JOINT REDUCTIONS OF MONOMIAL IDEALS AND MULTIPLICITY OF COMPLEX ANALYTIC MAPS

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Abstract. We characterize the joint reductions of a set of monomial ideals in the ring $\mathcal{O}_n$ of complex analytic functions defined in a neighbourhood of the origin in $\mathbb{C}^n$. We also define an integer $\sigma(I_1,\dots,I_n)$ attached to a family of ideals $I_1,\dots,I_n$ in a Noetherian local ring that extends the usual notion of mixed multiplicity. If $I_1,\dots,I_n$ are monomial ideals, then we also obtain a characterization of the families $g_1,\dots,g_n$ such that $g_i\in I_i$, for all $i=1,\dots,n$, and that $e(g_1,\dots,g_n)=\sigma(I_1,\dots,I_n)$.

1. Introduction

The computation of the integral closure of ideals is one of the central problems in commutative algebra (see [5], [9] or [26]). A key role in the context of this problem is played by the reductions of an ideal, which were defined by Northcott and Rees in [14] (see Section 2). These ideals are very useful in the computation of multiplicities of ideals. For instance, if I is an ideal of $\mathbb{C}[[x_1,\dots,x_n]]$ of finite colength generated by monomials, then the author obtained in [3] a canonical reduction of I that allowed to compute the multiplicity of I in an effective way (we refer [6] for a different approach to the computation of the multiplicity of a monomial ideal).

The notion of reduction of an ideal was generalized by Rees in [17] thus giving the notion of joint reduction of ideals. This notion simplifies the task of computing the mixed multiplicities of ideals, defined by Teissier and Risler in [21]. By a result of Swanson [20], joint reductions of ideals of finite colength are characterized via an equality of mixed multiplicities. This results extends the celebrated Rees' multiplicity theorem (see [9, p. 222]).

In Section 2 we define an integer attached to an ample class of n-tuples of ideals $I_1,\dots,I_n$ in a Noetherian local ring of dimension n (see Definition 2.4). This integer, that we denote by $\sigma(I_1,\dots,I_n)$, extends the notion of mixed multiplicity of ideals of finite colength defined by Teissier and Risler in [21]. However $\sigma(I_1,\dots,I_n)$ it is not defined for arbitrary n-tuples of ideals. We call $\sigma(I_1,\dots,I_n)$ the *σ-multiplicity* of $I_1,\dots,I_n$. In the study of this new invariant we apply results developed by Rees [17] and Swanson [20] concerning joint reductions, mixed multiplicities and integral closures of ideals.

Let us denote by $\mathcal{O}_n$ the ring of analytic function germs $f:(\mathbb{C}^n,0)\to\mathbb{C}$. Let $I_1,\dots,I_n$ be monomial ideals of $\mathcal{O}_n$. Then we give in Section 3 a combinatorial characterization of the joint reductions of $I_1,\dots,I_n$ (see Proposition 3.7). If we assume that $\sigma(I_1,\dots,I_n)<\infty$, then we

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will apply this result to characterize those analytic maps $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g_i \in I_i$, for all $i = 1, \dots, n$ and such that $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$ (see Theorem 3.10), where $e(g_1, \dots, g_n)$ is the Samuel multiplicity of the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. This characterization is expressed via the respective Newton polyhedra of $I_1, \dots, I_n$. The set of such maps is denoted by $\mathcal{R}(I_1, \dots, I_n)$.

If $I_1, \dots, I_n$ are monomial and integrally closed ideals of $\mathcal{O}_n$, then, at the end of the paper, we give a result where an important part of the integral closure of the ideals generated by the components of a map of $\mathcal{R}(I_1, \dots, I_n)$ is computed.

The results that we show in this article will be applied, in a subsequent work, to problems in singularity theory concerning invariants of analytic functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. This is the main reason that we fix the setup of this work in $\mathcal{O}_n$ instead of the ring of formal power series $\mathbb{C}[[x_1, \dots, x_n]]$.

2. Joint reductions of ideals and σ-multiplicity

Let R be a commutative ring. We denote by $\overline{I}$ the integral closure of an ideal I of R. If J and I are ideals of R such that $J \subseteq I$, then J is said to be a *reduction* of I if there exists an integer $r \geqslant 0$ such that $I^{r+1} = JI^r$. This definition is due to Northcott and Rees [14]. It is known that J is a reduction of I if and only if $\overline{I} = \overline{J}$ (see [9, p. 6]). The notion of reduction was generalized by Rees in [17] by defining the notion of joint reduction of a set of ideals.

Definition 2.1. [17] Let $I_1, \dots, I_p$ be ideals of R. Let $g_1, \dots, g_p$ be elements of R such that $g_i \in I_i$, for all $i = 1, \dots, p$. The p-tuple $(g_1, \dots, g_p)$ is termed a *joint reduction of* $(I_1, \dots, I_p)$ if and only if the ideal

$$g_1 I_2 \cdots I_p + g_2 I_1 I_3 \cdots I_p + \cdots + g_p I_1 \cdots I_{p-1}$$

is a reduction of $I_1 \cdots I_p$.

Let (R, m) be a Noetherian local ring of dimension n. If an ideal I of R is m-primary then we will also say that I has *finite colength*. If I is an ideal of R of finite colength then we denote by $e(I)$, or by $e(I; R)$, the multiplicity of I in the sense of Samuel (see [9, p. 214]). If $I_1, \dots, I_n$ are ideals of R of finite colength, we denote indistinctly by $e(I_1, \dots, I_n)$ or by $e(I_1, \dots, I_n; R)$ the mixed multiplicity of $I_1, \dots, I_n$ defined by Teissier and Risler in [21] (we also refer to [9, §17] or [20] for the definition and fundamental results concerning mixed multiplicities of ideals). We remark that if $I_1, \dots, I_n$ are all equal to a given ideal, say I, then $e(I_1, \dots, I_n) = e(I)$. We will need the following known result (see [9, p. 345] or [20, Lemma 2.4]).

Lemma 2.2. *Let R be a Noetherian local ring of dimension $n \geqslant 1$. Let $I_1, \dots, I_n$ be ideals of R of finite colength. Let $g_1, \dots, g_n$ be elements of R such that $g_i \in I_i$, for all $i = 1, \dots, n$, and that the ideal $\langle g_1, \dots, g_n \rangle$ has also finite colength. Then*

$$e(g_1, \dots, g_n) \geqslant e(I_1, \dots, I_n).$$

Rees proved in [16] that if $J \subseteq I$ are ideals of a quasi-unmixed Noetherian local ring R, then J is a reduction of I if and only if $e(I) = e(J)$ (see also [9, p. 222]). Moreover, Rees proved in

[17, Theorem 2.4] that if $(g_1, \ldots, g_n)$ is a joint reduction of $(I_1, \ldots, I_n)$, where $I_1, \ldots, I_n$ is a set of ideals of finite colength of a local Noetherian ring R, then $e(g_1, \ldots, g_n) = e(I_1, \ldots, I_n)$ (see also [9, p. 343]). The converse of this result is a nice result of Swanson that we now state.

Theorem 2.3. [20] *Let R be a quasi-unmixed Noetherian local ring. Let $I_1, \ldots, I_s$ be ideals and let g_i be an element of I_i, for $i = 1, \ldots, s$. Suppose that the ideals $I_1, \ldots, I_s$ and $\langle g_1, \ldots, g_s\rangle$ have the same height s and the same radical. If*

$$e(\langle g_1, \ldots, g_s\rangle R_{\mathfrak{p}}; R_{\mathfrak{p}}) = e(I_1 R_{\mathfrak{p}}, \ldots, I_s R_{\mathfrak{p}}; R_{\mathfrak{p}}),$$

for each prime ideal $\mathfrak{p}$ minimal over $\langle g_1, \ldots, g_s\rangle$, then $(g_1, \ldots, g_s)$ is a joint reduction of $(I_1, \ldots, I_s)$.

We now define an invariant, defined in terms of mixed multiplicities of ideals, that is attached to a set of ideals in a Noetherian local ring. The ideals we consider are not assumed to have finite colength. We denote by $\mathbb{Z}_+$ the set of non-negative integers.

Definition 2.4. Let (R, m) be a Noetherian local ring of dimension n. Let $I_1, \ldots, I_n$ be ideals of R. Then we define the *σ-multiplicity of* $I_1, \ldots, I_n$ as

$$\sigma(I_1, \ldots, I_n) = \max_{r \in \mathbb{Z}_+} e(I_1 + m^r, \ldots, I_n + m^r). \tag{1}$$

The set of integers $\{e(I_1 + m^r, \ldots, I_n + m^r) : r \in \mathbb{Z}_+\}$ is not bounded in general; therefore $\sigma(I_1, \ldots, I_n)$ is not always finite for any family of ideals $I_1, \ldots, I_n$. The finiteness of $\sigma(I_1, \ldots, I_n)$ is characterized in Proposition 2.9. We remark that if I_i has finite colength, for all $i = 1, \ldots, n$, then $\sigma(I_1, \ldots, I_n)$ equals the mixed multiplicity $e(I_1, \ldots, I_n)$.

Proposition 2.5. *Let (R, m) be a Noetherian local ring of dimension n. Let $I_1, \ldots, I_n$ be ideals of R such that $\sigma(I_1, \ldots, I_n) < \infty$ and let $g_1, \ldots, g_n$ be elements of R such that $g_i \in I_i$, for all $i = 1, \ldots, n$, and $\langle g_1, \ldots, g_n\rangle$ is an ideal of finite colength. Then $\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)$ if and only if there exists an integer $r_0 \geqslant 1$ such that $(g_1, \ldots, g_n)$ is a joint reduction of $(I_1 + m^r, \ldots, I_n + m^r)$, for all $r \geqslant r_0$.*

Proof. The *if* part follows as a direct consequence of the expression of mixed multiplicities as the multiplicity of a joint reduction (see the paragraph before Theorem 2.3).

Conversely, if $\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)$ then $(g_1, \ldots, g_n)$ is a joint reduction of $(I_1 + m^r, \ldots, I_n + m^r)$, for all $r \gg 0$, as a consequence of Theorem 2.3. □

By virtue of the previous result we give the following definition.

Definition 2.6. Let (R, m) be a local ring of dimension n and let $I_1, \ldots, I_n$ be ideals of R. Let $g_i \in I_i$, for $i = 1, \ldots, n$. We say that $g_1, \ldots, g_n$ is a *σ-joint reduction* of $(I_1, \ldots, I_n)$ when there exists an integer $r_0 \geqslant 1$ such that $(g_1, \ldots, g_n)$ is a joint reduction of $(I_1 + m^r, \ldots, I_n + m^r)$, for all $r \geqslant r_0$.

We will use the following auxiliary result, whose proof appears in [9, p. 134].

Lemma 2.7. *Let (R, m) be a Noetherian local ring and let I be an ideal of R. Then*

$$\overline{I} = \bigcap_{r \geqslant 1} \overline{I + m^r}.$$

Proposition 2.8. *Let (R, m) be a Noetherian local ring of dimension n and let $I_1, \dots, I_n$ be ideals of R. Let $g_i \in I_i$, for $i = 1, \dots, n$. If $g_1, \dots, g_n$ is a σ-joint reduction of $(I_1, \dots, I_n)$ then $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$.*

Proof. When $n = 1$ the result follows easily from Lemma 2.7. Let us suppose that $n \geqslant 2$. Let us define the ideals

$$\begin{aligned} P_r &= g_1(I_2 + m^r) \cdots (I_n + m^r) + \cdots + g_n(I_1 + m^r) \cdots (I_{n-1} + m^r) \\ Q_r &= (I_1 + m^r) \cdots (I_n + m^r). \end{aligned}$$

Then there exists an integer $r_0 \geqslant 1$ such that

$$\overline{Q_r} = \overline{P_r}, \quad \text{for all } r \geqslant r_0. \tag{2}$$

If $j, s \in \{1, \dots, n\}$, we define

$$L_j = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} I_{i_1} \cdots I_{i_j}, \qquad L_j^s = \sum_{\substack{1 \leqslant i_1 < \cdots < i_j \leqslant n \\ i_j \neq s}} I_{i_1} \cdots I_{i_j},$$

where in the definition of L_j^s we suppose that $j \leqslant n - 1$. Then, a simple computation shows that

$$L_n + m^{r(n-1)} L_1 \subseteq Q_r = L_n + m^r L_{n-1} + \cdots + m^{r(n-1)} L_1 + m^{rn} \subseteq L_n + m^{r+n-1} \tag{3}$$

and that

$$P_r = g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n + \sum_{i=1}^{n} g_i \Big(m^r L_{n-2}^i + \cdots + m^{(n-2)r} L_1^i + m^{(n-1)r} \Big). \tag{4}$$

Let J denote the ideal of R generated by $g_1, \dots, g_n$. Then

$$g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n + m^{(n-1)r} J \subseteq P_r \subseteq g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n + m^{r+n-1}. \tag{5}$$

Then, from Lemma 2.7 and the inclusions given in (3) and (5) we obtain the equalities

$$\overline{L_n} = \bigcap_{r \geqslant 1} \overline{Q_r}, \qquad \overline{g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n} = \bigcap_{r \geqslant 1} \overline{P_r}. \tag{6}$$

Therefore, from (2) we have

$$\overline{g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n} = \bigcap_{r \geqslant r_0} \overline{P_r} = \bigcap_{r \geqslant r_0} \overline{Q_r} = \overline{L_n} = \overline{I_1 \cdots I_n}.$$

This implies that $g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n$ is a reduction of $I_1 \cdots I_n$, or equivalently, that $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$. $\square$

In Example 2.10 we show that the converse of Proposition 2.8 does not hold in general.

Let (R, m) be a local ring of dimension n with $k = R/m$ an infinite field. Let $I_1, \dots, I_n$ be ideals of R. Let us consider a generating system $a_{i1}, \dots, a_{is_i}$ of I_i, for $i = 1, \dots, n$. Let $s = s_1 + \cdots + s_n$. We say that a property holds for *sufficiently general* elements of $I_1 \oplus \cdots \oplus I_n$ if there exists a non-empty Zariski-open set U in k^s such that all elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$ satisfy the said property provided that $g_i = \sum_j u_{ij} a_{ij}$, $i = 1, \dots, n$, where $(u_{11}, \dots, u_{1s_1}, \dots, u_{n1}, \dots, u_{ns_n}) \in U$.

Proposition 2.9. *Let $I_1, \dots, I_n$ be ideals of a Noetherian local ring (R, m) such that the residue field $k = R/m$ is infinite. Then $\sigma(I_1, \dots, I_n) < \infty$ if and only if there exist elements $g_i \in I_i$, for $i = 1, \dots, n$, such that $\langle g_1, \dots, g_n\rangle$ has finite colength and $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$. In this case, we have that $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$ for sufficiently general elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$.*

Proof. The *if* part is immediate. Let us suppose that $\sigma(I_1, \dots, I_n) < \infty$. Then there exists a positive integer r_0 such that

$$\sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r),$$

for all $r \geqslant r_0$. By the definition of joint reduction we have that if $(a_1, \dots, a_n)$ is a joint reduction of $(I_1 + m^r, \dots, I_n + m^r)$ and P denotes the ideal

$$a_1(I_2 + m^r) \cdots (I_n + m^r) + \cdots + a_n(I_1 + m^r) \cdots (I_{n-1} + m^r),$$

then

$$\overline{(I_1 + m^r) \cdots (I_n + m^r)} = \overline{P} \subseteq \overline{\langle a_1, \dots a_n\rangle}.$$

Therefore, we observe that there exists an integer $s \geqslant 1$ such that $m^s \subseteq \overline{\langle a_1, \dots, a_n\rangle}$, for all joint reduction $(a_1, \dots, a_n)$ of $(I_1 + m^{r_0}, \dots, I_n + m^{r_0})$. We can suppose that $s \geqslant r_0$.

By the theorem of existence of joint reductions (see [20, p. 4] or [9, p. 336]), let us consider elements $g_i \in I_i$, for $i = 1, \dots, n$, and elements $h_i \in m^{s+1}$, for $i = 1, \dots, n$, such that $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$ and that $(g_1+h_1, \dots, g_n+h_n)$ is a joint reduction of $(I_1 + m^{s+1}, \dots, I_n + m^{s+1})$. Let J be the ideal of R generated by $g_1 + h_1, \dots, g_n + h_n$. Then J has finite colength and $e(J) = e(I_1 + m^{s+1}, \dots, I_n + m^{s+1})$.

Since $s \geqslant r_0$, we have

$$e(I_1 + m^{r_0}, \dots, I_n + m^{r_0}) = e(I_1 + m^{s+1}, \dots, I_n + m^{s+1}) = e(J).$$

Then it follows that $(g_1 + h_1, \dots, g_n + h_n)$ is a joint reduction of $(I_1 + m^{r_0}, \dots, I_n + m^{r_0})$ by Theorem 2.3. But this implies that $m^s \subseteq \overline{J}$, by the definition of s.

Hence we have

$$\overline{J} \subseteq \overline{\langle g_1, \dots, g_n\rangle + m \cdot m^s} \subseteq \overline{\langle g_1, \dots, g_n\rangle + m \cdot J}.$$

By the integral Nakayama's Lemma (see [21, p. 324]), we deduce that

$$\overline{J} \subseteq \overline{\langle g_1, \dots, g_n\rangle}.$$

Then $\langle g_1, \dots, g_n\rangle$ has also finite colength. Moreover we have

$$\sigma(I_1, \dots, I_n) = e(J) \geqslant e(g_1, \dots, g_n) \geqslant e(I_1 + m^{r_0}, \dots, I_1 + m^{r_0}) = \sigma(I_1, \dots, I_n).$$

Hence we have

$$e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n). \tag{7}$$

By the construction of the elements $g_1, \dots, g_n$ that we have considered, we observe that equality (7) is satisfied for sufficiently general elements of $I_1 \oplus \cdots \oplus I_n$, as a consequence of the theorem of existence of joint reductions. □

By Proposition 2.9, if $\sigma(I_1, \dots, I_n) < \infty$ then $I_1 + \cdots + I_n$ is an ideal of finite colength in R. Obviously the converse does not hold. We also have that $e(I_1 + \cdots + I_n) \leqslant \sigma(I_1, \dots, I_n)$, by Lemma 2.2. As a consequence of Rees' multiplicity theorem (see [9, p. 222]) we have that $e(I_1 + \cdots + I_n) = \sigma(I_1, \dots, I_n)$ if and only if any n-tuple $(g_1, \dots, g_n)$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and satisfying the equality $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$ generates a reduction of $I_1 + \cdots + I_n$.

By Propositions 2.5 and 2.8 we have that $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$, where $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$. However, if $\sigma(I_1, \dots, I_n) < \infty$, not every joint reduction of $I_1, \dots, I_n$ generates an ideal of finite colength. Moreover, if I is the ideal generated by a joint reduction of $I_1, \dots, I_n$ and we suppose that I has finite colength then it does not hold in general that $e(I) = \sigma(I_1, \dots, I_n)$. Both facts are shown in the following example.

Example 2.10. Let us consider in $\mathcal{O}_3$ the ideals $I_1 = I_2 = \langle x, y\rangle$ and $I_3 = \langle z \rangle$ and the elements $g_1 = g_2 = x + y$ and $g_3 = z$, where we have fixed the coordinates x, y, z in $\mathbb{C}^3$. It is obvious that $\sigma(I_1, I_2, I_3) = 1$ and that (g_1, g_2, g_3) is a joint reduction of (I_1, I_2, I_3). However g_1, g_2, g_3 do not generate an ideal of finite colength of $\mathcal{O}_3$.

Let us consider the elements $g_1' = x + y + x^3$, $g_2' = x + y + y^3$, $g_3' = z$. Then we observe that (g_1', g_2', g_3') is also a joint reduction of (I_1, I_2, I_3). These elements generate an ideal of finite colength of $\mathcal{O}_3$ but $\sigma(I_1, I_2, I_3) = 1$ and $e(g_1', g_2', g_3') = 3$.

Let (R, m) be a Noetherian local ring of dimension n such that the residue field R/m is infinite. The mixed multiplicity of ideals, as introduced by Risler and Teissier [21] and studied by Rees [17] and Swanson [20], is defined for n ideals $I_1, \dots, I_n$ of finite colength in R. By the theorem of existence of joint reductions (see [9, p. 336]), we have

$$e(I_1, \dots, I_n) = e(g_1, \dots, g_n) \tag{8}$$

where $(g_1, \dots, g_n)$ is a sufficiently general element of $I_1 \oplus \cdots \oplus I_n$.

The number $e(I_1, \dots, I_n)$ is equal to the coefficient of the term $r_1 \cdots r_n$ in the homogeneous part of degree n of the polynomial that coincides with the length function $\ell(R/I_1^{r_1} \cdots I_n^{r_n})$ for $r_1, \dots, r_n \gg 0$. We observe that this function is well defined if and only if I_i has finite colength, for all $i = 1, \dots, n$. However, the multiplicity on the right hand side of (8) could be computed in cases where some of the ideals I_i has not finite colength. By Proposition 2.9, this multiplicity is expressed as a σ-multiplicity (see Definition 2.4).

If I, J are two ideals of finite colength of R, then we can define for all $i \in \{0, 1, \dots, n\}$ the multiplicity

$$e_i(I, J) = e(I, \dots, I, J, \dots, J), \tag{9}$$

where I is repeated $n-i$ times and J is repeated i times, for all $i = 0, 1, \dots, n$. If I and J are arbitrary ideals, we define analogously the number $\sigma_i(I, J)$ by replacing in (9) the mixed multiplicity $e(I, \dots, I, J, \dots, J)$ by $\sigma(I, \dots, I, J, \dots, J)$ (of course, for arbitrary ideals I and J the resulting numbers are not always finite for all $i = 0, 1, \dots, n$).

If J is an ideal of R, let $J^\infty = \{x \in R : x^s J = 0, \text{ for some } s \geqslant 1\}$. As can be seen in the paper [24] of Trung, there is also defined a family of mixed multiplicities $\{e_i(I|J) : i = 0, 1, \dots, r\}$ of a pair of ideals I, J, where I is assumed to have finite colength, J is an arbitrary ideal of R and $r = \dim(R/(0 : J^\infty)) - 1$. These numbers arise from the coefficients of the homogeneous part of highest degree of the polynomial that coincides with the length function of the bigraded ring

$$R(I|J) = \bigoplus_{(u,v)\in\mathbb{Z}_+^2} I^u J^v / I^{u+1} J^v.$$

We refer to [11], [24], [25] and [28] for the details about this definition.

Let $\ell(J)$ denote the analytic spread of J. The multiplicities $e_i(I|J)$ are not all positive for all $i = 0, 1, \dots, r$. In fact, Trung proved that $e_i(I|J) = 0$, for all $i \geqslant \ell(J)$ (see [24, Corollary 3.6]). Moreover, if $i \in 0, 1, \dots, \mathrm{ht}(J) - 1$, then it is proved in [24, Proposition 4.1] that

$$e_i(I|J) = e\big(a_1, \dots, a_{n-i}, b_1, \dots, b_i\big), \tag{10}$$

where $(a_1, \dots, a_{n-i}, b_1, \dots, b_i)$ is a sufficiently general element of $I \oplus \cdots \oplus I \oplus J \oplus \cdots \oplus J$. We remark that relation (10) shows that $e_i(I|J) = \sigma_i(I, J)$, for all $i \in 0, 1, \dots, \mathrm{ht}(J) - 1$, by Proposition 2.9. However, we show a simple example where the multiplicity on the right hand part of (10) can be positive for $i = \ell(J)$ and therefore it can be expressed as a σ-multiplicity.

Example 2.11. Let I, J be the ideals in $\mathcal{O}_3$ given by $I = \langle x, y, z\rangle$, $J = \langle x^2, y^2\rangle$. Then $\ell(J) = 2$ (see [3, Theorem 2.3]) and $\sigma_2(I, J) = \sigma(I, J, J) = 4$.

3. Mixed multiplicities and non-degeneracy

Throughout the remaining text, if no confusion arises, we will denote the maximal ideal of $\mathcal{O}_n$ by m instead of m_n. We say that an ideal I of $\mathcal{O}_n$ is a *monomial ideal* when I is generated by a family of monomials x^k such that $k \in \mathbb{Z}_+^n$, $k \neq 0$. Let $I_1, \dots, I_n$ be a sequence of monomial ideals in $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. In this section we characterize the sets of functions $g_1, \dots, g_n \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and that $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$. In order to show our results we will introduce first some definitions and notation.

Let $h \in \mathcal{O}_n$, let us suppose that the Taylor expansion of h around the origin is given by $h = \sum_k a_k x^k$. We define the *support* of h, denoted by $\mathrm{supp}(h)$, as the set of those $k \in \mathbb{Z}_+^n$ such that $a_k \neq 0$. If A is a compact subset of $\mathbb{R}_+^n$, then we denote by h_A the polynomial given by the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap A$. If $\mathrm{supp}(h) \cap A = \emptyset$, then we set $h_A = 0$. If I is a monomial ideal of $\mathcal{O}_n$, we define the *support* of I, denoted by $\mathrm{supp}(I)$, as the set of $k \in \mathbb{Z}_+^n$ such that $x^k \in I$.

We say that a subset Γ_+ of $\mathbb{R}_+^n$ is a *Newton polyhedron* when there exists some $B \subseteq \mathbb{Q}_+^n$ such that Γ_+ is equal to the convex hull in $\mathbb{R}_+^n$ of the set $\{k + v : k \in B,\, v \in \mathbb{R}_+^n\}$. In this case we say that Γ_+ is the *Newton polyhedron determined by* B and we also denote Γ_+ by $\Gamma_+(B)$. A Newton polyhedron Γ_+ is termed *convenient* when Γ_+ intersects each coordinate axis in a point different from the origin. In this case, we denote by $\mathrm{V}_n(\Gamma_+)$ the n-dimensional volume of the set $\mathbb{R}^n_+ \smallsetminus \Gamma_+$.

If $h \in \mathcal{O}_n$, the *Newton polyhedron of* h is defined as $\Gamma_+(h) = \Gamma_+\big(\mathrm{supp}(h)\big)$. Let J be an ideal of $\mathcal{O}_n$, let us suppose that J is generated by the elements $h_1, \dots, h_p$. Then the *Newton polyhedron of* J, denoted by $\Gamma_+(J)$, is defined as the convex hull of the union $\Gamma_+(h_1) \cup \cdots \cup \Gamma_+(h_p)$. It is easy to check that the definition of $\Gamma_+(J)$ does not depend on the chosen generating system of J.

If $\Gamma^1_+, \dots, \Gamma^p_+$ are Newton polyhedra in $\mathbb{R}^n_+$, then we define the *Minkowski sum* of $\Gamma^1_+, \dots, \Gamma^p_+$ as

$$\Gamma^1_+ + \cdots + \Gamma^p_+ = \{k_1 + \cdots + k_p : k_i \in \Gamma^i_+, \text{ for all } i = 1, \dots, p\}.$$

This set is again a Newton polyhedron, since it is known that $\Gamma^1_+ + \cdots + \Gamma^p_+ = \Gamma_+(I_1 \cdots I_p)$, whenever $\Gamma^i_+ = \Gamma_+(I_i)$, for some monomial ideal $I_i \in \mathcal{O}_n$, $i = 1, \dots, p$ (see for instance [7]).

Let us fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$. Given a vector $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ we define

$$\ell(v, \Gamma_+) = \min\big\{\langle v, k\rangle : k \in \Gamma_+\big\}.$$

We say that a subset Δ of Γ_+ is a *face* of Γ_+ if there exists a vector $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ such that Δ is expressed as

$$\Delta = \big\{k \in \Gamma_+ : \langle v, k\rangle = \ell(v, \Gamma_+)\big\}. \tag{11}$$

We will denote the set on the right hand side of (11) by $\Delta(v, \Gamma_+)$ and we will also say that Δ is the face of Γ_+ *supported* by v. We have that $\Delta(v, \Gamma_+)$ is a compact face of Γ_+ if and only if all components of v are non-zero. If I is an ideal of $\mathcal{O}_n$, then we denote by $\Gamma(I)$ the union of the compact faces of $\Gamma_+(I)$. Moreover, we will denote the face $\Delta(v, \Gamma_+(I))$, for a given $v \in \mathbb{R}^n \smallsetminus \{0\}$, by $\Delta(v, I)$.

Definition 3.1. Let $I_1, \dots, I_p$ be monomial ideals in $\mathcal{O}_n$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Let $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ and let $\Delta_i = \Delta(v, I_i)$, for all $i = 1, \dots, p$. We say that g satisfies the (K_v) *condition with respect to* $I_1, \dots, I_p$ when

$$\big\{x \in \mathbb{C}^n : (g_1)_{\Delta_1}(x) = \cdots = (g_p)_{\Delta_p}(x) = 0\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

Then the map g is termed *non-degenerate with respect to* $I_1, \dots, I_p$ when g satisfies the (K_v) condition with respect to $I_1, \dots, I_p$ for all $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$.

Under the conditions of the above definition, we observe that if there exists some $i_0 \in \{1, \dots, p\}$ such that g_{i_0} is equal to a monomial x^k, for some $k \in \mathbb{Z}^n_+$, $k \neq 0$, and $I_{i_0} = \langle x^k\rangle$, then the map g is automatically non-degenerate with respect to $I_1, \dots, I_p$.

If $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, we define $\mathbb{C}^n_L = \{x \in \mathbb{C}^n : x_i = 0, \text{ for all } i \notin L\}$. The set $\mathbb{R}^n_L$ is defined analogously. If $h \in \mathcal{O}_n$ and the Taylor expansion of h around the origin is given by $h = \sum_k a_k x^k$, we denote by h^L the function obtained as the sum of those terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap \mathbb{R}^n_L$. If $\mathrm{supp}(h) \cap \mathbb{R}^n_L = \emptyset$, then we set $h^L = 0$. If $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ, then we denote by g^L the map $(g_1^L,\dots,g_p^L):(\mathbb{C}^n_L,0)\to(\mathbb{C}^p,0)$. In some occasions we will identify $\mathbb{C}^n_L$ with $\mathbb{C}^r$, where $r=|L|$.

Let $L=\{i_1,\dots,i_r\}\subseteq\{1,\dots,n\}$, then we denote by $\mathcal{O}_{n,L}$ the subring of $\mathcal{O}_n$ generated by the functions of $\mathcal{O}_n$ depending, at most, on the variables $x_{i_1},\dots,x_{i_r}$. We denote by m_L the maximal ideal of $\mathcal{O}_{n,L}$. We observe that the map $\mathcal{O}_n\to\mathcal{O}_{n,L}$ given by $h\mapsto h^L$, $h\in\mathcal{O}_n$, is a ring epimorphism. If I is a monomial ideal of $\mathcal{O}_n$ then we denote by I^L the ideal of $\mathcal{O}_{n,L}$ generated by all monomials x^k such that $k\in\operatorname{supp}(I)\cap\mathbb{R}^n_L$. If $\operatorname{supp}(I)\cap\mathbb{R}^n_L=\emptyset$, then we set $I^L=0$.

Definition 3.2. Let $I_1,\dots,I_p$ be monomial ideals of $\mathcal{O}_n$ such that $I_1+\cdots+I_p$ is an ideal of finite colength in $\mathcal{O}_n$. Let $g:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be an analytic map germ such that $g_i\in I_i$, for all $i=1,\dots,p$. We say that g is *strongly non-degenerate with respect to* $I_1,\dots,I_p$ when for all $L\subseteq\{1,\dots,n\}$, $L\neq\emptyset$, the map $g^L:(\mathbb{C}^n_L,0)\to(\mathbb{C}^p,0)$ is non-degenerate with respect to the non-zero ideals of the sequence of ideals $I_1^L,\dots,I_p^L$.

We remark that, since we are assuming in the above definition that $I_1+\cdots+I_p$ is an ideal of finite colength, then the set of non-zero ideals in the sequence $I_1^L,\dots,I_p^L$ is non-empty, for all $L\subseteq\{1\dots,n\}$, $L\neq\emptyset$.

Under the conditions of Definition 3.2, we denote the set of analytic maps $g:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ such that $g_i\in I_i$, for all $i=1,\dots,p$, and such that g is strongly non-degenerate with respect to $I_1,\dots,I_p$ by $\mathcal{R}(I_1,\dots,I_p)$. Let us remark that if $g\in\mathcal{R}(I_1,\dots,I_p)$ then g_i does not need to have the same Newton polyhedron as I_i, for all $i=1,\dots,p$.

Example 3.3. Let us consider the ideals I_1,I_2,I_3 of $\mathcal{O}_3$ and the polynomials g_1',g_2',g_3' given in Example 2.10. Then we have that the map $g':(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ defined by $g'=(g_1',g_2',g_3')$ is non-degenerate with respect to I_1,I_2,I_3. If $L=\{1,2\}$, then $\{i: I_i^L\neq 0\}=\{1,2\}$. We observe that the map $h=((g_1')^L,(g_2')^L)$ is not non-degenerate with respect to I_1^L,I_2^L, since h does not satisfy the (K_v) condition for $v=(1,1)$. Therefore g' is not strongly non-degenerate with respect to I_1,I_2,I_3.

Remark 3.4. Let $\Gamma^1_+,\dots,\Gamma^p_+$ be a family of Newton polyhedra in $\mathbb{R}^n_+$. It is well known that if Δ is a compact face of $\Gamma^1_++\cdots+\Gamma^p_+$, then Δ is uniquely expressed as $\Delta_1+\cdots+\Delta_p$, where Δ_i is face of Γ^i_+, for all $i=1,\dots,p$. This expression is a consequence of the following relations:

$$\ell(v,\Gamma^1_++\cdots+\Gamma^p_+)=\ell(v,\Gamma^1_+)+\cdots+\ell(v,\Gamma^p_+)$$
$$\Delta(v,\Gamma^1_++\cdots+\Gamma^p_+)=\Delta(v,\Gamma^1_+)+\cdots+\Delta(v,\Gamma^p_+),$$

for all $v\in\mathbb{R}^n_+\smallsetminus\{0\}$. Therefore, under the hypothesis of Definition 3.1, the set of non-redundant (K_v) conditions that a non-degenerate map with respect to $I_1,\dots,I_p$ must satisfy is parameterized by the set of compact faces of $\Gamma_+(I_1)+\cdots+\Gamma_+(I_p)$. Hence the definition of strongly non-degenerate map with respect to $I_1,\dots,I_p$ consists of a finite set of conditions.

Here we recall the definition of Newton non-degenerate ideal (see [3] or [4]). Let I be an ideal of $\mathcal{O}_n$ and let $g_1,\dots,g_r$ be a generating system of I. Then the ideal I is said to be

Newton non-degenerate when for each compact face Δ of $\Gamma_+(I)$ we have

$$\big\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_r)_\Delta(x)\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

It is straightforward to see that this definition does not depend on the generating system of I. We observe that any monomial ideal is Newton non-degenerate. Moreover, we also have that an ideal I of $\mathcal{O}_n$ is Newton non-degenerate if and only if I admits a generating system $g_1, \dots, g_r$ such that the map $(g_1, \dots, g_r) : (\mathbb{C}^n, 0) \to (\mathbb{C}^r, 0)$ is non-degenerate with respect to $I, \dots, I$, with I repeated r times (see Definition 3.1). If I is an ideal of finite colength, then this condition is equivalent to saying that $(g_1, \dots, g_r) \in \mathcal{R}(I, \dots, I)$, where I is repeated r times (see also Corollary 3.8).

The next result shows a numerical characterization of the Newton non-degeneracy condition (we refer to [2] for the definition and characterization of the Newton non-degeneracy condition in the context of submodules of the free module $\mathcal{O}_n^p$, $p \geqslant 1$).

Theorem 3.5. [3, 4] *Let I be an ideal of $\mathcal{O}_n$ of finite colength. Then $e(I) \geqslant n!\mathrm{V}_n\big(\Gamma_+(I)\big)$ and equality holds if and only if I is a Newton non-degenerate ideal.*

Given an ideal J of $\mathcal{O}_n$ and a fixed coordinate system in $\mathbb{C}^n$, we denote by J_0 the ideal of $\mathcal{O}_n$ generated by all monomials x^k such that $k \in \Gamma_+(J)$. The ideal J_0 is integrally closed (see [9, p. 11] or [22]). Therefore, from the inclusions $J \subseteq \overline{J} \subseteq \overline{J_0} = J_0$, we deduce that $\Gamma_+(J) = \Gamma_+(\overline{J})$.

Proposition 3.6. *Let I be a Newton non-degenerate ideal of $\mathcal{O}_n$ and let $J \subseteq I$. Then the following conditions are equivalent:*

(1) *J is a reduction of I;*
(2) *J is Newton non-degenerate and $\Gamma_+(J) = \Gamma_+(I)$;*
(3) *there exists a generating system $g_1, \dots, g_r$ of J such that, for all compact face Δ of $\Gamma_+(I)$, we have*

$$\big\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_r)_\Delta(x)\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}. \tag{12}$$

Proof. We point out that the ideal I is not assumed to have finite colength. Let us see (1) $\Rightarrow$ (2). Suppose that J is a reduction of I. Then $\overline{I} = \overline{J}$ and, in particular, we have that $\Gamma_+(I) = \Gamma_+(J)$. Moreover we also deduce that

$$\overline{I + m^r} = \overline{\overline{I} + m^r} = \overline{\overline{J} + m^r} = \overline{J + m^r}, \tag{13}$$

for all $r \geqslant 1$. Using relation (13) and the fact that $\overline{I + m^r}$ is a monomial ideal of finite colength, it follows that

$$n!\mathrm{V}_n\big(\Gamma_+(J + m^r)\big) = n!\mathrm{V}_n\big(\Gamma_+(I + m^r)\big) = e(I + m^r) = e(J + m^r),$$

by Theorem 3.5. Therefore the ideal $J + m^r$ is Newton non-degenerate, for all $r \geqslant 1$, by virtue of Theorem 3.5. Let r_0 a positive integer such that each compact face Δ of $\Gamma_+(J)$ is a compact face of $\Gamma_+(J + m^r)$, for all $r \geqslant r_0$. Therefore, by writing down the condition that $J + m^r$ is Newton non-degenerate, for all $r \geqslant r_0$, we conclude that J is Newton non-degenerate.

Let us see (2) $\Rightarrow$ (1). We will see that item (2) implies that $\overline{I} = \overline{J}$. In particular, we will have that J is a reduction of I, since $J \subseteq I$ (see [9, p. 6]). As before, let us consider a big enough positive integer r_0 such that each compact face of $\Gamma_+(J)$ is a compact face of $\Gamma_+(J + m^r)$, for all $r \geqslant r_0$. Then we have that $J + m^r$ is Newton non-degenerate, for all $r \geqslant r_0$. Hence $e(J + m^r) = n!\mathrm{V}_n(\Gamma_+(J + m^r))$, for all $r \geqslant r_0$. This implies, by Rees' multiplicity theorem, that

$$\overline{J + m^r} = (J + m^r)_0 = (J_0 + m^r)_0 = \overline{J_0 + m^r}, \text{ for all } r \geqslant r_0.$$

By Lemma 2.7, we have

$$\overline{J} = \bigcap_{r \geqslant r_0} \overline{J + m^r} = \bigcap_{r \geqslant r_0} \overline{J_0 + m^r} = \overline{J_0} = J_0. \tag{14}$$

Since $J \subseteq I$ and $\Gamma_+(I) = \Gamma_+(J)$, then $\overline{J} \subseteq \overline{I} \subseteq I_0 = J_0$. Then relation (14) implies that $\overline{I} = \overline{J}$.

The implication (2) $\Rightarrow$ (3) is obvious. In order to see the implication (3) $\Rightarrow$ (2) it suffices to prove that $\Gamma_+(I) = \Gamma_+(J)$. Let $g_1, \dots, g_r$ be a generating system of J verifying the inclusion (12), for all compact face Δ of $\Gamma_+(I)$. In particular, if Δ is a vertex of $\Gamma_+(I)$, then this condition must be satisfied for Δ. This implies that if Δ is any vertex of $\Gamma_+(I)$, then some function $(g_i)_\Delta$ is not identically zero. Thus $\Gamma_+(I) \subseteq \Gamma_+(J)$. But since we assume that $J \subseteq I$, we have that $\Gamma_+(I) = \Gamma_+(J)$. □

The previous proposition gives the family of all reductions of a given monomial ideal. Rees and Sally [18] defined the *core* of an ideal I in a commutative ring as the intersection of all reductions of I; it is denoted by $\mathrm{core}(I)$. In particular, by Proposition 3.6, the computation of the core of a monomial and integrally closed ideal I in $\mathcal{O}_n$, or in $\mathbb{C}[[x_1, \dots, x_n]]$, reduces to compute the intersection of all ideals J of $\mathcal{O}_n$ such that $\Gamma_+(I) = \Gamma_+(J)$ and J is Newton non-degenerate. We remark that the study of the core of an ideal is quite an active research topic in commutative algebra (see for instance [10] or [15]).

In the next result we show a characterization of the joint reductions of a family of monomial ideals.

Proposition 3.7. *Let $I_1, \dots, I_p$ be monomial ideals of $\mathcal{O}_n$. Let $g_1, \dots, g_p \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Then the following conditions are equivalent:*

1. *$(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$;*
2. *the map $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is non-degenerate with respect to $I_1, \dots, I_p$.*

Proof. Let us consider the ideal J of $\mathcal{O}_n$ given by

$$J = g_1 I_2 \cdots I_p + g_2 I_1 I_3 \cdots I_p + \cdots + g_p I_1 \cdots I_{p-1}. \tag{15}$$

By Definition 2.1, we have that $(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$ if and only if J is a reduction of the monomial ideal $I_1 \cdots I_p$. Let I denote the ideal $I_1 \cdots I_p$, then $J \subseteq I$. Therefore, item (1) holds if and only if J satisfies item (3) of Proposition 3.6 with respect to $\Gamma_+(I)$.

Let $\Gamma_+ = \Gamma_+(I)$, we remark that Γ_+ is equal to the Minkowski sum $\Gamma_+(I_1) + \cdots + \Gamma_+(I_p)$. Let B denote the set $\{1, \dots, p\}$. From the definition of J we have that there exist finite subsets $S_1, \dots, S_p \subseteq \mathbb{Z}^n_+$ such that the set $\mathcal{J}$ of functions given by

$$\begin{aligned}\mathcal{J} = &\big\{g_1 x^{k_2+\cdots+k_p} : k_i \in S_i, i \in B, i \neq 1\big\} \cup \big\{g_2 x^{k_1+k_3+\cdots+k_p} : k_i \in S_i, i \in B, i \neq 2\big\} \cup \cdots \cup \\ &\cup \big\{g_p x^{k_1+\cdots+k_{p-1}} : k_i \in S_i, i \in B, i \neq p\big\}\end{aligned}$$

is a generating system of J. Let us fix a compact face Δ of $\Gamma_+(I)$. Then Δ is expressed univocally as $\Delta = \Delta_1 + \cdots + \Delta_p$, where Δ_i is a compact face of $\Gamma_+(I_i)$, for all $i = 1, \dots, p$.

If h is an element of $\mathcal{J}$, then there exists an $i_0 \in B$ such that $h = g_{i_0} x^{k_1+\cdots+k_{i_0-1}+k_{i_0+1}+\cdots+k_p}$, for some $k_i \in S_i$, $i \neq i_0$. Therefore h_Δ is expressed as

$$h_\Delta = \big(g_{i_0}\big)_{\Delta_{i_0}} \big(x^{k_1}\big)_{\Delta_1} \cdots \big(x^{k_{i_0-1}}\big)_{\Delta_{i_0-1}} \big(x^{k_{i_0+1}}\big)_{\Delta_{i_0+1}} \cdots \big(x^{k_p}\big)_{\Delta_p}.$$

Then the set of common zeros of $\{h_\Delta : h \in \mathcal{J}\}$ in $(\mathbb{C} \smallsetminus \{0\})^n$ is equal to the set of common zeros of $\{(g_i)_{\Delta_i} : i = 1, \dots, p\}$ in $(\mathbb{C} \smallsetminus \{0\})^n$. This fact shows that item (3) of Proposition 3.6 applied to the ideals J and I holds if and only if the map g is non-degenerate with respect to $I_1, \dots, I_p$. Thus the equivalence between (1) and (2) follows. □

Corollary 3.8. *Let $I_1, \dots, I_p$ be monomial ideals of finite colength of $\mathcal{O}_n$. Let $g_1, \dots, g_p \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Let $g = (g_1, \dots, g_p)$, then $g \in \mathcal{R}(I_1, \dots, I_p)$ if and only if g is non-degenerate with respect to $I_1, \dots, I_p$.*

Proof. The *only if* part is obvious. Let us suppose that g is non-degenerate with respect to $I_1, \dots, I_p$. Therefore $(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$, by Proposition 3.7. This means that J is a reduction of $I_1 \cdots I_p$, where J is the ideal defined in (15). In particular, for a given $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, we have that J^L is a reduction of $(I_1 \cdots I_p)^L = I_1^L \cdots I_p^L$, since reductions are stable under ring morphisms. Therefore $(g_1^L, \dots, g_p^L)$ is a joint reduction of $(I_1^L, \dots, I_p^L)$. We have that $I_i^L \neq 0$, for all $i = 1, \dots, p$, since each ideal I_i has finite colength. Then the result follows as a consequence of Proposition 3.7. □

Given an integer $r \geqslant 1$ and a subset $L \subseteq \{1, \dots, n\}$, we denote by $\delta_{L,r}$ the convex hull in $\mathbb{R}^n$ of $\{re_i : i \in L\}$, where $e_1, \dots, e_n$ denotes the canonical basis in $\mathbb{R}^n$.

If I is an ideal of $\mathcal{O}_n$, $I \neq 0$, then we denote by $\mathrm{ord}(I)$ the maximum of those integers $s \geqslant 1$ such that $I \subseteq m^s$.

Lemma 3.9. *Let $I_1, \dots, I_n$ be monomial ideals in $\mathcal{O}_n$ such that $I_1 + \cdots + I_n$ has finite colength. Let us consider, for a given integer $r \geqslant 1$, the ideal $Q_r = (I_1 + m^r) \cdots (I_n + m^r)$. Then, there exists an integer $r_0 \geqslant 1$ such that for all $r \geqslant r_0$ the following hold:*

1. *every compact face of $\Gamma_+(I_1 \cdots I_n)$ is a compact face of Q_r;*
2. *let Δ be a face of $\Gamma_+(Q_r)$ not intersecting $\Gamma(I_1 \cdots I_n)$, let us write Δ as $\Delta = \Delta_1 + \cdots + \Delta_n$, where Δ_i is a face of $I_i + m^r$, for all $i = 1, \dots, n$, and let $S = \{i : \Delta_i \cap \Gamma(I_i) \neq \emptyset\}$; then $S \neq \emptyset$ and there exists some $L \subsetneq \{1, \dots, n\}$ such that*

$$\Delta = \sum_{i \in S} \Delta_i + \big(n - |S|\big)\delta_{L,r},$$

and Δ_i is a face of $\Gamma(I_i^L)$ if $I_i^L \neq 0$.

Proof. Let us define, for a given integer $j \in \{1, \dots, n\}$, the ideal

$$L_j = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} I_{i_1} \cdots I_{i_j}.$$

Since the ideal L_1 has finite colength, then there exists an integer $r_0 \geqslant 1$ such that $m^{r_0} \subseteq L_1$. Then, for any integer $r \geqslant r_0$, we observe that Q_r is expressed as

$$Q_r = L_n + m^r L_{n-1} + \cdots + m^{r(n-1)} L_1. \tag{16}$$

Relation (16) shows that we can increase the integer r in order to have that any compact face of L_n is a compact face of Q_r. Then item (1) holds.

If $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ then we define $v_0 = \min_i v_i$. We also define $L(v) = \{i : v_i = v_0\}$. For any vector $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$ and any $r \geqslant 1$ we have

$$\ell(v, m^r) = r v_0 \quad \text{and} \quad \Delta(v, m^r) = \delta_{L(v), r}. \tag{17}$$

Let us suppose that $r > \mathrm{ord}(I_i^L)$, for all $i = 1, \dots, n$ and all $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$. Let $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$ and let $j \in \{1, \dots, n\}$ such that $I_j^{L(v)} \neq 0$. Then

$$\ell(v, I_j) \leqslant \ell(v, I_j^{L(v)}) = \mathrm{ord}(I_j^{L(v)}) v_0 < r v_0 = \ell(v, m^r).$$

In particular, there exists an integer $r_1 \geqslant r_0$ such that for all $r \geqslant r_1$ we have

$$\Delta(v, I_j + m^r) \cap \Delta(v, I_j) \neq \emptyset, \tag{18}$$

for all vector $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$ and all j such that $I_j^{L(v)} \neq 0$.

Let us consider an integer $r_2 \geqslant r_1$ such that each compact face of $\Gamma_+(I_i)$ is a compact face of $\Gamma_+(I_i + m^r)$, for all $i = 1, \dots, n$ and all $r \geqslant r_2$. Then the number of compact faces of $\Gamma_+(I_i + m^r)$ does not depend on r, if $r \geqslant r_2$, for all $i = 1, \dots, n$. In particular, there exists an integer $r_3 \geqslant r_2$ such that the number of compact faces of $\Gamma_+(Q_r)$ does not depend on r if $r \geqslant r_3$.

For each face Δ of $\Gamma_+(Q_{r_3})$, let us choose a vector v_Δ such that $\Delta = \Delta(v_\Delta, Q_{r_3})$. Let us consider the decomposition $\Delta = \Delta_1 + \cdots + \Delta_n$, where $\Delta_i = \Delta(v_\Delta, I_i + m^{r_3})$, for all $i = 1, \dots, n$.

Let us suppose that Δ is face of $\Gamma_+(Q_{r_3})$ such that $\Delta \cap \Gamma(I_1 \cdots I_n) = \emptyset$. Then the set $S = \{i : \Delta_i \cap \Gamma(I_i) \neq \emptyset\}$ is non-empty, by (16). Moreover, if L denotes the set $L(v_\Delta)$ we have that $\{j : I_j^L \neq 0\} \subseteq S$, by (18). In particular, if $i \notin S$ then $I_i^L = 0$ and $\Delta_i = \delta_{L, r_3}$, by (17).

We remark that, for a given $i \in \{1, \dots, n\}$, any face of $\Gamma_+(I_i + m^r)$, for $r \geqslant r_2$, is determined by its intersection with $\Gamma_+(I_i)$ and its intersection with the family of the coordinate axis. Then the vector v_Δ is integrated in a natural way in a family of vectors v_Δ^r, for $r \geqslant r_3$, satisfying

$$\begin{gathered}
\Delta(v_\Delta^r, I_i + m^r) \cap \Gamma(I_i) = \Delta_i \cap \Gamma(I_i), \ \text{ for all } i \in S \\
\Delta(v_\Delta^r, I_i + m^r) \cap \Gamma(m^r) = \delta_{L,r}, \ \text{ for all } i \notin S \\
L(v_\Delta^r) = L.
\end{gathered}$$

Then we can consider an integer $r_\Delta \geqslant r_3$ such that if $j \in S$ verifies that $I_j^L \neq 0$, then

$$\Delta(v_\Delta^r, I_j + m^r) \subseteq \Delta(v_\Delta^r, I_j) \cap \mathbb{R}^n_L,$$

for all $r \geqslant r_\Delta$. Hence, if $r \geqslant r_\Delta$, the face Δ is written as

$$\Delta = \sum_{i \in S} \Delta_i + (n - |S|)\delta_{L,r},$$

where Δ_j is a face of $\Gamma(I_j^L)$ for all $j \in S$ such that $I_j^L \neq 0$. □

Theorem 3.10. *Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$. Suppose that $\sigma(I_1, \dots, I_n) < \infty$. Let $g_1, \dots, g_n \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, n$. Then the following conditions are equivalent:*

(1) *the ideal $\langle g_1, \dots, g_n \rangle$ has finite colength and $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$;*
(2) *$g \in \mathcal{R}(I_1, \dots, I_n)$.*

Proof. Let g denote the map $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ given by $g = (g_1, \dots, g_n)$. For a given $r \geqslant 1$ we define the ideals

$$\begin{aligned} P_r &= g_1(I_2 + m^r) \cdots (I_n + m^r) + \cdots + g_n(I_1 + m^r) \cdots (I_{n-1} + m^r) \\ Q_r &= (I_1 + m^r) \cdots (I_n + m^r). \end{aligned}$$

Let us see that (1) implies (2). By Nakayama's Lemma we can suppose that g_i is a polynomial, for all $i = 1, \dots, n$. By Proposition 2.5, $(g_1, \dots, g_n)$ is a σ-joint reduction of $(I_1, \dots, I_n)$. In particular, it is a joint reduction of $(I_1, \dots, I_n)$, by Proposition 2.8. Therefore g is non-degenerate with respect to $(I_1, \dots, I_n)$, by Proposition 3.7.

Let r_0 be an integer such that P_r is a joint reduction of Q_r, for all $r \geqslant r_0$. Let us fix a subset $L \subsetneq \{1, \dots, n\}$, $L \neq \emptyset$, and an integer $r \geqslant r_0$. Since reductions are stable under ring morphisms, we have that P_r^L is a reduction of Q_r^L. Therefore the map g^L is non-degenerate with respect to $(I_1 + m^r)^L, \dots, (I_n + m^r)^L$, by Proposition 3.7. Let us remark that $(I_i + m^r)^L \neq 0$, for all $i = 1, \dots, n$.

Let $C = \{i : I_i^L \neq 0\}$. The condition $\sigma(I_1, \dots, I_n) < \infty$ implies that $I_1 + \cdots + I_n$ has finite colength and. Therefore $C \neq \emptyset$. Without loss of generality we can suppose that $C = \{1, \dots, s\}$, for some $1 \leqslant s \leqslant n$. We have to see that $(g_1^L, \dots, g_s^L) : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^s, 0)$ is non-degenerate with respect to $I_1^L, \dots, I_s^L$.

Since g_i is a polynomial, for all $i = 1, \dots, n$, let us assume that

$$\operatorname{supp}(g_i) \cap \Gamma_+(m^r) = \emptyset, \quad \text{for all } i = 1, \dots, n. \tag{19}$$

Let $H = (I_1^L + m_L^r) \cdots (I_s^L + m_L^r)$. Then $Q_r^L = H m_L^{r(n-s)}$. In particular, we have

$$\Gamma_+(Q_r^L) = \Gamma_+(H) + \Gamma_+(m_L^{r(n-s)}). \tag{20}$$

By Lemma 3.9 (1) we can suppose that r_0 is big enough in order to have that each compact face of $I_1^L \cdots I_s^L$ is a compact face of H. This fact together with (20) implies that if v is a vector of $(\mathbb{R}_+ \smallsetminus \{0\})^q$, where $q = |L|$, then the set

$$\Delta(v, I_1^L \cdots I_s^L) + \Delta(v, m_L^{r(n-s)}) \tag{21}$$

is a compact face of $\Gamma_+(Q_r^L)$.

By hypothesis the map g^L is non-degenerate with respect to $(I_1+m^r)^L, \dots, (I_n+m^r)^L$. Then g^L verifies the (K_v) condition with respect to these ideals (see Definition 3.1). Therefore, writing down this condition and considering (19) and (21), we have

$$\big\{x \in \mathbb{C}^n_L : (g_1^L)_{\Delta_1}(x) = \cdots = (g_s^L)_{\Delta_s}(x) = 0\big\} \subseteq \big\{x \in \mathbb{C}^n_L : \prod_{i\in L} x_i = 0\big\},$$

where $\Delta_i = \Delta(v, I_i^L)$, for all $i = 1, \dots, s$. This shows that the map $(g_1^L, \dots, g_s^L) : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^s, 0)$ is non-degenerate with respect to $I_1^L, \dots, I_s^L$. Then $g \in \mathcal{R}(I_1, \dots, I_n)$, since we started from an arbitrary $L \subsetneq \{1, \dots, n\}$.

Let us see that (2) implies (1). Let us suppose that $g \in \mathcal{R}(I_1, \dots, I_n)$. By Proposition 2.5 and Proposition 3.7, item (1) holds if and only if there exists an integer r_0 such that g is non-degenerate with respect to $I_1 + m^r, \dots, I_n + m^r$, for all $r \geqslant r_0$.

Let r_0 be an integer such that items (1) and (2) of Lemma 3.9 hold for all $r \geqslant r_0$. Let us fix an integer $r \geqslant r_0$ and let us fix a compact face Δ of $\Gamma_+(Q_r)$. Let us write Δ as $\Delta = \Delta_1 + \cdots + \Delta_n$, where Δ_i is a face of $\Gamma_+(I_i + m^r)$, for all $i = 1, \dots, n$. We have to see that

$$\big\{x \in \mathbb{C}^n : (g_1)_{\Delta_1}(x) = \cdots = (g_n)_{\Delta_n}(x) = 0\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}. \tag{22}$$

Let $\Delta' = \Delta \cap \Gamma_+(I_1 \cdots I_n)$ and let $\Delta_i' = \Delta_i \cap \Gamma_+(I_i)$, for all $i = 1, \dots, n$. If $\Delta' \neq \emptyset$, then $\Delta' = \Delta_1' + \cdots + \Delta_n'$ and $(g_i)_{\Delta_i} = (g_i)_{\Delta_i'}$, for all $i = 1, \dots, n$. Thus inclusion (22) holds, since g is non-degenerate with respect to $I_1, \dots, I_n$ by hypothesis.

Let us suppose that $\Delta' = \emptyset$. By Lemma 3.9, there exists a subset $L \subsetneq \{1, \dots, n\}$ such that, if $S = \{i : \Delta_i' \neq \emptyset\}$ and $C_L = \{i : I_i^L \neq 0\}$, then $C_L \subseteq S$ and Δ is written as

$$\Delta = \sum_{i\in S} \Delta_i + (n - |S|)\delta_{L,r}.$$

Let us suppose that $C_L = \{i_1, \dots, i_s\}$, for some $1 \leqslant i_1 < \cdots < i_s \leqslant n$, $s \leqslant t$, where $t = |S|$. Therefore we have

$$\Delta = \Delta^1 + \Delta^2,$$

where Δ^1 is a face of $m^{r(n-t)} I_{i_1}^L \cdots I_{i_s}^L$ and $\Delta^2 = \sum_{i \in S \smallsetminus C_L} \Delta_i$.

Then we observe that the set of common zeros of $(g_1)_{\Delta_1}, \dots, (g_n)_{\Delta_n}$ is contained in the set of common zeros of $(g_{i_1})_{\Delta'_{i_1}}, \dots, (g_{i_s})_{\Delta'_{i_s}}$.

Since Δ_i' is a face of I_i^L, for all $i \in C_L$, then $(g_i)_{\Delta_i'} = (g_i^L)_{\Delta_i'}$, for all $i \in C_L$. Then the inclusion (22) follows, since the map $(g_{i_1}^L, \dots, g_{i_s}^L) : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^s, 0)$ is non-degenerate with respect to $I_{i_1}^L, \dots, I_{i_s}^L$, by hypothesis. $\square$

Let us suppose that $I_1, \dots, I_n$ are ideals of finite colength of $\mathcal{O}_n$. Then Rees showed in [17] that the mixed multiplicity $e(I_1, \dots, I_n)$ can be computed in terms of Samuel mutiplicities via the following formula:

$$e(I_1, \dots, I_n) = \frac{1}{n!} \sum_{\substack{J \subseteq \{1,\dots,n\} \\ J \neq \emptyset}} (-1)^{n-|J|} e\Big(\prod_{j\in J} I_j\Big).$$

If we assume that I_i is a monomial ideal for all $i = 1, \dots, n$, then $e(\prod_{j\in J} I_j)$ can be computed effectively using [3], for all $J \subseteq \{1, \dots, n\}$, $J \neq \emptyset$. That is, we can apply [3, Theorem 5.1] to deduce that if f_J denotes the polynomial given by the sum of all x^k such that k is a vertex of $\Gamma_+(\prod_{j\in J} I_j)$, for all non-empty $J \subseteq \{1, \dots, n\}$, then

$$e(I_1, \dots, I_n) = \frac{1}{n!} \sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\emptyset}} (-1)^{n-|J|} \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle x_1\frac{\partial f_J}{\partial x_1}, \dots, x_n\frac{\partial f_J}{\partial x_n}\rangle}.$$

Thus we have an effective method to compute the mixed multiplicity $e(I_1, \dots, I_n)$ when I_i are monomial ideals of finite colength of $\mathcal{O}_n$. Let us suppose now that some of these ideals do not have finite colength but still $\sigma(I_1, \dots, I_n) < \infty$. Then, by the above discussion, the effective computation of $\sigma(I_1, \dots, I_n)$ reduces to compute some $r \geqslant 1$ such that $\sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)$. If $g = (g_1, \dots, g_n) \in \mathcal{R}(I_1, \dots, I_n)$, then we found in the proof of Theorem 3.10 that g is non-degenerate with respect to $I_1 + m^r, \dots, I_n + m^r$, when r is an integer such that $\Gamma_+(Q_r)$ satisfy conditions (1) and (2) of Lemma 3.9. Hence $e(g_1, \dots, g_n) = e(I_1 + m^r, \dots, I_n + m^r)$ and therefore $\sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)$. Obviously, the problem of finding an integer r satisfying these conditions is easy when $n = 2$, and needs a more careful analysis in higher dimensions.

To end the paper we show a result about the computation of the monomials which are integral over the ideal generated by the components of a given map of $\mathcal{R}(I_1, \dots, I_n)$.

Proposition 3.11. *Let $I_1, \dots, I_n$ monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. Let $g = (g_1, \dots, g_n) \in \mathcal{R}(I_1, \dots, I_n)$. Then*

$$I_1 \cap \dots \cap I_n \subseteq \overline{\langle g_1, \dots, g_n\rangle}.$$

Proof. Let J be the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. Let x^k be a monomial in $\mathcal{O}_n$. By Rees' multiplicity theorem we know that $x^k \in \overline{J}$ if and only if $e(J) = e(J, x^k)$ (see [9, p. 222]).

By a result of Northcott-Rees (see [9, p. 166] or [14]), we can consider general $\mathbb{C}$-linear combinations $h_1, \dots, h_n$ of $g_1, \dots, g_n, x^k$ such that the ideal H generated by $h_1, \dots, h_n$ is a reduction of $J + \langle x^k\rangle$. Then $e(H) = e(J, x^k)$. Therefore, let A be a squared matrix of size n with entries in $\mathbb{C}$ and let B be a row matrix with n columns with entries in $\mathbb{C}$ such that

$$\begin{bmatrix} g_1 & \cdots & g_n & x^k \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix}. \tag{23}$$

Since the coefficients of A are generic, we can suppose that A is invertible. In particular, multiplying both sides of (23) by A^{-1}, we obtain:

$$\begin{bmatrix} g_1 & \cdots & g_n & x^k \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \\ BA^{-1} \end{bmatrix} = \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix} A^{-1}, \tag{24}$$

where $\mathbf{I}_n$ denotes the identity matrix of size n. We observe that the entries of the left hand side of (24) are of the form $g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k$, for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Relation (24) implies that $H = \langle g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k\rangle$. Then, we obtain that

$$e(J) \geqslant e(J, x^k) = e(H) = e(g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k), \tag{25}$$

for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. If $x^k \in I_1 \cap \cdots \cap I_n$, then $e(g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k) \geqslant \sigma(I_1, \dots, I_n)$, by Lemma 2.2. But by Theorem 3.10, the equality $e(J) = \sigma(I_1, \dots, I_n)$ holds, since we assume that $g \in \mathcal{R}(I_1, \dots, I_n)$. Then (25) implies that $e(J) = e(J, x^k)$ and hence $x^k \in \overline{J}$. $\square$

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