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Category of morphisms of algebraic curves and a characterization of Prym varieties

BY

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Dedicated to Professor Shoshichi Kobayashi on the occasion of his sixtieth birthday

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Abstract. A characterization of arbitrary Prym varieties is established in terms of the multi-component KP equations. This characterization is obtained as a consequence of an equivalence between a category of covering morphisms of algebraic curves with arbitrary vector bundles on them, and a category consisting of points of the infinite-dimensional Grassmannian of vector valued functions together with commutative subalgebras of the Heisenberg algebras acting on the Grassmannian. The fully faithful functor provides a generalization of the Krichever map to the case of arbitrary covering morphisms of curves. The functor is also used to show that the objects of the above category containing line bundles with vanishing cohomologies give rise to a new class of maximal commutative algebras of ordinary differential operators with coefficients in matrix valued functions.

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0. Introduction.

0.1. From the geometric point of view, the *Kadomtsev-Petviashvili (KP) equations* are best understood as a set of commuting vector fields, or *flows*, defined on an infinite-dimensional Grassmannian [**S**]. The Grassmannian $Gr_1(\mu)$ is the set of vector subspaces W of the field $L = \mathbf{C}((z))$ of formal Laurent series in z such that the projection $W \longrightarrow \mathbf{C}((z))/\mathbf{C}[[z]]z$ is a Fredholm map of index μ. The commutative algebra $\mathbf{C}[z^{-1}]$ acts on L by multiplication, and hence it induces commuting flows on the Grassmannian. This very simple picture is nothing but the KP system written in the language of infinite-dimensional geometry. A striking fact is that every finite-dimensional orbit (or integral manifold) of these flows is canonically isomorphic to the Jacobian variety of an algebraic curve, and conversely, every Jacobian variety can be realized as a finite-dimensional orbit of the KP flows [**M1**]. This statement is equivalent to the claim that the KP equations characterize the Riemann theta functions associated with Jacobian varieties [**AD**].

If one generalizes the above Grassmannian to the Grassmannian $Gr_n(\mu)$ consisting of vector subspaces of $L^{\oplus n}$ with a Fredholm condition, then the formal loop algebra $gl(n, L)$ acts on it. In particular, the Borel subalgebra (one of the maximal commutative subalgebras) of the Heisenberg algebra acts on $Gr_n(\mu)$ with the center acting trivially. Let us call the system of vector fields coming from this action the *Heisenberg flows* on $Gr_n(\mu)$. Now one can ask a question: what are the finite-dimensional orbits of these Heisenberg flows, and what kind of geometric objects do they represent? Actually, this question was asked to one of the authors by Professor H. Morikawa as early as in 1984. In this paper, we give a complete answer to this question. Indeed, we shall prove (see 5.1 and 5.8)

THEOREM A. *A finite-dimensional orbit of the Heisenberg flows defined on the Grassmannian of vector valued functions corresponds to a covering morphism of algebraic curves, and the orbit itself is canonically isomorphic to the Jacobian variety of the curve upstairs. Moreover, the action of the traceless elements of the Borel subalgebra (the traceless Heisenberg flows) produces the Prym variety associated with this covering morphism as an orbit.*

REMARK: The relation between the Heisenberg algebras and the covering morphisms of algebraic curves has been pointed out in [**AB**].

0.2. Right after the publication of works ([**AD**], [**M1**], [**Sh1**]) on characterization of Jacobian varieties by means of integrable systems, it has become an important problem to seek for a similar characterization of Prym varieties. We establish in this paper a simple solution of this problem in terms of the *multi-component KP system* defined on a certain quotient space of the Grassmannian of vector valued functions.

Classically, the Prym varieties associated with degree two coverings of algebraic curves were introduced by Schottky and Jung in their approach to the Schottky problem [**SJ**]. The modern interests in Prym varieties were revived in [**Mum1**]. It promoted further studies of Prym varieties in the modern algebraic geometry setting [**Be**], [**DS**]. Recently, the Prym varieties of higher degree coverings have been used in the study of the generalized theta divisors on the moduli spaces of stable vector bundles over an algebraic curve [**BNR**]. A formula of [**BNR**] about the dimensions of the linear system of the generalized theta divisors provides a mathematical proof of the Verlinde formula in the level one case [**Bo**], which has an origin in conformal field theory. In this direction, an inequality generalizing the formula of [**BNR**] has been established in [**L**]. In the context of integrable systems, it has been discovered that Prym varieties of ramified double sheeted coverings of curves appear as solutions of the BKP system [**DJKM**]. Independently, a Prym variety of degree two covering with exactly two ramification points has been observed in the deformation theory of two-dimensional Schrödinger operators [**No**], [**NV**]. As far as the authors know, the only Prym varieties so far considered in the context of integrable systems are associated with ramified, double sheeted coverings of algebraic curves. Consequently, the attempts ([**Sh2**], [**T**]) of characterizing Prym varieties in terms of integrable systems are all restricted to these special Prym varieties.

Let us define the *quotient Grassmannian* $Z_n(0)$ as the quotient space of $Gr_n(0)$ by the diagonal action of $\left(1+\mathbf{C}[[z]]z\right)^{\times n}$. The *traceless n-component KP system* is defined by the action of the traceless diagonal matrices with entries in $\mathbf{C}[z^{-1}]$ on $Z_n(0)$. Since this system is a special case of the traceless Heisenberg flows, every finite-dimensional orbit of this system is a Prym variety. Conversely, an arbitrary Prym variety associated with a degree n covering morphism of algebraic curves can be realized as a finite-dimensional orbit. Thus a characterization theorem of Prym varieties follows (see 5.14):

THEOREM B. *An algebraic variety is isomorphic to the Prym variety associated with a degree n covering of an algebraic curve if and only if it can be realized as a finite-dimensional orbit of the traceless n-component KP system defined on the quotient Grassmannian $Z_n(0)$.*

0.3. An unexpected connection between moduli theory of algebraic curves and representation theory of Virasoro algebras has emerged through the study of the Grassmannian $Gr_1(0)$ of scalar valued functions of index 0 [**ADKP**], [BS], [**KNTY**], [**W1**]. The relation between algebraic geometry and the Grassmannian comes from the Krichever map of [**SW**], which assigns injectively a point of $Gr_1(0)$ to a set of geometric data consisting of an algebraic curve and a line bundle together with some local information. The Krichever correspondence was enlarged in [**M3**] to include arbitrary vector bundles on curves. The Grassmannian $Gr_1(0)$ appeared once again quite recently in connection with the mysterious interplay of matrix models and the KP system (see for example, [**Ko**], [**W2**]). In particular, it has been discovered that the partition function of the size $N \times N$ matrix model is a τ-function of the KP system, and hence it determines a unique point of $Gr_1(0)$ [**KM**]. Even though the question "why KP, and why the Grassmannian?" is not answered in the context of matrix models, one can speculate that the Grassmannian $Gr_n(0)$ of vector valued functions should play an important role in understanding the moduli spaces of covering morphisms of algebraic curves. We note here that the framework of $Gr_n(0)$ gives us all algebraic curves with ℓ-marked points $(1 \le \ell \le n)$, while the old $Gr_1(0)$ provides curves with only one marked point.

In this paper, we generalize the Krichever functor of [**M3**] so that we can deal with arbitrary covering morphisms of algebraic curves. Let $\mathbf{n} = (n_1, \cdots, n_\ell)$ denote an integral vector consisting of positive integers satisfying that $n = n_1 + \cdots + n_\ell$.

THEOREM C. *For each* $\mathbf{n}$, *the following two categories are equivalent:*

(1) *The category* $\mathcal{C}(\mathbf{n})$. *An object of this category consists of an arbitrary degree* n *morphism* $f : C_{\mathbf{n}} \longrightarrow C_0$ *of algebraic curves and an arbitrary vector bundle* $\mathcal{F}$ *on* $C_{\mathbf{n}}$. *The curve* C_0 *has a smooth marked point* p *with a local coordinate* y *around it. The curve* $C_{\mathbf{n}}$ *has* ℓ $(1 \leq \ell \leq n)$ *smooth marked points* $\{p_1, \cdots, p_\ell\} = f^{-1}(p)$ *with ramification index* n_j *at each point* p_j. *The curve* C_n *is further endowed with a local coordinate* y_j *and a local trivialization of* $\mathcal{F}$ *around* p_j.

(2) *The category* $\mathcal{S}(\mathbf{n})$. *An object of this category is a triple* $(A_0, A_{\mathbf{n}}, W)$ *consisting of a point* $W \in \bigcup_{\mu \in \mathbf{Z}} Gr_n(\mu)$, *a "large" subalgebra* $A_0 \subset \mathbf{C}((y))$ *for some* $y \in \mathbf{C}[[z]]$, *and another "large" subalgebra*

$$A_{\mathbf{n}} \subset \bigoplus_{j=1}^{\ell} \mathbf{C}((y^{1/n_j})) \cong \bigoplus_{j=1}^{\ell} \mathbf{C}((y_j)) \,.$$

In a certain matrix representation as subalgebras of the formal loop algebra $gl(n, \mathbf{C}((y)))$ *acting on the Grassmannian, they satisfy* $A_0 \subset A_{\mathbf{n}}$ *and* $A_{\mathbf{n}} \cdot W \subset W$.

The precise statement of this theorem is given in Section 3, and its proof is completed in Section 4. One of the motivations of introducing a category rather than just a set is because we need not only a set-theoretical bijection of objects but also a canonical correspondence of the morphisms in the proof of the claim that every Prym variety can be realized as a finite-dimensional orbit of the traceless multi-component KP system on the quotient Grassmannian.

0.4. The motivation of extending the framework of the original Krichever map to include arbitrary vector bundles on curves of [**M3**] was to establish a complete geometric classification of all the commutative algebras consisting of ordinary differential operators with coefficients in scalar valued functions. If we apply the functor of Theorem C in this direction, then we obtain (6.14), (6.15):

THEOREM D. *Every object of the category* $\mathcal{C}(\mathbf{n})$ *with a smooth curve* $C_{\mathbf{n}}$ *and a line bundle* $\mathcal{F}$ *on* $C_{\mathbf{n}}$ *satisfying the cohomology vanishing condition*

$$H^0(C_{\mathbf{n}}, \mathcal{F}) = H^1(C_{\mathbf{n}}, \mathcal{F}) = 0$$

gives rise to a maximal commutative algebra consisting of ordinary differential operators with coefficients in $n \times n$ *matrix valued functions.*

The only commutative algebras of matrix ordinary differential operators known before are constructed from locally cyclic coverings of curves, i.e. a morphism $f : C \longrightarrow C_0$ such that there is a point $p \in C_0$ where $f^{-1}(p)$ consists of one point [**Na**, Appendix]. Since we can use arbitrary coverings of curves, the algebras we obtain in this paper form a far larger class of totally new examples. As a key step from algebraic geometry of curves and vector bundles to the differential operator algebra with matrix coefficients, we prove the following (6.6):

THEOREM E. *The big-cell of the Grassmannian $Gr_n(0)$ is canonically identified with the group of monic invertible pseudodifferential operators with matrix coefficients.*

Only the case of $n = 1$ of this statement was known before. With this identification, we can translate the flows on the Grassmannian associated with an arbitrary commutative subalgebra of the loop algebras into an integrable system of nonlinear partial differential equations. The unique solvability of these systems can be shown by using the generalized Birkhoff decomposition of [**M2**].

0.5. This paper is organized as follows. In Section 1, we review some standard facts about Prym varieties. The Heisenberg flows are introduced in Section 2. Since we do not deal with any central extensions in this paper, we shall not use the Heisenberg algebras in the main text. All we need are the maximal commutative subalgebras of the formal loop algebras. Accordingly, the action of the Borel subalgebras will be replaced by the action of the full maximal commutative algebras defined on certain quotient spaces of the Grassmannian. This turns out to be more natural because of the coordinate-free nature of the flows on the quotient spaces. The two categories we work with are defined in Section 3, where a generalization of the Krichever functor is given. In Section 4, we give the construction of the geometric data out of the algebraic data consisting of commutative algèbras and a point of the Grassmannian. The finite-dimensional orbits of the Heisenberg flows are studied in Section 5, in which the characterization theorem of Prym varieties is proved. Section 6 is devoted to explaining the relation of the entire theory with the ordinary differential operators with matrix coefficients.

The results we obtain in Sections 3, 4, and 6 (except for 6.15, where we need zero characteristic) hold for an arbitrary field k. In Sections 1 and 5 (except for 5.1, which is true for any field), we work with the field C of complex numbers.

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1. Covering morphisms of curves and Prym varieties.

We begin with defining Prym varieties in the most general setting, and then introduce *locally cyclic coverings* of curves, which play an important role in defining the category of arbitrary covering morphisms of algebraic curves in Section 3.

1.1. DEFINITION. *Let $f : C \longrightarrow C_0$ be a covering morphism of degree n between smooth algebraic curves C and C_0, and let $N_f : Jac(C) \longrightarrow Jac(C_0)$ be the norm homomorphism between the Jacobian varieties, which assigns to an element $\sum_q n_q \cdot q \in Jac(C)$ its image $\sum_q n_q \cdot f(q) \in Jac(C_0)$. This is a surjective homomorphism, and hence the kernel $Ker(N_f)$ is an abelian subscheme of $Jac(C)$ of dimension $g(C) - g(C_0)$, where $g(C)$ denotes the genus of the curve C. We call this kernel the* Prym variety *associated with the morphism f, and denote it by $Prym(f)$.*

1.2. REMARK: Usually the Prym variety of a covering morphism f is defined to be the connected component of the kernel of the norm homomorphism containing 0. Since any two connected components of $\mathrm{Ker}(N_f)$ are translations of each other in $\mathrm{Jac}(C)$, there is no harm to call the whole kernel the Prym variety. If the pull-back homomorphism $f^* : \mathrm{Jac}(C_0) \longrightarrow \mathrm{Jac}(C)$ is injective, then the norm homomorphism can be identified with the transpose of f^*, and hence its kernel is connected. So in this situation, our definition coincides with the usual one. We will give a class of coverings where the norm homomorphisms are injective (see 1.7).

1.3. REMARK: Let $R \subset C$ be the ramification divisor of the morphism f of (1.1) and $\mathcal{O}_C(R)$ the locally free sheaf associated with R. Then it can be shown that for any line bundle $\mathcal{L}$ on C, we have $N_f(\mathcal{L}) = \det(f_*\mathcal{L}) \otimes \det\left(f_*\mathcal{O}_C(R)\right)$. Thus up to a translation, the norm homomorphism can be identified with the map assigning the determinant of the direct image to the line bundle on C. Therefore, one can talk about the Prym varieties in $\mathrm{Pic}^d(C)$ for an arbitrary d, not just in $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$.

When the curves C and C_0 are singular, we replace the Jacobian variety $\mathrm{Jac}(C)$ by the generalized Jacobian, which is the connected component of $H^1(C, \mathcal{O}_C^*)$ containing the structure sheaf. By taking the determinant of the direct image sheaf, we can define a map of the generalized Jacobian of C into $H^1(C_0, \mathcal{O}_{C_0}^*)$. The fiber of this map is called the *generalized* Prym variety associated with the morphism f.

1.4. REMARK: According to our definition (1.1), the Jacobian variety of an arbitrary algebraic curve C can be viewed as a Prym variety. Indeed, for a nontrivial morphism of C onto $\mathbf{P}^1$, the induced norm homomorphism is the zero-map. Thus the class of Prym varieties contains Jacobians as a subclass. Of course there are infinitely many ways to realize $\mathrm{Jac}(C)$ as a Prym variety in this manner.

Let us consider the polarizations of Prym varieties. Let Θ_C and Θ_{C_0} be the Riemann theta divisors on $\mathrm{Jac}(C)$ and $\mathrm{Jac}(C_0)$, respectively. Then the restriction of Θ_C to $\mathrm{Prym}(f)$ gives an ample divisor H on $\mathrm{Prym}(f)$. However, this is never a principal polarization. In fact, it is of type $(1, \cdots, 1, n, \cdots, n)$, where the entry n is repeated $g(C_0)$-times. There is a natural homomorphism $\psi : \mathrm{Jac}(C_0) \times \mathrm{Prym}(f) \longrightarrow \mathrm{Jac}(C)$ which assigns $f^*\mathcal{L} \otimes \mathcal{M}$ to $(\mathcal{L}, \mathcal{M}) \in \mathrm{Jac}(C_0) \times \mathrm{Prym}(f)$. This is an isogeny, and the pull-back of Θ_C under this homomorphism is given by

$$\psi^*\mathcal{O}_{\mathrm{Jac}(C)}(\Theta_C) \cong \mathcal{O}_{\mathrm{Jac}(C_0)}(n\Theta_{C_0}) \otimes \mathcal{O}_{\mathrm{Prym}(f)}(H) .$$

In Section 3, we define a category of covering morphisms of algebraic curves. As a *morphism* between the covering morphisms, we use the following special coverings:

1.5. DEFINITION. *A degree r morphism $\alpha : C \longrightarrow C_0$ of algebraic curves is said to be a **locally cyclic covering** if there is a point $p \in C_0$ such that $\alpha^*(p) = r \cdot q$ for some $q \in C$.*

1.6. PROPOSITION. *Every smooth projective curve C has infinitely many smooth locally cyclic coverings of an arbitrary degree.*

PROOF: We use the theory of spectral curves to prove this statement. For a detailed account of spectral curves, we refer to [**BNR**] and [**H**].

Let us take a line bundle $\mathcal{L}$ over C of sufficiently large degree. For such $\mathcal{L}$ we can choose sections $s_i \in H^0(C, \mathcal{L}^i)$, $i = 1, 2, \cdots, r$, satisfying the following conditions:

(1) All s_i's have a common zero point, say $p \in C$, i.e., $s_i \in H^0(C, \mathcal{L}^i(-p))$, $i = 1, 2, \cdots, r$;

(2) $s_r \notin H^0(C, \mathcal{L}^r(-2p))$.

Now consider the sheaf $\mathcal{R}$ of symmetric $\mathcal{O}_C$-algebras generated by $\mathcal{L}^{-1}$. As an $\mathcal{O}_C$-module this algebra can be written as

$$\mathcal{R} = \bigoplus_{i=0}^{\infty} \mathcal{L}^{-i} .$$

In order to construct a locally cyclic covering of C, we take the ideal $\mathcal{I}_s$ of the algebra $\mathcal{R}$ generated by the image of the sum of the homomorphisms $s_i : \mathcal{L}^{-r} \longrightarrow \mathcal{L}^{-r+i}$. We define $C_s = \mathrm{Spec}(\mathcal{R}/\mathcal{I}_s)$, where $s = (s_1, s_2, \cdots, s_r)$. Then C_s is a spectral curve, and the natural projection $\pi : C_s \longrightarrow C$ gives a degree r covering of C. For sufficiently general sections s_i with properties (1) and (2), we may also assume the following (see [**BNR**]):

(3) The spectral curve C_s is integral, i.e. reduced and irreducible.

We claim here that C_s is smooth in a neighborhood of the inverse image of p. Indeed, let us take a local parameter y of C around p and a local coordinate x in the fiber direction of the total space of the line bundle $\mathcal{L}$. Then the local Jacobian criterion for smoothness in a neighborhood of $\pi^{-1}(p)$ states that the following system

$$\begin{cases} x^r + s_1(y)x^{r-1} + \cdots + s_r(y) = 0 \\ rx^{r-1} + s_1(y)(r-1)x^{r-2} + \cdots + s_{r-1}(y) = 0 \\ s_1(y)'x^{r-1} + s_2(y)'x^{r-2} + \cdots + s_r(y)' = 0 \end{cases}$$

of equations in (x, y) has no solutions. But this is clearly the case in our situation because of the conditions (1), (2) and (3). Thus we have verified the claim. It is also clear that $\pi^*(p) = r \cdot q$, where q is the point of C_s defined by $x^r = 0$ and $y = 0$. Then by taking the normalization of C_s we obtain a smooth locally cyclic covering of C. This completes the proof.

1.7. PROPOSITION. *Let $\alpha : C \longrightarrow C_0$ be a locally cyclic covering of degree r. Then the induced homomorphism $\alpha^* : Jac(C_0) \longrightarrow Jac(C)$ of Jacobians is injective. In particular, the Prym variety $Prym(\alpha)$ associated with the morphism α is connected.*

PROOF: Let us suppose in contrary that $\mathcal{L} \not\cong \mathcal{O}_{C_0}$ and $\alpha^*\mathcal{L} \cong \mathcal{O}_C$ for some $\mathcal{L} \in \mathrm{Jac}(C_0)$. Then by the projection formula we have $\mathcal{L} \otimes \alpha_*\mathcal{O}_C \cong \alpha_*\mathcal{O}_C$. Taking determinants on both sides we see that $\mathcal{L}$ is an r-torsion point in $\mathrm{Jac}(C_0)$, i.e. $\mathcal{L}^r \cong \mathcal{O}_{C_0}$. Let m be the smallest positive integer satisfying that $\mathcal{L}^m \cong \mathcal{O}_{C_0}$. Let us consider the spectral curve

$$C' = \mathrm{Spec}(\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}/\mathcal{I}_s)$$

defined by the line bundle $\mathcal{L}$ and its sections

$$s = (s_1, s_2, \cdots, s_{m-1}, s_m) = (0, 0, \cdots, 0, 1) \in \bigoplus_{i=1}^{m} H^0(C_0, \mathcal{L}^i) .$$

It is easy to verify that C' is an unramified covering of C_0 of degree m. Now we claim that the morphism $\alpha : C \longrightarrow C_0$ factors through C', but this leads to a contradiction to our assumption that α is a locally cyclic covering.

The construction of such a morphism $f : C \longrightarrow C'$ over C_0 amounts to defining an $\mathcal{O}_{C_0}$-algebra homomorphism

$$f^\sharp : \bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}/\mathcal{I}_s \longrightarrow \alpha_* \mathcal{O}_C . \tag{1.8}$$

In order to give (1.8), it is sufficient to define an $\mathcal{O}_{C_0}$-module homomorphism ϕ : $\mathcal{L}^{-1} \longrightarrow \alpha_* \mathcal{O}_C$ such that $\phi^{\otimes m} : \mathcal{L}^{-m} \cong \mathcal{O}_{C_0} \longrightarrow \alpha_* \mathcal{O}_C$ is the inclusion map induced by α. Since we have

$$H^0(C, \mathcal{O}_C) \cong H^0(C_0, \alpha_* \mathcal{C}_C) \cong H^0(C_0, \mathcal{L} \otimes \alpha_* \mathcal{O}_C) \cong H^0(C_0, \mathcal{L}^m \otimes \alpha_* \mathcal{O}_C) ,$$

the existence of the desired ϕ is obvious. This completes the proof.

2. The Heisenberg flows on the Grassmannian of vector valued functions.

In this section, we define the Grassmannians of vector valued functions and introduce various vector fields (or flows) on them. Let k be an arbitrary field, $k[[z]]$ the ring of formal power series in one variable z defined over k, and $L = k((z))$ the field of fractions of $k[[z]]$. An element of L is a formal Laurent series in z with a pole of finite order. We call $y = y(z) \in L$ an element of *order* m if $y \in k[[z]]z^{-m} \setminus k[[z]]z^{-m+1}$. Consider the infinite-dimensional vector space $V = L^{\oplus n}$ over k. It has a natural filtration by the (pole) order

$$\cdots \subset F^{(m-1)}(V) \subset F^{(m)}(V) \subset F^{(m+1)}(V) \subset \cdots ,$$

where we define

$$F^{(m)}(V) = \left\{ \sum_{j=0}^{\infty} a_j z^{-m+j} \;\middle|\; a_j \in k^{\oplus n} \right\} . \tag{2.1}$$

In particular, we have $F^{(m)}(V)/F^{(m-1)}(V) \cong k^{\oplus n}$ for all $m \in \mathbb{Z}$. The filtration satisfies

$$\bigcup_{m=-\infty}^{\infty} F^{(m)}(V) = V \quad \text{and} \quad \bigcap_{m=-\infty}^{\infty} F^{(m)}(V) = \{0\} ,$$

and hence it determines a topology in V. In Section 4, we will introduce other filtrations of V in order to define algebraic curves and vector bundles on them. The current filtration (2.1) is used only for the purpose of defining the Grassmannian as a pro-algebraic variety (see for example [**KUS**]).

2.2. DEFINITION. *For every integer μ, the following set is called the index μ Grassmannian of vector valued functions of size n:*

$$Gr_n(\mu) = \{W \subset V \mid \gamma_W \text{ is Fredholm of index } \mu\} ,$$

where $\gamma_W : W \longrightarrow V/F^{(-1)}(V)$ is the natural projection.

Let $N_W = \{\mathrm{ord}_z(v) \mid v \in W\}$. Then the Fredholm condition implies that N_W is bounded from below and contains all sufficiently large positive integers. But of course, this condition of N_W does not imply the Fredholm property of γ_W when $n > 1$.

2.3. REMARK: We have used $F^{(-1)}(V)$ in the above definition as a reference open set for the Fredholm condition. This is because it becomes the natural choice in Section 6 when we deal with the differential operator action on the Grassmannian. From purely algebro-geometric point of view, $F^{(0)}(V)$ can also be used (see 4.6).

The *big-cell* $Gr_n^+(0)$ of the Grassmannian of vector valued functions of size n is the set of vector subspaces $W \subset V$ such that γ_W is an isomorphism. For every point $W \in Gr_n(\mu)$, the tangent space at W is naturally identified with the space of continuous homomorphism of W into V/W:

$$T_W Gr_n(\mu) = \mathrm{Hom}_{\mathrm{cont}}(W, V/W) .$$

Let us define various vector fields on the Grassmannians. Since the formal loop algebra $gl(n, L)$ acts on V, every element $\xi \in gl(n, L)$ defines a homomorphism

$$W \longrightarrow V \xrightarrow{\xi} V \longrightarrow V/W , \tag{2.4}$$

which we shall denote by $\Psi_W(\xi)$. Thus the association

$$Gr_n(\mu) \ni W \longmapsto \Psi_W(\xi) \in \mathrm{Hom}_{\mathrm{cont}}(W, V/W) = T_W Gr_n(\mu)$$

determines a vector field $\Psi(\xi)$ on the Grassmannian. For a subset $\Xi \subset gl(n, L)$, we use the notations $\Psi_W(\Xi) = \{\Psi_W(\xi) \mid \xi \in \Xi\}$ and $\Psi(\Xi) = \{\Psi(\xi) \mid \xi \in \Xi\}$.

2.5. DEFINITION. *A smooth subvariety X of $Gr_n(\mu)$ is said to be an **orbit** (or the **integral manifold**) of the flows of $\Psi(\Xi)$ if the tangent space $T_W X$ of X at W is equal to $\Psi_W(\Xi)$ as a subspace of the whole tangent space $T_W Gr_n(\mu)$ for every point $W \in X$.*

2.6. REMARK: There is a far larger algebra than the loop algebra, the algebra $gl(n, E)$ of *pseudodifferential operators* with matrix coefficients, acting on V. We will come back to this point in Section 6.

Let us choose a monic element

$$y = z^r + \sum_{m=1}^{\infty} c_m z^{r+m} \in L \tag{2.7}$$

of order $-r$ and consider the following $n \times n$ matrix

$$h_n(y) = \begin{pmatrix} 0 & & & & 0 & y \\ 1 & 0 & & & & 0 \\ & 1 & \ddots & & & \\ & & \ddots & 0 & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix} \tag{2.8}$$

satisfying that $h_n(y)^n = y \cdot I_n$, where I_n is the identity matrix of size n. We denote by $H_{(n)}(y)$ the algebra generated by $h_n(y)$ over $k((y))$, which is a maximal commutative subalgebra of the formal loop algebra $gl(n, k((y)))$. Obviously, we have a natural $k((y))$-algebra isomorphism

$$H_{(n)}(y) \cong k((y))[x]/(x^n - y) \cong k((y^{1/n})) ,$$

where x is an indeterminant.

2.9. Definition. *For every integral vector $\mathbf{n} = (n_1, n_2, \cdots, n_\ell)$ of positive integers n_j such that $n = n_1 + n_2 + \cdots + n_\ell$ and a monic element $y \in L$ of order $-r$, we define a maximal commutative $k((y))$-subalgebra of $gl(n, k((y))$ by*

$$H_{\mathbf{n}}(y) = \bigoplus_{j=1}^{\ell} H_{(n_j)}(y) \cong \bigoplus_{j=1}^{\ell} k((y^{1/n_j})) ,$$

where each $H_{(n_j)}(y)$ is embedded by the disjoint principal diagonal blocks:

$$\begin{pmatrix} H_{(n_1)}(y) & & & \\ & H_{(n_2)}(y) & & \\ & & \ddots & \\ & & & H_{(n_\ell)}(y) \end{pmatrix} .$$

*The algebra $H_{\mathbf{n}}(y)$ is called the **maximal commutative algebra of type** $\mathbf{n}$ associated with the variable y.*

As a module over the field $k((y))$, the algebra $H_{\mathbf{n}}(y)$ has dimension n.

2.10. Remark: The lifting of the algebra $H_{\mathbf{n}}(y)$ to the central extension of the formal loop algebra $gl(n, k((y)))$ is the Heisenberg algebra associated with the conjugacy class of the Weyl group of $gl(n,k)$ determined by the integral vector $\mathbf{n}$ ([**FLM**], [**Ka**], [**PS**]). The word *Heisenberg* in the following definition has its origin in this context.

2.11. DEFINITION. *The set of commutative vector fields $\Psi(H_{\mathbf{n}}(y))$ defined on $Gr_n(\mu)$ is called the Heisenberg flows of type $\mathbf{n} = (n_1, n_2, \cdots, n_\ell)$ and rank r associated with the algebra $H_{\mathbf{n}}(y)$ and the coordinate y of (2.7). Let $H_{\mathbf{n}}(y)_0$ denote the subalgebra of $H_{\mathbf{n}}(y)$ consisting of the traceless elements. The system of vector fields $\Psi(H_{\mathbf{n}}(y)_0)$ is called the traceless Heisenberg flows. The set of commuting vector fields $\Psi(k((y)))$ on $Gr_n(\mu)$ is called the r-reduced KP system (or the r-reduction of the KP system) associated with the coordinate y. The usual KP system is defined to be the 1-reduced KP system with the choice of $y = z$. The Heisenberg flows associated with $H_{(1,\cdots,1)}(z)$ of type $(1, \cdots, 1)$ is called the n-component KP system.*

2.12. REMARK: As we shall see in Section 4, the $H_{\mathbf{n}}(y)$-action on V is equivalent to the component-wise multiplication of (4.1) to (4.4). From this point of view, the Heisenberg flows of type $\mathbf{n}$ and rank r are contained in the ℓ-component KP system. What is important in our presentation as the Heisenberg flows is the new algebro-geometric interpretation of the orbits of these systems defined on the (quotient) Grassmannian which can be seen only through the right choice of the coordinates.

2.13. REMARK: The traceless Heisenberg flows of type $\mathbf{n} = (2)$ and rank one are known to be equivalent to the BKP system. As we shall see later in this paper, these flows produce the Prym variety associated with a double sheeted covering of algebraic curves with at least one ramification point. This explains why the BKP system is related only with these very special Prym varieties.

The flows defined above are too large from the geometric point of view. The action of the negative order elements of $gl(n, L)$ should be considered trivial in order to give a direct connection between the orbits of these flows and the Jacobian varieties. Thus it is more convenient to define these flows on certain quotient spaces. So let

$$H_{\mathbf{n}}(y)^- = H_{\mathbf{n}}(y) \cap gl(n, k[[y]]y) \tag{2.14}$$

and define an abelian group

$$\Gamma_{\mathbf{n}}(y) = \exp\bigl(H_{\mathbf{n}}(y)^-\bigr) = I_n + H_{\mathbf{n}}(y)^- \ . \tag{2.15}$$

This group is isomorphic to an affine space, and acts on the Grassmannian without fixed points. This can be verified as follows. Suppose we have $g \cdot W = W$ for some $g = I_n + h \in \Gamma_{\mathbf{n}}(y)$ and $W \in Gr_n(\mu)$. Then $h \cdot W \subset W$. Since h is a nonnilpotent element of negative order, by iterating the action of h on W, we get a contradiction to the Fredholm condition of γ_W.

2.16. DEFINITION. *The quotient Grassmannian of type* $\mathbf{n}$, *index* μ *and rank* r *associated with the algebra* $H_{\mathbf{n}}(y)$ *is the quotient space*

$$Z_{\mathbf{n}}(\mu, y) = Gr_n(\mu)/\Gamma_{\mathbf{n}}(y) \ .$$

We denote by $Q_{\mathbf{n},y} : Gr_{\mathbf{n}}(\mu) \longrightarrow Z_{\mathbf{n}}(\mu, y)$ the canonical projection.

Since $\Gamma_{\mathbf{n}}(y)$ is an affine space acting on the Grassmannian without fixed points, the affine principal fiber bundle $Q_{\mathbf{n},y} : Gr_{\mathbf{n}}(\mu) \longrightarrow Z_{\mathbf{n}}(\mu, y)$ is trivial. If the Grassmannian is modeled on a complex Hilbert space, then one can introduce a Kähler structure on it, which gives rise to a canonical connection on the principal bundle $Q_{\mathbf{n},y}$. In that case, there is a standard way of defining vector fields on the quotient Grassmannian by using the connection. In our case, however, since the Grassmannian $Gr_{\mathbf{n}}(\mu)$ is modeled over $k((z))$, we cannot use these technique of infinite-dimensional complex geometry. Because of this reason, instead of defining vector fields on the quotient Grassmannian, we give directly a definition of orbits on $Z_{\mathbf{n}}(\mu, y)$ in the following manner.

2.17. DEFINITION. *A subvariety $\overline{X}$ of the quotient Grassmannian $Z_{\mathbf{n}}(\mu, y)$ is said to be an orbit of the Heisenberg flows associated with $H_{\mathbf{n}}(y)$ if the pull-back $Q_{\mathbf{n},y}^{-1}(\overline{X})$ is an orbit of the Heisenberg flows on the Grassmannian $Gr_{\mathbf{n}}(\mu)$.*

Here, we note that because of the commutativity of the algebra $H_{\mathbf{n}}(y)$ and the group $\Gamma_{\mathbf{n}}(y)$, the Heisenberg flows on the Grassmannian "descend" to the quotient Grassmannian. Thus for the flows generated by subalgebras of $H_{\mathbf{n}}(y)$, we can safely talk about the *induced flows* on the quotient Grassmannian.

2.18. DEFINITION. *An orbit X of the vector fields $\Psi(\Xi)$ on the Grassmannian $Gr_{\mathbf{n}}(\mu)$ is said to be of finite type if $\overline{X} = Q_{\mathbf{n},y}(X)$ is a finite-dimensional subvariety of the quotient Grassmannian $Z_{\mathbf{n}}(\mu, y)$.*

In Section 5, we study algebraic geometry of finite type orbits of the Heisenberg flows and establish a characterization of Prym varieties in terms of these flows. The actual system of nonlinear partial differential equations corresponding to these vector fields are derived in Section 6, where the unique solvability of the initial value problem of these nonlinear equations is shown by using a theorem of [**M2**].

3. The Krichever functor for covering morphisms of algebraic curves.

The original Krichever correspondence of [**Kr**] is a construction of an exact solution of the entire KP system out of a set of algebro-geometric data consisting of curves and line bundles on them. This correspondence was formulated as a map of the set of these geometric data into the Grassmannian by Segal and Wilson [**SW**]. Its generalization to the geometric data containing arbitrary vector bundles on curves was discovered in [**M3**]. In order to deal with arbitrary covering morphisms of algebraic curves, we have to enlarge the framework of the Krichever functor of [**M3**].

3.1. DEFINITION. *A set of geometric data of a covering morphism of algebraic curves of type $\mathbf{n}$, index μ and rank r is the collection*

$$\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}, \phi) \rangle$$

of the following objects:

(1) $\mathbf{n} = (n_1, n_2, \cdots, n_\ell)$ *is an integral vector of positive integers* n_j *such that* $n = n_1 + n_2 + \cdots + n_\ell$.

(2) $C_\mathbf{n}$ *is a reduced algebraic curve defined over* k, *and* $\Delta = \{p_1, p_2, \cdots, p_\ell\}$ *is a set of* ℓ *smooth rational points of* $C_\mathbf{n}$.

(3) $\Pi = (\pi_1, \cdots, \pi_\ell)$ *consists of a cyclic covering morphism* $\pi_j : U_{oj} \longrightarrow U_j$ *of degree* r *which maps the formal completion* U_{oj} *of the affine line* $\mathbb{A}^1_k$ *along the origin onto the formal completion* U_j *of the curve* $C_\mathbf{n}$ *along* p_j.

(4) $\mathcal{F}$ *is a torsion free sheaf of rank* r *defined over* $C_\mathbf{n}$ *satisfying that*

$$\mu = \dim_k H^0(C_\mathbf{n}, \mathcal{F}) - \dim_k H^1(C_\mathbf{n}, \mathcal{F}) .$$

(5) $\Phi = (\phi_1, \cdots, \phi_\ell)$ *consists of an* $\mathcal{O}_{U_j}$*-module isomorphism*

$$\phi_j : \mathcal{F}_{U_j} \xrightarrow{\sim} \pi_{j*}\big(\mathcal{O}_{U_{oj}}(-1)\big) ,$$

where $\mathcal{F}_{U_j}$ *is the formal completion of* $\mathcal{F}$ *along* p_j. *We identify* ϕ_j *and* $c_j \cdot \phi_j$ *for every nonzero constant* $c_j \in k^*$.

(6) C_0 *is an integral curve with a marked smooth rational point* p.

(7) $f : C_\mathbf{n} \longrightarrow C_0$ *is a finite morphism of degree* n *of* $C_\mathbf{n}$ *onto* C_0 *such that* $f^{-1}(p) = \{p_1, \cdots, p_\ell\}$ *with ramification index* n_j *at each point* p_j.

(8) $\pi : U_o \longrightarrow U_p$ *is a cyclic covering morphism of degree* r *which maps the formal completion* U_o *of the affine line* $\mathbb{A}^1_k$ *at the origin onto the formal completion* U_p *of the curve* C_0 *along* p.

(9) $\pi_j : U_{oj} \longrightarrow U_j$ *and the formal completion* $f_j : U_j \longrightarrow U_p$ *of the morphism* f *at* p_j *satisfy the commutativity of the diagram*

$$\begin{tikzcd} U_{oj} \arrow[r, "\pi_j"] \arrow[d, "\psi_j"'] & U_j \arrow[d, "f_j"] \\ U_o \arrow[r, "\pi"'] & U_p, \end{tikzcd}$$

where $\psi_j : U_{oj} \longrightarrow U_o$ *is a cyclic covering of degree* n_j.

(10) $\phi : (f_*\mathcal{F})_{U_p} \xrightarrow{\sim} \pi_*\Big(\bigoplus_{j=1}^{\ell} \psi_{j*}\big(\mathcal{O}_{U_{oj}}(-1)\big)\Big)$ *is an* $\big(f_*\mathcal{O}_{C_\mathbf{n}}\big)_{U_p}$*-module isomorphism of the sheaves on the formal scheme* U_p *which is compatible with the datum* Φ *upstairs.*

Here we note that we have an isomorphism $\psi_{j*}\big(\mathcal{O}_{U_{oj}}(-1)\big) \cong \mathcal{O}_{U_o}(-1)^{\oplus n_j}$ as an $\mathcal{O}_{U_o}$-module.

Recall that the original Krichever functor is really a cohomology functor. In order to see what kind of algebraic data come up from our geometric data, let us apply the cohomology functor to them. We choose a coordinate z on the formal scheme U_o and fix it once for all. Then we have $U_o = \mathrm{Spec}\big(k[[z]]\big)$. Since $\psi_j : U_{oj} \longrightarrow U_o$ is a cyclic covering of degree n_j, we can identify $U_{oj} = \mathrm{Spec}\big(k[[z^{1/n_j}]]\big)$ so that ψ_j is given by $z = \big(z^{1/n_j}\big)^{n_j} = z_j^{n_j}$, where $z_j = z^{1/n_j}$ is a coordinate of U_{oj}. The morphism π determines a coordinate

$$y = z^r + \sum_{m=1}^{\infty} c_m z^{r+m}$$

on U_p. We also choose a coordinate $y_j = y^{1/n_j}$ of U_j in which the morphism f_j can be written as $y = \big(y^{1/n_j}\big)^{n_j} = y_j^{n_j}$. Out of the geometric data, we can assign a vector subspace W of V by

$$\begin{aligned} W &= \phi(H^0(C_0 \setminus \{p\}, f_*\mathcal{F})) \\ &\subset H^0\Big(U_p \setminus \{p\}, \pi_* \bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1))\Big) \\ &= H^0\Big(U_o \setminus \{o\}, \bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1))\Big) \\ &\cong H^0(U_o \setminus \{o\}, \bigoplus_{j=1}^{\ell} \mathcal{O}_{U_o}(-1)^{\oplus n_j}) \\ &\cong H^0(U_o \setminus \{o\}, \mathcal{O}_{U_o}(-1)^{\oplus n}) = k((z))^{\oplus n} = V\,. \end{aligned} \tag{3.2}$$

Here, we have used the convention of [**M3**] that

$$\begin{aligned} H^0(C_0 \setminus \{p\}, \mathcal{O}_{C_0}) &= \varinjlim_m H^0\big(C_0, \mathcal{O}_{C_0}(m \cdot p)\big) \\ H^0(U_o \setminus \{o\}, \mathcal{O}_{U_o}) &= \varinjlim_m H^0\big(U_o, \mathcal{O}_{U_o}(m)\big) = k((z))\,, \end{aligned}$$

etc. The coordinate ring of the curve C_0 determines a scalar diagonal stabilizer algebra

$$\begin{aligned} A_0 &= \pi^*\big(H^0(C_0 \setminus \{p\}, \mathcal{O}_{C_0})\big) \\ &\subset \pi^*\big(H^0(U_p \setminus \{p\}, \mathcal{O}_{U_p})\big) \\ &\subset H^0\big(U_o \setminus \{o\}, \mathcal{O}_{U_o}\big) \\ &= L \subset gl(n, L) \end{aligned} \tag{3.3}$$

satisfying that $A_0 \cdot W \subset W$, where L is identified with the set of scalar matrices in $gl(n, L)$. The rank of W over A_0 is $r \cdot n$, which is equal to the rank of $f_*\mathcal{F}$. Note that we have also an inclusion

$$A_0 \cong H^0(C_0 \setminus \{p\}, \mathcal{O}_{C_0}) \subset H^0(U_p \setminus \{p\}, \mathcal{O}_{U_p}) = k((y))$$

by the coordinate y. As in [**M3**, Section 2 and 3], we can use the formal patching $C_0 = (C_0 \setminus \{p\}) \cup U_p$ to compute the cohomology group

$$\begin{aligned} H^1(C_0, \mathcal{O}_{C_0}) &\cong \frac{H^0(U_p \setminus \{p\}, \mathcal{O}_{U_p})}{H^0(C_0 \setminus \{p\}, \mathcal{O}_{C_0}) + H^0(U_p, \mathcal{O}_{U_p})} \\ &\cong \frac{k((y))}{A_0 + k[[y]]}\,. \end{aligned} \tag{3.4}$$

Thus the cokernel of the projection $\gamma_{A_0} : A_0 \longrightarrow k((y))/k[[y]]$ has finite dimension. The function ring

$$A_{\mathbf{n}} = H^0(C_{\mathbf{n}} \setminus \Delta, \mathcal{O}_{C_{\mathbf{n}}}) \subset \bigoplus_{j=1}^{\ell} H^0(U_j \setminus \{p_j\}, \mathcal{O}_{U_j})$$

also acts on V and satisfies that $A_{\mathbf{n}} \cdot W \subset W$, because we have a natural injective isomorphism

$$\begin{aligned}
A_{\mathbf{n}} = H^0(C_{\mathbf{n}} \setminus \Delta, \mathcal{O}_{C_{\mathbf{n}}}) &\cong H^0\big(C_0 \setminus \{p\}, f_*\mathcal{O}_{C_{\mathbf{n}}}\big) \\
&\subset H^0\big(U_p \setminus \{p\}, (f_*\mathcal{O}_{C_{\mathbf{n}}})_{U_p}\big) \\
&= H^0\big(U_p \setminus \{p\}, \bigoplus_{j=1}^{\ell} f_{j*}\mathcal{O}_{U_j}\big) \\
&= \bigoplus_{j=1}^{\ell} k((y))\left[h_{n_j}(y)\right] \\
&= H_{\mathbf{n}}(y) \subset gl\big(n, k((y))\big) \ ,
\end{aligned} \tag{3.5}$$

where $h_{n_j}(y)$ is the block matrix of (2.8) and $H_{\mathbf{n}}(y)$ is the maximal commutative subalgebra of $gl\big(n, k((y))\big)$ of type $\mathbf{n}$. In order to see the action of $A_{\mathbf{n}}$ on W more explicitly, we first note that the above isomorphism is given by the identification $y^{1/n_j} = h_{n_j}(y)$. Since the formal completion $\mathcal{F}_{U_j}$ of the vector bundle $\mathcal{F}$ at the point p_j is a free $\mathcal{O}_{U_j}$-module of rank r, let us take a basis $\{e_1, e_2, \cdots, e_r\}$ for the free $H^0(U_j, \mathcal{O}_{U_j})$-module $H^0(U_j, \mathcal{F}_{U_j})$. The direct image sheaf $f_{j*}\mathcal{F}_{U_j}$ is a free $\mathcal{O}_{U_p}$-module of rank $n_j \cdot r$, so we can take a basis of sections

$$\left\{y^{\alpha/n_j} \otimes e_\beta\right\}_{0 \le \alpha < n_j, 1 \le \beta \le r} \tag{3.6}$$

for the free $H^0(U_p, \mathcal{O}_{U_p})$-module $H^0(U_p, f_{j*}\mathcal{F}_{U_j})$. Since $H^0(U_j, \mathcal{F}_{U_j}) = H^0(U_p, f_{j*}\mathcal{F}_{U_j})$, $H^0(U_j, \mathcal{O}_{U_j}) = H^0(U_p, f_{j*}\mathcal{O}_{U_j})$ acts on the basis (3.6) by the matrix $h_{n_j}(y) \otimes I_r$, where I_r is the identity matrix acting on $\{e_1, e_2, \cdots, e_r\}$. This can be understood by observing that the action of $y^{1/n}$ on the vector

$$(c_0, c_1, \cdots, c_{n-1}) = \sum_{\alpha=0}^{n-1} c_\alpha y^{\alpha/n}$$

is given by the action of the block matrix $h_n(y)$.

3.7. REMARK: From the above argument, it is clear that the role which our π and ϕ play is exactly the same as that of the *parabolic structure* of [**Mum2**]. The advantage of using π and ϕ rather than the parabolic structure lies in their functoriality. Indeed, the parabolic structure does not transform functorially under morphisms of curves, while our data naturally do (see 3.14).

The algebra $H_{\mathbf{n}}(y)$ has two different presentations in terms of geometry. We have used

$$H_{\mathbf{n}}(y) \cong H^0(U_p \setminus \{p\}, (f_* \mathcal{O}_{C_{\mathbf{n}}})_{U_p}) = \bigoplus_{j=1}^{\ell} k((y))[h_{n_j}(y)] \subset gl(n, k((y)))$$

in (3.5). In this presentation, an element of $H_{\mathbf{n}}(y)$ is an $n \times n$ matrix acting on $V \cong H^0(U_p \setminus \{p\}, (f_* \mathcal{F})_{U_p})$. The other geometric interpretation is

$$H_{\mathbf{n}}(y) \cong H^0(U_p \setminus \{p\}, \bigoplus_{j=1}^{\ell} f_{j*} \mathcal{O}_{U_j}) \cong \bigoplus_{j=1}^{\ell} H^0(U_j \setminus \{p_j\}, \mathcal{O}_{U_j}) = \bigoplus_{j=1}^{\ell} k((y_j)) .$$

In this presentation, the algebra $H_{\mathbf{n}}(y)$ acts on

$$\begin{aligned} V &\cong H^0\left(U_p \setminus \{p\}, \pi_* \bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1))\right) \\ &\cong \bigoplus_{j=1}^{\ell} H^0(U_{oj} \setminus \{o\}, \mathcal{O}_{U_{oj}}(-1)) \\ &= \bigoplus_{j=1}^{\ell} k((z_j)) \end{aligned}$$

by the component-wise multiplication of y_j to z_j. We will come back to this point in (4.4).

The pull-back through the morphism f gives an embedding $A_0 \subset A_{\mathbf{n}}$. As an A_0-module, $A_{\mathbf{n}}$ is torsion free of rank n, because C_0 is integral and the morphism f is of degree n. Using the formal patching $C_{\mathbf{n}} = (C_{\mathbf{n}} \setminus \Delta) \cup U_1 \cup \cdots \cup U_\ell$, we can compute the cohomology

$$\begin{aligned} H^1(C_{\mathbf{n}}, \mathcal{O}_{C_{\mathbf{n}}}) &\cong \frac{\bigoplus_{j=1}^{\ell} H^0(U_j \setminus \{p_j\}, \mathcal{O}_{U_j})}{H^0(C_{\mathbf{n}} \setminus \Delta, \mathcal{O}_{C_{\mathbf{n}}}) + \bigoplus_{j=1}^{\ell} H^0(U_j, \mathcal{O}_{U_j})} \\ &\cong \frac{\bigoplus_{j=1}^{\ell} k((y^{1/n_j}))}{A_{\mathbf{n}} + \bigoplus_{j=1}^{\ell} k[[y^{1/n_j}]]} \\ &\cong \frac{H_{\mathbf{n}}(y)}{A_{\mathbf{n}} + H_{\mathbf{n}}(y) \cap gl(n, k[[y]])} . \end{aligned} \tag{3.8}$$

This shows that the projection

$$\gamma_{A_{\mathbf{n}}} : A_{\mathbf{n}} \longrightarrow \frac{H_{\mathbf{n}}(y)}{H_{\mathbf{n}}(y) \cap gl(n, k[[y]])}$$

has a finite-dimensional cokernel. These discussions motivate the following definition:

3.9. DEFINITION. *A triple $(A_0, A_\mathbf{n}, W)$ is said to be a set of algebraic data of type* $\mathbf{n}$, *index* μ, *and rank* r *if the following conditions are satisfied:*

(1) *W is a point of the Grassmannian $Gr_\mathbf{n}(\mu)$ of index μ of the vector valued functions of size n.*
(2) *The type $\mathbf{n}$ is an integral vector $(n_1, \cdots, n_\ell)$ consisting of positive integers such that $n = n_1 + \cdots + n_\ell$.*
(3) *There is a monic element $y \in L = k((z))$ of order $-r$ such that A_0 is a subalgebra of $k((y))$ containing the field k.*
(4) *The cokernel of the projection $\gamma_{A_0} : A_0 \longrightarrow k((y))/k[[y]]$ has finite dimension.*
(5) *$A_\mathbf{n}$ is a subalgebra of the maximal commutative algebra $H_\mathbf{n}(y) \subset gl(n, k((y)))$ of type $\mathbf{n}$ such that the projection*

$$\gamma_{A_\mathbf{n}} : A_\mathbf{n} \longrightarrow \frac{H_\mathbf{n}(y)}{H_\mathbf{n}(y) \cap gl(n, k[[y]])}$$

has a finite-dimensional cokernel.
(6) *There is an embedding $A_0 \subset A_\mathbf{n}$ as the scalar diagonal matrices, and as an A_0-module (which is automatically torsion free), $A_\mathbf{n}$ has rank n over A_0.*
(7) *The algebra $A_\mathbf{n} \subset gl(n, k((y)))$ stabilizes $W \subset V$, i.e. $A_\mathbf{n} \cdot W \subset W$.*

The homomorphisms γ_{A_0} and $\gamma_{A_\mathbf{n}}$ satisfy the Fredholm condition because (7) implies that they have finite-dimensional kernels. Now we can state

3.10. PROPOSITION. *For every set of geometric data of* (3.1), *there is a unique set of algebraic data of* (3.9) *having the same type, index and rank.*

PROOF: We have already constructed the triple $(A_0, A_\mathbf{n}, W)$ out of the geometric data in (3.2), (3.3) and (3.5) which satisfies all the conditions in (3.9) but (1). The only remaining thing we have to show is that the vector subspace W of (3.2) is indeed a point of the Grassmannian $Gr_\mathbf{n}(\mu)$. To this end, we need to compute the cohomology of $f_*\mathcal{F}$ by using the formal patching $C_0 = \mathrm{Spec}(A_0) \cup U_p$ (for more detail, see [M3]). Noting the identification

$$\bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1)) \cong \mathcal{O}_{U_o}(-1)^{\oplus n}$$

as in (3.2), we can show that

$$\begin{aligned} H^0(C_0, f_*\mathcal{F}) &= H^0(C_0 \setminus \{p\}, f_*\mathcal{F}) \cap H^0(U_p, f_*\mathcal{F}_{U_p}) \\ &\cong W \cap H^0\big(U_p, \pi_*(\mathcal{O}_{U_o}(-1)^{\oplus n})\big) \\ &\cong W \cap H^0\big(U_o, \mathcal{O}_{U_o}(-1)^{\oplus n}\big) \\ &\cong W \cap \big(k[[z]]z\big)^{\oplus n} \\ &= \mathrm{Ker}(\gamma_W)\,, \end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
H^1(C_0, f_*\mathcal{F}) &\cong \frac{H^0(U_p \setminus \{p\}, f_*\mathcal{F})}{H^0(C_0 \setminus \{p\}, f_*\mathcal{F}) + H^0(U_p, f_*\mathcal{F}_{U_p})} \\
&\cong \frac{H^0(U_p \setminus \{p\}, \pi_*(\mathcal{O}_{U_o}(-1)^{\oplus n}))}{W + H^0(U_p, \pi_*(\mathcal{O}_{U_o}(-1)^{\oplus n}))} \\
&\cong \frac{H^0(U_o \setminus \{o\}, \mathcal{O}_{U_o}(-1)^{\oplus n})}{W + H^0(U_o, \mathcal{O}_{U_o}(-1)^{\oplus n})} \qquad (3.12)\\
&\cong \frac{k((z))^{\oplus n}}{W + \left(k[[z]]z\right)^{\oplus n}} \\
&= \mathrm{Coker}(\gamma_W)\ ,
\end{aligned}
$$

where γ_W is the canonical projection of (2.2). Since f is a finite morphism, we have $H^i(C_0, f_*\mathcal{F}) \cong H^i(C_n, \mathcal{F})$. Thus

$$
(3.13)\quad \mu = \dim_k H^0(C_{\mathbf{n}}, \mathcal{F}) - \dim_k H^1(C_{\mathbf{n}}, \mathcal{F}) = \dim_k \mathrm{Ker}(\gamma_W) - \dim_k \mathrm{Coker}(\gamma_W)\ ,
$$

which shows that W is indeed a point of $Gr_n(\mu)$. This completes the proof.

This proposition gives a generalization of the Krichever map to the case of covering morphisms of algebraic curves. We can make the above map further into a functor, which we shall call the *Krichever functor for covering morphisms*. The categories we use are the following:

3.14. DEFINITION. *The category $\mathcal{C}(\mathbf{n})$ of geometric data of a fixed type $\mathbf{n}$ consists of the set of geometric data of type $\mathbf{n}$ and arbitrary index μ and rank r as its object. A morphism between two objects*

$$
\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}, \phi)\rangle
$$

of type $\mathbf{n}$, index μ and rank r and

$$
\langle f' : (C'_{\mathbf{n}}, \Delta', \Pi', \mathcal{F}', \Phi') \longrightarrow (C'_0, p', \pi', f'_*\mathcal{F}', \phi')\rangle
$$

of the same type $\mathbf{n}$, index μ' and rank r' is a triple (α, β, λ) of morphisms satisfying the following conditions:

(1) *$\alpha : C'_0 \longrightarrow C_0$ is a locally cyclic covering of degree s of the base curves such that $\alpha^*(p) = s \cdot p'$, and π and π' are related by $\pi = \widehat{\alpha} \circ \pi'$ with the morphism $\widehat{\alpha}$ of formal schemes induced by α.*

(2) *$\beta : C'_{\mathbf{n}} \longrightarrow C_{\mathbf{n}}$ is a covering morphism of degree s such that $\Delta' = \beta^{-1}(\Delta)$, and the following diagram*

$$
\begin{tikzcd}
C'_{\mathbf{n}} \arrow[r, "\beta"] \arrow[d, "f'"'] & C_{\mathbf{n}} \arrow[d, "f"] \\
C'_0 \arrow[r, "\alpha"'] & C_0
\end{tikzcd}
$$

commutes.

(3) *The morphism* $\widehat{\beta}_j : U'_j \longrightarrow U_j$ *of formal schemes induced by* β *at each* p'_j *satisfies* $\pi_j = \widehat{\beta}_j \circ \pi'_j$ *and the commutativity of*

$$\begin{array}{ccccc} U_{oj} & \xrightarrow{\pi'_j} & U'_j & \xrightarrow{\widehat{\beta}_j} & U_j \\ {\scriptstyle \psi_j}\big\downarrow & & {\scriptstyle f'_j}\big\downarrow & & {\scriptstyle f_j}\big\downarrow \\ U_o & \xrightarrow[\pi']{} & U'_{p'} & \xrightarrow[\widehat{\alpha}]{} & U_p. \end{array}$$

(4) $\lambda : \beta_* \mathcal{F}' \longrightarrow \mathcal{F}$ *is an injective* $\mathcal{O}_{C_n}$*-module homomorphism such that its completion* λ_j *at each point* p_j *satisfies commutativity of*

$$\begin{array}{ccc} (\beta_* \mathcal{F}')_{U_j} & \xrightarrow{\lambda_j} & \mathcal{F}_{U_j} \\ {\scriptstyle \widehat{\beta}_j(\phi'_j)}\big\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\big\downarrow{\scriptstyle \phi_j} \\ \widehat{\beta}_{j*}\pi'_{j*}\mathcal{O}_{U_{oj}}(-1) & = & \pi_{j*}\mathcal{O}_{U_{oj}}(-1). \end{array}$$

In particular, each λ_j *is an isomorphism.*

3.15. REMARK: From (3) above, we have $r = s \cdot r'$. The condition (4) above implies that $\mathcal{F}/\beta_*\mathcal{F}'$ is a torsion sheaf on C_n whose support does not intersect with Δ.

One can show by using (1.6) that there are many nontrivial morphisms among the sets of geometric data with different ranks.

3.16. DEFINITION. *The category* $\mathcal{S}(\mathbf{n})$ *of algebraic data of type* $\mathbf{n}$ *has the stabilizer triples* $(A_0, A_\mathbf{n}, W)$ *of (3.9) of type* $\mathbf{n}$ *and arbitrary index* μ *and rank* r *as its objects. Note that for every object* $(A_0, A_\mathbf{n}, W)$*, we have the commutative algebras* $k((y))$ *and* $H_\mathbf{n}(y)$ *associated with it. A morphism between two objects* $(A_0, A_\mathbf{n}, W)$ *and* $(A'_0, A'_\mathbf{n}, W')$ *is a triple* $(\iota, \epsilon, \omega)$ *of injective homomorphisms satisfying the following conditions:*

(1) $\iota : A_0 \hookrightarrow A'_0$ *is an inclusion compatible with the inclusion* $k((y)) \subset k((y'))$ *defined by a power series*

$$y = y(y') = y'^s + a_1 y'^{s+1} + a_2 y'^{s+2} + \cdots .$$

(2) $\epsilon : A_\mathbf{n} \longrightarrow A'_\mathbf{n}$ *is an injective homomorphism satisfying the commutativity of the diagram*

$$\begin{array}{ccc} A_\mathbf{n} & \xrightarrow{\epsilon} & A'_\mathbf{n} \\ \big\downarrow & & \big\downarrow \\ H_\mathbf{n}(y) & \xrightarrow[\varepsilon]{} & H_\mathbf{n}(y'), \end{array}$$

where the vertical arrows are the inclusion maps, and

$$\mathcal{E} : H_{\mathbf{n}}(y) \cong \bigoplus_{j=1}^{\ell} k((y^{1/n_j})) \longrightarrow \bigoplus_{j=1}^{\ell} k((y'^{1/n_j})) \cong H_{\mathbf{n}}(y')$$

is an injective homomorphism defined by the Puiseux expansion

$$y^{1/n_j} = y(y')^{1/n_j} = y'^{s/n_j} + b_1 y'^{(s+1)/n_j} + b_2 y'^{(s+2)/n_j} + \cdots$$

of (1) *for every* n_j. *Note that neither* ϵ *nor* $\mathcal{E}$ *is an inclusion map of subalgebras of* $gl(n, L)$.

(3) $\omega : W' \longrightarrow W$ *is an injective* $A_{\mathbf{n}}$*-module homomorphism. We note that* W' *has a natural* $A_{\mathbf{n}}$*-module structure by the homomorphism* ϵ. *As in* (2), ω *is not an inclusion map of the vector subspaces of* V.

3.17. THEOREM. *There is a fully-faithful functor*

$$\kappa_{\mathbf{n}} : \mathcal{C}(\mathbf{n}) \xrightarrow{\sim} \mathcal{S}(\mathbf{n})$$

between the category of geometric data and the category of algebraic data. An object of $\mathcal{C}(\mathbf{n})$ *of index* μ *and rank* r *corresponds to an object of* $\mathcal{S}(\mathbf{n})$ *of the same index and rank.*

PROOF: The association of $(A_0, A_{\mathbf{n}}, W)$ to the geometric data has been done in (3.2), (3.3), (3.5) and (3.10). Let (α, β, λ) be a morphism between two sets of geometric data as in (3.14). We use the notations $U_j^* = U_j \setminus \{p_j\}$ and $U_p^* = U_p \setminus \{p\}$. The homomorphism ι is defined by the commutative diagram

$$\begin{array}{ccccc}
A_0 & \xrightarrow{\sim} & H^0(C_0 \setminus \{p\}, \mathcal{O}_{C_0}) & \longrightarrow & H^0(U_p^*, \mathcal{O}_{U_p}) \\
{\scriptstyle\iota}\big\downarrow & & {\scriptstyle\alpha^*}\big\downarrow & & {\scriptstyle\widehat{\alpha}^*}\big\downarrow \\
A_0' & \xrightarrow{\sim} & H^0(C_0' \setminus \{p'\}, \mathcal{O}_{C_0'}) & \longrightarrow & H^0(U'^*_{p'}, \mathcal{O}_{U'_{p'}}).
\end{array}$$

Similarly,

$$\begin{array}{ccccc}
A_{\mathbf{n}} & \xrightarrow{\sim} & H^0(C_{\mathbf{n}} \setminus \Delta, \mathcal{O}_{C_{\mathbf{n}}}) & \longrightarrow & \bigoplus_{j=1}^{\ell} H^0(U_j^*, \mathcal{O}_{U_j}) \\
{\scriptstyle\epsilon}\big\downarrow & & {\scriptstyle\beta^*}\big\downarrow & & {\scriptstyle\oplus\widehat{\beta}_j^*}\big\downarrow \\
A_{\mathbf{n}}' & \xrightarrow{\sim} & H^0(C_{\mathbf{n}}' \setminus \Delta', \mathcal{O}_{C_{\mathbf{n}}'}) & \longrightarrow & \bigoplus_{j=1}^{\ell} H^0(U'^*_j, \mathcal{O}_{U'_j})
\end{array}$$

defines the homomorphism ϵ. Finally,

$$\begin{array}{ccccc}
W' & = & W' & \xrightarrow{\omega} & W \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
H^0(C_0' \setminus \{p'\}, f'_*\mathcal{F}') & \underset{\sim}{\xrightarrow{\alpha_*}} & H^0(C_0 \setminus \{p\}, f_*\beta_*\mathcal{F}') & \xrightarrow{f_*(\lambda)} & H^0(C_0 \setminus \{p\}, f_*\mathcal{F}) \\
f'^* \downarrow \wr & & f^* \downarrow \wr & & f^* \downarrow \wr \\
H^0(C_n' \setminus \Delta', \mathcal{F}') & \underset{\sim}{\xrightarrow{\beta_*}} & H^0(C_n \setminus \Delta, \beta_*\mathcal{F}') & \xrightarrow{\lambda} & H^0(C_n \setminus \Delta, \mathcal{F}) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_j H^0(U'^*_j, \mathcal{F}'_{U'_j}) & \underset{\sim}{\xrightarrow{\oplus \widehat{\beta}_{j*}}} & \bigoplus_j H^0(U^*_j, \widehat{\beta}_{j*}\mathcal{F}'_{U'_j}) & \xrightarrow{\oplus \lambda_j} & \bigoplus_j H^0(U^*_j, \mathcal{F}_{U_j})
\end{array}$$

determines the homomorphism ω.

In order to establish that the two categories are equivalent, we need the inverse construction. The next section is entirely devoted to the proof of this claim.

The following proposition and its corollary about the geometric data of rank one are crucial when we study geometry of orbits of the Heisenberg flows in Section 5.

3.18. PROPOSITION. *Suppose we have two sets of geometric data of rank one having exactly the same constituents except for the sheaf isomorphisms (Φ, ϕ) for one and (Φ', ϕ') for the other. Let (A_0, A_n, W) and (A_0, A_n, W') be the corresponding algebraic data, where A_0 and A_n are common in both of the triples because of the assumption. Then there is an element $g \in \Gamma_n(y)$ of (2.15) such that $W' = g \cdot W$.*

PROOF: Recall that

$$\phi : (f_*\mathcal{F})_{U_p} \xrightarrow{\sim} \pi_*\Big(\bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1))\Big)$$

is an $(f_*\mathcal{O}_{C_n})_{U_p}$-module isomorphism. Thus,

$$g = \phi' \circ \phi^{-1} : \pi_*\Big(\bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1))\Big) \xrightarrow{\sim} \pi_*\Big(\bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1))\Big)$$

is also an $(f_*\mathcal{O}_{C_n})_{U_p}$-module isomorphism. Note that we have identified $(f_*\mathcal{O}_{C_n})_{U_p}$ as a subalgebra of $H_n(y)$ in (3.5). Indeed, this subalgebra is $H_n(y) \cap gl(n, k[[y]])$. Therefore, the invertible $n \times n$ matrix

$$g \in k^{*\oplus n} + gl(n, k[[y]]y) = k^{*\oplus n} + gl(n, k[[z]]z)$$

commutes with $H_{\mathbf{n}}(y) \cap gl(n, k[[y]])$, where k^* denotes the set of nonzero constants and $k^{*\oplus n}$ the set of invertible constant diagonal matrices. We recall that $k[[z]] = k[[y]]$, because y has order -1. The commutativity of g and $H_{\mathbf{n}}(y) \cap gl(n, k[[y]])$ immediately implies that g commutes with all of $H_{\mathbf{n}}(y)$. But since $H_{\mathbf{n}}(y)$ is a *maximal* commutative subalgebra of $gl(n, k((y)))$, it implies that $g \in \Gamma_{\mathbf{n}}(y)$. Here we note that $\phi'_j \circ \phi_j{}^{-1}$ is exactly the j-th block of size $n_j \times n_j$ of the $n \times n$ matrix g, and that we can normalize the leading term of $\phi'_j \circ \phi_j{}^{-1}$ to be equal to I_{n_j} by the definition (5) of (3.1). Thus the leading term of g can be normalized to I_n. From the construction of (3.2), we have $W' = g \cdot W$. This completes the proof.

3.19. COROLLARY. *The Krichever functor induces a bijective correspondence between the collection of geometric data*

$$\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}) \rangle$$

of type $\mathbf{n}$, *index* μ, *and rank one, and the triple of algebraic data* $(A_0, A_{\mathbf{n}}, \overline{W})$ *of type* $\mathbf{n}$, *index* μ, *and rank one satisfying the same conditions of* (3.9) *except that* $\overline{W}$ *is a point of the quotient Grassmannian* $Z_{\mathbf{n}}(\mu, y)$.

PROOF: Note that the datum Φ is indeed the block decomposition of the datum of ϕ. Thus taking the quotient space of the Grassmannian by the group action of $\Gamma_{\mathbf{n}}(y)$ exactly corresponds to eliminating the data Φ and ϕ from the set of geometric data of (3.1).

4. The inverse construction.

Let $W \in Gr_{\mathbf{n}}(\mu)$ be a point of the Grassmannian and consider a commutative subalgebra A of $gl(n, L)$ such that $A \cdot W \subset W$. Since the set of vector fields $\Psi(A)$ has W as a fixed point, we call such an algebra a commutative *stabilizer* algebra of W. In the previous work [**M3**], the algebro-geometric structures of arbitrary commutative stabilizers were determined for the case of the Grassmannian $Gr_1(\mu)$ of scalar valued functions. In the context of the current paper, the Grassmannian is enlarged, and consequently there are far larger varieties of commutative stabilizers. However, it is not the purpose of this paper to give the complete geometric classification of arbitrary stabilizers. We restrict ourselves to studying *large* stabilizers in connection with Prym varieties, which will be the central theme of the next section. A stabilizer is said to be large if it corresponds to a finite-dimensional orbit of the Heisenberg flows on the quotient Grassmannian. The goal of this section is to recover the geometric data out of a point of the Grassmannian together with a large stabilizer.

Choose an integral vector $\mathbf{n} = (n_1, n_2, \cdots, n_\ell)$ with $n = n_1 + \cdots + n_\ell$ and a monic element y of order $-r$ as in (2.7), and consider the formal loop algebra $gl(n, k((y)))$ acting on the vector space $V = L^{\oplus n}$. Let us denote $y_j = h_{n_j}(y) = y^{1/n_j}$. We introduce a new filtration

$$\cdots \subset H_{\mathbf{n}}(y)^{(rm-r)} \subset H_{\mathbf{n}}(y)^{(rm)} \subset H_{\mathbf{n}}(y)^{(rm+r)} \subset \cdots$$

in the maximal commutative algebra

$$H_{\mathbf{n}}(y) \cong \bigoplus_{j=1}^{\ell} k((y))[y^{1/n_j}] \cong \bigoplus_{j=1}^{\ell} k((y^{1/n_j})) = \bigoplus_{j=1}^{\ell} k((y_j)) \tag{4.1}$$

by defining

$$H_{\mathbf{n}}(y)^{(rm)} = \left\{ (a_1(y_1), \cdots, a_\ell(y_\ell)) \,\middle|\, \max\left[\operatorname{ord}_{y_1}(a_1), \cdots, \operatorname{ord}_{y_\ell}(a_\ell)\right] \leq m \right\}, \tag{4.2}$$

where $\operatorname{ord}_{y_j}(a_j)$ is the order of $a_j(y_j) \in k((y_j))$ with respect to the variable y_j. Accordingly, we can introduce a filtration in V which is compatible with the action of $H_{\mathbf{n}}(y)$ on V. In order to define the new filtration in V geometrically, let us start with $U_o = \operatorname{Spec}(k[[z]])$ and $U_p = \operatorname{Spec}(k[[y]])$. The inclusion $k[[y]] \subset k[[z]]$ given by $y = y(z) = z^r + c_1 z^{r+1} + c_2 z^{r+2} + \cdots$ defines a morphism $\pi : U_o \longrightarrow U_p$. Let $U_j = \operatorname{Spec}(k[[y_j]])$. The identification $y_j = y^{1/n_j}$ gives a cyclic covering $f_j : U_j \longrightarrow U_p$ of degree n_j. Correspondingly, the covering $\psi_j : U_{oj} \longrightarrow U_o$ of degree n_j of (9) of (3.1) is given by $k[[z]] \subset k[[z^{1/n_j}]]$. Thus we have a commutative diagram

$$\begin{tikzcd}
k[[z^{1/n_j}]] & k[[y^{1/n_j}]] \arrow[l, "\pi_j^*"'] \\
k[[z]] \arrow[u, "\psi_j^*"] & k[[y]] \arrow[l, "\pi^*"] \arrow[u, "f_j^*"']
\end{tikzcd}$$

of inclusions, where π_j^* is defined by the Puiseux expansion

$$y_j = y^{1/n_j} = y(z)^{1/n_j} = z^{r/n_j} + a_1 z^{(r+1)/n_j} + a_2 z^{(r+2)/n_j} + \cdots \tag{4.3}$$

of $y(z)$. Recall that in order to distinguish from $U_o = \operatorname{Spec}(k[[z]])$, we have introduced the notation $U_{oj} = \operatorname{Spec}(k[[z^{1/n_j}]])$ for the cyclic covering of U_o. The above diagram corresponds to the geometric diagram of covering morphisms

$$\begin{tikzcd}
U_{oj} \arrow[r, "\pi_j"] \arrow[d, "\psi_j"'] & U_j \arrow[d, "f_j"] \\
U_o \arrow[r, "\pi"'] & U_p .
\end{tikzcd}$$

We denote $U_o^* = U_o \setminus \{o\}$, $U_{oj}^* = U_{oj} \setminus \{o\}$, $U_p^* = U_p \setminus \{p\}$, and $U_j^* = U_j \setminus \{p_j\}$ as before. The $k((y))$-algebra $H_{\mathbf{n}}(y)$ is identified with the $H^0(U_p^*, \mathcal{O}_{U_p})$-algebra

$$H_{\mathbf{n}}(y) = H^0(U_p^*, \bigoplus_{j=1}^{\ell} f_{j*}\mathcal{O}_{U_j}) \cong \bigoplus_{j=1}^{\ell} H^0(U_j^*, \mathcal{O}_{U_j}) .$$

Corresponding to this identification, the vector space $V = L^{\oplus n}$ as a module over $L = H^0(U_o^*, \mathcal{O}_{U_o})$ is identified with

$$(4.4)\quad V = H^0\left(U_o^*, \bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1))\right) \cong \bigoplus_{j=1}^{\ell} H^0(U_{oj}^*, \mathcal{O}_{U_{oj}}(-1)) \cong \bigoplus_{j=1}^{\ell} k((z^{1/n_j}))\,.$$

The $H_{\mathbf{n}}(y)$-module structure of V is given by the pull-back $\bigoplus_{j=1}^{\ell} \pi_j^*$, which is nothing but the component-wise multiplication of $k((y^{1/n_j}))$ to $k((z^{1/n_j}))$ through (4.3) for each j. Define a new variable by $z_j = z^{1/n_j}$. We note from (4.3) that $y_j = y_j(z^{1/n_j}) = y_j(z_j)$ is of order $-r$ with respect to z_j. Now we can introduce a new filtration

$$\cdots \subset V^{(m-1)} \subset V^{(m)} \subset V^{(m+1)} \subset \cdots$$

in V by defining
(4.5)

$$V^{(m)} = \left\{ (v_1(z_1), \cdots, v_\ell(z_\ell)) \in \bigoplus_{j=1}^{\ell} k((z_j)) \;\middle|\; \max\left[\mathrm{ord}_{z_1}(v_1), \cdots, \mathrm{ord}_{z_\ell}(v_\ell)\right] \le m \right\},$$

where $\mathrm{ord}_{z_j}(v_j)$ denotes the order of $v_j = v_j(z_j)$ with respect to z_j.

4.6. Remark: The filtration (4.5) is different from (2.1) in general. However, we always have $V^{(0)} = F^{(0)}(V)$ and $V^{(-1)} = F^{(-1)}(V)$. This is one of the reasons why we have chosen $F^{(-1)}(V)$ instead of an arbitrary $F^{(\nu)}(V)$ in the definition of the Grassmannian in (2.2).

It is clear from (4.2) and (4.5) that $H_{\mathbf{n}}(y)^{(rm_1)} \cdot V^{(m_2)} \subset V^{(rm_1+m_2)}$, and hence V is a filtered $H_{\mathbf{n}}(y)$-module. With these preparation, we can state the inverse construction theorem.

4.7. Theorem. *A triple* $(A_0, A_{\mathbf{n}}, W)$ *of algebraic data of* (3.9) *determines a unique set of geometric data*

$$\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}, \phi)\rangle\,.$$

Proof: The proof is divided into four parts.

(I) *Construction of the curve* C_0 *and the point* p*:* Let us define $A_0^{(rm)} = A_0 \cap k[[y]]y^{-m}$, which consists of elements of A_0 of order at most m with respect to the variable y. This gives a filtration of A_0:

$$\cdots \subset A_0^{(rm-r)} \subset A_0^{(rm)} \subset A_0^{(rm+r)} \subset \cdots\,.$$

Using the finite-dimensionality of the cokernel (4) of (3.9), we can show that A_0 has an element of order m (with respect to y) for every large integer $m \in \mathbb{N}$, i.e.

$$(4.8)\qquad \dim_k A_0^{(rm)}/A_0^{(rm-r)} = 1 \quad \text{for all} \quad m >> 0\,.$$

Since $A_0 \cdot W \subset W$, the Fredholm condition of W implies that $A_0^{(rm)} = 0$ for all $m < 0$. Note that A_0 is a subalgebra of a field, and thus it is an integral domain. Therefore, the complete algebraic curve $C_0 = \mathrm{Proj}(grA_0)$ defined by the graded algebra

$$grA_0 = \bigoplus_{m=0}^{\infty} A_0^{(rm)}$$

is integral. We claim that C_0 is a one-point completion of the affine curve $\mathrm{Spec}(A_0)$. In order to prove the claim, let w denote the homogeneous element of degree one given by the image of the element $1 \in A_0^{(0)}$ under the inclusion $A_0^{(0)} \subset A_0^{(r)}$. Then the homogeneous localization $(grA_0)_{((w))}$ is isomorphic to A_0. Thus the principal open subset $D^+(w)$ defined by the element w is isomorphic to the affine curve $\mathrm{Spec}(A_0)$. The complement of $\mathrm{Spec}(A_0)$ in C_0 is the closed subset defined by (w), which is nothing but the projective scheme

$$\mathrm{Proj}\left(\bigoplus_{m=0}^{\infty} A_0^{(rm)}/A_0^{(rm-r)}\right)$$

given by the associated graded algebra of grA_0. Take a monic element $a_m \in A_0^{(rm)} \setminus A_0^{(rm-r)}$ for every $m >> 0$, whose existence is assured by (4.8). Since $a_i \cdot a_j \equiv a_{i+j} \mod A_0^{(ri+rj-r)}$, the map

$$\zeta : \bigoplus_{m=0}^{\infty} A_0^{(rm)}/A_0^{(rm-r)} \rightarrow k[x] ,$$

which assigns x^m to each a_m for $m >> 0$ and 0 otherwise, is a well-defined homomorphism of graded rings, where x is an indeterminant. In fact, ζ is an isomorphism in large degrees, and hence we have

$$\mathrm{Proj}\left(\bigoplus_{m=0}^{\infty} A_0^{(rm)}/A_0^{(rm-r)}\right) \cong \mathrm{Proj}(k[x]) = p .$$

This proves the claim.

Next we want to show that the added point p is a smooth rational point of C_0. To this end, it is sufficient to show that the formal completion of the structure sheaf of C_0 along p is isomorphic to a formal power series ring. Let us consider $(grA_0)/(w^n)$. The degree m homogeneous piece of this ring is given by $A_0^{(rm)}/(w^n A_0^{(rm-rn)})$, which is isomorphic to $k \cdot a_m \oplus k \cdot a_{m-1}w \oplus \cdots \oplus k \cdot a_{m-n+1}w^{n-1}$ for all $m > n >> 0$. From this we conclude that

$$gr(A_0)/(w^n) \cong k[x, w]/(w^n)$$

in large degrees for $n >> 0$. Therefore, taking the homogeneous localization at the ideal (w), we have

$$\big(gr(A_0)/(w^n)\big)_{((w))} \cong k[w/x]/\big((w/x)^n\big)$$

for $n >> 0$. Letting $n \to \infty$ and taking the inverse limit of this inverse system, we see that the formal completion of the structure sheaf of C_0 along p is indeed isomorphic to the formal power series ring $k[[w/x]]$. We can also present an affine local neighborhood of the point p. Let $a = a(y) \in A_0$ be a monic, nonconstant element with the lowerest order. It is unique up to the addition of a constant: $a(y) \mapsto a(y) + c$. This element defines a principal open subset $D^+(a)$ corresponding to the ring

$$
\begin{aligned}
(grA_0)_{(a)} &= grA_0\left[a^{-1}\right]_0 \\
&= \{a^{-i}b \mid b \in A_0,\ i \geq 0,\ \mathrm{ord}_y(b) - i \cdot \mathrm{ord}_y(a) \leq 0\} \\
&\subset k[[y]]\,.
\end{aligned}
\tag{4.9}
$$

Since the formal completion of C_0 along p coincides with that of $D^+(a)$ at p, and since the structure sheaf of the latter is $k[[y]]$ by (4.9), we have obtained that $k[[w/x]] = k[[y]]$. Thus y is indeed a formal parameter of the curve C_0 at p.

(II) *Construction of* $C_{\mathbf{n}}$ *and* Δ*:* Since $A_{\mathbf{n}} \subset H_{\mathbf{n}}(y)$, it has a filtration $A_{\mathbf{n}}^{(rm)} = A_{\mathbf{n}} \cap H_{\mathbf{n}}(y)^{(rm)}$ induced by (4.2). The Fredholm condition of W again implies that $A_{\mathbf{n}}^{(rm)} = 0$ for all $m < 0$. So let us define $C_{\mathbf{n}} = \mathrm{Proj}(grA_{\mathbf{n}})$, where

$$grA_{\mathbf{n}} = \bigoplus_{m=0}^{\infty} A_{\mathbf{n}}^{(rm)}\,.$$

This is a complete algebraic curve and has an affine part $\mathrm{Spec}(A_{\mathbf{n}})$. The complement $C_{\mathbf{n}} \setminus \mathrm{Spec}(A_{\mathbf{n}})$ is given by the projective scheme

$$\mathrm{Proj}\left(\bigoplus_{m=0}^{\infty} A_{\mathbf{n}}^{(rm)}/A_{\mathbf{n}}^{(rm-r)}\right)\,.$$

The finite-dimensionality (5) of (3.9) implies that for every ℓ-tuple $(\nu_1, \cdots, \nu_\ell)$ of positive integers satisfying that $\nu_j >> 0$, the stabilizer algebra $A_{\mathbf{n}}$ has an element of the form

$$\left(a_1(y_1),\ \cdots,\ a_\ell(y_\ell)\right) \in A_{\mathbf{n}} \subset \bigoplus_{j=1}^{\ell} k((y_j))$$

such that the order of $a_j(y_j)$ with respect to y_j is equal to ν_j for all $j = 1, \cdots, \ell$. Thus for all sufficiently large integer $m \in \mathbb{N}$, we have an isomorphism

$$A_{\mathbf{n}}^{(rm)}/A_{\mathbf{n}}^{(rm-r)} \cong k^{\oplus \ell}\,.$$

Actually, by choosing a basis of $A_{\mathbf{n}}^{(rm)}/A_{\mathbf{n}}^{(rm-r)}$ for each $m >> 0$, we can prove in the similar way as in the scalar case that the associated graded algebra $\bigoplus_{m=0}^{\infty} A_{\mathbf{n}}^{(rm)}/A_{\mathbf{n}}^{(rm-r)}$ is isomorphic to the graded algebra $\bigoplus_{j=0}^{\ell} k[x_j]$ in sufficiently large degrees, where x_j's are independent variables. The projective scheme of the latter graded algebra is an

ℓ-point scheme. Therefore, the curve $C_{\mathbf{n}}$ is an ℓ-point completion of the affine curve $\mathrm{Spec}(A_{\mathbf{n}})$. Let

$$\Delta = \{p_1, p_2, \cdots, p_\ell\} = \mathrm{Proj}\left(\bigoplus_{m=0}^{\infty} A_{\mathbf{n}}^{(rm)}/A_{\mathbf{n}}^{(rm-r)}\right) .$$

We have to show that these points are smooth and rational. To this end, we investigate the completion of $C_{\mathbf{n}}$ along the subscheme $\{p_1, p_2, \cdots, p_\ell\}$. Let u be the homogeneous element of degree one in $A_{\mathbf{n}}^{(r)}$ given by the image of $1 \in A_{\mathbf{n}}^{(0)}$ under the inclusion map $A_{\mathbf{n}}^{(0)} \subset A_{\mathbf{n}}^{(r)}$. Then the closed subscheme (the added points) is exactly the one defined by the principal homogeneous ideal (u). We can prove, in a similar way as in (I), that

$$gr(A_{\mathbf{n}})/(u^n) \cong (\bigoplus_{j=1}^{\ell} k[x_j])[u]/(u^n) \cong \bigoplus_{j=1}^{\ell} \left(k[x_j, u_j]/(u_j^n)\right)$$

in large degrees for $n >> 0$, where x_j's and u_j's are independent variables. Letting $n \to \infty$ and taking the inverse limit, we conclude that the formal completion of the structure sheaf of $C_{\mathbf{n}}$ along the subscheme $\{p_1, p_2, \cdots, p_\ell\}$ is isomorphic to the direct sum $\bigoplus_{j=1}^{\ell} k[[u_j/x_j]]$. Thus all of these ℓ points are smooth and rational. By considering the adic-completion of the ring

$$(A_{\mathbf{n}})_p = \left\{a^{-i}h \mid h \in {A_{\mathbf{n}}}^{(rm)},\ i \geq 0,\ m - i \cdot \mathrm{ord}_y(a) \leq 0\right\} ,$$

where a is an in (4.9), we can show that $k[[u_j/x_j]] = k[[y_j]]$. So y_j can be viewed as a formal parameter of $C_{\mathbf{n}}$ around the point p_j.

(III) *Construction of the morphism f:* The inclusion map $A_0 \hookrightarrow A_{\mathbf{n}}$ gives rise to an inclusion

$$\bigoplus_{q=0}^{\infty} A_0^{(rq)} \subset \bigoplus_{m=0}^{\infty} A_{\mathbf{n}}^{(rm)} , \tag{4.10}$$

because we have $A_0^{(rq)} \subset A_{\mathbf{n}}^{(rm)}$ for all $m \geq q \cdot \max[n_1, \cdots, n_\ell]$. It defines a finite surjective morphism $f : C_{\mathbf{n}} \longrightarrow C_0$. Using the formal parameter y_j, we know that the morphism $f_j : U_j \longrightarrow U_p$ of the formal completion U_j of $C_{\mathbf{n}}$ along p_j induced by $f : C_{\mathbf{n}} \longrightarrow C_o$ is indeed the cyclic covering morphism defined by $y = y_j^{n_j}$. Since $H_{\mathbf{n}}(y)$ is a free $k((y))$-module of dimension n and since the algebras A_0 and $A_{\mathbf{n}}$ satisfy the Fredholm condition described in (4), (5) and (7) of (3.9), $A_{\mathbf{n}}$ is a torsion free module of rank n over A_0. Thus the morphism f has degree n.

(IV) *Construction of the sheaf $\mathcal{F}$:* We introduce a filtration in $W \subset V$ induced by (4.5). The $A_{\mathbf{n}}$-module structure of W is compatible with the $H_{\mathbf{n}}(y) = \bigoplus_{j=1}^{\ell} k((y_j))$-action on $V = \bigoplus_{j=1}^{\ell} k((z_j))$. Note that we have $A_{\mathbf{n}}^{(rm_1)} \cdot W^{(m_2)} \subset W^{(rm_1+m_2)}$, and hence $\bigoplus_{m=-\infty}^{\infty} W^{(m)}$ is a graded module over $grA_{\mathbf{n}}$. Let $\mathcal{F}$ be the sheaf corresponding to the shifted graded module $\left(\bigoplus_{m=-\infty}^{\infty} W^{(m)}\right)(-1)$, where this shifting by -1 comes from our convention of (2.2). This sheaf is an extension of the sheaf $W^{\sim}$ defined on the affine curve Spec$(A_{\mathbf{n}})$. The graded module $\left(\bigoplus_{m=-\infty}^{\infty} W^{(m)}\right)(-1)$ is also a graded module over grA_0 by (4.10). It gives rise to a torsion-free sheaf on C_0, which is nothing but $f_*\mathcal{F}$. Let us define

$$W_p = \left\{a^{-i}w \mid w \in W^{(m)},\ i \geq 0,\ m - i \cdot r \cdot \mathrm{ord}_y(a) \leq -1\right\} ,$$

where a is as in (4.9). Then W_p is an $(A_0)_p$-module of rank $r \cdot n = r \sum n_j$. The formal completion $(f_*\mathcal{F})_{U_p}$ of $f_*\mathcal{F}$ at the point p is given by the $k[[y]]$-module $W_p \otimes_{(A_0)_p} k[[y]]$, and the isomorphism

$$W_p \otimes_{(A_0)_p} k[[y]] \cong \bigoplus_{j=1}^{\ell} k[[z_j]]z_j \tag{4.11}$$

gives rise to the sheaf isomorphism

$$\phi : (f_*\mathcal{F})_{U_p} \xrightarrow{\sim} \pi_* \bigoplus_{j=1}^{\ell} \psi_{j*}(\mathcal{O}_{U_{oj}}(-1))$$

and its diagonal blocks $\Phi = (\phi_1, \cdots, \phi_\ell)$. Since $f_*\mathcal{F}$ has rank $r \cdot n$ over $\mathcal{O}_{C_0}$ from (4.11) and $A_{\mathbf{n}}$ has rank n over A_0, the sheaf $\mathcal{F}$ on $C_{\mathbf{n}}$ must have rank r. The cohomology calculation of (3.11), (3.12) and (3.13) shows that the Euler characteristic of $\mathcal{F}$ is equal to μ. Thus we have constructed all of the ingredients of the geometric data of type $\mathbf{n}$, index μ, and rank r. This completes the proof of (4.7).

In order to complete the proof of the categorical equivalence of (3.17), we have to construct a triple (α, β, λ) out of the homomorphisms $\iota : A_0 \hookrightarrow A_0'$, $\epsilon : A_{\mathbf{n}} \longrightarrow A'_{\mathbf{n}}$, and $\omega : W' \longrightarrow W$. Let s be the rank of A_0' as an A_0-module. The injection ι is associated with the inclusion $k((y)) \subset k((y'))$, and the coordinate y has order $-s$ with respect to y'. Therefore, we have $r = s \cdot r'$. Recall that the filtration we have introduced in A_0 is defined by the order with respect to y. The homomorphism ι induces an injective homomorphism

$$grA_0 = \bigoplus_{m=0}^{\infty} A_0^{(rm)} \longrightarrow \bigoplus_{m=0}^{\infty} {A'}_0^{(s \cdot r' m)} \subset \bigoplus_{m=0}^{\infty} {A'}_0^{(r' m)} = grA'_0 ,$$

which then defines a morphism $\alpha : C'_0 \longrightarrow C_0$.

Note that the homomorphism ϵ comes from the inclusion $k((y_j)) \subset k((y_j'))$ for every j. By the Puiseux expansion, we see that every $y_j = y^{1/n_j}$ has order $-s$ as an element of $k((y_j')) = k((y'^{1/n_j}))$. Thus we have

$$grA_{\mathbf{n}} = \bigoplus_{m=0}^{\infty} A_{\mathbf{n}}^{(rm)} \longrightarrow \bigoplus_{m=0}^{\infty} {A'}_{\mathbf{n}}^{(s \cdot r' m)} \subset \bigoplus_{m=0}^{\infty} {A'}_{\mathbf{n}}^{(r' m)} = grA'_{\mathbf{n}} ,$$

and this homomorphism defines $\beta : C'_{\mathbf{n}} \longrightarrow C_{\mathbf{n}}$.

Finally, the homomorphism λ can be constructed as follows. Note that ω gives an inclusion $W'^{(m)} \subset W^{(m)}$ as subspaces of $\bigoplus_{j=1}^{\ell} k((z_j))$ for every $m \in \mathbf{Z}$. Thus we have an inclusion map

$$\bigoplus_{m=-\infty}^{\infty} W'^{(m)} \subset \bigoplus_{m=-\infty}^{\infty} W^{(m)} ,$$

which is clearly a $grA_{\mathbf{n}}$-module homomorphism. Thus it induces an injective homomorphism $\lambda : \beta_* \mathcal{F}' \longrightarrow \mathcal{F}$.

One can check that the construction we have given in Section 4 is indeed the inverse of the map we defined in Section 3. Thus we have completed the entire proof of the categorical equivalence (3.17).

5. A characterization of arbitrary Prym varieties.

In this section, we study the geometry of finite type orbits of the Heisenberg flows, and establish a simple characterization theorem of arbitrary Prym varieties. Consider the Heisenberg flows associated with $H_{\mathbf{n}}(y)$ on the quotient Grassmannian $Z_{\mathbf{n}}(\mu, y)$ and assume that the flows produce a finite-dimensional orbit at a point $\overline{W} \in Z_{\mathbf{n}}(\mu, y)$. Then this situation corresponds to the geometric data of (3.1):

5.1. PROPOSITION. *Let $W \in Gr_{\mathbf{n}}(\mu)$ be a point of the Grassmannian at which the Heisenberg flows of type $\mathbf{n}$ and rank r associated with $H_{\mathbf{n}}(y)$ generate an orbit of finite type. Then W gives rise to a set of geometric data*

$$\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, \pi, f_* \mathcal{F}, \phi) \rangle$$

of type $\mathbf{n}$, index μ, and rank r.

PROOF: Let $X_{\mathbf{n}}$ be the orbit of the Heisenberg flows starting at W, and consider the r-reduced KP flows associated with $k((y))$. The finite-dimensionality of $\overline{X}_{\mathbf{n}} = Q_{\mathbf{n},y}(X_{\mathbf{n}})$ implies that the r-reduced KP flows also produce a finite type orbit X_0 at W. Let $A_0 = \{a \in k((y)) \mid a \cdot W \subset W\}$ and $A_{\mathbf{n}} = \{h \in H_{\mathbf{n}}(y) \mid h \cdot W \subset W\}$ be the stabilizer subalgebras, which satisfy $A_0 \subset A_{\mathbf{n}}$. From the definition of the vector fields (2.5), an element of $k((y))$ gives the zero tangent vector at W if and only if it is in A_0. Similarly, for an element $b \in H_{\mathbf{n}}(y)$, $\Psi_W(b) = 0$ if and only if $b \in A_{\mathbf{n}}$. Thus the tangent spaces of these orbits are given by

$$T_W X_0 \cong k((y))/A_0 \quad \text{and} \quad T_W X_{\mathbf{n}} \cong H_{\mathbf{n}}(y)/A_{\mathbf{n}} .$$

Therefore, going down to the quotient Grassmannian, the tangent spaces of $\overline{X}_{\mathbf{n}}$ and $\overline{X}_0 = Q_{\mathbf{n},y}(X_0)$ are now given by

$$T_{\overline{W}} \overline{X}_0 \cong \frac{k((y))}{A_0 + k((y)) \cap gl(n, k[[y]]y)} = \frac{k((y))}{A_0 \oplus k[[y]]y}$$

and

$$T_{\overline{W}}\overline{X}_{\mathbf{n}} \cong \frac{H_{\mathbf{n}}(y)}{A_{\mathbf{n}} + H_{\mathbf{n}}(y) \cap gl(n, k[[y]]y)} = \frac{H_{\mathbf{n}}(y)}{A_{\mathbf{n}} \oplus H_{\mathbf{n}}(y)^{-}} ,$$

where $\overline{W} = Q_{\mathbf{n},y}(W)$, and $H_{\mathbf{n}}(y)^{-}$ is defined in (2.14). Since both of the above sets are finite-dimensional, the triple $(A_0, A_{\mathbf{n}}, W)$ satisfies the cokernel conditions (4) and (5) of (3.9). The rank condition (6) of (3.9) is a consequence of the fact that $H_{\mathbf{n}}(y)$ has dimension n over $k((y))$. Therefore, applying the inverse construction of the Krichever functor to the triple, we obtain a set of geometric data. This completes the proof.

Since $k \subset A_0 \subset A_{\mathbf{n}}$, from (3.4) and (3.8) we obtain

$$T_{\overline{W}}\overline{X}_0 \cong \frac{k((y))}{A_0 \oplus k[[y]]y} \cong H^1(C_0, \mathcal{O}_{C_0}) \tag{5.2}$$

and

$$T_{\overline{W}}\overline{X}_{\mathbf{n}} \cong \frac{H_{\mathbf{n}}(y)}{A_{\mathbf{n}} \oplus H_{\mathbf{n}}(y)^{-}} \cong H^1(C_{\mathbf{n}}, \mathcal{O}_{C_{\mathbf{n}}}) . \tag{5.3}$$

Thus we know that the genera of C_0 and $C_{\mathbf{n}}$ are equal to the dimension of the orbits $\overline{X}_0$ and $\overline{X}_{\mathbf{n}}$ on the quotient Grassmannian, respectively. However, we cannot conclude that these orbits are actually Jacobian varieties. The difference of the orbits and the Jacobians lies in the deformation of the data (Φ, ϕ). In order to give a surjective map from the Jacobians to these orbits, we have to eliminate these unwanted information by using (3.19). Therefore, in the rest of this section, we have to assume that the point $W \in Gr_{\mathbf{n}}(\mu)$ gives rise to a rank one triple $(A_0, A_{\mathbf{n}}, W)$ of algebraic data from the application of the Heisenberg flows associated with $H_{\mathbf{n}}(y)$ and an element $y \in L$ of order -1.

In order to deal with Jacobian varieties, we further assume that the field k is the field $\mathbf{C}$ of complex numbers in what follows in this section. The computation (5.3) shows that every element of $H^1(C_{\mathbf{n}}, \mathcal{O}_{C_{\mathbf{n}}})$ is represented by

$$\sum_{j=1}^{\ell} \sum_{i=-\infty}^{\infty} t_{ij} y_j^{-i} \in \bigoplus_{j=1}^{\ell} \mathbf{C}((y_j)) = H_{\mathbf{n}}(y) . \tag{5.4}$$

The Heisenberg flows at W are given by the equations

$$\frac{\partial W}{\partial t_{ij}} = y_j^{-i} \cdot W = \left(h_{n_j}(y)\right)^{-i} \cdot W , \tag{5.5}$$

where $h_{n_j}(y)$ acts on W through the block matrix

$$\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & h_{n_j}(y) & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} ,$$

and the index i runs over all of $\mathbb{Z}$. The formal integration

$$W(t) = \exp\left(\sum_{j=1}^{\ell}\sum_{i=-\infty}^{\infty} t_{ij}y_j^{-i}\right)\cdot W \tag{5.6}$$

of the system (5.5) shows that the stabilizers A_0 and $A_\mathbf{n}$ of $W(t)$ do not deform as t varies, because the exponential factor

$$e(t) = \exp\left(\sum_{j=1}^{\ell}\sum_{i=-\infty}^{\infty} t_{ij}y_j^{-i}\right) \tag{5.7}$$

commutes with the algebra $H_\mathbf{n}(y)$. Note that half of the exponential factor

$$\exp\left(\sum_{j=1}^{\ell}\sum_{i=-\infty}^{-1} t_{ij}y_j^{-i}\right)$$

is an element of $\Gamma_\mathbf{n}(y)$.

5.8. THEOREM. *Let $y \in L$ be a monic element of order -1 and $X_\mathbf{n}$ a finite type orbit of the Heisenberg flows on $Gr_\mathbf{n}(\mu)$ associated with $H_\mathbf{n}(y)$ starting at W. As we have seen in* (5.1), *the orbit $X_\mathbf{n}$ gives rise to a set of geometric data*

$$\langle f : (C_\mathbf{n}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}, \phi)\rangle \ .$$

Then the projection image $\overline{X}_\mathbf{n}$ of this orbit by $Q_{\mathbf{n},y} : Gr_\mathbf{n}(\mu) \longrightarrow Z_\mathbf{n}(\mu, y)$ is canonically isomorphic to the Jacobian variety $\mathrm{Jac}(C_\mathbf{n})$ of the curve $C_\mathbf{n}$ with $\overline{W} = Q_{\mathbf{n},y}(W)$ as its origin. Moreover, the orbit $\overline{X}_0$ of the KP system (written in terms of the variable y) defined on the quotient Grassmannian $Z_\mathbf{n}(\mu, y)$ is isomorphic to the deformation space

$$\{\mathcal{N} \otimes f_*\mathcal{F} \mid \mathcal{N} \in \mathrm{Jac}(C_0)\} \ .$$

Thus we have a finite covering $\mathrm{Jac}(C_0) \longrightarrow \overline{X}_0$ of the orbit, which is indeed isomorphic if $f_*\mathcal{F}$ is a general vector bundle on C_0.

PROOF: Even though the formal integration (5.6) is not well-defined as a point of the Grassmannian, we can still apply the same construction of Section 4 to the algebraic data $(A_0, A_\mathbf{n}, W(t))$ understanding that the exponential matrix $e(t)$ of (5.7) is an extra factor of degree 0. Of course the curves, points, and the covering morphism $f : C_\mathbf{n} \longrightarrow C_0$ are the same as before. Therefore, we obtain

$$\langle f : (C_\mathbf{n}, \Delta, \Pi, \mathcal{F}(t), \Phi(t)) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}(t), \phi(t))\rangle \ ,$$

where the line bundle $\mathcal{F}(t)$ comes from the $A_{\mathbf{n}}$-module $W(t)$. We do not need to specify the data $\Phi(t)$ and $\phi(t)$ here, because they will disappear anyway by the trick of (3.19). On the curve $C_{\mathbf{n}}$, the formal expression $e(t)$ makes sense because of the homomorphism

$$\exp : H^1(C_{\mathbf{n}}, \mathcal{O}_{C_{\mathbf{n}}}) \ni \sum_{j=1}^{\ell} \sum_{i=-\infty}^{\infty} t_{ij} y_j^{-i} \longmapsto [e(t)] = \mathcal{L}(t) \in \mathrm{Jac}(C_{\mathbf{n}}) \subset H^1(C_{\mathbf{n}}, \mathcal{O}^*_{C_{\mathbf{n}}}) \,,$$

where $\mathcal{L}(t)$ is the line bundle of degree 0 corresponding to the cohomology class $[e(t)] \in H^1(C_{\mathbf{n}}, \mathcal{O}^*_{C_{\mathbf{n}}})$. Thus the sheaf we obtain from $W(t) = e(t) \cdot W$ is $\mathcal{F}(t) = \mathcal{L}(t) \otimes \mathcal{F}$. Now consider the projection image $(A_0, A_{\mathbf{n}}, \overline{W}(t))$ of the algebraic data by $Q_{\mathbf{n},y}$. Then it corresponds to the data

$$\left\langle f : \left(C_{\mathbf{n}}, \Delta, \Pi, \mathcal{L}(t) \otimes \mathcal{F}\right) \longrightarrow \left(C_0, p, \pi, f_*(\mathcal{L}(t) \otimes \mathcal{F})\right)\right\rangle \tag{5.9}$$

by (3.19). Since $\exp : H^1(C_{\mathbf{n}}, \mathcal{O}_{C_{\mathbf{n}}}) \longrightarrow \mathrm{Jac}(C_{\mathbf{n}})$ is surjective, we can define a map assigning (5.9) to every point $\mathcal{L}(t) \in \mathrm{Jac}(C_{\mathbf{n}})$ of the Jacobian. Through the Krichever functor, it gives indeed the desired identification of $\mathrm{Jac}(C_{\mathbf{n}})$ and the orbit $\overline{X}_{\mathbf{n}}$:

$$\mathrm{Jac}(C_{\mathbf{n}}) \ni \mathcal{L}(t) \longmapsto (5.9) \longmapsto \overline{W}(t) \in \overline{X}_{\mathbf{n}} \,.$$

The KP system in the y-variable at $\overline{W} \in Z_{\mathbf{n}}(\mu, y)$ is given by the equation

$$\frac{\partial \overline{W}}{\partial s_m} = y^{-m} \cdot \overline{W} \,.$$

The formal integration

$$\overline{W}(s) = \exp\left(\sum_{m=1}^{\infty} s_m y^{-m}\right) \cdot \overline{W}$$

corresponds to

$$\left\langle f : \left(C_{\mathbf{n}}, \Delta, \Pi, (f^*\mathcal{N}(s)) \otimes \mathcal{F}\right) \longrightarrow \left(C_0, p, \pi, \mathcal{N}(s) \otimes f_*\mathcal{F}\right)\right\rangle \,,$$

where $\mathcal{N}(s) \otimes f_*\mathcal{F}$ is the vector bundle corresponding to the A_0-module $\overline{W}(s)$. From (5.2), we have a surjective map of $H^1(C_0, \mathcal{O}_{C_0})$ onto the Jacobian variety $\mathrm{Jac}(C_0) \subset H^1(C_0, \mathcal{O}^*_{C_0})$ defined by

$$\exp : H^1(C_0, \mathcal{O}_{C_0}) \ni \sum_{m=1}^{\infty} s_m y^{-m} \longmapsto \left[\exp\Big(\sum_{m=1}^{\infty} s_m y^{-m}\Big)\right] = \mathcal{N}(s) \in \mathrm{Jac}(C_0) \,.$$

Thus the orbit $\overline{X}_0$ coincides with the deformation space $\mathcal{N}(s) \otimes f_*\mathcal{F}$, which is covered by $\mathrm{Jac}(C_0)$. The last statement of the theorem follows from a result of [L]. This completes the proof.

Let $(\eta_1, \cdots, \eta_\ell)$ be the transition function of $\mathcal{F}$ defined on $U_j \backslash \{p_j\}$, where $\eta_j \in \mathbb{C}((y_j))$. Then the family $\mathcal{F}(t)$ of line bundles on $C_{\mathbf{n}}$ is given by the transition function

$$\left(\exp\left(\sum_{i=1}^{\infty} t_{i1} y_j^{-i}\right) \cdot \eta_1, \; \cdots, \; \exp\left(\sum_{i=1}^{\infty} t_{i\ell} y_\ell^{-i}\right) \cdot \eta_\ell\right) ,$$

and similarly, the line bundle $\mathcal{L}(t)$ is given by

$$\left(\exp\left(\sum_{i=1}^{\infty} t_{i1} y_j^{-i}\right), \; \cdots, \; \exp\left(\sum_{i=1}^{\infty} t_{i\ell} y_\ell^{-i}\right)\right) .$$

Here, we note that the nonnegative powers of $y_j = h_{n_j}(y)$ do not contribute to these transition functions.

Recall that $H_{\mathbf{n}}(y)_0$ denotes the subalgebra of $H_{\mathbf{n}}(y)$ consisting of the traceless elements.

5.10. THEOREM. *In the same situation as above, the projection image $\overline{X} \subset Z_{\mathbf{n}}(\mu, y)$ of the orbit X of the traceless Heisenberg flows $\Psi(H_{\mathbf{n}}(y)_0)$ starting at $\overline{W}$ is canonically isomorphic to the Prym variety associated with the covering morphism $f : C_{\mathbf{n}} \longrightarrow C_0$.*

PROOF: Because of (1.3), the locus of $\mathcal{L}(t) \in \mathrm{Jac}(C_{\mathbf{n}})$ such that

$$\det\left(f_*\big(\mathcal{L}(t) \otimes \mathcal{F}\big)\right) = \det(f_*\mathcal{F})$$

is the Prym variety $\mathrm{Prym}(f)$ associated with the covering morphism f. So let us compute the factor

$$\mathcal{D}(t) = \det\left(f_*(\mathcal{L}(t) \otimes \mathcal{F})\right) \otimes \det(f_*\mathcal{F})^{-1} , \tag{5.11}$$

which is a line bundle of degree 0 defined on C_0. We use the transition function η of $f_*\mathcal{F}$ defined on $U_p \setminus \{p\}$ written in terms of the basis (3.6). Since $f_*\mathcal{F}(t)$ is defined by the A_0-module structure of $W(t) = e(t) \cdot W$, its transition function is given by

$$\exp\begin{pmatrix} \sum_{i=1}^{\infty} t_{i1}\big(h_{n_1}(y)\big)^{-i} & & & \\ & \sum_{i=1}^{\infty} t_{i2}\big(h_{n_2}(y)\big)^{-i} & & \\ & & \ddots & \\ & & & \sum_{i=1}^{\infty} t_{i\ell}\big(h_{n_\ell}(y)\big)^{-i} \end{pmatrix} \cdot \eta ,$$

where the $n \times n$ matrix acts on the y^{α/n_j}-part of the basis of (3.6) in an obvious way. Let us denote the above matrix by

$$T(t) = \begin{pmatrix} \sum_{i=1}^{\infty} t_{i1}\big(h_{n_1}(y)\big)^{-i} & & & \\ & \sum_{i=1}^{\infty} t_{i2}\big(h_{n_2}(y)\big)^{-i} & & \\ & & \ddots & \\ & & & \sum_{i=1}^{\infty} t_{i\ell}\big(h_{n_\ell}(y)\big)^{-i} \end{pmatrix} .$$

Then, it is clear that $\mathcal{D}(t) \cong [\exp \operatorname{trace} T(t)] \in H^1(C_0, \mathcal{O}^*_{C_0})$. From this expression, we see that if $\mathcal{L}(t)$ stays on the orbit $\overline{X}$ of the traceless Heisenberg flows, then $\mathcal{D}(t) \cong \mathcal{O}_{C_0}$. Namely, $\overline{X} \subset \operatorname{Prym}(f)$.

Conversely, take a point $\overline{W}(t) \in \overline{X}_{\mathbf{n}}$ of the orbit of the Heisenberg flows defined on the quotient Grassmannian $Z_{\mathbf{n}}(\mu, y)$. It corresponds to a unique element $\mathcal{L}(t) \in \operatorname{Jac}(C_{\mathbf{n}})$ by (5.8). Now suppose that the factor $\mathcal{D}(t)$ of (5.11) is the trivial bundle on C_0. Then it implies that $[\operatorname{trace} T(t)] = 0$ as an element of $H^1(C_0, \mathcal{O}_{C_0})$. In particular, $\operatorname{trace} T(t)$ acts on $\overline{W}$ trivially from (5.2). Therefore, $\overline{W}(t)$ is on the orbit of the flows defined by

$$T(t) - I_n \cdot \frac{1}{n} \operatorname{trace} T(t) ,$$

which are clearly traceless. In other words, $\overline{W}(t) \in \overline{X}$. Thus $\operatorname{Prym}(f) \subset \overline{X}$. This completes the proof.

5.12. REMARK: Let us observe the case when the curve C_0 downstairs happens to be a $\mathbf{P}^1$. First of all, we note that the r-reduced KP system associated with y is nothing but the trace part of the Heisenberg flows defined by $H_{\mathbf{n}}(y)$. Because of the second half statement of (5.8), the trace part of the Heisenberg flows acts on the point $\overline{W} \in Z_{\mathbf{n}}(\mu, y)$ trivially. Therefore, the orbit $\overline{X}_{\mathbf{n}}$ of the entire Heisenberg flows coincides with the orbit $\overline{X}$ of the traceless part of the flows. Of course, this reflects the fact that every Jacobian variety is a Prym variety associated with a covering over $\mathbf{P}^1$. Thus the characterization theorem of Prym varieties we are presenting below contains the characterization of Jacobians of [**M1**] as a special case.

Now consider the most trivial maximal commutative algebra $H = H_{(1,\cdots,1)}(z) = \mathbf{C}((z))^{\oplus n}$. We define the group $\Gamma_{(1,\cdots,1)}(z)$ following (2.15), and denote by

$$Z_n(\mu) = Z_{(1,\cdots,1)}(\mu, z) = Gr_n(\mu)/\Gamma_{(1,\cdots,1)}(z) \tag{5.13}$$

the corresponding quotient Grassmannian. On this space the algebra H acts, and gives the n-component KP system. Let H_0 be the traceless subalgebra of H, and consider the traceless n-component KP system on the quotient Grassmannian $Z_n(\mu)$.

5.14. THEOREM. *Every finite-dimensional orbit of the traceless n-component KP system defined on the quotient Grassmannian $Z_n(\mu)$ of (5.13) is canonically isomorphic to a (generalized) Prym variety. Conversely, every Prym variety associated with a degree n covering morphism of smooth curves can be realized in this way.*

PROOF: The first half part has been already proved. So start with the Prym variety $\operatorname{Prym}(f)$ associated with a degree n covering morphism $f : C \longrightarrow C_0$ of smooth curves. Without loss of generality, we can assume that C_0 is connected. Choose a point p of C_0 outside of the branching locus so that its preimage $f^{-1}(p)$ consists of n distinct points of C, and supply the necessary geometric objects to make the situation into the geometric data

$$\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}, \phi) \rangle$$

of (3.1) of rank one and type $\mathbf{n} = (1, \cdots, 1)$ with $C = C_{\mathbf{n}}$. The data give rise to a unique triple $(A_0, A_{\mathbf{n}}, W)$ of algebraic data by the Krichever functor. We can choose $\pi = id$ so that the maximal commutative subalgebra we have here is indeed $H = H_{(1,\cdots,1)}(z)$. Define $A_0' = \{a \in \mathbb{C}((z)) \mid a \cdot W \subset W\}$ and $A' = \{h \in H \mid h \cdot W \subset W\}$, which satisfy $A_0 \subset A_0'$ and $A_{\mathbf{n}} \subset A'$, and both have finite codimensions in the larger algebras. From the triple of the algebraic data (A_0', A', W), we obtain a set of geometric data

$$\langle f' : (C_{\mathbf{n}}', \Delta', \Pi', \mathcal{F}', \Phi') \longrightarrow (C_0', p', \pi', f_*'\mathcal{F}', \phi') \rangle \ .$$

The morphism (α, β, id) between the two sets of data consists of a morphism $\alpha : C_0' \longrightarrow C_0$ of the base curves and $\beta : C' \longrightarrow C_{\mathbf{n}}$. Obviously, these morphisms are birational, and hence, they have to be an isomorphism, because C_0 and $C_{\mathbf{n}}$ are smooth. Going back to the algebraic data by the Krichever functor, we obtain $A_0 = A_0'$ and $A_{\mathbf{n}} = A'$. Thus the orbit of the traceless n-component KP system starting at $\overline{W}$ defined on the quotient Grassmannian $Z_{\mathbf{n}}(\mu)$ is indeed the Prym variety of the covering morphism f. This completes the proof of the characterization theorem.

5.15. REMARK: In the above proof, we need the full information of the functor, not just the set-theoretical bijection of the objects. We use a similar argument once again in (6.15).

5.16. REMARK: The determinant line bundle DET over $Gr_{\mathbf{n}}(0)$ is defined by

$$DET_W = \left(\bigwedge^{\max} \operatorname{Ker}(\gamma_W) \right)^* \bigotimes \bigwedge^{\max} \operatorname{Coker}(\gamma_W) \ .$$

The canonical section of the DET bundle defines the determinant divisor Y of $Gr_{\mathbf{n}}(0)$, whose support is the complement of the big-cell $Gr_{\mathbf{n}}^+(0)$. Note that the action of $\Gamma_{\mathbf{n}}(y)$ preserves the big-cell. So we can define the *big-cell* of the quotient Grassmannian by $Z_{\mathbf{n}}^+(0, y) = Gr_{\mathbf{n}}^+(0)/\Gamma_{\mathbf{n}}(y)$. The determinant divisor also descends to a divisor $Y/\Gamma_{\mathbf{n}}(y)$, which we also call the determinant divisor of the quotient Grassmannian. Consider a point $W \in Gr_{\mathbf{n}}(0)$ at which the Heisenberg flows of rank one produce a finite type orbit $X_{\mathbf{n}}$. The geometric data corresponding to this situation consists of a curve $C_{\mathbf{n}}$ of genus $g = \dim_{\mathbb{C}} \overline{X}_{\mathbf{n}}$ and a line bundle $\mathcal{F}$ of degree $g-1$ because of the Riemann-Roch formula

$$\dim_{\mathbb{C}} H^0(C_{\mathbf{n}}, \mathcal{F}) - \dim_{\mathbb{C}} H^1(C_{\mathbf{n}}, \mathcal{F}) = \deg(\mathcal{F}) - r(g-1) \ .$$

Thus we have an *equality* $\overline{X}_{\mathbf{n}} = \operatorname{Pic}^{g-1}(C_{\mathbf{n}})$ from the proof of (5.8). The intersection of $\overline{X}_{\mathbf{n}}$ with the determinant divisor of $Z_{\mathbf{n}}(0, y)$ coincides with the theta divisor Θ which gives the principal polarization of $\operatorname{Pic}^{g-1}(C_{\mathbf{n}})$. However, the restriction of this divisor to the Prym variety does not give a principal polarization as we have noted in Section 1.

5.17. REMARK: From the expression of (5.9), we can see that a finite-dimensional orbit of the Heisenberg flows of rank one defined on the quotient Grassmannian gives a family of deformations $f_*(\mathcal{L}(t) \otimes \mathcal{F})$ of the vector bundle $f_*\mathcal{F}$ on C_0. It is an interesting question to ask what kind of deformations does this family produce. More generally, we can ask the following question: For a given curve and a family of vector bundles on it, can one find a point W of the Grassmannian $Gr_{\mathbf{n}}(\mu)$ and a suitable Heisenberg flows such that the orbit starting from W contains the original family?

It is known that for every vector bundle $\mathcal{V}$ of rank n on a smooth curve C_0, there is a degree n covering $f : C \longrightarrow C_0$ and a line bundle $\mathcal{F}$ on C such that $\mathcal{V}$ is isomorphic to the direct image sheaf $f_*\mathcal{F}$. We can supply suitable local data so that we have a set of geometric data

$$\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}) \longrightarrow (C_0, p, \pi, f_*\mathcal{F})\rangle$$

with $C_{\mathbf{n}} = C$. Let $(A_0, A_{\mathbf{n}}, \overline{W})$ be the triple of algebraic data corresponding to the above geometric situation with a point $\overline{W} \in Z_{\mathbf{n}}(\mu, z)$, where μ is the Euler characteristic of the original bundle $\mathcal{V}$. Now the problem is to compare the family of deformations given by (5.9) and the original family.

The only thing we can say about this question at the present moment is the following. If the original vector bundle is a general stable bundle, then one can find a set of geometric data and a corresponding point $\overline{W}$ of a quotient Grassmannian such that there is a dominant and generically finite map of a Zariski open subset of the orbit of the Heisenberg flows starting from $\overline{W}$ into the moduli space of stable vector bundles of rank n and degree $\mu + n\big(g(C_0) - 1\big)$ over the curve C_0. Note that this statement is just an interpretation of a theorem of [**BNR**] into our language using (5.8).

As in the proof of (5.14), the Heisenberg flows can be replaced by the n-component KP flows if we choose the point $p \in C_0$ away from the branching locus of f. Thus one may say that the n-component KP system can produce general vector bundles of rank n defined on an arbitrary smooth curve in its orbit.

6. Commuting ordinary differential operators with matrix coefficients.

In this section, we work with an arbitrary field k again. Let us denote by

$$E = \big(k[[x]]\big)((\partial^{-1})) \tag{6.1}$$

the set of all pseudodifferential operators with coefficients in $k[[x]]$, where $\partial = d/dx$. This is an associative algebra and has a natural filtration

$$E^{(m)} = \big(k[[x]]\big)[[\partial^{-1}]] \cdot \partial^m$$

defined by the *order* of the operators. We can identify $k((z))$ with the set of pseudodifferential operators with constant coefficients by the *Fourier transform* $z = \partial^{-1}$:

$$L = k((z)) = k((\partial^{-1})) \subset E\,.$$

There is also a canonical projection

$$\rho : E \longrightarrow E/Ex \cong k((\partial^{-1})) = L\,, \tag{6.2}$$

where Ex is the left-maximal ideal of E generated by x. In an explicit form, this projection is given by

$$\rho : E \ni P = \sum_{m \in \mathbb{Z}} \partial^m \cdot a_m(x) \longmapsto \sum_{m \in \mathbb{Z}} a_m(0) z^{-m} \in L\,. \tag{6.3}$$

It is obvious from (6.2) that L is a left E-module. The action is given by $P \cdot v = P \cdot \rho(Q) = \rho(PQ)$, where $v \in L = E/Ex$ and $Q \in E$ is a representative of the equivalence class such that $\rho(Q) = v$. The well-definedness of this action is easily checked. We also use the notations

$$\begin{cases} D = \big(k[[x]]\big)[\partial] \\ E^{(-1)} = \big(k[[x]]\big)[[\partial^{-1}]] \cdot \partial^{-1}\,, \end{cases}$$

which are the set of linear ordinary differential operators and the set of pseudodifferential operators of negative order, respectively. Note that there is a natural left $\big(k[[x]]\big)$-module direct sum decomposition

$$E = D \oplus E^{(-1)}\,. \tag{6.4}$$

According to this decomposition, we write $P = P^+ \oplus P^-$, $P \in E$, $P^+ \in D$, and $P^- \in E^{(-1)}$.

Now consider the matrix algebra $gl(n, E)$ defined over the noncommutative algebra E, which is the algebra of pseudodifferential operators with coefficients in matrix valued functions. This algebra acts on our vector space $V = L^{\oplus n} \cong \big(E/Ex\big)^{\oplus n}$ from the left. In particular, every element of $gl(n, E)$ gives rise to a vector field on the Grassmannian $Gr_n(\mu)$ via (2.4). The decomposition (6.4) induces

$$V = k[z^{-1}]^{\oplus n} \oplus \big(k[[z]] \cdot z\big)^{\oplus n}$$

after the identification $z = \partial^{-1}$, and the base point $k[z^{-1}]^{\oplus n}$ of the Grassmannian $Gr_n(0)$ of index 0 is the residue class of $D^{\oplus n}$ in $E^{\oplus n}$ via the projection $E^{\oplus n} \longrightarrow E^{\oplus n}/\big(E^{(-1)}\big)^{\oplus n}$. Therefore, the $gl(n, D)$-action on V preserves $k[z^{-1}]^{\oplus n}$. The following proposition shows that the converse is also true:

6.5. PROPOSITION. *A pseudodifferential operator $P \in gl(n, E)$ with matrix coefficients is a differential operator, i.e. $P \in gl(n, D)$, if and only if*

$$P \cdot k[z^{-1}]^{\oplus n} \subset k[z^{-1}]^{\oplus n}\,.$$

PROOF: The case of $n = 1$ of this proposition was established in [**M3**, Lemma 6.2]. So let us assume that $P = \big(P_{\mu\nu}\big) \in gl(n, E)$ preserves the base point $k[z^{-1}]^{\oplus n}$. If we apply the matrix P to the vector subspace

$$0 \oplus \cdots \oplus 0 \oplus k[z^{-1}] \oplus 0 \oplus \cdots \oplus 0 \subset k[z^{-1}]^{\oplus n}$$

with only nonzero entries in the ν-th position, then we know that $P_{\mu\nu} \in E$ stabilizes $k[z^{-1}]$ in L. Thus $P_{\mu\nu}$ is a differential operator, i.e. $P \in gl(n, D)$. This completes the proof.

Since differential operators preserve the base point of the Grassmannian $Gr_n(0)$, the negative order pseudodifferential operators should give the most part of $Gr_n(0)$. In fact, we have

6.6. THEOREM. *Let $S \in gl(n, E)$ be a monic zero-th order pseudodifferential operator of the form*

$$S = I_n + \sum_{m=1}^{\infty} s_m(x)\partial^{-m} , \tag{6.7}$$

where $s_m(x) \in gl(n, k[[x]])$. Then the map

$$\sigma : \Sigma \ni S \longmapsto W = S^{-1} \cdot k[z^{-1}]^{\oplus n} \in Gr_n^+(0)$$

gives a bijective correspondence between the set Σ of pseudodifferential operators of the form of (6.7) *and the big-cell $Gr_n^+(0)$ of the index 0 Grassmannian.*

PROOF: Since S is invertible of order 0, we have $S^{-1} \cdot V = V$ and $S^{-1} \cdot V^{(-1)} = V^{(-1)}$, where $V^{(-1)} = F^{(-1)}(V) = \big(k[[z]]z\big)^{\oplus n}$. Thus $V = S^{-1} \cdot k[z^{-1}]^{\oplus n} \oplus V^{(-1)}$, which shows that σ maps into the big-cell.

The injectivity of σ is easy: if $S_1^{-1} \cdot k[z^{-1}]^{\oplus n} = S_2^{-1} \cdot k[z^{-1}]^{\oplus n}$, then $S_1 S_2^{-1} \cdot k[z^{-1}]^{\oplus n} = k[z^{-1}]^{\oplus n}$. It means, by (6.5), that $S_1 S_2^{-1}$ is a differential operator. Since $S_1 S_2^{-1}$ has the same form of (6.7), the only possibility is that $S_1 S_2^{-1} = I_n$, which implies the injectivity of σ.

In order to establish surjectivity, take an arbitrary point W of the big-cell $Gr^+(0)$. We can choose a basis $\langle \mathbf{w}_j^\mu \rangle_{1 \le j \le n, 0 \le \mu}$ for the vector space W in the form

$$\mathbf{w}_j^\mu = \mathbf{e}_j z^{-\mu} + \sum_{\nu=1}^{\infty} \sum_{i=1}^{n} \mathbf{e}_i w_{j\nu}^{i\mu} z^\nu ,$$

where $\mathbf{e}_j$ is the elementary column vector of size n and $w_{j\nu}^{i\mu} \in k$. Our goal is to construct an operator $S \in \Sigma$ such that $S^{-1} \cdot k[z^{-1}]^{\oplus n} = W$. Let us put $S^{-1} = \big(S_j^i\big)_{1 \le i,j \le n}$ with

$$S_j^i = \delta_j^i + \sum_{\nu=1}^{\infty} \partial^{-\nu} \cdot s_{j\nu}^i(x) .$$

Since every coefficient $s_{j\nu}^i(x)$ of S^{-1} is a formal power series in x, we can construct the operator by induction on the power of x. So let us assume that we have constructed $s_{j\nu}^i(x)$ modulo $k[[x]]x^\mu$. We have to introduce one more equation of order μ in order to determine the coefficient of x^μ in $s_{j\nu}^i(x)$, which comes from the equation

$$S^{-1} \cdot \mathbf{e}_j z^{-\mu} = \text{ a linear combination of } \mathbf{w}_i^\nu .$$

For the purpose of finding a consistent equation, let us compute the left-hand side by using the projection ρ of (6.3):

$$
\begin{aligned}
S^{-1}\cdot \mathbf{e}_j z^{-\mu} &= \sum_{i=1}^{n} \mathbf{e}_i \cdot S_j^i \cdot z^{-\mu} \\
&= \mathbf{e}_j z^{-\mu} + \rho\left(\sum_{\nu=1}^{\infty}\sum_{i=1}^{n} \partial^{-\nu}\cdot s_{j\nu}^{i}(x)\mathbf{e}_i \cdot \partial^{\mu}\right) \\
&= \mathbf{e}_j z^{-\mu} + \rho\left(\sum_{\nu=1}^{\infty}\sum_{i=1}^{n}\sum_{m=0}^{\mu}(-1)^m\binom{\mu}{m}\mathbf{e}_i\cdot\partial^{\mu-\nu-m}\cdot {s_{j\nu}^{i}}^{(m)}(x)\right) \\
&= \mathbf{e}_j z^{-\mu} + \sum_{\nu=1}^{\infty}\sum_{i=1}^{n}\sum_{m=0}^{\mu}(-1)^m\binom{\mu}{m}{s_{j\nu}^{i}}^{(m)}(0)\cdot \mathbf{e}_i z^{-\mu+\nu+m} \\
&= \mathbf{e}_j z^{-\mu} + \sum_{\alpha=1}^{\mu-1}\sum_{m=0}^{\alpha}\sum_{i=1}^{n}(-1)^m\binom{\mu}{m}{s_{j\alpha-m}^{i}}^{(m)}(0)\cdot \mathbf{e}_i z^{-\mu+\alpha} \\
&\quad + \sum_{m=0}^{\mu-1}\sum_{i=1}^{n}(-1)^m\binom{\mu}{m}{s_{j\mu-m}^{i}}^{(m)}(0)\cdot \mathbf{e}_i \\
&\quad + \sum_{\beta=1}^{\infty}\sum_{m=0}^{\mu}\sum_{i=1}^{n}(-1)^m\binom{\mu}{m}{s_{j\beta+\mu-m}^{i}}^{(m)}(0)\cdot \mathbf{e}_i z^{\beta}\,.
\end{aligned}
$$

Thus we see that the equation

$$
\begin{aligned}
S^{-1}\cdot \mathbf{e}_j z^{-\mu} = \mathbf{w}_j^{\mu} &+ \sum_{\alpha=1}^{\mu-1}\sum_{m=0}^{\alpha}\sum_{i=1}^{n}(-1)^m\binom{\mu}{m}{s_{j\alpha-m}^{i}}^{(m)}(0)\cdot \mathbf{w}_i^{-\mu+\alpha} \\
&+ \sum_{m=0}^{\mu-1}\sum_{i=1}^{n}(-1)^m\binom{\mu}{m}{s_{j\mu-m}^{i}}^{(m)}(0)\cdot \mathbf{w}_i^{0}
\end{aligned}
\tag{6.8}
$$

is the identity for the coefficients of $\mathbf{e}_i z^{-\nu}$ for all i and $\nu \geq 0$, and determines $s_{j\beta}^{i}(0)^{(\mu)}$ uniquely, because the coefficient of $s_{j\beta}^{i}(0)^{(\mu)}$ in the equation is $(-1)^{\mu}$. Thus by solving (6.8) for all j and $\mu \geq 0$ inductively, we can determine the operator S uniquely, which satisfies the desired property by the construction. This completes the proof.

Using this identification of $Gr^{+}(0)$ and Σ, we can translate the Heisenberg flows defined on the big-cell into a system of nonlinear partial differential equations. Since we are not introducing any analytic structures in Σ, we cannot talk about a Lie group structure in it. However, the exponential map

$$\exp : E^{(-1)} \longrightarrow I_n + E^{(-1)} = \Sigma$$

is well-defined and surjective, and hence we can regard $E^{(-1)}$ as the *Lie algebra* of the infinite-dimensional group Σ. Simbolically, we have an identification

$$E^{(-1)} = \mathrm{Lie}(\Sigma) = T_{I_n}\Sigma = S^{-1}\cdot T_S\Sigma$$

for every $S \in \Sigma$. The equation

$$\frac{\partial W(t)}{\partial t_{ij}} = \left(h_{n_j}(y)\right)^{-i} \cdot W(t) \tag{5.5}$$

is an equation of tangent vectors at the point $W(t)$. We now identify the variable y of (2.7) with a pseudodifferential operator

$$y = \partial^{-r} + \sum_{m=1}^{\infty} c_m \partial^{-r-m} \tag{6.9}$$

with coefficients in k. Then the block matrix $h_{n_j}(y)$ of (5.5) is identified with an element of $gl(n, E)$. Let $W(t)$ be a solution of (5.5) which lies in $Gr_n^+(0)$, where $t = (t_{ij})$. Writing $W(t) = S(t)^{-1} \cdot k[z^{-1}]^{\oplus n}$, the tangent vector of the left-hand side of (5.5) is given by

$$\frac{\partial W(t)}{\partial t_{ij}} = \frac{\partial S(t)^{-1}}{\partial t_{ij}},$$

which then gives an element

$$S(t) \cdot \frac{\partial S(t)^{-1}}{\partial t_{ij}} = -\frac{\partial S(t)}{\partial t_{ij}} \cdot S(t)^{-1} \in E^{(-1)}.$$

The tangent vector of the right-hand side of (5.5) is $\left(h_{n_j}(t)\right)^{-i} \in \mathrm{Hom}_{\mathrm{cont}}(W, V/W)$, which gives rise to a tangent vector $S(t) \cdot \left(h_{n_j}(t)\right)^{-i} \cdot S(t)^{-1}$ at the base point $k[z^{-1}]^{\oplus n}$ of the big-cell by the diagram

$$\begin{array}{ccccccc}
k[z^{-1}]^{\oplus n} & \longrightarrow & V & \xrightarrow{S \cdot h \cdot S^{-1}} & V & \longrightarrow & V/k[z^{-1}]^{\oplus n} \\
{\scriptstyle S^{-1}}\downarrow & & {\scriptstyle S^{-1}}\downarrow & & \downarrow{\scriptstyle S^{-1}} & & \downarrow{\scriptstyle S^{-1}} \\
W & \longrightarrow & V & \xrightarrow[h]{} & V & \longrightarrow & V/W,
\end{array}$$

where we denote $W = W(t)$, $S = S(t)$ and $h = \left(h_{n_j}(t)\right)^{-i}$. Since the base point is preserved by the differential operators, the equation of the tangent vectors reduces to an equation

$$\frac{\partial S(t)}{\partial t_{ij}} \cdot S(t)^{-1} = -\left(S(t) \cdot \left(h_{n_j}(y)\right)^{-i} \cdot S(t)^{-1}\right)^{-} \tag{6.10}$$

in the Lie algebra $E^{(-1)}$ level, where $(\bullet)^-$ denotes the negative order part of the operator by (6.4). We call this equation the *Heisenberg KP system*. Note that the above equation is trivial for negative i because of (6.9). In terms of the operator

$$P(t) = S(t) \cdot y^{-1} \cdot I_n \cdot S(t)^{-1} \in gl(n, E)$$

whose leading term is $I_n \cdot \partial^r$, the equation (6.10) becomes a more familiar *Lax equation*

$$\frac{\partial P(t)}{\partial t_{ij}} = \left[\left(S(t) \cdot \left(h_{n_j}(y)\right)^{-i} \cdot S(t)^{-1} \right)^+ , \; P(t) \right] .$$

In particular, the Heisenberg KP system describes the infinitesimal isospectral deformations of the operator $P = P(0)$. Note that if one chooses $y = z = \partial^{-1}$ in (6.9), then the above Lax equation for the case of $n = 1$ becomes the original KP system. We can solve the initial value problem of the Heisenberg KP system (6.10) by the *generalized Birkhoff decomposition* of [M2]:

$$\exp\left(\sum_{j=1}^{\ell} \sum_{i=1}^{\infty} t_{ij} \left(h_{n_j}(y)\right)^{-i} \right) \cdot S(0)^{-1} = S(t)^{-1} \cdot Y(t) , \tag{6.11}$$

where $Y(t)$ is an invertible differential operator of infinite order defined in [M2]. In order to see that the $S(t)$ of (6.11) gives a solution of (6.10), we differentiate the equation (6.11) with respect to t_{ij}. Then we have

$$S(t) \cdot \left(h_{n_j}(y)\right)^{-i} \cdot S(t)^{-1} = - \frac{\partial S(t)}{\partial t_{ij}} \cdot S(t)^{-1} + \frac{\partial Y(t)}{\partial t_{ij}} \cdot Y(t)^{-1} ,$$

whose negative order terms are nothing but the Heisenberg KP system (6.10). It shows that the Heisenberg KP system is a completely integrable system of nonlinear partial differential equations.

Now, consider a set of geometric data

$$\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}, \phi) \rangle$$

such that $H^0(C_{\mathbf{n}}, \mathcal{F}) = H^1(C_{\mathbf{n}}, \mathcal{F}) = 0$. Then by the Krichever functor of (3.17), it gives rise to a triple $(A_0, A_{\mathbf{n}}, W)$ satisfying that $W \in Gr^+(0)$. By (6.6), there is a monic zero-th order pseudodifferential operator S such that $W = S^{-1} \cdot k[z^{-1}]^{\oplus n}$. Using the identification (6.9) of the variable y as the pseudodifferential operator with constant coefficients, we can define two commutative subalgebras of $gl(n, E)$ by

$$\begin{cases} B_0 = S \cdot A_0 \cdot S^{-1} \\ B_{\mathbf{n}} = S \cdot A_{\mathbf{n}} \cdot S^{-1} . \end{cases} \tag{6.12}$$

The inclusion relation $A_0 \subset k((y))$ gives us $B_0 \subset k((P^{-1}))$, where $P = S \cdot y^{-1} \cdot I_n \cdot S^{-1} \in gl(n, E)$. Since A_0 and $A_{\mathbf{n}}$ stabilize W, we know that B_0 and $B_{\mathbf{n}}$ stabilize $k[z^{-1}]^{\oplus n}$. Therefore, these algebras are commutative algebras of ordinary differential operators with matrix coefficients!

6.13. Definition. *We denote by* $C^+(\mathbf{n}, 0, r)$ *the set of objects*

$$\langle f : (C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, \pi, f_*\mathcal{F}, \phi) \rangle$$

of the category $\mathcal{C}(\mathbf{n})$ of index 0 and rank r such that

$$H^0(C_\mathbf{n}, \mathcal{F}) = H^1(C_\mathbf{n}, \mathcal{F}) = 0 .$$

The set of pairs (B_0, B) of commutative algebras satisfying the following conditions is denoted by $\mathcal{D}(n,r)$:

(1) $k \subset B_0 \subset B \subset gl(n, D)$.
(2) B_0 *and* B *are commutative* k*-algebras.*
(3) *There is an operator* $P \in gl(n, E)$ *whose leading term is* $I_n \cdot \partial^r$ *such that* $B_0 \subset k((P^{-1}))$.
(4) *The projection map* $B_0 \longrightarrow k((P^{-1}))/k[[P^{-1}]]$ *is Fredholm.*
(5) B *has rank* n *as a torsion free module over* B_0.

Using this definition, we can summarize

6.14. PROPOSITION. *The construction* (6.12) *gives a canonical map*

$$\chi_{\mathbf{n},r} : \mathcal{C}^+(\mathbf{n}, 0, r) \longrightarrow \mathcal{D}(n, r)$$

for every r and a positive integral vector $\mathbf{n} = (n_1, \cdots, n_\ell)$ with $n = n_1 + \cdots + n_\ell$.

If the field k is of characteristic zero, then we can construct maximal commutative algebras of ordinary differential operators with coefficients in matrix valued functions as an application of the above proposition.

6.15. THEOREM. *Every set*

$$\langle f : (C_\mathbf{n}, \Delta, \Pi, \mathcal{F}, \Phi) \longrightarrow (C_0, p, id, f_*\mathcal{F}, \phi)\rangle$$

of geometric data with a smooth curve $C_\mathbf{n}$, $\pi = id$ and a line bundle $\mathcal{F}$ satisfying that $H^0(C_\mathbf{n}, \mathcal{F}) = H^1(C_\mathbf{n}, \mathcal{F}) = 0$ gives rise to a maximal commutative subalgebra $B_\mathbf{n} \subset gl(n, D)$ by $\chi_{\mathbf{n},1}$.

PROOF: Let $(B_0, B_\mathbf{n})$ be the image of $\chi_{\mathbf{n},1}$ applied to the above object, and $(A_0, A_\mathbf{n}, W)$ the stabilizer data corresponding to the geometric data. Recall that $B_0 = S \cdot A_0 \cdot S^{-1}$, where S is the operator corresponding to W. Since $r = 1$ in our case, (4.8) implies the existence of an element $a \in A_0$ of the form

$$a = a(z^{-1}) = z^{-m} + c_2 z^{-m+2} + c_3 z^{-m+3} + \cdots \in A_0 \subset k((z)) .$$

We call a pseudodifferential operator $a(\partial) \cdot I_n \in gl(n, E)$ a *normalized* scalar diagonal operator of order m with constant coefficients. Here, we need

6.16. LEMMA. *Let $K \in gl(n, E)$ be a normalized scalar diagonal operator of order $m > 0$ with constant coefficients and $Q = (Q_{ij})$ an arbitrary element of $gl(n, E)$. If Q and K commute, then every coefficient of Q is a constant matrix.*

PROOF: Let $K = a(\partial) \cdot I_n$ for some $a(\partial) \in k((\partial^{-1}))$. It is well known that there is a monic zero-th order pseudodifferential operator $S_0 \in E$ such that

$$S_0^{-1} \cdot a(\partial) \cdot S_0 = \partial^m .$$

Since $a(\partial)$ is a constant coefficient operator, we can show that (see **[M3]**)

$$S_0^{-1} \cdot k((\partial^{-1})) \cdot S_0 = k((\partial^{-1})) .$$

Going back to the matrix case, we have

$$0 = (S_0 \cdot I_n)^{-1} \cdot [Q, K] \cdot (S_0 \cdot I_n) = \left[(S_0 \cdot I_n)^{-1} \cdot Q \cdot (S_0 \cdot I_n),\ \partial^m \cdot I_n\right] .$$

In characteristic zero, commutativity with ∂^m implies commutativity with ∂. Thus each matrix component $S_0^{-1} \cdot Q_{ij} \cdot S_0$ commutes with ∂, and hence $S_0^{-1} \cdot Q_{ij} \cdot S_0 \in k((\partial^{-1}))$. Therefore, $Q_{ij} \in k((\partial^{-1}))$. This completes the proof of lemma.

Now, let $B \supset B_{\mathbf{n}}$ be a commutative subalgebra of $gl(n, D)$ containing $B_{\mathbf{n}}$. Since $B_0 = S \cdot A_0 \cdot S^{-1}$ and $B_0 \subset B$, every element of B commutes with $S \cdot a(\partial) \cdot I_n \cdot S^{-1}$. Then by the lemma, we have

$$A = S^{-1} \cdot B \cdot S \subset gl(n, k((\partial^{-1}))) .$$

Note that the algebra A stabilizes $W = S^{-1} \cdot k[z^{-1}]^{\oplus n}$. Since $H_{\mathbf{n}}(z)$ can be generated by $A_{\mathbf{n}}$ over $k((z)) = k((\partial^{-1}))$, every element of A commutes with $H_{\mathbf{n}}(z)$. Therefore, we have $A \subset H_{\mathbf{n}}(z)$ because of the maximality of $H_{\mathbf{n}}(z)$. Thus we obtain another triple (A_0, A, W) of stabilizer data of the same type $\mathbf{n}$. The inclusion $A_{\mathbf{n}} \longrightarrow A$ gives rise to a birational morphism $\beta : C \longrightarrow C_{\mathbf{n}}$. Since we are assuming that the curve $C_{\mathbf{n}}$ is nonsingular, β has to be an isomorphism, which then implies that $A = A_{\mathbf{n}}$. Therefore, we have $B = B_{\mathbf{n}}$. This completes the proof of maximality of $B_{\mathbf{n}}$.

6.17. REMARK: There are other maximal commutative subalgebras in $gl(n, D)$ than what we have constructed in (6.15). It corresponds to the fact that the algebras $H_{\mathbf{n}}(z)$ are not the only maximal commutative subalgebras of the formal loop algebra $gl(n, k((z)))$.

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