THE RESTRICTION HOMOMORPHISM

$\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X)$

FOR G A COMPACT LIE GROUP

by

Sören Illman

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

Department of Mathematics
University of Helsinki
Hallituskatu 15
00100 Helsinki
Finland

MPI/89 – 89

The restriction homomorphism $Res^G_H : Wh_G(X) \longrightarrow Wh_H(X)$ for G a compact Lie group

Sören Illman

Let G be a compact Lie group and let H be a closed subgroup of G. The main objective of this paper is to establish a construction of the following form. Given a finite G–CW complex X there exist a finite H–CW complex $R_H X$ and an H–homotopy equivalence

$$\eta : X \longrightarrow R_H X \tag{1}$$

such that this construction is unique up to simple H–homotopy type; i.e., if $\eta' : X \longrightarrow R'_H X$ is another choice then $\eta' \circ \eta^{-1} : R_H X \longrightarrow R'_H X$ is a simple H–homotopy equivalence. (Here η^{-1} denotes an H–homotopy inverse of η.) The notion of equivariant simple–homotopy equivalence is as defined in [3]. The operation of restricting the transformation group G, of an arbitrary G–CW complex, to a closed subgroup H of G is treated in [8], and in fact we use the same construction in this paper. Our main task in this paper is then to prove that in the case of a finite G–CW complex X the construction has the additional property that it produces a finite H–CW complex $R_H X$ which is uniquely determined up to simple H–homotopy type. Here we should recall the following two facts. First, the H–CW $R_H X$ is in general not H–homeomorphic to the H–space X. For more details on this see [8], in particular the example given in [8, Section 2]. Secondly we recall that equivariant Whitehead torsion is <u>not</u> an equivariant topological invariant. This means that there may exist two finite H–CW complex structures Y_1 and Y_2 on the same H–space Y such that the identity map of Y, $id_Y : Y_1 \longrightarrow Y_2$ is not a simple H–homotopy equivalence between the finite H–CW complexes Y_1 and Y_2. Thus we see that although one cannot construct an H–CW complex structure on the H–space X itself, one can establish a result that in fact is more precise and says something more, namly one can construct a finite H–CW complex $R_H X$ which represents a unique simple H–homotopy type. This construction also has the property that $\dim(R_H X)^K = \dim X^K$, for every closed subgroup K of H and moreover the H–isotropy types occuring in the H–spaces X and $R_H X$ are exactly the same. We call such an H–homotpy equivalence $\eta : X \longrightarrow R_H X$ a preferred H–reduction of X. The existence of such a class of preferred H–reductions of X is proved in Section 6.

Then we go on to establish that if X and X_1 are finite G–CW complexes of the same simple G–homotopy type then the finite H–CW complexes $R_H X$ and $R_H X_1$ have the same simple H–homotopy type. A result of this form (see Corollary 8.2) leads to the definition of a well–defined homomorphism $\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow W_H(X)$. Recall that an element of the equivariant Whitehead group $\mathrm{Wh}_G(X)$ is given as an equivalence class $s_G(V,X)$ of a finite G–CW pair (V,X), where the inclusion $X \hookrightarrow V$ is a G–homotopy equivalence, and the relation is the one of formal G–deformations of V rel X (see [3]). The definition of Res_H^G can then be given by $\mathrm{Res}_H^G(s_G(V,X)) = (R_H V, R_H X) \in \mathrm{Wh}_H(R_H X)$ and $\mathrm{Wh}_H(R_H X)$ can moreover be interpreted as a group that we denote by $\mathrm{Wh}_H(X)$. We prove that $\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X)$ is a homomorphism and we also establish the basic fact that for any G–homotopy equivalence $f : X \longrightarrow X_1$ we have that the H–equivariant Whitehead torsion of the induced H–homotopy equivalence $R_H f : R_H X \longrightarrow R_H X_1$ is given by

$$\tau(R_H f) = \mathrm{Res}_H^G(\tau(f)) . \qquad (2)$$

The main result of this paper, Theorem 6.1, which proves the existence of preferred H–reductions , was announced in [7, Theorem C], and the corresponding restriction homomorphism $\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X)$ is discussed in Section 2 of [7]. (In [7] we denoted $R_H X$ by $\mathrm{esh}_H(X)$.)

A different approach to the restriction homomorphism between equivariant Whitehead groups is given in the forthcoming book by W. Lück [9].

We shall in a later paper prove the transitivity property $\mathrm{Res}_K^H \circ \mathrm{Res}_H^G = \mathrm{Res}_K^G$, where $K < H < G$. Curiously enough this transitivity property is a non–trivial fact.

The reader should be advised that the present form of the paper at hand is a somewhat preliminary one.

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1. Preliminaries

Let G be a compact Lie group. We will consider the basic properties of G–CW complexes as well–known and use them without any further reference. For example the fact that a G–CW pair (X,A) has the G–homotopy extension property and the equivariant skeletal approximation theorem, also in its relative form, are used freely in this paper. The following statement concerning different choices of characteristic G–maps for a G–cell c does perhaps not appear in the literature, so we present it here since we have explicit use of it in this paper.

Lemma 1.1. Suppose that $\xi,\xi' : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (c,\dot{c}) \hookrightarrow (X^n,X^{n-1})$ are two characteristic G–maps for some G–cell c of X . Then there exists $g_0 \in N(P)$ and an isometry $a : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that the G–maps
$\xi : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (X^n,X^{n-1})$ and
$(a \times g_0) \circ \xi' : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (X^n,X^{n-1})$ are G–homotopic . In particular we have that the G–maps $\xi| : S^{n-1} \times G/P \longrightarrow X^{n-1}$ and
$(a \times g_0) \circ \xi'| : S^{n-1} \times G/P \longrightarrow X^{n-1}$ are G–homotopic .

In Lemma 1.1 one can always choose $a : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ to be either the identity map on $\mathbb{R}^n$ or the isometry given by $a(x_1, \dots ,x_n) = (x_1, \dots ,x_{n-1}, -x_n)$. These two alternatives then give a map from S^{n-1} to itself of degree 1 or degree –1 , respectively. The map $g_0 : G/P \longrightarrow G/P$, given by $gP \longmapsto gg_0P$ for all $gP \in G/P$, is a real analytic G–isomorphism of the real analytic G–manifold G/P to itself. The proof of Lemma 1.1 is easy and left to the reader.

We will also have use of the following weakened version of the notion of skeletal map.

Definition 1.2. Let (A,B) be a finite G–CW pair and let X be a finite G–CW complex. We say that a G–map $\varphi : B \longrightarrow X$ is (A,B)–skeletal if for every G–cell c, say of dimension m, in A–B, we have $\varphi(\dot{c}) \subset X^{m-1}$.

Observe that if $\varphi : B \longrightarrow X$ is an (A,B)–skeletal G–map then the adjunction space $X \cup_\varphi A$ is a finite G–CW complex.

In [10] Matumoto and Shiota show that one can associate to any compact smooth G–manifold a well–defined simple G–homotopy type. In this paper we only use a very special case of this result, namely the following one. Let H be a closed subgroup of G and consider the standard action of H by multiplication from the left on a homogeneous space G/P, where P denotes a closed subgroup of G. Then one can give the compact H–manifold G/P a well–defined simple H–homotopy type in the following way. By a well–known theorem, due independently to Mostow [11] and Palais [12], there exists a linear representation space $\mathbb{R}^n(\rho)$ for G, where $\rho : G \longrightarrow O(n)$, and a point $v \in \mathbb{R}^n(\rho)$ such that $G_v = P$. Then the G–orbit through v is a real analytic G–submanifold of $\mathbb{R}^n(\rho)$ which is G–isomorphic to G/P. We now consider $\mathbb{R}^n(\rho)$ as a linear representation space of the closed subgroup H; i.e., we consider the linear representation space $\mathbb{R}^n(\rho|H)$. It is a well–known result that the orbit space $\mathbb{R}^n(\rho|H)/H$ can be considered as a semi–algebraic subset of some euclidean space $\mathbb{R}^k$. Since the H–manifold G/P is a real analytic H–submanifold of $\mathbb{R}^n(\rho|H)$ it follows that the orbit space (G/P)/H, where $(G/P)/H \subset \mathbb{R}^n(\rho|H)/H \subset \mathbb{R}^k$, is a subanalytic subset of $\mathbb{R}^k$.

Let F be a finite H–CW complex such that the orbit space F/H is a finite simplicial complex. We say that an H–homeomorphism

$$u : F \xrightarrow{\cong} G/P$$

is a distinguished H–triangulation of the H–manifold G/P if the induced map $\bar{u} : F/H \longrightarrow (G/P)/H$ is a subanalytic triangulation of the subanalytic set $(G/P)/H$. The existence of distinguished H–triangulations is a consequence of the result concerning existence of subanalytic triangulations of subanalytic sets, due to Hironaka [2] and Hardt [1], together with the lifting procedure of Matumoto and Illman, see [4], which gives an H–CW structure on an H–space whose orbit space has a well–behaved triangulation. We can now state the result that gives a well–defined simple H–homotopy type to the H–manifold G/P.

Theorem 1.3. There exists a distinguished H–triangulation $u : F \longrightarrow G/P$ of the H–manifold G/P. If $u : F \longrightarrow G/P$ and $u' : F \longrightarrow G/P$ are distinguished H–triangulations of the H–manifold G/P, then $u' \circ u^{-1} : F \longrightarrow F'$ is a simple H–homotopy equivalence.

The uniqueness part of Theorem 1.2 follows from the fact that two subanalytic triangulations of a subanalytic set have a common subanalytic subdivision, see Hironaka [2] and Hardt [1], and the fact that any H–equivariant subdivision map of finite H–CW complexes is a simple H–homotopy equivalence, see [6, Theorem 12.2].

We will have use of the following result.

Lemma 1.4. Let $f : Y \longrightarrow Z$ be a simple H–homotopy equivalence between finite H–CW complexes, and let L be an ordinary finite CW complex. Then $id \times f : L \times Y \longrightarrow L \times Z$ is a simple H–homotopy equivalence.

This is an immediate consequence of the product formula for equivariant Whitehead torsion [5], but one can also easily give a direct proof.

Furthermore we will need the following.

Lemma 1.5. Suppose that (L,L_0) is a finite CW pair such that L collapses to L_0. Let F be a finite H–CW complex. Then the H–CW complex $L \times F$ H–collapses to $L_0 \times F$.

The proof of the above lemma is easy and left to the reader.

Definition 1.6. Let U be an arbitrary H–space and let Y_1 and Y_2 be finite H–CW complexes. We say that two H–maps $f_1 : U \longrightarrow Y_1$ and $f_2 : U \longrightarrow Y_2$ are s–equivalent if there exists a simple H–homotopy equivalence $\sigma : Y_1 \longrightarrow Y_2$ such that $\sigma \circ f_1$ is H–homotopic to f_2.

If (U,U_0) is an arbitrary H–pair and (C_1,D_1) and (C_2,D_2) are finite H–CW pairs we say that the H–maps $f_1 : (U,U_0) \longrightarrow (C_1,D_1)$ and $f_2 : (U,U_0) \longrightarrow (C_2,D_2)$ are s–equivalent, as maps of pairs, if there exists an H–map $\alpha : (C_1,D_1) \longrightarrow (C_2,D_2)$ such that the maps $\alpha \circ f_1$ and $f_2 : (U,U_0) \longrightarrow (C_2,D_2)$ are H–homotopic as maps of pairs and $\alpha : C_1 \longrightarrow C_2$ and $\alpha| : D_1 \longrightarrow D_2$ are simple H–homotopy equivalences.

2. Background information on equivariant homotopy type of adjunction spaces

Let H denote an arbitrary compact Lie group. (In this section and in Section 3 the role of the transformation group H is completely formal, and hence H could as well be any locally compact group.) By X and Y we denote arbitrary H–spaces and (A,B) and (C,D) denote H–pairs , which have the H–homotpy extension property, and where B and D are closed in A and C , respectively.

The map $k(\Phi)$ in Lemma 2.1 is defined as the composite map

$$X \cup_{\varphi_0} A \xrightarrow{\cong} X \cup_\Phi (A\times\{0\} \cup B\times I) \xrightarrow{\bar{i}_0} X \cup_\Phi (A\times I) \xrightarrow{\bar{r}_1} X \cup_\Phi (A\times\{1\} \cup B\times I) \xrightarrow{\cong} X \cup_{\varphi_1} A$$

where the first and last map are natural G–homeomorphisms, which we shall use as identifications. Since $A \times \{0\} \cup B \times I$ and $A \times \{1\} \cup B \times I$ are strong H–deformation retractions of $A \times I$ it follows that both $\bar{i}_0$ and $\bar{r}_1$ are H–homotopy equivalences. (A more detailed discussion of the H–homotopy equivalence $k(\Phi)$ can be found in [8, Section 3].) In this section we simply state the results, and leave the proofs to the reader.

All results in this section are well known and they have easy proofs. (Proofs of Lemma 2.4 and 2.5 are given in [8, Section 3].)

Lemma 2.1. Suppose that the H–maps $\varphi_0, \varphi_1 : B \longrightarrow X$ are H–homotopic and that $\Phi : B \times I \longrightarrow X$ is an H–homotopy from φ_0 to φ_1 . Then

$$k(\Phi) : X \cup_{\varphi_0} A \longrightarrow X \cup_{\varphi_1} A$$

is an H–homotopy equivalence. Furthermore $k(\Phi)|X = id_X$ and $k(\Phi^{-1})$ is an H–homotopy inverse of $k(\Phi)$ rel X .

Lemma 2.2. Suppose that $\Phi, \Phi' : B \times I \longrightarrow X$ are two H–homotopies from φ_0 to φ_1 such that Φ and Φ' are H–homotopic rel $B \times \dot{I}$. Then the two H–homotopy equivalences

$$k(\Phi), k(\Phi') : X \cup_{\varphi_0} A \longrightarrow X \cup_{\varphi_1} A$$

are H–homotopic rel X . By abuse of notation we denote the conclusion of Lemma 2.2 simply by $k(\Phi) = k(\Phi')$.

If $\Phi : B \times I \longrightarrow X$ is an H–homotopy from φ_0 to φ_1 and $\Omega : B \times I \longrightarrow X$ is an H–homotopy from φ_1 to φ_2 the join $\Phi * \Omega : B \times I \longrightarrow X$ is an H–homotopy from φ_0 to φ_2 .

Lemma 2.3. We have

$$k(\Phi * \Omega) = k(\Omega) \circ k(\Phi) : X \cup_{\varphi_0} A \longrightarrow X \cup_{\varphi_2} A .$$

If $f : X \longrightarrow Y$ is an H–map we let

$$\bar{f} : X \cup_{\varphi} A \longrightarrow Y \cup_{f\varphi} A$$

be the H–map induced by f and by the identity map on A. We call $\bar{f}$ the canonical extension of f. We have $\bar{f} \mid X = f$.

<u>Lemma 2.4</u>. Suppose that the H–maps $f_0, f_1 : X \longrightarrow Y$ are H–homotopic and that $F : X \times I \longrightarrow Y$ is an H–homotopy from f_0 to f_1. Then the diagram

$$\begin{array}{ccc} X \cup_{\varphi} A & \xrightarrow{\ \bar{f}_0\ } & Y \cup_{f_0\varphi} A \\ & \underset{\bar{f}_1}{\searrow} & \big\downarrow k(\theta) \\ & & Y \cup_{f_1} A \end{array}$$

is H–homotopy commutative. Here $\theta = F \circ (\varphi \times \mathrm{id}) : B \times I \longrightarrow Y$, and $k(\theta)$ denotes the corresponding H–homotopy equivalence given by Lemma 2.1.

<u>Lemma 2.5</u>. If $f : X \longrightarrow Y$ is an H–homotopy equivalence then so is its canonical extension $\bar{f} : X \cup_{\varphi} A \longrightarrow Y \cup_{f\varphi} A$.

<u>Lemma 2.6</u>. Let $\Phi : B \times I \longrightarrow X$ be an H–homotopy from φ_0 to φ_1, and let $f : X \longrightarrow Y$ be an H–map. Then the diagram

$$\begin{array}{ccc} X \cup_{\varphi_0} A & \xrightarrow{\ \bar{f}\ } & Y \cup_{f\varphi_0} A \\ k(\Phi) \big\downarrow & & \big\downarrow k(f \circ \Phi) \\ X \cup_{\varphi_1} A & \xrightarrow{\ \bar{f}\ } & Y \cup_{f\varphi_1} A \end{array}$$

is H–homotopy commutative.

If $\alpha : (A,B) \longrightarrow (C,D)$ is an H–map we define

$$\bar{\alpha} : X \cup_{\psi\alpha|} A \longrightarrow X \cup_{\psi} C$$

to be the H–map induced by the identity map on X and the H–map α on A. Observe that $\bar{\alpha}|X = \mathrm{id}_X$.

<u>Lemma 2.7</u>. Suppose that the H–maps $\alpha_0, \alpha_1 : (A,B) \longrightarrow (C,D)$ are H–homotopic (as maps of pairs) and that $\Lambda : (A,B) \times I \longrightarrow (C,D)$ is an H–homotopy from α_0 to α_1. Then the diagram

$$\begin{array}{ccc} X \cup_{\psi\alpha_0} A & \xrightarrow{\ \bar{\alpha}_0\ } & X \cup_{\psi} C \\ {\scriptstyle k(\Gamma)}\big\downarrow & \nearrow {\scriptstyle \bar{\alpha}_1} & \\ X \cup_{\psi\alpha_1|} A & & \end{array}$$

is H–homotopy commutative. Here $\Gamma = \psi \circ (\Lambda|) : B \times I \longrightarrow X$, and $k(\Gamma)$ denotes the corresponding H–homotpy equivalence as given by Lemma 2.1.

<u>Lemma 2.8</u>. Suppose that $\alpha : (A,B) \longrightarrow (C,D)$ is an H–homotopy equivalence of H–pairs. Then $\bar{\alpha} : X \cup_{\psi\alpha|} A \longrightarrow X \cup_{\psi} C$ is an H–homotopy equivalence.

<u>Lemma 2.9</u>. Suppose that $\Psi : D \times I \longrightarrow X$ is an H–homotopy from ψ_0 to ψ_1 and that $\alpha : (A,B) \longrightarrow (C,D)$ is an H–map. Then the diagram

$$\begin{array}{ccc} X \cup_{\psi_0 \alpha|} A & \xrightarrow{\;\bar{\alpha}\;} & Y \cup_{\psi_0} C \\ {\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle k} \\ X \cup_{\psi_1 \alpha|} A & \xrightarrow{\;\bar{\alpha}\;} & X \cup_{\psi_1 \alpha|} C \end{array}$$

is H–homotopy commutative. Here $k = k(\Psi)$ and $k' = k(\Psi \circ (\alpha| \times \text{id}))$.

<u>Lemma 2.10</u>. Let $f : X \longrightarrow Y$ and $\alpha : (A,B) \longrightarrow (C,D)$ be H–maps. Then the diagram

$$\begin{array}{ccc} X \cup_{\psi\alpha|} A & \xrightarrow{\;\bar{f}\;} & Y \cup_{f\psi\alpha|} A \\ {\scriptstyle \bar{\alpha}}\downarrow & & \downarrow{\scriptstyle \bar{\alpha}} \\ X \cup_{\psi} C & \xrightarrow{\;\bar{f}\;} & Y \cup_{f\psi} C \end{array}$$

commutes.

3. k–equivalence and λ–maps

Assume that we are given a diagram of the form

$$(S) \qquad \begin{array}{ccc} A & \xrightarrow{\quad\alpha\quad} & C \\ \cup & & \cup \\ B & \xrightarrow{\quad\alpha|\quad} & D \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\mu} \\ X & \xrightarrow{\quad\eta\quad} & Y \end{array}$$

where $\alpha : (A,B) \longrightarrow (C,D)$ is an H–homotopy equivalence of pairs and $\eta : X \longrightarrow Y$ is an H–homotpy equivalence and where the lower square of the diagram is H–homotopy commutative. Let $\Omega : B \times I \longrightarrow Y$ be some H–homotopy from $\eta \circ \varphi$ to $\mu \circ \alpha|$. Then we can form the composite map

$$\lambda(S;\Omega) : X \cup_{\varphi} A \xrightarrow{\bar{\eta}} Y \cup_{\eta\varphi} A \xrightarrow{k(\Omega)} Y \cup_{\mu\alpha|} A \xrightarrow{\bar{\alpha}} Y \cup_{\mu} C .$$

The maps $\bar{\eta}$, $k(\Omega)$ and $\bar{\alpha}$ are H–homotopy equivalences by Lemma 2.5, 2.1 and 2.8, respectively. Furthermore we have $\bar{\eta}|X = \eta$, $k(\Omega)|X = id_X$ and $\bar{\alpha}|X = id_X$, and hence $\lambda(S;\Omega)|X = \eta$. Also recall that for a fixed H–homotopy Ω the map $k(\Omega)$ is determined up to H–homotopy rel X . Thus the diagram (S) and a fixed choice of an H–homotopy $\Omega : B \times I \longrightarrow Y$ from $\eta \circ \varphi$ to $\mu \circ \alpha|$ gives us an H–homotopy equivalence

$$\lambda(S;\Omega) : X \cup_{\varphi} A \longrightarrow Y \cup_{\mu} C$$

such that $\lambda(S;\Omega)\,|\,X = id_X$ and $\lambda(S;\Omega)$ is uniquely determined up to H–homotopy rel X .

We shall show below that for two different choices of H–homotopics Ω and Ω' from $\eta \circ \varphi$ to $\mu \circ \alpha|$ the corresponding H–homotopy equivalences $\lambda(S;\Omega)$ and $\lambda(S;\Omega')$ are k–equivalent in the sense defined below.

Definition 3.1. We say that two H–maps $f_1 : U \longrightarrow Y \cup_{\mu_1} C$ and $f_2 : U \longrightarrow Y \cup_{\mu_2} C$ are k–equivalent if there exists an H–homotopy $\Lambda : D \times I \longrightarrow Y$ from μ_1 to μ_2 such that the diagram

$$\begin{array}{ccc} & & Y \cup_{\mu_1} C \\ & \overset{f_1}{\nearrow} & \\ U & & \downarrow k(\Lambda) \\ & \underset{f_2}{\searrow} & \\ & & Y \cup_{\mu_2} C \end{array}$$

is H–homotopy commutative.

Lemma 3.2. The k–equivalence relation is both symmetric and transitiv.

Proof. This follows immediately from Lemma 2.1 and Lemma 2.3.

□

Lemma 3.3. Let $f_1 : U \longrightarrow Y \cup_{\mu_1} C$ and $f_2 : U \longrightarrow Y \cup_{\mu_2} C$ be k–equivalent H–maps . Then we have:

(a) If $\theta : Y \longrightarrow Z$ is an H–map the H–maps $\bar{\theta} \circ f_1 : U \longrightarrow Z \cup_{\theta\mu_1} C$ and $\bar{\theta} \circ f_2 : U \longrightarrow Z \cup_{\theta\mu_2} C$ are k–equivalent.

(b) If $\Omega : D \times I \longrightarrow Y$ is an H–homotopy from μ_2 to μ_3 we have that the H–maps $f_1 : U \longrightarrow Y \cup_{\mu_1} C$ and $k(\Omega) \circ f_\theta : U \longrightarrow Y \cup_{\mu_3} C$ are k–equivalent .

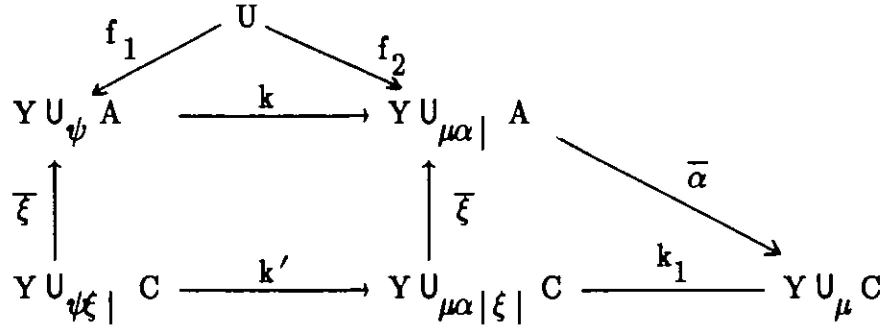
Proof. (a) is an immediate consequence of Lemma 2.6, and (b) follows from Lemma 2.3.

□

In Lemma 3.4 below U and Y denote arbitrary H–spaces . By $\alpha : (A,B) \longrightarrow (C,D)$ we denote an H–homotopy equivalence of pairs and $\xi : (C,D) \longrightarrow (A,B)$ is an H–homotopy inverse of α . We let $\psi : B \longrightarrow Y$ and $\mu : D \longrightarrow Y$ denote arbitrary H–maps .

Lemma 3.4. Suppose that $f_1 : U \longrightarrow Y \cup_\psi A$ and $f_2 : U \longrightarrow Y \cup_{\mu\alpha|} A$ are k–equivalent H–maps . Then the H–maps $(\bar{\xi})^{-1} \circ f_1 : U \longrightarrow Y \cup_{\psi\xi|} A$ and $\bar{\alpha} \circ f_2 : U \longrightarrow Y \cup_\mu C$ are k–equivalent .

Proof. We consider the following diagram



Here $k = k(\Lambda)$, where $\Lambda : B \times I \longrightarrow Y$ is some H–homotopy from ψ to $\mu\alpha|$, and the upper triangle is H–homotopy commutative. Let $k' = k(\Lambda \circ (\xi| \times id))$, then the left hand side square is H–homotopy commutative by Lemma 2.9. Let $\Phi : (C,D) \times I \longrightarrow (C,D)$ be an H–homotopy from $\alpha \circ \xi : (C,D) \longrightarrow (C,D)$ to the identity map. Then $\mu \circ \Phi| : D \times I \longrightarrow Y$ is an H–homotpy from $\mu\alpha|\xi|$ to μ, and we set $k_1 = k(\mu \circ \Phi)$. Since $\overline{\alpha} \circ \overline{\xi} = \overline{\alpha\xi}$ it follows by Lemma 2.7 that the lower right hand side triangle is H–homotopy commutative. Thus the above diagram is H–homotopy commutative. By Lemma 2.3 we have that $k_1 \circ k' = k(\Gamma)$, where $\Gamma : D \times I \longrightarrow Y$ is an H–homotopy from $\psi\xi|$ to μ . It now follows that $k(\Gamma) \circ (\overline{\xi})^{-1} \circ f_1$ is H–homotopic to $\overline{\alpha} \circ f_2$; i.e., the H–map $(\overline{\xi})^{-1} \circ f_1$ is k–equivalent to $\overline{\alpha} \circ f_2$.

□

Lemma 3.5. The H–homotopy equivalences

$$(\overline{\xi})^{-1} \circ \overline{\eta} : X \cup_{\varphi} A \longrightarrow Y \cup_{\nu} C$$

and

$$\lambda(S;\Omega) : X \cup_{\varphi} A \longrightarrow Y \cup_{\mu} C$$

are k–equivalent .

Proof. Since $\overline{\eta} : X \cup_{\varphi} A \longrightarrow Y \cup_{\eta\varphi} A$ is k–equivalent to $k(\Omega) \circ \overline{\eta} : X \cup_{\varphi} A \longrightarrow Y \cup_{\mu\alpha|} A$ we have by Lemma 3.4 that $(\overline{\xi})^{-1} \circ \overline{\eta} : X \cup_{\varphi} A \longrightarrow Y \cup_{\eta\varphi\xi|} C = Y \cup_{\nu} C$ is k–equivalent to $\overline{\alpha} \circ k(\Omega) \circ \overline{\eta} = \lambda(S;\Omega) : X \cup_{\varphi} A \longrightarrow Y \cup_{\mu} C$.

□

Corollary 3.6. Let $\Omega,\Omega' : B \times I \longrightarrow Y$ be two H–homotopics from $\eta \circ \varphi$ to $\mu \circ \alpha|$. Then the H–homotopy equivalences $\lambda(S;\Omega)$ and $\lambda(S;\Omega')$ are k–equivalent .

Proof. This is an immediate consequence of Lemma 3.5, since k–equivalence is a transitiv relation.

□

Given the H–homotopy commutative diagram (S) we will by

$$\lambda(S) : X \cup_{\varphi} A \longrightarrow Y \cup_{\mu} C$$

denote an H–homotopy equivalence of the form $\lambda(S;\Omega)$. Consequently $\lambda(S)$ denotes an H–homotopy equivalence, uniquely determined up to k–equivalence, by the diagram S . We sometimes call $\lambda(S)$ the λ–map induced by the diagram (S). On some occasions we will denote the λ–map induced by (S) by

$$\lambda(\eta,\alpha) : X \cup_{\varphi} A \longrightarrow Y \cup_{\mu} C ,$$

and such a map is uniquely determined up to k–equivalence.

Furthermore the following holds. Suppose

$$(S_1) \qquad \begin{array}{ccc} A & \xrightarrow{\alpha_1} & C \\ \cup & & \cup \\ B & \xrightarrow{\alpha_1|} & D \\ \varphi\big\downarrow & & \big\downarrow\mu_1 \\ X & \xrightarrow{\eta_1} & Y \end{array}$$

is an H–homotopy commutative diagram such that $\alpha_1 : (A,B) \longrightarrow (C,D)$ is H–homotopic to the H–map $\alpha : (A,B) \longrightarrow (C,D)$ in (S) and such that $\eta_1 : X \longrightarrow Y$ is H–homotopic to $\eta : X \longrightarrow Y$ in (S) . Then we have

Corollary 3.7. The map

$$\lambda(S_1) = \lambda(\eta_1,\alpha_1) : X \cup_\varphi A \longrightarrow Y \cup_{\mu_1} C$$

is k–equivalent to $\lambda(S) = \lambda(\eta,\alpha)$.

Proof. This follows from Lemma 3.5.

□

Let

$$(S_1) \qquad \begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ \cup & & \cup \\ B & \xrightarrow{\alpha|} & D \\ \varphi\big\downarrow & & \big\downarrow\mu \\ X & \xrightarrow{\eta} & Y \end{array}$$

and

(S_2)

$$\begin{array}{ccc} C & \xrightarrow{\ \gamma\ } & E \\ \cup & & \cup \\ D & \xrightarrow{\ \gamma|\ } & F \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\omega} \\ Y & \xrightarrow{\ \theta\ } & Z \end{array}$$

be H–homotopy commutative diagrams, where $\alpha : (A,B) \longrightarrow (C,D)$ and $\gamma : (C,D) \longrightarrow (E,F)$ are H–homotopy equivalences of pairs, and η and θ are H–homotopy equivalences, and φ, μ and ω are H–maps . Let $\Omega_1 : B \times I \longrightarrow Y$ be an H–homotopy from $\eta \circ \varphi$ to $\mu \circ \alpha|$, and $\Omega_2 : D \times I \longrightarrow Z$ an H–homotopy from $\theta \circ \mu$ to $\omega \circ \gamma|$. This gives us the λ–maps

$$\lambda(S_1;\Omega_1) : X \cup_{\varphi} A \longrightarrow Y \cup_{\mu} C$$

and

$$\lambda(S_2;\Omega_2) : Y \cup_{\mu} C \longrightarrow Z \cup_{\omega} E$$

The "composite" of the diagrams (S_1) and (S_2) gives us the H–homotopy commutative diagram

$$(S)\qquad \begin{array}{ccc} A & \xrightarrow{\gamma\circ\alpha} & E \\ \cup & & \cup \\ B & \xrightarrow{\gamma\circ\alpha|} & F \\ \varphi\big\downarrow & & \big\downarrow\omega \\ X & \xrightarrow{\theta\circ\eta} & Z \end{array}$$

Let $\Omega : B \times I \longrightarrow Z$ be any H–homotopy from $\theta \circ \eta \circ \varphi$ to $\omega \circ (\gamma \circ \alpha|)$.

Lemma 3.8. The composite map

$$\lambda(S_2;\Omega_2) \circ \lambda(S_1;\Omega_1) : X \cup_\varphi A \longrightarrow Z \cup_\omega E$$

is k–equivalent to $\lambda(S;\Omega) : X \cup_\varphi A \longrightarrow Z \cup_\omega E$.

Proof. Let $\xi : (C,D) \longrightarrow (A,B)$ be an H–homotopy inverse of α , and let $\zeta : (E,F) \longrightarrow (C,D)$ be an H–homotopy inverse of γ .

Proof. By Lemma 3.5 $(\overline{\xi})^{-1} \circ \overline{\eta}$ is k–equivalent to $\lambda(S_1;\Omega_1)$. Hence Lemma 3.3 a and b imply that $\overline{\theta} \circ (\overline{\xi})^{-1} \circ \overline{\eta}$ is k–equivalent to $k(\Omega_2) \circ \overline{\theta} \circ \lambda(S_1;\Omega_1)$. Applying Lemma 3.4 to these k–equivalent H–maps we conclude that $(\overline{\zeta})^{-1} \circ \overline{\theta} \circ (\overline{\xi})^{-1} \circ \overline{\eta}$ is k–equivalent to $\overline{\gamma} \circ k(\Omega_2) \circ \overline{\theta} \circ \lambda(S_1;\Omega_1) = \lambda(S_2;\Omega_2) \circ \lambda(S_1;\Omega_1)$. But $(\overline{\zeta})^{-1} \circ \overline{\theta} \circ (\overline{\xi})^{-1} \circ \overline{\eta} = (\overline{\xi \circ \zeta})^{-1} \circ \overline{\theta \circ \eta}$, by Lemma 2.10, and by Lemma 3.5 this map is k–equivalent to $\lambda(S;\Omega)$. Thus we have shown that $\lambda(S_2;\Omega_2) \circ \lambda(S_1;\Omega_1)$ is k–equivalent to $\lambda(S;\Omega)$.

□

4. Equivariant simple–homotopy type of adjunction spaces

Proposition 4.1. Let Y be a finite H–CW complex and (C,D) a finite H–CW pair. Suppose that $\mu_0,\mu_1 : D \longrightarrow Y$ are skeletal H–maps that are H–homotopic and that $\Lambda : D \times I \longrightarrow Y$ is an H–homotopy from μ_0 to μ_1. Then

$$k = k(\Lambda) : Y \cup_{\mu_0} C \longrightarrow Y \cup_{\mu_1} C$$

is a simple H–homotopy equivalence.

Proof. By the relative equivariant skeletal approximation theorem Λ is H–homotopic rel $D \times \dot{I}$ to a skeletal H–map $\tilde{\Lambda} : D \times I \longrightarrow Y$. Then $\tilde{\Lambda}$ is a skeletal H–homotopy from μ_0 to μ_1. By Lemma 2.2 $k(\Lambda)$ is H–homotopic (in fact rel Y) to $k(\tilde{\Lambda})$. Hence it is enough to prove that $k(\tilde{\Lambda})$ is a simple H–homotopy equivalence. The adjunction space $Y \cup_{\tilde{\Lambda}} (C \times I)$ is a finite H–CW complex. Since $C \times I$ H–collapses to $C \times \{0\} \cup D \times I$ (see [3, Corollary II. 1.10]) it follows, by [3, Lemma II. 1.6], that $Y \cup_{\tilde{\Lambda}} (C \times I)$ H–collapses to $Y \cup_{\tilde{\Lambda}} (C \times \{0\} \cup D \times I) = Y \cup_{\mu_0} C$. In the same way we see that $Y \cup_{\tilde{\Lambda}} (C \times I)$ H–collapses to $Y \cup_{\mu_1} C$. Hence, in the composite map

$$k(\tilde{\Lambda}) : Y \cup_{\mu_0} C \xrightarrow{\ \bar{i}_0\ } Y \cup_{\tilde{\Lambda}} (C \times I) \xrightarrow{\ \bar{r}_1\ } Y \cup_{\mu_1} C$$

the inclusion $\bar{i}_0$ is an H–expansion and $\bar{r}_1$ is an H–collapse. Thus $k(\tilde{\Lambda})$ is a formal

H–deformation (rel Y) and in particular $k(\tilde{\Lambda})$ is a simple H–homotopy equivalence.

□

Corollary 4.2. Let Y and (C,D) and $\mu_0, \mu_1 : D \longrightarrow Y$ be as in Proposition 4.1. Suppose that $f_0 : U \longrightarrow Y \cup_{\mu_0} C$ and $f_1 : U \longrightarrow Y \cup_{\mu_1} C$ are k–equivalent H–maps . Then f_0 and f_1 are s–equivalent.

□

Propositions 4.3 and 4.4 below have proofs that are very similar to each other. Of these the proof of Proposition 4.4 is the more enlightening one, and also the somewhat more complicated one. Hence we give the proof of Proposition 4.4 in detail and leave the proof of Proposition 4.3 to the reader.

Proposition 4.3. Let Y and Y′ be finite H–CW complexes and let (C,D) be a finite H–CW pair, and let $\mu : D \longrightarrow Y$ be a skeletal H–map . Suppose that $\sigma : Y \longrightarrow Y'$ is a skeletal simple H–homotopy equivalence. Then its canonical extension $\bar{\sigma} : Y \cup_{\mu} C \longrightarrow Y' \cup_{\sigma\mu} C$ is a simple H–homotopy equivalence.

Proposition 4.4. Let Y be a finite H–CW complex, and let (C′,D′) and (C,D) be finite H–CW pairs, and let $\mu : D \longrightarrow Y$ be a skeletal H–map . Suppose that $\gamma : (C',D') \longrightarrow (C,D)$ is a skeletal H–map such that $\gamma : C' \longrightarrow C$ and $\gamma| : D' \longrightarrow D$ are simple H–homotopy equivalences. Then $\bar{\gamma} : Y \cup_{\mu\gamma|} C' \longrightarrow Y \cup_{\mu} C$ is a simple H–homotopy equivalence.

Proof of Proposition 4.4. We consider the diagram

$$\begin{array}{ccc}
Y \cup_{(\mu\gamma|)_{\{1\}}} (C' \times \{0\} \cup D' \times I) & \xrightarrow{\overline{\gamma \times \{0\} \cup \gamma| \times I}} & Y \cup_{\mu_{\{1\}}} (C \times \{0\} \cup D \times I) \\
{\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\
Y \cup_{(\mu\gamma|)_{\{1\}}} (C' \times I) & \xrightarrow{\overline{\gamma \times I}} & Y \cup_{\mu_{\{1\}}} (C \times I) \\
{\scriptstyle r'}\downarrow & & \downarrow{\scriptstyle r} \\
Y \cup_{\mu\gamma|} C' & \xrightarrow{\overline{\gamma}} & Y \cup_{\mu} C
\end{array} .$$

Here $\mu_{\{1\}} : D \times \{1\} \longrightarrow Y$, is given by $\mu_{\{1\}}(d,1) = \mu(d)$ for all $d \in D$, and $(\mu\gamma|)_{\{1\}} : D' \times \{1\} \longrightarrow Y$ is defined similarly.

The inclusions $C \times \{0\} \cup D \times I \hookrightarrow C \times I$ and $C' \times \{0\} \cup D' \times I \hookrightarrow D' \times I$ are H–expansions, see [3, Corollary II. 1.10]. Hence it follows by [3, Lemma II. 1.6] that the inclusions i and i' in the above diagram are H–expansions. The upper square of the above diagram clearly commutes.

In the lower square of the above diagram r and r' denote the retractions induced by the standard projections $C \times I \longrightarrow C \times \{1\} = C$ and $C' \times I \longrightarrow C' \times \{1\} = C'$, respectively. With this choice of retractions r and r' the lower square clearly commutes. Since $C \times I$ H–collapses to $C \times \{1\}$, see [3, Corollary II. 1.10], it follows by [3, Lemma II. 1.6] that $Y \cup_{\mu_{\{1\}}} (C \times I)$ H–collapses to $Y \cup_\mu C$. Therefore any H–retraction from $Y \cup_{\mu_{\{1\}}} (C \times I)$ onto $Y \cup_\mu C$ is a simple H–homotopy equivalence. In particular r is a simple H–homotopy equivalence, and the same argument shows that r' is a simple H–homotopy equivalence.

Thus we have shown that the maps $r \circ i$ and $r' \circ i'$ are simple H–homotopy equivalences. Therefore, in order to prove that $\bar{\gamma}$ is a simple H–homotopy equivalence, it is enough to show that the map $\tilde{\gamma} = \overline{\gamma \times \{0\} \cup \gamma| \times I}$, at the top of the diagram, is a simple H–homotopy equivalence. This we do in the following way.

In order to simplify the notation we denote

$$W' = Y \cup_{(\mu\gamma|)_{\{1\}}} (C' \times \{0\} \cup D' \times I)$$

and

$$W = Y \cup_{\mu_{\{1\}}} (C \times \{0\} \cup D \times I) .$$

We shall prove that

$$\tilde{\gamma} = \overline{\gamma \times \{0\} \cup \gamma| \times I} : W' \longrightarrow W$$

is a simple H–homotopy equivalence. Let W^* denote the H–equivariant subdivision of W obtained by subdividing the unit interval $I = [0,1]$ at the point 1/2. Then

$$W_0^* = C \times \{0\} \cup D \times [0,1/2]$$

and

$$W_1^* = Y \cup_{\mu_{\{1\}}} (D \times [1/2,1])$$

are H–subcomplexes of W^* . Furthermore we have that

$$W_0^* \cup W_1^* = W^*$$

and

$$W_0^* \cap W_1^* = D \times \{1/2\} .$$

In complete analogy with the above we define an H–equivariant subdivision W'^* of W' , and obtain H–subcomplexes $W_0'^*$ and $W_1'^*$ of W'^* such that

$$W_0'^* \cup W_1'^* = W'^*$$

and

$$W_0'^* \cap W_1'^* = D' \times \{1/2\} .$$

We claim that $\tilde{\gamma}| : W_0'^* \longrightarrow W_0^*$ is a simple H–homotopy equivalence. In order to see this we consider the commutative diagram

$$\begin{array}{ccc} C' \times \{0\} & \xrightarrow{\gamma \times \{0\}} & C \times \{0\} \\ {\scriptstyle r_0'} \uparrow & & \uparrow {\scriptstyle r_0} \\ W_0'^* & \xrightarrow{\tilde{\gamma}|} & W_0^* \end{array}$$

where r_0 denotes the restriction induced by the standard projection $D \times [0,1/2] \longrightarrow D \times \{0\}$ and r_0' is defined similarly. Then r_0 and r_0' are simple H–homotopy equivalences, and since $\gamma : C' \longrightarrow C$ is a simple H–homotopy equivalence, by assumption, the claim follows.

The map $\tilde{\gamma}| : W_1'^* \longrightarrow W_1^*$ is also a simple H–homotopy equivalence. This follows from the fact that in the commutative diagram

$$\begin{array}{ccc} W_1'^* & \xrightarrow{\tilde{\gamma}|} & W_1^* \\ {\scriptstyle r_1'}\downarrow & & \downarrow{\scriptstyle r_1} \\ Y & \xrightarrow{id} & Y \end{array}$$

the retractions r_1 and r_1', induced by the standard projections $D \times [1/2,1] \longrightarrow D \times \{1\} = D$ and $D' \times [1/2,1] \longrightarrow D' \times \{1\} = D'$, are simple H–homotopy equivalences.

Furthermore we have that $\tilde{\gamma}| : W_0'^* \cap W_1'^* \longrightarrow W_0^* \cap W_1^*$ is a simple H–homotopy equivalence since it equals $\gamma| \times \{1/2\} : D' \times \{1/2\} \longrightarrow D \times \{1/2\}$, which is a simple H–homotopy equivalence by assumption.

By the sum theorem for equivariant Whitehead torsion, see [3, Theorem II. 3.12], it now follows that $\tilde{\gamma} : W'^* \longrightarrow W^*$ is a simple H–homotopy equivalence. Furthermore, the G–equivariant subdivsion maps $j : W^* \longrightarrow W$ and $j' : W'^* \longrightarrow W'$ are simple G–homotopy equivalences by [6, Theorem 12.2]. Hence $\tilde{\gamma} : W' \longrightarrow W$ is a simple H–homotopy equivalence.

□

We shall now combine the results of this section into one theorem. In Theorem 4.5 below Y_i denote finite H–CW complexes, (C_i, D_i) denote finite H–CW pairs and $\mu_i : D_i \longrightarrow Y_i$ are skeletal H–maps , i = 1,2 .

Theorem 4.5. Suppose that

$$\text{(S)} \qquad \begin{array}{ccc} C_1 & \xrightarrow{\gamma} & C_2 \\ \cup & & \cup \\ D_1 & \xrightarrow{\gamma|} & D_2 \\ \mu_1 \downarrow & & \downarrow \mu_2 \\ Y_1 & \xrightarrow{\sigma} & Y_2 \end{array}$$

is an H–homotopy commutative diagram, where $\gamma : C_1 \longrightarrow C_2$, $\gamma| : D_1 \longrightarrow D_2$ and $\sigma : Y_1 \longrightarrow Y_2$ are simple H–homotopy equivalences. Then

$$\lambda(S) : Y_1 \cup_{\mu_1} C_1 \longrightarrow Y_2 \cup_{\mu_2} C_2$$

is a simple H–homotopy equivalence.

Proof. By Corollary 3.7, Corollary 4.2 and the equivariant skeletal approximation theorem, we can assume that γ and σ are skeletal h–maps. Let $\Omega : D_1 \times I \longrightarrow Y_2$ be an H–homotopy from $\sigma \circ \mu_1$ to $\mu_2 \circ \gamma|$. Then

$$\lambda(S) : Y_1 \cup_{\mu_1} C_1 \xrightarrow{\bar{\sigma}} Y_2 \cup_{\sigma\mu_1} C_1 \xrightarrow{k(\Omega)} Y_2 \cup_{\mu_2\gamma|} C_1 \xrightarrow{\bar{\gamma}} Y_2 \cup_{\mu_2} C_2$$

is a simple H–homotopy equivalence, since $\overline{\sigma}$, $k(\Omega)$ and $\overline{\gamma}$ are simple H–homotopy equivalences by Propositions 4.3, 4.1 and 4.4, respectively.

□

5. A conveniant lemma

In Lemma 5.1 below X denotes an H–space, (A,B) is an H–pair, which has the H–homotopy extension property and where B is closed in A, and $\psi : B \longrightarrow X$ is an H–map. Furthermore, Y_i are finite H–CW complexes, (C_i,D_i) are finite H–CW pairs and $\mu_i : D_i \longrightarrow Y_i$ denote skeletal H–maps, $i = 1,2$.

Lemma 5.1. Suppose that

$$(S_i) \qquad \begin{array}{ccc} A & \xrightarrow{\alpha_i} & C_i \\ \cup & & \cup \\ B & \xrightarrow{\alpha_i|} & D_i \\ \psi\downarrow & & \downarrow\mu_i \\ X & \xrightarrow{\theta_i} & Y_i \end{array} \qquad i = 1,2$$

are two H–homotopy commutative diagrams such that $\theta_i : X \longrightarrow Y_i$, $i = 1,2$, are s–equivalent H–homotopy equivalences and $\alpha_i : (A,B) \longrightarrow (C_i,D_i)$, $i = 1,2$, are s–equivalent, as maps of pairs, H–homotopy equivalences of pairs. Then the λ–maps $\lambda(S_i) : X \cup_\psi A \longrightarrow Y_i \cup_{\mu_i} C_i$, $i = 1,2$, induced by the diagrams (S_1) and (S_2), are s–equivalent H–homotopy equivalences. In fact, if $\sigma : Y_1 \longrightarrow Y_2$ is a simple H–homotopy equivalence such that $\sigma \circ \theta_1$ is H–homotopic to θ_2, then, there exists a simple H–homotopy equivalence $\Sigma : Y_1 \cup_{\mu_1} C_1 \longrightarrow Y_2 \cup_{\mu_2} C_2$ which extends σ and $\Sigma \circ \lambda(S_1)$ is H–homotopic to $\lambda(S_2)$.

Proof. Let $\sigma : Y_1 \longrightarrow Y_2$ be a simple H–homotopy equivalence such that $\sigma \circ \theta_1$ is H–homotopic to θ_2. Suppose that $\gamma : (C_1,D_1) \longrightarrow (C_2,D_2)$ is an H–map such that $\gamma : C_1 \longrightarrow C_2$ and $\gamma\,| : D_1 \longrightarrow D_2$ are simple H–homotopy equivalences, and $\gamma \circ \alpha_1$ is H–homotopic to α_2 as maps of pairs. By Corollary 3.7 we have that the diagram (S_2) and the diagram

$$(S_2^*) \qquad \begin{array}{ccc} A & \xrightarrow{\gamma \circ \alpha_1} & C_2 \\ \cup & & \cup \\ B & \xrightarrow{\gamma \circ \alpha_1 |} & D_2 \\ \psi \big\downarrow & & \big\downarrow \mu_2 \\ X & \xrightarrow{\sigma \circ \theta_1} & Y_2 \end{array}$$

induce λ–maps that are k–equivalent to each other. Furthermore we have by Lemma 3.8 that $\lambda(S_2^*) : X \cup_\psi A \longrightarrow Y_2 \cup_{\mu_2} D_2$ is k–equivalent to the composite map $\lambda(S) \circ \lambda(S_1)$, where (S) denotes the diagram

$$(S) \qquad \begin{array}{ccc} C_1 & \xrightarrow{\gamma} & C_2 \\ \cup & & \cup \\ D_1 & \xrightarrow{\gamma|} & D_2 \\ \mu_1 \big\downarrow & & \big\downarrow \mu_2 \\ Y_1 & \xrightarrow{\sigma} & Y_2 \end{array}$$

Thus $\lambda(S_2)$ is k–equivalent to $\lambda(S) \circ \lambda(S_1)$.

In particular we have by Corollary 4.2 that $\lambda(S_2)$ and $\lambda(S) \circ \lambda(S_1)$ are s–equivalent . Since $\lambda(S)$ is a simple H–homotopy equivalence by Theorem 4.5 it follows that $\lambda(S_1)$ and $\lambda(S_2)$ are s–equivalent .

□

6. Preferred H–reductions

Theorem 6.1. Let G be a compact Lie group and H a closed subgroup of G . Then, given a finite G–CW complex X , there exist a finite H–CW complex $R_H X$ and an H–homotopy equivalence

$$\eta : X \longrightarrow R_H X$$

such that this construction is unique up to a simple H–homotopy equivalence.

The last statement in Theorem 3.1 means that if one by some other choices in the construction arrives at the finite H–CW complex $R'_H X$ and the H–homotopy equivalence $\eta' : X \longrightarrow R'_H X$, then, η and η' are s–equivalent H–maps ; i.e., the map

$$\eta' \circ \eta^{-1} : R_H X \longrightarrow R'_H X$$

is a simple H–homotopy equivalence. (Here η^{-1} denotes an H–homotopy inverse of η .)

Proof. The proof is by induction on the number of G–cells in X . Assume first that X consists of one 0–dimensional G–cell , say $X = G/P$. By Theorem 1.3 there exists a distinguished H–triangulation $\gamma : C \xrightarrow{\cong} G/P$ of the H–manifold G/P . We define $R_H(G/P) = C$ and $\eta = \gamma^{-1} : G/P \longrightarrow R_H(G/P)$. Then $R_H(G/P)$ is a finite H–CW complex and η is an H–homeomorphism and hence in particular an H–homotopy equivalence. If $\gamma' : C' \longrightarrow G/P$ is another distinguished H–triangulation of G/P there exists by Theorem 1.3 a skeletal simple H–homotopy equivalence $\sigma : C \longrightarrow C'$ such that $\gamma \simeq \gamma' \circ \sigma$. Hence the H–maps η and η' , where

$\eta' = (\gamma')^{-1} : G/P \longrightarrow C' = R'_H(G/P)$, are s–equivalent.

We now assume by induction that X is a finite G–CW complex, which is obtained from a subcomplex X_0 by adjoining one G–cell , and that we have given a construction of a finite H–CW complex $R_H X_0$ and an H–homotopy equivalence $\theta : X_0 \longrightarrow R_H X_0$, such that the construction is unique up to a simple H–homotopy equivalence. Suppose that $X = X_0 \cup c$, where c is a G–cell , of say dimension n and type G/P . Let

$$\xi : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (c,\dot{c}) \hookrightarrow (X,X_0)$$

be a characteristic G–map for c and let

$$\psi = \xi | : S^{n-1} \times G/P \longrightarrow X_0$$

denote the corresponding attaching map. We choose a distinguished H–triangulation

$$\gamma : (C,D) \xrightarrow{\cong} (D^n \times G/P, S^{n-1} \times G/P)$$

of the H–pair $(D^n \times G/P, S^{n-1} \times G/P)$, see Theorem 1.3, and denote

$$\delta = \gamma | : D \longrightarrow S^{n-1} \times G/P .$$

We now consider the diagram

$$(*)\qquad \begin{array}{ccc} D^n \times G/P & \xrightarrow{\ \alpha\ } & C \\ \cup & & \cup \\ S^{n-1} \times G/P & \xrightarrow{\ \beta\ } & D \\ \psi\downarrow & & \downarrow\mu \\ X_0 & \xrightarrow{\ \theta\ } & R_H X_0 \end{array}$$

where $\alpha = \gamma^{-1}$ and $\beta = \delta^{-1}$, and hence $\beta = \alpha | S^{n-1} \times G/P$. Furthermore $\mu : D \longrightarrow R_H X_0$ is a skeletal H–map such that the diagram (*) is H–homotopy commutative; i.e., μ is a skeletal H–approximation of the H–map $\theta \circ \psi \circ \delta : D \longrightarrow R_H X_0$. We now define $\eta : X \longrightarrow R_H X$ to be the composite map

$$(1)\qquad \eta : X \xrightarrow{(\xi)^{-1}} X_0 \cup_\psi (D^n \times G/p) \xrightarrow{\lambda(*)} R_H X_0 \cup_\mu C := R_H X$$

where $\lambda(*)$ is the λ–map induced by the diagram (*). Then η is an H–homotopy equivalence and $\eta | X_0 = \theta : X_0 \longrightarrow R_H X_0$. Furthermore we know by Corollary 3.6 that $\lambda(*)$ is uniquely determined up to k–equivalence by the diagram (*). The adjunction space $R_H X = R_H X_0 \cup_\mu C$ is a finite H–CW complex since $R_H X_0$ is a finite H–CW complex and (C,D) is a finite H–CW pair and $\mu : D \longrightarrow R_H X_0$ is a skeletal H–map.

If we are at the inductive level choose $\theta' : X_0 \longrightarrow R'_H X_0$, then we have by the inductive assumption that the H–homotopy equivalences $\theta : X_0 \longrightarrow R_H X_0$ and $\theta' : X_0 \longrightarrow R'_H X_0$ are s–equivalent. In case we choose another distinguished H–triangulation $\gamma' : (C',D') \longrightarrow (D^n \times G/P, S^{n-1} \times G/P)$ of the H–pair $(D^n \times G/P, S^{n-1} \times G/P)$ then we have by Theorem 1.3 and Lemma 1.4 that the H–homotopy equivalences (in fact H–homeomorphisms)

$\alpha : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (C,D)$ and
$\alpha' = (\gamma')^{-1} : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (C',D')$ are s–equivalent as maps of pairs. Now consider the H–homotopy commutative diagram

$$(*') \qquad \begin{array}{ccc} D^n \times G/P & \xrightarrow{\alpha'} & C' \\ \cup & & \cup \\ S^{n-1} \times G/P & \xrightarrow{\alpha'|} & D' \\ \psi \downarrow & & \downarrow \mu' \\ X_0 & \xrightarrow{\theta'} & R'_H X_0 \end{array}$$

where μ' is a skeletal H–approximation of the H–map $\theta' \circ \psi \circ (\gamma' |) : D' \longrightarrow R'_H X_0$. By Lemma 5.1 any λ–map

$$\lambda(*') : X \cup_{\psi} (D^n \times G/P) \longrightarrow R'_H X_0 \cup_{\mu'} C' = R'_H X$$

induced by $(*')$ is s–equivalent to $\lambda(*)$.

It remains to prove that if we choose another characteristic G–map $\xi' : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (c,\dot{c}) \hookrightarrow (X,X_0)$ and use this one instead in forming the map in (1) then the resulting map is s–equivalent to the one in (1). This follows from Lemma 1.1 by arguments analogous to the one above.

The choice of a G–subcomplex X_0 of X such that X is obtained from X_0 by adjoining one G–cell corresponds to the choice of one specific filtration of X by an increasing sequence of G–subcomplexes each obtained from the preceeding one by adjoining one G–cell . The construction of $\eta : X \longrightarrow R_H X$ is independent up to simple

H–homotopy equivalence of this choice of filtration, and we leave the verification of this fact to the reader. This completes the proof of Theorem 6.1.

□

It follows from the construction of the finite H–CW complex $R_H X$ that it also satisfies the following properties

(i) $\dim(R_H X)^K = \dim X^K$, for every $K < G$.

(ii) The H–isotropy types occurring in $R_H X$ and in X are exactly the same.

Compare with Theorem A in [8].

Definition 6.2. Let X be a finite G–CW complex. A preferred H–reduction of X consists of a finite H–CW complex Y and an H–homotopy equivalence

$$\theta : X \longrightarrow Y$$

such that θ is s–equivalent to an H–homotopy equivalence $\eta : X \longrightarrow R_H X$ constructed in Theorem 6.1 and Y satisfies conditions (i) and (ii) above.

With this terminology we can say that Theorem 6.1 proves the existence of preferred H–reductions for any finite G–CW complex X. Observe that the construction given in Theorem 6.1 is such that if (V,X) is a finite G–CW pair and $\eta : X \longrightarrow R_H X$ is any preferred H–reduction of X then there exists a preferred H–reduction $\tilde{\eta} : V \longrightarrow R_H V$ of V such that $R_H X \subset R_H V$ and $\tilde{\eta} | X = \eta$, and hence we obtain a preferred H–reduction $\tilde{\eta} : (V,X) \longrightarrow (R_H V, R_H X)$ of the G–CW pair (V,X).

The following Lemma is easy to prove, and we leave the details to the reader.

Lemma 6.3. Let $\theta : X \longrightarrow Y$ be a preferred H–reduction of the finite G–CW complex X, and let L be an ordinary finite CW complex. Then $id \times \theta : L \times X \longrightarrow L \times Y$ is a preferred H–reduction of the finite G–CW complex $L \times X$.

7. A key property of preferred H–reductions

Let X be a finite G–CW complex and let (A,B) be a finite G–CW pair, and assume that $\varphi : B \longrightarrow X$ is an (A,B)–skeletal G–map . Then $X \cup_\varphi A$ is a finite G–CW complex.

Suppose that

$$\eta : X \longrightarrow R_H X$$

is a preferred H–reduction of X , and that

$$\theta : (A,B) \longrightarrow (R_H A, R_H B)$$

is a preferred H–reduction of (A,B) . Let $\mu : R_H B \longrightarrow R_H X$ be a skeletal H–approximation of the H–map $\eta \circ \varphi \circ (\theta|)^{-1}$.

Proposition 7.1. The H–map

$$\lambda(\eta,\theta) : X \cup_\varphi A \longrightarrow R_H X \cup_\mu R_H A$$

is a preferred H–reduction of the finite G–CW complex $X \cup_\varphi A$.

Proof. The proof is by induction on the number of G–cells in $A-B$. Let A_0 be a G–subcomplex of A such that $B \subset A_0$ and such that $A-A_0$ consists of one G–cell c , say of dimension n and type G/P . Let

$$\xi : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (c,\dot{c}) \hookrightarrow (A,A_0)$$

be a characteristic G–map for c , and let

$$\psi = \xi| : S^{n-1} \times G/P \longrightarrow A_0$$

be the corresponding attaching G–map for c . Then

$$\bar{\xi} : A_0 \cup_\psi (D^n \times G/P) \xrightarrow{\cong} A$$

is a G–homeomorphism and $\bar{\xi}|A_0 = \mathrm{id}_{A_0}$.

Next we note the following. If $\theta : (A,B) \longrightarrow (R_H A, R_H B)$ and $\theta' : (A,B) \longrightarrow (R_H' A, R_H' B)$ are preferred H–reductions of (A,B) , then it follows by Lemma 5.1 that the H–maps $\lambda(\eta,\theta) : X \cup_\varphi A \longrightarrow R_H X \cup_\mu R_H A$ and $\lambda(\eta,\theta') : X \cup_\varphi A \longrightarrow R_H X \cup_{\mu'} R_H' A$ are s–equivalent, and hence if one of them is a preferred H–reduction of $X \cup_\varphi A$ then so is also the other one. Thus in order to prove Proposition 7.1 it is enough to find an appropriate H–reduction $\theta : (A,B) \longrightarrow (R_H A, R_H B)$ for which we can show that $\lambda(\eta,\theta) : X \cup_\varphi A \longrightarrow R_H X \cup_\mu R_H A$ is a preferred H–reduction of $X \cup_\varphi A$.

So let us first exhibit a preferred H–reduction of (A,B) , with which it will be convenient to work. Let

$$\theta| : B \longrightarrow R_H B$$

be a preferred H–reduction of B and extend this to a preferred H–reduction

$$\theta_0 : A_0 \longrightarrow R_H A_0$$

where $R_H B \subset R_H A_0$. Let $u : F \longrightarrow G/P$ be a distinguished H–triangulation of the H–manifold G/P , and let

$$\kappa = \mathrm{id} \times u^{-1} : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (C,D)$$

be the corresponding preferred H–reduction of $(D^n \times G/P, S^{n-1} \times G/P)$, where we have denoted $(C,D) = (D^n \times F, S^{n-1} \times F)$, see Lemma 6.3. In order to shorten the notation we denote in the following

$$(M,N) = (D^n \times G/P, S^{n-1} \times G/P) .$$

The H–map

(1) $$\theta : A \xrightarrow[\cong]{(\xi)^{-1}} A_0 \cup_\psi M \xrightarrow{\bar\theta_0} R_H A_0 \cup_{\theta_0 \psi} M \xrightarrow{\bar\kappa} R_H A_0 \cup_\omega C \xrightarrow{k'} R_H A_0 \cup_{\tilde\omega} C$$

where $\omega = (\theta|) \circ \psi \circ (\kappa|)^{-1}$ and $\tilde\omega$ is a skeletal H–approximation of ω , is a preferred H–reduction of A . Observe that $\theta | A_0 = \theta_0$ and $\theta | B = \theta|$.

We shall show that with θ as in (1) the map

(2) $$\lambda(\eta,\theta) : X \cup_\varphi A \longrightarrow R_H X \cup_\mu (R_H A_0 \cup_{\tilde\omega} C)$$

is a preferred H–reduction of $X \cup_\varphi A$. This map is the composite map

(3)

$$\lambda(\eta,\theta) : X \cup_\varphi A \xrightarrow{\bar{\eta}} R_H X \cup_{\eta\varphi} A \xrightarrow{k(\Omega)} R_H X \cup_{\mu\theta|} A \xrightarrow{\bar{\theta}} R_H X \cup_\mu (R_H A \cup_{\tilde{\omega}} C) .$$

We will prove that $\lambda(\eta,\theta)$ in (3) is a preferred H–reduction of $X \cup_\varphi A$, by comparing this map with another H–map which we know is a preferred H–reduction .

We now consider the H–map

(4) $$\lambda_0 = \lambda(\eta,\theta_0) : X \cup_\varphi A_0 \longrightarrow R_H X \cup_\mu R_H A_0 .$$

In case A–B consists of one G–cell ; i.e., when $A_0 = B$, the map λ_0 in (4) equals $\eta : X \longrightarrow R_H X$, a preferred H–reduction of X . In case $B \subsetneq A_0$ we have that the number of G–cells in A_0–B is one less than in A–B , and hence we will have, by the inductive assumption that Proposition 7.1 holds in this case, that λ_0 is a preferred H–reduction of $X \cup_\varphi A_0$.

We shall now extend λ_0 to a preferred H–reduction of $X \cup_\varphi A$. We have that $X \cup_\varphi A$ is obtained from $X \cup_\varphi A_0$ by adjoining a G–cell of dimension n and type G/P , with characteristic G–map

$$\xi : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (A,A_0) \longrightarrow (X \cup_\varphi A, X \cup_\varphi A_0)$$

and corresponding attaching G–map equal to

$$\psi : S^{n-1} \times G/P \longrightarrow A_0 \longrightarrow X \cup_\varphi A_0 .$$

Then

$$\xi : (X \cup_\varphi A_0) \cup_\psi (D^n \times G/P) \xrightarrow{\cong} X \cup_\varphi A$$

is a G–homeomorphism. In this situation we know that the composite map

$$\sigma : X \cup_\varphi A \xrightarrow[\cong]{(\xi)^{-1}} (X \cup_\varphi A_0) \cup_\psi M \xrightarrow{\bar{\lambda}_0} (R_H X \cup_\mu R_H A_0) \cup_{\lambda_0 \psi} M$$

(5)

$$\xrightarrow{\bar{\kappa}} (R_H X \cup_\mu R_H A_0) \cup_\nu C \xrightarrow{k'} (R_H X \cup_\mu R_H A_0) \cup_{\tilde{\nu}} C$$

where $\nu = \lambda_0 \circ \psi \circ (\kappa|)^{-1}$ and $\tilde{\nu}$ is a skeletal H–approximation of ν, is a preferred H–reduction of $X \cup_\varphi A$.

Now inserting θ from (1) into the H–map $\lambda(\eta,\theta)$ in (3) and inserting the map λ_0 from (4) into the H–map σ in (5) one checks that $\lambda(\eta,\theta)$ and σ are k–equivalent and hence s–equivalent . In the case where $A_0 = B$ this establishes the start of the induction, namely that $\lambda(\eta,\theta)$ is a preferred H–reduction of $X \cup_\varphi A$ in the case when A–B consists of one G–cell . This having been established we have by the inductive assumption that λ_0 in (4) is a preferred H–induction of $X \cup_\varphi A_0$, and the above conclusion then gives that $\lambda(\eta,\theta)$ is a preferred H–reduction of $X \cup_\varphi A$ since $\lambda(\eta,\theta)$ is s–equivalent to σ in (5) which is a preferred H–induction of $X \cup_\varphi A$. □

8. Definition and basic properties of the restriction homomorphism $\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X)$.

Lemma 8.1. Suppose that (X,X_0) is a finite G–CW pair such that X_0 is an elementary G–collapse of X. Then $R_H X$ H–collapses to $R_H X_0$.

Proof. The assumption that X_0 is an elementary G–collapse of X means that

$$X = X_0 \cup_\varphi (I^n \times G/P)$$

where $P < G$ and $\varphi : J^{n-1} \times G/P \longrightarrow X_0$ is a G–map such that $\varphi(J^{n-1} \times G/P) \subset X_0^{n-1}$ and $\varphi(\dot{I}^{n-1} \times G/P) \subset X_0^{n-2}$, see [3, p. 13]. In the terminology of Definition 1.2 the conditions on the map φ mean that $\varphi : J^{n-1} \times G/P \longrightarrow X_0$ is a $(I^n \times G/P, J^{n-1} \times G/P)$ – skeletal G–map.

Let $w : F \longrightarrow G/P$ be a distinguished H–triangulation of the H–manifold G/P. Then $\zeta = w^{-1} : G/P \longrightarrow F$ is a preferred H–reduction of the 0–dimensional G–cell G/P. By Lemma 6.3

$$\alpha : \mathrm{id} \times \zeta : I^n \times G/P \longrightarrow I^n \times F$$

and

$$\beta = \mathrm{id} \times \zeta : J^{n-1} \times G/P \longrightarrow J^{n-1} \times F$$

are preferred H–reductions of $I^n \times G/P$ and $J^{n-1} \times G/P$, respectively. We denote $C = I^n \times F$ and $D = J^{n-1} \times F$.

Let $\theta : X_0 \longrightarrow R_H X_0$ be a preferred H–reduction of X_0. We consider the H–homotopy commutative diagram

$$\begin{array}{ccc}
I^n \times G/P & \xrightarrow{\alpha} & C \\
\cup & & \cup \\
J^{n-1} \times G/P & \xrightarrow{\beta} & D \\
\varphi \downarrow & & \downarrow \mu \\
X_0 & \xrightarrow{\theta} & R_H X_0
\end{array}$$

where μ is a skeletal H–approximation of the H–map $\theta \circ \varphi \circ \beta^{-1} : D \longrightarrow R_H X_0$. By Proposition 7.1 the λ–map

$$\lambda(S) : X_0 \cup_\varphi (I^n \times G/P) \longrightarrow R_H X_0 \cup_\mu C$$

induced by the diagram (S) is a preferred H–reduction of $X = X_0 \cup_\varphi (I^n \times G/P)$. Thus we may take

$$(R_H X, R_H X_0) = (R_H X_0 \cup_\mu C, R_H X_0).$$

Since I^n collapses to J^{n-1} (in fact by one elementary collapse) it follows by Lemma 1.5 that $C = I^n \times F$ H–collapses to $D = J^{n-1} \times F$. Hence we have by [3, Lemma II. 1.6] that $R_H X = R_H X_0 \cup_\mu C$ H–collapses to $R_H X_0 \cup_\mu D = R_H X_0$.

□

Corollary 8.2. Suppose that (V,X) and (U,X) are finite G–CW pairs such that there is a formal G–deformation rel X from V to U. Then there is a formal H–deformation rel R_HX from R_HV to R_HU.

□

As a consequence of the above corollary we obtain a well–defined map

$$\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(R_HX)$$

by defining

$$\mathrm{Res}_H^G(s_G(V,X)) = s_H(R_HV,R_HX)$$

for every $s_G(V,X) \in \mathrm{Wh}_G(X)$. The fact that this gives a well–defined map is an immediate consequence of Corollary 8.2. If $s_G(V,X) = s_G(U,X) \in \mathrm{Wh}_G(X)$ then there is a formal G–deformation rel X from V to U, and hence by Corollary 8.2 there is a formal H–deformation rel R_HX from R_HV to R_HU which means that $s_H(R_HV,R_HX) = s_H(R_HU,R_HX) \in \mathrm{Wh}_H(R_HX)$.

Lemma 8.3. The map $\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(R_HX)$ is a homomorphism.

Proof. Let $s_G(V_1,X) \in \mathrm{Wh}_G(X)$ and $s_G(V_2,X) \in \mathrm{Wh}_G(X)$. Then

$$s_G(V_1,X) + s_G(V_2,X) = s_G(V_1 \cup_X V_2,X).$$

The space $V_1 \cup_X V_2$; i.e., then union of V_1 and V_2 along X , can also be considered as the adjunction space obtained by adjoining V_2 to V_1 by the inclusion map $i : X \longrightarrow V_1$. Therefore we have by Proposition 7.1 that

$$R_H(V_1 \cup_X V_2) = R_H V_1 \cup_{R_H X} R_H V_2 .$$

Thus we obtain

$$\mathrm{Res}_H^G(s_G(V_1,X) + s_G(V_2,X)) = \mathrm{Res}_H^G(s_G(V_1 \cup_X V_2,X))$$

$$= s_H(R_H(V_1 \cup_X V_2),R_H X) = s_H(R_H V_1 \cup_{R_H X} R_H V_2,R_H X)$$

$$= s_H(R_H V_1,R_H X) + s_H(R_H V_2,R_H X)$$

$$= \mathrm{Res}_H^G(s_G(V_1,X)) + \mathrm{Res}_H^G(s_G(V_2,X)) \in \mathrm{Wh}_H(R_H X) .$$

□

We shall investigate the fact that $\mathrm{Wh}_H(R_H X)$ is independent of the choice of preferred H–reduction $\eta : X \longrightarrow R_H X$. This will give us a group $\mathrm{Wh}_H(X)$ and a homomorphism $\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X)$.

Let

$$\eta : X \longrightarrow R_H X$$

and

$$\eta' : X \longrightarrow R'_H X$$

be preferred H–reductions of X . Then

$$\sigma = \eta' \circ \eta^{-1} : R_H X \longrightarrow R'_H X$$

is a simple H–homotopy equivalence. In particular σ is an H–homotopy equivalence and hence induces an isomorphism

$$\sigma_* : Wh_H(R_H X) \xrightarrow{\cong} Wh_H(R'_H X) .$$

Let (V,X) be a finite G–CW pair, where X is a strong deformation retract of V , representing the element $s_G(V,X) \in Wh_G(X)$. Let

$$\tilde{\eta} : (V,X) \longrightarrow (R_H V, R_H X)$$

be a preferred H–reduction of (V,X) extending the given preferred H–reduction η of X . Let

$$\tilde{\eta}' : (V,X) \longrightarrow (R'_H V, R'_H X)$$

be a preferred H–reduction of (V,X) extending the given preferred H–reduction η' of X . Then we have, by the definition above, that

$$\mathrm{Res}^G_H(s_G(V,X)) = s_H(R_H V, R_H X) \in Wh_H(R_H X)$$

and

$$\mathrm{Res}'^{G}_{H}(s_G(V,X)) = s_H(R'_H, R'_H X) \in \mathrm{Wh}_H(R'_H X)\ .$$

We now have a commutative diagram

$$\begin{array}{ccc} R_H X & \xrightarrow{\ \sigma\ } & R'_H X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ R_H V & \xrightarrow{\ \Sigma\ } & R'_H V \end{array}$$

where σ and Σ are simple H–homotopy equivalences. We now compute the H–equivariant Whitehead torsion of the composite map $\Sigma \circ i = i' \circ \sigma$. By the formula for the equivariant Whitehead torsion of a composite map, [3, Proposition II. 3.8], we obtain

$$\tau(\Sigma \circ i) = \tau(i) + i_*^{-1}\tau(\Sigma) \in \mathrm{Wh}_H(R_H X)$$

and

$$\tau(i' \circ \sigma) = \tau(\sigma) + \sigma_*^{-1}\tau(i') \in \mathrm{Wh}_H(R_H X)\ .$$

Thus

$$\tau(i) + i_*^{-1}\tau(\Sigma) = \tau(\sigma) + \sigma_*^{-1}\tau(i^1) \in \mathrm{Wh}_H(R_H X)\ .$$

Since σ and Σ are simple H–homotopy equivalences we have $\tau(\Sigma) = \tau(\sigma) = 0$ and hence

$$\tau(i) = \sigma_*^{-1}\tau(i') \in Wh_H(R_H X)$$

i.e.,

$$\tau(i') = \sigma_*\tau(i) \in Wh_H(R'_H X) .$$

But by [3, Lemma II. 3.11] we have

$$\tau(i) = s_H(R_H V, R_H X) \in Wh_H(R_H X)$$

and

$$\tau(i') = s_H(R'_H V, R'_H X) \in Wh_H(R'_X X) .$$

Thus the above formula reads

$$s_H(R'_H V, R'_H X) = \sigma_* s_H(R_H V, R_H X)$$

i.e.,

$$\mathrm{Res}'^G_H(s_G(V,X)) = \sigma_* \mathrm{Res}^G_H(s_G(V,X)) .$$

This shows that the diagram

$$\begin{array}{ccc} & \nearrow^{\mathrm{Res}_H^G} & \mathrm{Wh}_H(R_H X) \\ \mathrm{Wh}_G(X) & & \downarrow \sigma_* \\ & \searrow_{\mathrm{Res}'{}_H^G} & \mathrm{Wh}_H(R'_H X) \end{array}$$

commutes. Hence we may identify $\mathrm{Wh}_H(R_H X)$ with $\mathrm{Wh}_H(R'_H X)$ through σ_*, and call this group $\mathrm{Wh}_H(X)$, and then the restriction homomorphisms Res_H^G and $\mathrm{Res}'{}_H^G$, determine the same homomorphism

$$\mathrm{Res}_H^G : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}_H(X) .$$

To conclude this paper we will investigate the behaviour of the equivariant Whitehead torsion under the restriction homomorphism. Suppose that $f : X \longrightarrow X_1$ is a G–homotopy equivalence between finite G–CW complexes. Let $\eta : X \longrightarrow R_H X$ and $\eta_1 : X_1 \longrightarrow R_H X_1$ be preferred H–reductions of X and X_1, repectively. Let $R_H f : R_H X \longrightarrow R_H X_1$ be the H–homotopy equivalence induced by f, i.e., we have the H–homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X_1 \\ \eta \downarrow & & \downarrow \eta_1 \\ R_H X & \xrightarrow{\quad R_H f \quad} & R_H X_1 \end{array}$$

and $R_H f = \eta_1 \circ f \circ \eta^{-1}$, where η^{-1} is some H–homotopy inverse of η. We may assume that $R_H f$ is skeletal.

We claim that the H–equivariant Whitehead torsion of $R_H f$ is given by

$$\tau(R_H f) = \mathrm{Res}_H^G(\tau(f)) \in Wh_H(R_H X) .$$

We may assume that f is skeletal and the G–equivariant Whitehead torsion of f is then given by

$$\tau(f) = s_G(M_f, X) \in Wh_G(X)$$

where M_f denotes the mapping cylinder of X , see [3, page 23 and Proposition II. 3.5]. Similarly we have that

$$\tau(R_H f) = s_H(M_{R_H f}, R_H X) \in Wh_H(R_H X) .$$

By Lemma 6.3 we have that $\alpha = \eta \times \mathrm{id} : X \times I \longrightarrow R_H X \times I$ is a preferred H–reduction of X × I . We now consider the H–homotopy commutative diagram

$$(S) \qquad \begin{array}{ccc} X \times I & \xrightarrow{\alpha} & R_H X \times I \\ \cup & & \cup \\ X \times \{1\} & \xrightarrow{\alpha|} & R_H X \times \{1\} \\ {\scriptstyle f_{\{1\}}}\downarrow & & \downarrow {\scriptstyle (R_H f)_{\{1\}}} \\ X_1 & \xrightarrow{\eta_1} & R_H X_1 \end{array}$$

By Proposition 7.1 we have that

$$\lambda(S) : X_1 \cup_{f_{\{1\}}} (X \times I) \longrightarrow R_H X_1 \cup_{(R_H f)_{\{1\}}} (R_H X \times I)$$

is a preferred H–reduction of $M_f = X_1 \cup_{f_{\{1\}}} (X \times I)$; i.e.

$$\lambda(S) : M_f \longrightarrow M_{R_H f}$$

is a preferred H–reduction of M_f. This means that we have

$$R_H(M_f) \equiv M_{R_H f} .$$

Thus we obtain

$$\mathrm{Res}^G_H(\tau(f)) = \mathrm{Res}^G_H(s_G(M_f,X)) = s_H(R_H(M_f),R_HX)$$

$$= s_H(M_{R_H f},R_HX) = \tau(R_Hf) \in \mathrm{Wh}_H(R_HX) .$$

We may write this as

$$\tau(R_Hf) = \mathrm{Res}^G_H(\tau(f)) \in \mathrm{Wh}_H(X) .$$

In particular it follows from this formula that if $f : X \longrightarrow X_1$ is a simple G–homotopy equivalence then the induced H–homotopy equivalence $R_Hf : R_HX \longrightarrow R_HX_1$ is also a simple H–homotopy equivalence.

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