)(g, h'g) =$$

$$\sum_{g\in C}\sum_{h'\in S_n} \rho(h')(g, h'g) = \frac{n!}{\rho(1_n)}\Delta_2(e_\rho) \cdot \Delta_1(\underline{C})$$

whence the required statement follows. ∎

Below given three irreducible characters $\rho, \mu, \nu \in \mathsf{Irr}(S_n)$ we set

$$q(\rho, \mu, \nu) = \frac{\rho(2_n)}{\rho(1_n)} + \frac{\mu(2_n)}{\mu(1_n)} + \frac{\nu(2_n)}{\nu(1_n)}$$

where $2_n = (1, 2)$ is the transposition in S_n.

Theorem 5.2. *Let $V = V_{n,1}$ and $\mathcal{A} = \mathcal{A}_n$. Then*

$$V \cap \mathcal{A} = \bigoplus_{\pi:\, e_\pi \in V} \mathcal{A}_\pi.$$

Moreover, $e_\pi = e_{\rho,\mu,\nu} \in V$ if and only if $q(\rho, \mu, \nu) = -1$.

Proof. Set $\zeta = \zeta_{n,1}$ and $T = T_n$. Then by (11) we have

$$\zeta = \binom{n}{2}\underline{1}_n + \Delta_1(\underline{T}) + \Delta_2(\underline{T}) + \Delta_3(\underline{T}).$$

By Lemma 4.3 given a character $\pi \in \mathsf{Irr}(\mathcal{A})$ there exist characters $\rho, \mu, \nu \in \mathsf{Irr}(S_n)$ such that $e_\pi = \Delta_1(e_\rho) \cdot \Delta_2(e_\mu) \cdot \Delta_3(e_\nu)$. Now by Lemma 5.1 we have

$$\zeta \cdot e_\pi = \left(\binom{n}{2}\underline{1}_n + \Delta_1(\underline{T}) + \Delta_2(\underline{T}) + \Delta_3(\underline{T})\right) \cdot \Delta_1(e_\rho) \cdot \Delta_2(e_\mu) \cdot \Delta_3(e_\nu) =$$

$$\binom{n}{2} e_\pi + \Delta_1(\underline{T}e_\rho)\cdot\Delta_2(e_\mu)\cdot\Delta_3(e_\nu) + \Delta_1(e_\rho)\cdot\Delta_2(\underline{T}e_\mu)\cdot\Delta_3(e_\nu) + \Delta_1(e_\rho)\cdot\Delta_2(e_\mu)\cdot\Delta_3(\underline{T}e_\nu) =$$

[2]Although this statement is a direct consequence of (7.11), [1], we give here a direct proof to make the text self-contained.

$$\binom{n}{2}e_\pi + \frac{|T|\rho(2_n)}{\rho(1_n)}e_\pi + \frac{|T|\mu(2_n)}{\mu(1_n)}e_\pi + \frac{|T|\nu(2_n)}{\nu(1_n)}e_\pi = \binom{n}{2}(1+q(\rho,\mu,\nu))\,e_\pi.$$

This proves the second statement, and by Theorem 4.5 also the first one. ∎

As an immediate consequence of Theorem 5.2 and Lemma 4.3 we obtain the following statement.

Corollary 5.3. *The linear space $V_{n,1}$ is generated as a right ideal of the algebra $\mathbb{Q}S_n^2$ by the set of all elements $\Delta_1(e_\rho)\cdot\Delta_2(e_\mu)\cdot\Delta_3(e_\nu)$ where $\rho,\mu,\nu \in \mathsf{Irr}(S_n)$ are such that $k^\rho_{\mu,\nu} \neq 0$ and $q(\rho,\mu,\nu) = -1$.* ∎

Example: The symmetric group S_4 has 5 irreducible representations; denote them by χ_i, $i = 1,\dots,5$. The values $\chi_i(2_n)$ and $\chi_i(1_n)$ for all i are given in the following table (here and below we use tables from [10]):

i	1	2	3	4	5
$\chi_i(1_n)$	1	3	2	3	1
$\chi_i(2_n)$	1	1	0	-1	-1
$\boldsymbol{\lambda}_{\chi_i}$	$[4]$	$[3,1]$	$[2,2]$	$[2,1,1]$	$[1^4]$

A direct check shows that $q(\chi_i,\chi_j,\chi_k) = -1$ for $1 \le i \le j \le k \le 5$ only if (i,j,k) is one of the following triples: $(1,5,5)$, $(3,3,5)$, $(2,4,5)$ and $(4,4,4)$. In all these cases $k^{\chi_k}_{\chi_i,\chi_j} = 1$. Thus

$$\begin{aligned} V_{4,1} \quad = \quad & V_{1,5,5} \oplus V_{5,1,5} \oplus V_{5,5,1} \oplus \\ & V_{3,3,5} \oplus V_{3,5,3} \oplus V_{5,3,3} \oplus \\ & V_{2,4,5} \oplus V_{2,5,4} \oplus V_{4,2,5} \oplus V_{4,5,2} \oplus V_{5,2,4} \oplus V_{5,4,2} \oplus \\ & V_{4,4,4} \end{aligned}$$

where $V_{i,j,k}$ is the right ideal $e_{\chi_i,\chi_j,\chi_k}\mathbb{Q}(S_4 \times S_4)$. The dimensions of the ideals in the first row are 1, in the second - 4, in the third - 9 and in the fourth - 27. ∎

We make use the remark given in the proof of Theorem 3.5 of the paper [19]. Let $\rho \in \mathsf{Irr}(S_n)$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\rho$. Then

$$\frac{\rho(2_n)}{\rho(1_n)} = \frac{1}{\binom{n}{2}} \sum_i \left(\binom{\boldsymbol{\lambda}_i}{2} - \binom{\boldsymbol{\lambda}'_i}{2} \right) \tag{23}$$

where $\boldsymbol{\lambda}_i$ (resp. $\boldsymbol{\lambda}'_i$) is the size of the ith row (resp. the ith column) of $\boldsymbol{\lambda}$. The negative part of the sum in the right-hand side of this equality is more or equal than the ratio $-\boldsymbol{\lambda}'_1/n$: this bound is obtained by maximizing $\sum_i (\boldsymbol{\lambda}'_i)^2$ provided that $\boldsymbol{\lambda}'_1 \ge \boldsymbol{\lambda}'_2 \ge \cdots \ge 0$ and $\sum_i \boldsymbol{\lambda}'_i = n$. Applying this bound to each summand of $q(\rho,\mu,\nu)$ we come to the following statement.

Lemma 5.4. *Given $\rho,\mu,\nu \in \mathsf{Irr}(S_n)$ we have $q(\rho,\mu,\nu) = -1$ only if at least one of the diagrams $\boldsymbol{\lambda}_\rho$, $\boldsymbol{\lambda}_\mu$, $\boldsymbol{\lambda}_\nu$ has at least $\lfloor n/3 \rfloor$ rows.* ∎

It is not so difficult to construct infinite families of triples $\rho,\mu,\nu \in \mathsf{Irr}(S_n)$ for which $q(\rho,\mu,\nu) = -1$. For example a straightforward computation by formula (23) shows that if all the diagrams $\boldsymbol{\lambda}_\rho$, $\boldsymbol{\lambda}_\mu$, $\boldsymbol{\lambda}_\nu$ are hooks (resp. rectangles), then an infinite family is defined by the condition $r(\rho)+r(\mu)+r(\nu) = 2n+1$ (resp. the conditions $r(\rho) = r(\mu) = n/2$ and $r(\nu)(r(\nu)-3) = n$). However, even in these rather simple cases almost nothing is known on the Kronecker coefficients $k^\nu_{\rho,\mu}$ (see [3]).

We complete the section by giving an explicit construction of a family of cubic determinants of defect 1. Take an arbitrary partition $[n] = I \cup J \cup K$, and denote by λ_I, λ_J and λ_K the partitions of $[n]$ into $|I|+1$, $|J|+1$ and $|K|+1$ classes containing as a class respectively the complements to the sets I, J and K. The following lemma shows that the cubic determinant corresponding to the element

$$D(I,J,K) = \Delta_1(\chi_I) \cdot \Delta_2(\chi_J) \cdot \Delta_3(\chi_K) \tag{24}$$

with

$$\chi_I = \sum_{g \in S_{\lambda_I}} (-1)^{\mathsf{sgn}(g)} g, \quad \chi_J = \sum_{g \in S_{\lambda_J}} (-1)^{\mathsf{sgn}(g)} g, \quad \chi_K = \sum_{g \in S_{\lambda_K}} (-1)^{\mathsf{sgn}(g)} g,$$

is of defect 1. (It is not difficult to check that this element is not zero.)

Lemma 5.5. *For a partition $[n] = I \cup J \cup K$ set $\chi = D(I,J,K)$. Then*

(1) $\chi \in V_{n,1}$,
(2) $\chi^{\lambda_1,\lambda_2,\lambda_3} = 0$ *for all* $\lambda_1, \lambda_2, \lambda_3 \in \Lambda(n,r)$ *with* $r < 2n/3$.

Proof. By the definition of $V_{n,1}$ to prove statement (1) it suffices to check that given a transposition $t = (i,j) \in S_n$ we have

$$\Delta_2(\underline{T}) \cdot \Delta_3(\underline{T}) \cdot \chi = 0$$

where $T = \{1_n, t\}$. Suppose first that $\{i,j\} \not\subset J \cup K$, say i does not belong to $J \cup K$. Then given a permutation $f \in S_{\lambda_I}$ the element j is not contained in one of the sets $J^{f^{-1}}$ or $K^{f^{-1}}$. In the former case, $\{i,j\} \cap J = \emptyset$, and hence $(f + tf)\chi_J = 0$, whereas in the latter case $\{i,j\} \cap K = \emptyset$, and hence $(f+tf)\chi_K = 0$. Thus in any case

$$\Delta_2(\underline{T}) \cdot \Delta_3(\underline{T}) \cdot \Delta_1(f) \cdot \Delta_2(\chi_J) \cdot \Delta_3(\chi_K) = \Delta_2((f+tf)\chi_J) \cdot \Delta_3(f+tf)\chi_K) = 0$$

for all $f \in S_{\lambda_I}$, and we are done. To complete the proof of (1) let us assume that $\{i,j\} \subset J \cup K$. Then by a direct computation we obtain

$$\Delta_2(\underline{T}) \cdot \Delta_3(\underline{T}) \cdot \chi_I = \chi' \cdot \Delta_1(\underline{T} \cdot \chi_I) = 0,$$

because of $\underline{T} \cdot \chi_I = 0$ where $\chi' = (1_n, 1_n) + (t, 1_n)$.

To prove statement (2) suppose that $\lambda_1, \lambda_2, \lambda_3 \in \Lambda(n,r)$ are partitions such that $\chi^{\lambda_1,\lambda_2,\lambda_3} \neq 0$. It is easily seen that

$$\chi^{\lambda_1,\lambda_2,\lambda_3} = \Delta_1(\underline{S}_{\lambda_1} \chi_I) \cdot \Delta_2(\chi_J \underline{S}_{\lambda_2}) \cdot \Delta_3(\chi_K \underline{S}_{\lambda_3}),$$

and hence

$$\underline{S}_{\lambda_1} \cdot \chi_I \neq 0, \qquad \chi_J \cdot \underline{S}_{\lambda_2} \neq 0, \qquad \chi_K \cdot \underline{S}_{\lambda_3} \neq 0. \tag{25}$$

On the other hand, let a transposition t and the set T be as above. Then obviously $\underline{T} \cdot \chi_I = \chi_I \cdot \underline{T} = 0$ whenever $\{i,j\} \cap I = \emptyset$. Since in the latter case $S_{\lambda_1} = S_{\lambda_1} T$, we conclude by (25) that at least one of any two distinct elements i,j from the same class of the partition λ_1 belongs to the set I. It follows that a class of λ_1 of size a intersects the set I in at least $a-1$ elements. Therefore $r(\lambda_1) \geq n - |I|$. Similarly, one can prove that $r(\lambda_2) \geq n - |J|$ and $r(\lambda_3) \geq n - |K|$. This implies that

$$3r = r(\lambda_1) + r(\lambda_2) + r(\lambda_3) \geq 3n - (|I| + |J| + |K|) = 2n.$$

Thus $r \geq 2n/3$ as required. ∎

6. Applications to the border rank of a cubic tensor

In this section we return to the questions discussed in the introduction. The following metascheme (based on Lemmas 2.2, 3.1 and 3.3) provides a tool to prove that given a positive integer r the border rank $\underline{\mathsf{rk}}(A)$ of a cubic $n \times n \times n$ tensor A is at least r:

- choose a positive integer $N \geq n$,
- find $\chi \in V_{N,N-r}$ such that $\chi^{\lambda_1,\lambda_2,\lambda_3} \neq 0$ for some $\lambda_1, \lambda_2, \lambda_3 \in \Lambda(N,n)$,
- set $P = (F_N^{-1}(\chi))^{\lambda_1,\lambda_2,\lambda_3}$ (depolarization),
- if $P(A) \neq 0$, then $\underline{\mathsf{rk}}(A) \geq r+1$.

Let us apply this metascheme to the two following concrete examples. The first one shows, in particular, that the bound from the second statement of Lemma 5.5 is attained, whereas the second one gives a low bound for the border rank of the matrix multiplication tensor.

Example 1. Given a positive odd integer $n = 2m+1$ we define an $n \times n \times n$ zero-one cubic tensor $A = A_n$ such that $A_{i,j,k} = 1$ if and only if $(i,j,k) \in Q_n$ where

$$\begin{aligned} Q_n \;=\; & \{(n,i,i) : && 1 \leq i \leq m\} \cup \\ & \{(j,n,j+m) : && 1 \leq j \leq m\} \cup \\ & \{(k,k,n) : && m < k < n\}. \end{aligned}$$

We claim that

$$\underline{\mathsf{rk}}(A_n) = \mathsf{rk}(A_n) = 3m = \frac{3(n-1)}{2}. \tag{26}$$

Indeed, according to our metascheme take $r = 3m-1$ and $N = 3m$. Then by statement (1) of Lemma 5.5 (with $n = N$) the linear space $V_{N,N-r} = V_{N,1}$ contains the nonzero element $\chi = D(I,J,K)$ where

$$I = \{1,\ldots,m\}, \quad J = \{m+1,\ldots,2m\}, \quad K = \{2m+1,\ldots,3m\}.$$

For $i = 1,2,3$ set λ_i to be the partition in $\Lambda(3m, 2m+1)$ of the shape $[m, 1^{2m}]$ the unique nonsingleton class of which coincides with the set I, J and K respectively. Then to prove that $\underline{\mathsf{rk}}(A) \geq r+1$ it suffices to verify that $P(A) \neq 0$ where $P(X)$ is the $(\lambda_1,\lambda_2,\lambda_3)$-contraction of the matching polynomial $F_N^{-1}(\chi)$. For this purpose choose the indices of the variables X_{ijk} of the polynomial $P(X)$ so that

$$c_1(i) = \begin{cases} n, & \text{if } i \in I, \\ i-m, & \text{if } i \in J \cup K, \end{cases} \qquad c_3(k) = \begin{cases} k, & \text{if } k \in I \cup J, \\ n, & \text{if } k \in K, \end{cases}$$

and

$$c_2(j) = \begin{cases} j, & \text{if } j \in I, \\ n, & \text{if } j \in J, \\ j-2m, & \text{if } j \in K, \end{cases}$$

where $c_a(b)$ denotes the number of the class of the partition λ_a that contains b (see (6)). Due to (2) and (6) the polynomial $P(X)$ can be written in the form

$$P(X) = \sum_{g \in S_N^2} \chi(g) M_g(X), \qquad M_g(X) = \prod_{i=1}^{n} X_{c_1(i)\, c_2(i^{g_1})\, c_3(i^{g_2})}, \tag{27}$$

where $\chi(g) \in \{0, \pm 1\}$ is the coefficient of χ at g. It immediately follows from the definitions of the tensor A and the numbers $c_a(b)$ that $M_g(A) \neq 0$ only if the permutations g_1 and g_2 leave the sets I, J, K fixed (as sets) and

$$g_1^I = g_2^I, \qquad g_2^J = 1_J, \qquad g_1^K = 1_K$$

where 1_J and 1_K are the identical permutations of J and K respectively. Denote the set of all such permutations by H. Then the latter conditions imply that (a) the coefficient $\chi(g)$ is nonzero and does not depend on $g \in H$ (see (24)), and (b) the monomial $M_g(X)$ does not depend on $g \in H$, and hence $M_g(A) = 1$. Therefore

$$P(A) = \sum_{g \in H} \chi(g) M_g(A) = \chi(g_0)|H| \neq 0$$

where g_0 is an arbitrary element from H. According to our metascheme this means that $\underline{\mathsf{rk}}(A) \geq 3m$. The converse inequality holds because $\underline{\mathsf{rk}}(A) \leq \mathsf{rk}(A)$, and $\mathsf{rk}(A)$ does not exceed the number of nonzero entries of A that is $|Q_n| = 3m$.

Example 2. Set M_n to be the structure tensor of $n \times n$-matrix multiplication. As it was mentioned in the introduction $\underline{\mathsf{rk}}(M_n) \geq (2 - \epsilon) \cdot n^2$ [16]. At present we can not improve this bound, but we can easily apply our metascheme to obtain the lower bound

$$\underline{\mathsf{rk}}(M_n) \geq \frac{5}{4} n^2. \tag{28}$$

Without loss of generality we can assume that $n = 2m$ is even. Set $r = 5m^2 - 1$ and $N = 5m^2$. By statement (1) of Lemma 5.5 (with $n = N$) the linear space $V_{N,N-r} = V_{N,1}$ contains the nonzero element $\chi = D(I, J, K)$ where

$$I = \{1, \ldots, m^2\}, \qquad J = \{m^2 + 1, \ldots, 3m^2\}, \qquad K = \{3m^2 + 1, \ldots, 5m^2\}.$$

Now, to prove inequality (28) it suffices to find some partitions λ_1, λ_2 and λ_3 in the set $\Lambda(5m^2, 4m^2)$ such that $P(M) \neq 0$ where $M = M_n$ and $P(X)$ is the $(\lambda_1, \lambda_2, \lambda_3)$-contraction of the matching polynomial $F_N^{-1}(\chi)$. In fact, it is enough for our purposes that $\lambda_i \in \Lambda(5m^2, n'_i)$ for some $n'_i \in [4m^2]$ (in this case not all variables X_{ijk} occur in the polynomial $P(X)$). To construct the partitions choose arbitrarily three bijections:

$$\begin{array}{rlcl} g_1: & [4m^2] & \to & [2m] \times [2m], \\ g_2: & [3m^2] & \to & [2m] \times [m] \cup [m] \times (m, 2m], \\ g_3: & [3m^2] & \to & [2m] \times (m, 2m] \cup (m, 2m] \times [m], \end{array}$$

and define three maps $f_i : [5m^2] \to [2m] \times [2m]$, $i = 1, 2, 3$, such that

$$f_1(i) = \begin{cases} (\lceil \frac{i}{m} \rceil, \lceil \frac{i}{m} \rceil), & \text{if } i \in I, \\ g_1(i), & \text{if } i \in J \cup K, \end{cases}$$
$$f_2(j) = \begin{cases} (\lceil \frac{j-m^2}{m} \rceil, \lceil \frac{j-m^2}{m} \rceil), & \text{if } j \in J, \\ g_2(j), & \text{if } j \in I \cup K, \end{cases}$$
$$f_3(k) = \begin{cases} (\lceil \frac{k-3m^2}{m} \rceil, \lceil \frac{k-3m^2}{m} \rceil), & \text{if } k \in K, \\ g_3(k), & \text{if } k \in I \cup J, \end{cases}$$

For $a = 1, 2, 3$ we define the required partition $\lambda_a \in \Lambda(5m^2)$ by the condition that $x, y \in [5m^2]$ belong the same class of λ_a if and only if $f_a(x) = f_a(y)$. From the definition it immediately follows that

$$\begin{array}{rcl}
\lambda_1 \in \Lambda(5m^2, 4m^2) & \text{and} & [\lambda_1] = [(m+1)^m, 1^{4m^2-m}], \\
\lambda_2 \in \Lambda(5m^2, 3m^2+m) & \text{and} & [\lambda_2] = [(m+1)^m, m^m, 1^{3m^2-m}], \\
\lambda_3 \in \Lambda(5m^2, 3m^2+m) & \text{and} & [\lambda_3] = [m^{2m}, 1^{3m^2}].
\end{array}$$

Due to (2) and (6) the polynomial $P(X)$ can be written in the form (27). It immediately follows from the definitions of the tensor M that given $g \in S_N \times S_N$, we have $M_g(M) \neq 0$ only if $(c_1(i), c_2(i^{g_1}), c_3(i^{g_2})) = (uv, vw, wu)$ for some elements $u, v, w \in [2m]$ where $uv = (u, v)$, $vw = (v, w)$ and $wu = (w, u)$. In this case the above triple obviously belongs to one of the following sets:

$$\begin{array}{rcl}
\{(uv, vu, uu) & : & uv \in [m] \times [2m]\}, \\
\{(uv, vv, vu) & : & uv \in (m, 2m] \times [2m]\}, \\
\{(uu, uw, wu) & : & uw \in [m] \times (m, 2m]\}.
\end{array}$$

Now, denote by H the set of all $g \in S_N^2$ for which $M_g(M) \neq 0$. Then the argument as in Example 1 shows that (a) the coefficient $\chi(g)$ is nonzero and does not depend on $g \in H$, and (b) the monomial $M_g(X)$ does not depend on the element $g \in H$. Therefore $P(M) \neq 0$, and hence $\underline{\mathsf{rk}}(M) \geq 5m^2$.

Remark. The above proof shows that the structure tensor of the $2m \times 2m$-matrix multiplication contains a $4m^2 \times (3m^2 + m) \times (3m^2 + m)$ subtensor the border rank of which is at least $5m^2$.

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