# JOINT REDUCTIONS OF MONOMIAL IDEALS AND MULTIPLICITY OF COMPLEX ANALYTIC MAPS

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Abstract. We characterize the joint reductions of a set of monomial ideals in the ring $\mathcal{O}_n$ of complex analytic functions defined in a neighbourhood of the origin in $\mathbb{C}^n$. We also define an integer $\sigma(I_1,\dots,I_n)$ attached to a family of ideals $I_1,\dots,I_n$ in a Noetherian local ring that extends the usual notion of mixed multiplicity. If $I_1,\dots,I_n$ are monomial ideals, then we also obtain a characterization of the families $g_1,\dots,g_n$ such that $g_i\in I_i$, for all $i=1,\dots,n$, and that $e(g_1,\dots,g_n)=\sigma(I_1,\dots,I_n)$.

## 1. Introduction

The computation of the integral closure of ideals is one of the central problems in commutative algebra (see [5], [9] or [26]). A key role in the context of this problem is played by the reductions of an ideal, which were defined by Northcott and Rees in [14] (see Section 2). These ideals are very useful in the computation of multiplicities of ideals. For instance, if $I$ is an ideal of $\mathbb{C}[[x_1,\dots,x_n]]$ of finite colength generated by monomials, then the author obtained in [3] a canonical reduction of $I$ that allowed to compute the multiplicity of $I$ in an effective way (we refer [6] for a different approach to the computation of the multiplicity of a monomial ideal).

The notion of reduction of an ideal was generalized by Rees in [17] thus giving the notion of joint reduction of ideals. This notion simplifies the task of computing the mixed multiplicities of ideals, defined by Teissier and Risler in [21]. By a result of Swanson [20], joint reductions of ideals of finite colength are characterized via an equality of mixed multiplicities. This results extends the celebrated Rees' multiplicity theorem (see [9, p. 222]).

In Section 2 we define an integer attached to an ample class of $n$-tuples of ideals $I_1,\dots,I_n$ in a Noetherian local ring of dimension $n$ (see Definition 2.4). This integer, that we denote by $\sigma(I_1,\dots,I_n)$, extends the notion of mixed multiplicity of ideals of finite colength defined by Teissier and Risler in [21]. However $\sigma(I_1,\dots,I_n)$ it is not defined for arbitrary $n$-tuples of ideals. We call $\sigma(I_1,\dots,I_n)$ the *$\sigma$-multiplicity* of $I_1,\dots,I_n$. In the study of this new invariant we apply results developed by Rees [17] and Swanson [20] concerning joint reductions, mixed multiplicities and integral closures of ideals.

Let us denote by $\mathcal{O}_n$ the ring of analytic function germs $f:(\mathbb{C}^n,0)\to\mathbb{C}$. Let $I_1,\dots,I_n$ be monomial ideals of $\mathcal{O}_n$. Then we give in Section 3 a combinatorial characterization of the joint reductions of $I_1,\dots,I_n$ (see Proposition 3.7). If we assume that $\sigma(I_1,\dots,I_n)<\infty$, then we

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will apply this result to characterize those analytic maps $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g_i \in I_i$, for all $i = 1, \dots, n$ and such that $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$ (see Theorem 3.10), where $e(g_1, \dots, g_n)$ is the Samuel multiplicity of the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. This characterization is expressed via the respective Newton polyhedra of $I_1, \dots, I_n$. The set of such maps is denoted by $\mathcal{R}(I_1, \dots, I_n)$.

If $I_1, \dots, I_n$ are monomial and integrally closed ideals of $\mathcal{O}_n$, then, at the end of the paper, we give a result where an important part of the integral closure of the ideals generated by the components of a map of $\mathcal{R}(I_1, \dots, I_n)$ is computed.

The results that we show in this article will be applied, in a subsequent work, to problems in singularity theory concerning invariants of analytic functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. This is the main reason that we fix the setup of this work in $\mathcal{O}_n$ instead of the ring of formal power series $\mathbb{C}[[x_1, \dots, x_n]]$.

## 2. Joint reductions of ideals and $\sigma$-multiplicity

Let $R$ be a commutative ring. We denote by $\overline{I}$ the integral closure of an ideal $I$ of $R$. If $J$ and $I$ are ideals of $R$ such that $J \subseteq I$, then $J$ is said to be a *reduction* of $I$ if there exists an integer $r \geqslant 0$ such that $I^{r+1} = JI^r$. This definition is due to Northcott and Rees [14]. It is known that $J$ is a reduction of $I$ if and only if $\overline{I} = \overline{J}$ (see [9, p. 6]). The notion of reduction was generalized by Rees in [17] by defining the notion of joint reduction of a set of ideals.

**Definition 2.1.** [17] Let $I_1, \dots, I_p$ be ideals of $R$. Let $g_1, \dots, g_p$ be elements of $R$ such that $g_i \in I_i$, for all $i = 1, \dots, p$. The $p$-tuple $(g_1, \dots, g_p)$ is termed a *joint reduction of* $(I_1, \dots, I_p)$ if and only if the ideal

$$g_1 I_2 \cdots I_p + g_2 I_1 I_3 \cdots I_p + \cdots + g_p I_1 \cdots I_{p-1}$$

is a reduction of $I_1 \cdots I_p$.

Let $(R, m)$ be a Noetherian local ring of dimension $n$. If an ideal $I$ of $R$ is $m$-primary then we will also say that $I$ has *finite colength*. If $I$ is an ideal of $R$ of finite colength then we denote by $e(I)$, or by $e(I; R)$, the multiplicity of $I$ in the sense of Samuel (see [9, p. 214]). If $I_1, \dots, I_n$ are ideals of $R$ of finite colength, we denote indistinctly by $e(I_1, \dots, I_n)$ or by $e(I_1, \dots, I_n; R)$ the mixed multiplicity of $I_1, \dots, I_n$ defined by Teissier and Risler in [21] (we also refer to [9, §17] or [20] for the definition and fundamental results concerning mixed multiplicities of ideals). We remark that if $I_1, \dots, I_n$ are all equal to a given ideal, say $I$, then $e(I_1, \dots, I_n) = e(I)$. We will need the following known result (see [9, p. 345] or [20, Lemma 2.4]).

**Lemma 2.2.** *Let $R$ be a Noetherian local ring of dimension $n \geqslant 1$. Let $I_1, \dots, I_n$ be ideals of $R$ of finite colength. Let $g_1, \dots, g_n$ be elements of $R$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and that the ideal $\langle g_1, \dots, g_n \rangle$ has also finite colength. Then*

$$e(g_1, \dots, g_n) \geqslant e(I_1, \dots, I_n).$$

Rees proved in [16] that if $J \subseteq I$ are ideals of a quasi-unmixed Noetherian local ring $R$, then $J$ is a reduction of $I$ if and only if $e(I) = e(J)$ (see also [9, p. 222]). Moreover, Rees proved in

[17, Theorem 2.4] that if $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$, where $I_1, \dots, I_n$ is a set of ideals of finite colength of a local Noetherian ring $R$, then $e(g_1, \dots, g_n) = e(I_1, \dots, I_n)$ (see also [9, p. 343]). The converse of this result is a nice result of Swanson that we now state.

**Theorem 2.3.** [20] *Let $R$ be a quasi-unmixed Noetherian local ring. Let $I_1, \dots, I_s$ be ideals and let $g_i$ be an element of $I_i$, for $i = 1, \dots, s$. Suppose that the ideals $I_1, \dots, I_s$ and $\langle g_1, \dots, g_s\rangle$ have the same height $s$ and the same radical. If*

$$e(\langle g_1, \dots, g_s\rangle R_{\mathfrak{p}}; R_{\mathfrak{p}}) = e(I_1 R_{\mathfrak{p}}, \dots, I_s R_{\mathfrak{p}}; R_{\mathfrak{p}}),$$

*for each prime ideal $\mathfrak{p}$ minimal over $\langle g_1, \dots, g_s\rangle$, then $(g_1, \dots, g_s)$ is a joint reduction of $(I_1, \dots, I_s)$.*

We now define an invariant, defined in terms of mixed multiplicities of ideals, that is attached to a set of ideals in a Noetherian local ring. The ideals we consider are not assumed to have finite colength. We denote by $\mathbb{Z}_+$ the set of non-negative integers.

**Definition 2.4.** Let $(R, m)$ be a Noetherian local ring of dimension $n$. Let $I_1, \dots, I_n$ be ideals of $R$. Then we define the *$\sigma$-multiplicity of* $I_1, \dots, I_n$ as

$$\sigma(I_1, \dots, I_n) = \max_{r\in\mathbb{Z}_+} e(I_1 + m^r, \dots, I_n + m^r). \tag{1}$$

The set of integers $\{e(I_1 + m^r, \dots, I_n + m^r) : r \in \mathbb{Z}_+\}$ is not bounded in general; therefore $\sigma(I_1, \dots, I_n)$ is not always finite for any family of ideals $I_1, \dots, I_n$. The finiteness of $\sigma(I_1, \dots, I_n)$ is characterized in Proposition 2.9. We remark that if $I_i$ has finite colength, for all $i = 1, \dots, n$, then $\sigma(I_1, \dots, I_n)$ equals the mixed multiplicity $e(I_1, \dots, I_n)$.

**Proposition 2.5.** *Let $(R, m)$ be a Noetherian local ring of dimension $n$. Let $I_1, \dots, I_n$ be ideals of $R$ such that $\sigma(I_1, \dots, I_n) < \infty$ and let $g_1, \dots, g_n$ be elements of $R$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and $\langle g_1, \dots, g_n\rangle$ is an ideal of finite colength. Then $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$ if and only if there exists an integer $r_0 \geqslant 1$ such that $(g_1, \dots, g_n)$ is a joint reduction of $(I_1 + m^r, \dots, I_n + m^r)$, for all $r \geqslant r_0$.*

*Proof.* The *if* part follows as a direct consequence of the expression of mixed multiplicities as the multiplicity of a joint reduction (see the paragraph before Theorem 2.3).

Conversely, if $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$ then $(g_1, \dots, g_n)$ is a joint reduction of $(I_1 + m^r, \dots, I_n + m^r)$, for all $r \gg 0$, as a consequence of Theorem 2.3. □

By virtue of the previous result we give the following definition.

**Definition 2.6.** Let $(R, m)$ be a local ring of dimension $n$ and let $I_1, \dots, I_n$ be ideals of $R$. Let $g_i \in I_i$, for $i = 1, \dots, n$. We say that $g_1, \dots, g_n$ is a *$\sigma$-joint reduction* of $(I_1, \dots, I_n)$ when there exists an integer $r_0 \geqslant 1$ such that $(g_1, \dots, g_n)$ is a joint reduction of $(I_1 + m^r, \dots, I_n + m^r)$, for all $r \geqslant r_0$.

We will use the following auxiliary result, whose proof appears in [9, p. 134].

**Lemma 2.7.** *Let $(R, m)$ be a Noetherian local ring and let $I$ be an ideal of $R$. Then*

$$\overline{I} = \bigcap_{r \geqslant 1} \overline{I + m^r}.$$

**Proposition 2.8.** *Let $(R, m)$ be a Noetherian local ring of dimension $n$ and let $I_1, \dots, I_n$ be ideals of $R$. Let $g_i \in I_i$, for $i = 1, \dots, n$. If $g_1, \dots, g_n$ is a $\sigma$-joint reduction of $(I_1, \dots, I_n)$ then $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$.*

*Proof.* When $n = 1$ the result follows easily from Lemma 2.7. Let us suppose that $n \geqslant 2$. Let us define the ideals

$$\begin{aligned} P_r &= g_1(I_2 + m^r) \cdots (I_n + m^r) + \cdots + g_n(I_1 + m^r) \cdots (I_{n-1} + m^r) \\ Q_r &= (I_1 + m^r) \cdots (I_n + m^r). \end{aligned}$$

Then there exists an integer $r_0 \geqslant 1$ such that

$$\overline{Q_r} = \overline{P_r}, \quad \text{for all } r \geqslant r_0. \tag{2}$$

If $j, s \in \{1, \dots, n\}$, we define

$$L_j = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} I_{i_1} \cdots I_{i_j}, \qquad L_j^s = \sum_{\substack{1 \leqslant i_1 < \cdots < i_j \leqslant n \\ i_j \neq s}} I_{i_1} \cdots I_{i_j},$$

where in the definition of $L_j^s$ we suppose that $j \leqslant n - 1$. Then, a simple computation shows that

$$L_n + m^{r(n-1)} L_1 \subseteq Q_r = L_n + m^r L_{n-1} + \cdots + m^{r(n-1)} L_1 + m^{rn} \subseteq L_n + m^{r+n-1} \tag{3}$$

and that

$$P_r = g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n + \sum_{i=1}^n g_i \left( m^r L_{n-2}^i + \cdots + m^{(n-2)r} L_1^i + m^{(n-1)r} \right). \tag{4}$$

Let $J$ denote the ideal of $R$ generated by $g_1, \dots, g_n$. Then

$$g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n + m^{(n-1)r} J \subseteq P_r \subseteq g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n + m^{r+n-1}. \tag{5}$$

Then, from Lemma 2.7 and the inclusions given in (3) and (5) we obtain the equalities

$$\overline{L_n} = \bigcap_{r \geqslant 1} \overline{Q_r}, \qquad \overline{g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n} = \bigcap_{r \geqslant 1} \overline{P_r}. \tag{6}$$

Therefore, from (2) we have

$$\overline{g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n} = \bigcap_{r \geqslant r_0} \overline{P_r} = \bigcap_{r \geqslant r_0} \overline{Q_r} = \overline{L_n} = \overline{I_1 \cdots I_n}.$$

This implies that $g_1 L_{n-1}^1 + \cdots + g_n L_{n-1}^n$ is a reduction of $I_1 \cdots I_n$, or equivalently, that $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$. □

In Example 2.10 we show that the converse of Proposition 2.8 does not hold in general.

Let $(R,m)$ be a local ring of dimension $n$ with $k=R/m$ an infinite field. Let $I_1,\dots,I_n$ be ideals of $R$. Let us consider a generating system $a_{i1},\dots,a_{is_i}$ of $I_i$, for $i=1,\dots,n$. Let $s=s_1+\cdots+s_n$. We say that a property holds for *sufficiently general* elements of $I_1\oplus\cdots\oplus I_n$ if there exists a non-empty Zariski-open set $U$ in $k^s$ such that all elements $(g_1,\dots,g_n)\in I_1\oplus\cdots\oplus I_n$ satisfy the said property provided that $g_i=\sum_j u_{ij}a_{ij}$, $i=1,\dots,n$, where $(u_{11},\dots,u_{1s_1},\dots,u_{n1},\dots,u_{ns_n})\in U$.

**Proposition 2.9.** *Let $I_1,\dots,I_n$ be ideals of a Noetherian local ring $(R,m)$ such that the residue field $k=R/m$ is infinite. Then $\sigma(I_1,\dots,I_n)<\infty$ if and only if there exist elements $g_i\in I_i$, for $i=1,\dots,n$, such that $\langle g_1,\dots,g_n\rangle$ has finite colength and $\sigma(I_1,\dots,I_n)=e(g_1,\dots,g_n)$. In this case, we have that $\sigma(I_1,\dots,I_n)=e(g_1,\dots,g_n)$ for sufficiently general elements $(g_1,\dots,g_n)\in I_1\oplus\cdots\oplus I_n$.*

*Proof.* The *if* part is immediate. Let us suppose that $\sigma(I_1,\dots,I_n)<\infty$. Then there exists a positive integer $r_0$ such that

$$\sigma(I_1,\dots,I_n)=e(I_1+m^r,\dots,I_n+m^r),$$

for all $r\geqslant r_0$. By the definition of joint reduction we have that if $(a_1,\dots,a_n)$ is a joint reduction of $(I_1+m^r,\dots,I_n+m^r)$ and $P$ denotes the ideal

$$a_1(I_2+m^r)\cdots(I_n+m^r)+\cdots+a_n(I_1+m^r)\cdots(I_{n-1}+m^r),$$

then

$$\overline{(I_1+m^r)\cdots(I_n+m^r)}=\overline{P}\subseteq\overline{\langle a_1,\dots a_n\rangle}.$$

Therefore, we observe that there exists an integer $s\geqslant 1$ such that $m^s\subseteq\overline{\langle a_1,\dots,a_n\rangle}$, for all joint reduction $(a_1,\dots,a_n)$ of $(I_1+m^{r_0},\dots,I_n+m^{r_0})$. We can suppose that $s\geqslant r_0$.

By the theorem of existence of joint reductions (see [20, p. 4] or [9, p. 336]), let us consider elements $g_i\in I_i$, for $i=1,\dots,n$, and elements $h_i\in m^{s+1}$, for $i=1,\dots,n$, such that $(g_1,\dots,g_n)$ is a joint reduction of $(I_1,\dots,I_n)$ and that $(g_1+h_1,\dots,g_n+h_n)$ is a joint reduction of $(I_1+m^{s+1},\dots,I_n+m^{s+1})$. Let $J$ be the ideal of $R$ generated by $g_1+h_1,\dots,g_n+h_n$. Then $J$ has finite colength and $e(J)=e(I_1+m^{s+1},\dots,I_n+m^{s+1})$.

Since $s\geqslant r_0$, we have

$$e(I_1+m^{r_0},\dots,I_n+m^{r_0})=e(I_1+m^{s+1},\dots,I_n+m^{s+1})=e(J).$$

Then it follows that $(g_1+h_1,\dots,g_n+h_n)$ is a joint reduction of $(I_1+m^{r_0},\dots,I_n+m^{r_0})$ by Theorem 2.3. But this implies that $m^s\subseteq\overline{J}$, by the definition of $s$.

Hence we have

$$\overline{J}\subseteq\overline{\langle g_1,\dots,g_n\rangle+m\cdot m^s}\subseteq\overline{\langle g_1,\dots,g_n\rangle+m\cdot J}.$$

By the integral Nakayama's Lemma (see [21, p. 324]), we deduce that

$$\overline{J}\subseteq\overline{\langle g_1,\dots,g_n\rangle}.$$

Then $\langle g_1,\dots,g_n\rangle$ has also finite colength. Moreover we have

$$\sigma(I_1,\dots,I_n)=e(J)\geqslant e(g_1,\dots,g_n)\geqslant e(I_1+m^{r_0},\dots,I_1+m^{r_0})=\sigma(I_1,\dots,I_n).$$

Hence we have

$$e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n). \tag{7}$$

By the construction of the elements $g_1, \dots, g_n$ that we have considered, we observe that equality (7) is satisfied for sufficiently general elements of $I_1 \oplus \cdots \oplus I_n$, as a consequence of the theorem of existence of joint reductions. □

By Proposition 2.9, if $\sigma(I_1, \dots, I_n) < \infty$ then $I_1 + \cdots + I_n$ is an ideal of finite colength in $R$. Obviously the converse does not hold. We also have that $e(I_1 + \cdots + I_n) \leqslant \sigma(I_1, \dots, I_n)$, by Lemma 2.2. As a consequence of Rees' multiplicity theorem (see [9, p. 222]) we have that $e(I_1 + \cdots + I_n) = \sigma(I_1, \dots, I_n)$ if and only if any $n$-tuple $(g_1, \dots, g_n)$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and satisfying the equality $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$ generates a reduction of $I_1 + \cdots + I_n$.

By Propositions 2.5 and 2.8 we have that $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$, where $(g_1, \dots, g_n)$ is a joint reduction of $(I_1, \dots, I_n)$. However, if $\sigma(I_1, \dots, I_n) < \infty$, not every joint reduction of $I_1, \dots, I_n$ generates an ideal of finite colength. Moreover, if $I$ is the ideal generated by a joint reduction of $I_1, \dots, I_n$ and we suppose that $I$ has finite colength then it does not hold in general that $e(I) = \sigma(I_1, \dots, I_n)$. Both facts are shown in the following example.

**Example 2.10.** Let us consider in $\mathcal{O}_3$ the ideals $I_1 = I_2 = \langle x, y \rangle$ and $I_3 = \langle z \rangle$ and the elements $g_1 = g_2 = x + y$ and $g_3 = z$, where we have fixed the coordinates $x, y, z$ in $\mathbb{C}^3$. It is obvious that $\sigma(I_1, I_2, I_3) = 1$ and that $(g_1, g_2, g_3)$ is a joint reduction of $(I_1, I_2, I_3)$. However $g_1, g_2, g_3$ do not generate an ideal of finite colength of $\mathcal{O}_3$.

Let us consider the elements $g_1' = x + y + x^3$, $g_2' = x + y + y^3$, $g_3' = z$. Then we observe that $(g_1', g_2', g_3')$ is also a joint reduction of $(I_1, I_2, I_3)$. These elements generate an ideal of finite colength of $\mathcal{O}_3$ but $\sigma(I_1, I_2, I_3) = 1$ and $e(g_1', g_2', g_3') = 3$.

Let $(R, m)$ be a Noetherian local ring of dimension $n$ such that the residue field $R/m$ is infinite. The mixed multiplicity of ideals, as introduced by Risler and Teissier [21] and studied by Rees [17] and Swanson [20], is defined for $n$ ideals $I_1, \dots, I_n$ of finite colength in $R$. By the theorem of existence of joint reductions (see [9, p. 336]), we have

$$e(I_1, \dots, I_n) = e(g_1, \dots, g_n) \tag{8}$$

where $(g_1, \dots, g_n)$ is a sufficiently general element of $I_1 \oplus \cdots \oplus I_n$.

The number $e(I_1, \dots, I_n)$ is equal to the coefficient of the term $r_1 \cdots r_n$ in the homogeneous part of degree $n$ of the polynomial that coincides with the length function $\ell(R/I_1^{r_1} \cdots I_n^{r_n})$ for $r_1, \dots, r_n \gg 0$. We observe that this function is well defined if and only if $I_i$ has finite colength, for all $i = 1, \dots, n$. However, the multiplicity on the right hand side of (8) could be computed in cases where some of the ideals $I_i$ has not finite colength. By Proposition 2.9, this multiplicity is expressed as a $\sigma$-multiplicity (see Definition 2.4).

If $I, J$ are two ideals of finite colength of $R$, then we can define for all $i \in \{0, 1, \dots, n\}$ the multiplicity

$$e_i(I, J) = e(I, \dots, I, J, \dots, J), \tag{9}$$

where $I$ is repeated $n-i$ times and $J$ is repeated $i$ times, for all $i = 0, 1, \dots, n$. If $I$ and $J$ are arbitrary ideals, we define analogously the number $\sigma_i(I, J)$ by replacing in (9) the mixed multiplicity $e(I, \dots, I, J, \dots, J)$ by $\sigma(I, \dots, I, J, \dots, J)$ (of course, for arbitrary ideals $I$ and $J$ the resulting numbers are not always finite for all $i = 0, 1, \dots, n$).

If $J$ is an ideal of $R$, let $J^\infty = \{x \in R : x^s J = 0, \text{ for some } s \geqslant 1\}$. As can be seen in the paper [24] of Trung, there is also defined a family of mixed multiplicities $\{e_i(I|J) : i = 0, 1, \dots, r\}$ of a pair of ideals $I, J$, where $I$ is assumed to have finite colength, $J$ is an arbitrary ideal of $R$ and $r = \dim(R/(0 : J^\infty)) - 1$. These numbers arise from the coefficients of the homogeneous part of highest degree of the polynomial that coincides with the length function of the bigraded ring

$$R(I|J) = \bigoplus_{(u,v)\in\mathbb{Z}^2_+} I^u J^v / I^{u+1} J^v.$$

We refer to [11], [24], [25] and [28] for the details about this definition.

Let $\ell(J)$ denote the analytic spread of $J$. The multiplicities $e_i(I|J)$ are not all positive for all $i = 0, 1, \dots, r$. In fact, Trung proved that $e_i(I|J) = 0$, for all $i \geqslant \ell(J)$ (see [24, Corollary 3.6]). Moreover, if $i \in 0, 1, \dots, \mathrm{ht}(J) - 1$, then it is proved in [24, Proposition 4.1] that

$$e_i(I|J) = e\big(a_1, \dots, a_{n-i}, b_1, \dots, b_i\big), \tag{10}$$

where $(a_1, \dots, a_{n-i}, b_1, \dots, b_i)$ is a sufficiently general element of $I \oplus \cdots \oplus I \oplus J \oplus \cdots \oplus J$. We remark that relation (10) shows that $e_i(I|J) = \sigma_i(I, J)$, for all $i \in 0, 1, \dots, \mathrm{ht}(J) - 1$, by Proposition 2.9. However, we show a simple example where the multiplicity on the right hand part of (10) can be positive for $i = \ell(J)$ and therefore it can be expressed as a $\sigma$-multiplicity.

**Example 2.11.** Let $I$, $J$ be the ideals in $\mathcal{O}_3$ given by $I = \langle x, y, z\rangle$, $J = \langle x^2, y^2\rangle$. Then $\ell(J) = 2$ (see [3, Theorem 2.3]) and $\sigma_2(I, J) = \sigma(I, J, J) = 4$.

## 3. Mixed multiplicities and non-degeneracy

Throughout the remaining text, if no confusion arises, we will denote the maximal ideal of $\mathcal{O}_n$ by $m$ instead of $m_n$. We say that an ideal $I$ of $\mathcal{O}_n$ is a *monomial ideal* when $I$ is generated by a family of monomials $x^k$ such that $k \in \mathbb{Z}^n_+$, $k \neq 0$. Let $I_1, \dots, I_n$ be a sequence of monomial ideals in $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. In this section we characterize the sets of functions $g_1, \dots, g_n \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and that $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$. In order to show our results we will introduce first some definitions and notation.

Let $h \in \mathcal{O}_n$, let us suppose that the Taylor expansion of $h$ around the origin is given by $h = \sum_k a_k x^k$. We define the *support* of $h$, denoted by $\mathrm{supp}(h)$, as the set of those $k \in \mathbb{Z}^n_+$ such that $a_k \neq 0$. If $A$ is a compact subset of $\mathbb{R}^n_+$, then we denote by $h_A$ the polynomial given by the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap A$. If $\mathrm{supp}(h) \cap A = \emptyset$, then we set $h_A = 0$. If $I$ is a monomial ideal of $\mathcal{O}_n$, we define the *support* of $I$, denoted by $\mathrm{supp}(I)$, as the set of $k \in \mathbb{Z}^n_+$ such that $x^k \in I$.

We say that a subset $\Gamma_+$ of $\mathbb{R}^n_+$ is a *Newton polyhedron* when there exists some $B \subseteq \mathbb{Q}^n_+$ such that $\Gamma_+$ is equal to the convex hull in $\mathbb{R}^n_+$ of the set $\{k + v : k \in B, v \in \mathbb{R}^n_+\}$. In this case we say that $\Gamma_+$ is the *Newton polyhedron determined by* $B$ and we also denote $\Gamma_+$ by $\Gamma_+(B)$. A Newton polyhedron $\Gamma_+$ is termed *convenient* when $\Gamma_+$ intersects each coordinate axis in a point different from the origin. In this case, we denote by $\mathrm{V}_n(\Gamma_+)$ the $n$-dimensional volume of the set $\mathbb{R}^n_+ \smallsetminus \Gamma_+$.

If $h \in \mathcal{O}_n$, the *Newton polyhedron of* $h$ is defined as $\Gamma_+(h) = \Gamma_+\big(\mathrm{supp}(h)\big)$. Let $J$ be an ideal of $\mathcal{O}_n$, let us suppose that $J$ is generated by the elements $h_1, \dots, h_p$. Then the *Newton polyhedron of* $J$, denoted by $\Gamma_+(J)$, is defined as the convex hull of the union $\Gamma_+(h_1) \cup \cdots \cup \Gamma_+(h_p)$. It is easy to check that the definition of $\Gamma_+(J)$ does not depend on the chosen generating system of $J$.

If $\Gamma^1_+, \dots, \Gamma^p_+$ are Newton polyhedra in $\mathbb{R}^n_+$, then we define the *Minkowski sum* of $\Gamma^1_+, \dots, \Gamma^p_+$ as

$$\Gamma^1_+ + \cdots + \Gamma^p_+ = \{k_1 + \cdots + k_p : k_i \in \Gamma^i_+, \text{ for all } i = 1, \dots, p\}.$$

This set is again a Newton polyhedron, since it is known that $\Gamma^1_+ + \cdots + \Gamma^p_+ = \Gamma_+(I_1 \cdots I_p)$, whenever $\Gamma^i_+ = \Gamma_+(I_i)$, for some monomial ideal $I_i \in \mathcal{O}_n$, $i = 1, \dots, p$ (see for instance [7]).

Let us fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$. Given a vector $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ we define

$$\ell(v, \Gamma_+) = \min\big\{\langle v, k\rangle : k \in \Gamma_+\big\}.$$

We say that a subset $\Delta$ of $\Gamma_+$ is a *face* of $\Gamma_+$ if there exists a vector $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ such that $\Delta$ is expressed as

$$\Delta = \big\{k \in \Gamma_+ : \langle v, k\rangle = \ell(v, \Gamma_+)\big\}. \tag{11}$$

We will denote the set on the right hand side of (11) by $\Delta(v, \Gamma_+)$ and we will also say that $\Delta$ is the face of $\Gamma_+$ *supported* by $v$. We have that $\Delta(v, \Gamma_+)$ is a compact face of $\Gamma_+$ if and only if all components of $v$ are non-zero. If $I$ is an ideal of $\mathcal{O}_n$, then we denote by $\Gamma(I)$ the union of the compact faces of $\Gamma_+(I)$. Moreover, we will denote the face $\Delta(v, \Gamma_+(I))$, for a given $v \in \mathbb{R}^n \smallsetminus \{0\}$, by $\Delta(v, I)$.

**Definition 3.1.** Let $I_1, \dots, I_p$ be monomial ideals in $\mathcal{O}_n$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Let $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ and let $\Delta_i = \Delta(v, I_i)$, for all $i = 1, \dots, p$. We say that $g$ satisfies the $(K_v)$ *condition with respect to* $I_1, \dots, I_p$ when

$$\big\{x \in \mathbb{C}^n : (g_1)_{\Delta_1}(x) = \cdots = (g_p)_{\Delta_p}(x) = 0\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

Then the map $g$ is termed *non-degenerate with respect to* $I_1, \dots, I_p$ when $g$ satisfies the $(K_v)$ condition with respect to $I_1, \dots, I_p$ for all $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$.

Under the conditions of the above definition, we observe that if there exists some $i_0 \in \{1, \dots, p\}$ such that $g_{i_0}$ is equal to a monomial $x^k$, for some $k \in \mathbb{Z}^n_+$, $k \neq 0$, and $I_{i_0} = \langle x^k\rangle$, then the map $g$ is automatically non-degenerate with respect to $I_1, \dots, I_p$.

If $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, we define $\mathbb{C}^n_L = \{x \in \mathbb{C}^n : x_i = 0, \text{ for all } i \notin L\}$. The set $\mathbb{R}^n_L$ is defined analogously. If $h \in \mathcal{O}_n$ and the Taylor expansion of $h$ around the origin is given by $h = \sum_k a_k x^k$, we denote by $h^L$ the function obtained as the sum of those terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap \mathbb{R}^n_L$. If $\mathrm{supp}(h) \cap \mathbb{R}^n_L = \emptyset$, then we set $h^L = 0$. If $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ, then we denote by $g^L$ the map $(g_1^L, \dots, g_p^L) : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^p, 0)$. In some occasions we will identify $\mathbb{C}^n_L$ with $\mathbb{C}^r$, where $r = |L|$.

Let $L = \{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$, then we denote by $\mathcal{O}_{n,L}$ the subring of $\mathcal{O}_n$ generated by the functions of $\mathcal{O}_n$ depending, at most, on the variables $x_{i_1}, \dots, x_{i_r}$. We denote by $m_L$ the maximal ideal of $\mathcal{O}_{n,L}$. We observe that the map $\mathcal{O}_n \to \mathcal{O}_{n,L}$ given by $h \mapsto h^L$, $h \in \mathcal{O}_n$, is a ring epimorphism. If $I$ is a monomial ideal of $\mathcal{O}_n$ then we denote by $I^L$ the ideal of $\mathcal{O}_{n,L}$ generated by all monomials $x^k$ such that $k \in \operatorname{supp}(I) \cap \mathbb{R}^n_L$. If $\operatorname{supp}(I) \cap \mathbb{R}^n_L = \emptyset$, then we set $I^L = 0$.

**Definition 3.2.** Let $I_1, \dots, I_p$ be monomial ideals of $\mathcal{O}_n$ such that $I_1 + \cdots + I_p$ is an ideal of finite colength in $\mathcal{O}_n$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that $g_i \in I_i$, for all $i = 1, \dots, p$. We say that $g$ is *strongly non-degenerate with respect to* $I_1, \dots, I_p$ when for all $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, the map $g^L : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^p, 0)$ is non-degenerate with respect to the non-zero ideals of the sequence of ideals $I_1^L, \dots, I_p^L$.

We remark that, since we are assuming in the above definition that $I_1 + \cdots + I_p$ is an ideal of finite colength, then the set of non-zero ideals in the sequence $I_1^L, \dots, I_p^L$ is non-empty, for all $L \subseteq \{1 \dots, n\}$, $L \neq \emptyset$.

Under the conditions of Definition 3.2, we denote the set of analytic maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ such that $g_i \in I_i$, for all $i = 1, \dots, p$, and such that $g$ is strongly non-degenerate with respect to $I_1, \dots, I_p$ by $\mathcal{R}(I_1, \dots, I_p)$. Let us remark that if $g \in \mathcal{R}(I_1, \dots, I_p)$ then $g_i$ does not need to have the same Newton polyhedron as $I_i$, for all $i = 1, \dots, p$.

**Example 3.3.** Let us consider the ideals $I_1, I_2, I_3$ of $\mathcal{O}_3$ and the polynomials $g_1', g_2', g_3'$ given in Example 2.10. Then we have that the map $g' : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ defined by $g' = (g_1', g_2', g_3')$ is non-degenerate with respect to $I_1, I_2, I_3$. If $L = \{1, 2\}$, then $\{i : I_i^L \neq 0\} = \{1, 2\}$. We observe that the map $h = ((g_1')^L, (g_2')^L)$ is not non-degenerate with respect to $I_1^L, I_2^L$, since $h$ does not satisfy the $(K_v)$ condition for $v = (1, 1)$. Therefore $g'$ is not strongly non-degenerate with respect to $I_1, I_2, I_3$.

**Remark 3.4.** Let $\Gamma_+^1, \dots, \Gamma_+^p$ be a family of Newton polyhedra in $\mathbb{R}^n_+$. It is well known that if $\Delta$ is a compact face of $\Gamma_+^1 + \cdots + \Gamma_+^p$, then $\Delta$ is uniquely expressed as $\Delta_1 + \cdots + \Delta_p$, where $\Delta_i$ is face of $\Gamma_+^i$, for all $i = 1, \dots, p$. This expression is a consequence of the following relations:

$$\ell(v, \Gamma_+^1 + \cdots + \Gamma_+^p) = \ell(v, \Gamma_+^1) + \cdots + \ell(v, \Gamma_+^p)$$
$$\Delta(v, \Gamma_+^1 + \cdots + \Gamma_+^p) = \Delta(v, \Gamma_+^1) + \cdots + \Delta(v, \Gamma_+^p),$$

for all $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$. Therefore, under the hypothesis of Definition 3.1, the set of non-redundant $(K_v)$ conditions that a non-degenerate map with respect to $I_1, \dots, I_p$ must satisfy is parameterized by the set of compact faces of $\Gamma_+(I_1) + \cdots + \Gamma_+(I_p)$. Hence the definition of strongly non-degenerate map with respect to $I_1, \dots, I_p$ consists of a finite set of conditions.

Here we recall the definition of Newton non-degenerate ideal (see [3] or [4]). Let $I$ be an ideal of $\mathcal{O}_n$ and let $g_1, \dots, g_r$ be a generating system of $I$. Then the ideal $I$ is said to be

*Newton non-degenerate* when for each compact face $\Delta$ of $\Gamma_+(I)$ we have

$$\big\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_r)_\Delta(x)\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

It is straightforward to see that this definition does not depend on the generating system of $I$. We observe that any monomial ideal is Newton non-degenerate. Moreover, we also have that an ideal $I$ of $\mathcal{O}_n$ is Newton non-degenerate if and only if $I$ admits a generating system $g_1, \dots, g_r$ such that the map $(g_1, \dots, g_r) : (\mathbb{C}^n, 0) \to (\mathbb{C}^r, 0)$ is non-degenerate with respect to $I, \dots, I$, with $I$ repeated $r$ times (see Definition 3.1). If $I$ is an ideal of finite colength, then this condition is equivalent to saying that $(g_1, \dots, g_r) \in \mathcal{R}(I, \dots, I)$, where $I$ is repeated $r$ times (see also Corollary 3.8).

The next result shows a numerical characterization of the Newton non-degeneracy condition (we refer to [2] for the definition and characterization of the Newton non-degeneracy condition in the context of submodules of the free module $\mathcal{O}_n^p$, $p \geqslant 1$).

**Theorem 3.5.** [3, 4] *Let $I$ be an ideal of $\mathcal{O}_n$ of finite colength. Then $e(I) \geqslant n!\mathrm{V}_n\big(\Gamma_+(I)\big)$ and equality holds if and only if $I$ is a Newton non-degenerate ideal.*

Given an ideal $J$ of $\mathcal{O}_n$ and a fixed coordinate system in $\mathbb{C}^n$, we denote by $J_0$ the ideal of $\mathcal{O}_n$ generated by all monomials $x^k$ such that $k \in \Gamma_+(J)$. The ideal $J_0$ is integrally closed (see [9, p. 11] or [22]). Therefore, from the inclusions $J \subseteq \overline{J} \subseteq \overline{J_0} = J_0$, we deduce that $\Gamma_+(J) = \Gamma_+(\overline{J})$.

**Proposition 3.6.** *Let $I$ be a Newton non-degenerate ideal of $\mathcal{O}_n$ and let $J \subseteq I$. Then the following conditions are equivalent:*

(1) *$J$ is a reduction of $I$;*
(2) *$J$ is Newton non-degenerate and $\Gamma_+(J) = \Gamma_+(I)$;*
(3) *there exists a generating system $g_1, \dots, g_r$ of $J$ such that, for all compact face $\Delta$ of $\Gamma_+(I)$, we have*

$$\big\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_r)_\Delta(x)\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}. \tag{12}$$

*Proof.* We point out that the ideal $I$ is not assumed to have finite colength. Let us see (1) $\Rightarrow$ (2). Suppose that $J$ is a reduction of $I$. Then $\overline{I} = \overline{J}$ and, in particular, we have that $\Gamma_+(I) = \Gamma_+(J)$. Moreover we also deduce that

$$\overline{I + m^r} = \overline{\overline{I} + m^r} = \overline{\overline{J} + m^r} = \overline{J + m^r}, \tag{13}$$

for all $r \geqslant 1$. Using relation (13) and the fact that $\overline{I + m^r}$ is a monomial ideal of finite colength, it follows that

$$n!\mathrm{V}_n\big(\Gamma_+(J + m^r)\big) = n!\mathrm{V}_n\big(\Gamma_+(I + m^r)\big) = e(I + m^r) = e(J + m^r),$$

by Theorem 3.5. Therefore the ideal $J + m^r$ is Newton non-degenerate, for all $r \geqslant 1$, by virtue of Theorem 3.5. Let $r_0$ a positive integer such that each compact face $\Delta$ of $\Gamma_+(J)$ is a compact face of $\Gamma_+(J + m^r)$, for all $r \geqslant r_0$. Therefore, by writing down the condition that $J + m^r$ is Newton non-degenerate, for all $r \geqslant r_0$, we conclude that $J$ is Newton non-degenerate.

Let us see (2) $\Rightarrow$ (1). We will see that item (2) implies that $\overline{I} = \overline{J}$. In particular, we will have that $J$ is a reduction of $I$, since $J \subseteq I$ (see [9, p. 6]). As before, let us consider a big enough positive integer $r_0$ such that each compact face of $\Gamma_+(J)$ is a compact face of $\Gamma_+(J + m^r)$, for all $r \geqslant r_0$. Then we have that $J + m^r$ is Newton non-degenerate, for all $r \geqslant r_0$. Hence $e(J + m^r) = n!\mathrm{V}_n(\Gamma_+(J + m^r))$, for all $r \geqslant r_0$. This implies, by Rees' multiplicity theorem, that

$$\overline{J + m^r} = (J + m^r)_0 = (J_0 + m^r)_0 = \overline{J_0 + m^r}, \text{ for all } r \geqslant r_0.$$

By Lemma 2.7, we have

$$\overline{J} = \bigcap_{r \geqslant r_0} \overline{J + m^r} = \bigcap_{r \geqslant r_0} \overline{J_0 + m^r} = \overline{J_0} = J_0. \tag{14}$$

Since $J \subseteq I$ and $\Gamma_+(I) = \Gamma_+(J)$, then $\overline{J} \subseteq \overline{I} \subseteq I_0 = J_0$. Then relation (14) implies that $\overline{I} = \overline{J}$.

The implication (2) $\Rightarrow$ (3) is obvious. In order to see the implication (3) $\Rightarrow$ (2) it suffices to prove that $\Gamma_+(I) = \Gamma_+(J)$. Let $g_1, \dots, g_r$ be a generating system of $J$ verifying the inclusion (12), for all compact face $\Delta$ of $\Gamma_+(I)$. In particular, if $\Delta$ is a vertex of $\Gamma_+(I)$, then this condition must be satisfied for $\Delta$. This implies that if $\Delta$ is any vertex of $\Gamma_+(I)$, then some function $(g_i)_\Delta$ is not identically zero. Thus $\Gamma_+(I) \subseteq \Gamma_+(J)$. But since we assume that $J \subseteq I$, we have that $\Gamma_+(I) = \Gamma_+(J)$. □

The previous proposition gives the family of all reductions of a given monomial ideal. Rees and Sally [18] defined the *core* of an ideal $I$ in a commutative ring as the intersection of all reductions of $I$; it is denoted by core($I$). In particular, by Proposition 3.6, the computation of the core of a monomial and integrally closed ideal $I$ in $\mathcal{O}_n$, or in $\mathbb{C}[[x_1, \dots, x_n]]$, reduces to compute the intersection of all ideals $J$ of $\mathcal{O}_n$ such that $\Gamma_+(I) = \Gamma_+(J)$ and $J$ is Newton non-degenerate. We remark that the study of the core of an ideal is quite an active research topic in commutative algebra (see for instance [10] or [15]).

In the next result we show a characterization of the joint reductions of a family of monomial ideals.

**Proposition 3.7.** *Let $I_1, \dots, I_p$ be monomial ideals of $\mathcal{O}_n$. Let $g_1, \dots, g_p \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Then the following conditions are equivalent:*

(1) *$(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$;*
(2) *the map $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is non-degenerate with respect to $I_1, \dots, I_p$.*

*Proof.* Let us consider the ideal $J$ of $\mathcal{O}_n$ given by

$$J = g_1 I_2 \cdots I_p + g_2 I_1 I_3 \cdots I_p + \cdots + g_p I_1 \cdots I_{p-1}. \tag{15}$$

By Definition 2.1, we have that $(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$ if and only if $J$ is a reduction of the monomial ideal $I_1 \cdots I_p$. Let $I$ denote the ideal $I_1 \cdots I_p$, then $J \subseteq I$. Therefore, item (1) holds if and only if $J$ satisfies item (3) of Proposition 3.6 with respect to $\Gamma_+(I)$.

Let $\Gamma_+ = \Gamma_+(I)$, we remark that $\Gamma_+$ is equal to the Minkowski sum $\Gamma_+(I_1) + \cdots + \Gamma_+(I_p)$. Let $B$ denote the set $\{1, \dots, p\}$. From the definition of $J$ we have that there exist finite subsets $S_1, \dots, S_p \subseteq \mathbb{Z}^n_+$ such that the set $\mathcal{J}$ of functions given by

$$\mathcal{J} = \left\{ g_1 x^{k_2+\cdots+k_p} : k_i \in S_i, i \in B, i \neq 1 \right\} \cup \left\{ g_2 x^{k_1+k_3+\cdots+k_p} : k_i \in S_i, i \in B, i \neq 2 \right\} \cup \cdots \cup \\ \cup \left\{ g_p x^{k_1+\cdots+k_{p-1}} : k_i \in S_i, i \in B, i \neq p \right\}$$

is a generating system of $J$. Let us fix a compact face $\Delta$ of $\Gamma_+(I)$. Then $\Delta$ is expressed univocally as $\Delta = \Delta_1 + \cdots + \Delta_p$, where $\Delta_i$ is a compact face of $\Gamma_+(I_i)$, for all $i = 1, \dots, p$.

If $h$ is an element of $\mathcal{J}$, then there exists an $i_0 \in B$ such that $h = g_{i_0} x^{k_1+\cdots+k_{i_0-1}+k_{i_0+1}+\cdots+k_p}$, for some $k_i \in S_i$, $i \neq i_0$. Therefore $h_\Delta$ is expressed as

$$h_\Delta = \left(g_{i_0}\right)_{\Delta_{i_0}} \left(x^{k_1}\right)_{\Delta_1} \cdots \left(x^{k_{i_0-1}}\right)_{\Delta_{i_0-1}} \left(x^{k_{i_0+1}}\right)_{\Delta_{i_0+1}} \cdots \left(x^{k_p}\right)_{\Delta_p}.$$

Then the set of common zeros of $\{h_\Delta : h \in \mathcal{J}\}$ in $(\mathbb{C} \smallsetminus \{0\})^n$ is equal to the set of common zeros of $\{(g_i)_{\Delta_i} : i = 1, \dots, p\}$ in $(\mathbb{C} \smallsetminus \{0\})^n$. This fact shows that item (3) of Proposition 3.6 applied to the ideals $J$ and $I$ holds if and only if the map $g$ is non-degenerate with respect to $I_1, \dots, I_p$. Thus the equivalence between (1) and (2) follows. □

**Corollary 3.8.** *Let $I_1, \dots, I_p$ be monomial ideals of finite colength of $\mathcal{O}_n$. Let $g_1, \dots, g_p \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, p$. Let $g = (g_1, \dots, g_p)$, then $g \in \mathcal{R}(I_1, \dots, I_p)$ if and only if $g$ is non-degenerate with respect to $I_1, \dots, I_p$.*

*Proof.* The *only if* part is obvious. Let us suppose that $g$ is non-degenerate with respect to $I_1, \dots, I_p$. Therefore $(g_1, \dots, g_p)$ is a joint reduction of $(I_1, \dots, I_p)$, by Proposition 3.7. This means that $J$ is a reduction of $I_1 \cdots I_p$, where $J$ is the ideal defined in (15). In particular, for a given $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, we have that $J^L$ is a reduction of $(I_1 \cdots I_p)^L = I_1^L \cdots I_p^L$, since reductions are stable under ring morphisms. Therefore $(g_1^L, \dots, g_p^L)$ is a joint reduction of $(I_1^L, \dots, I_p^L)$. We have that $I_i^L \neq 0$, for all $i = 1, \dots, p$, since each ideal $I_i$ has finite colength. Then the result follows as a consequence of Proposition 3.7. □

Given an integer $r \geqslant 1$ and a subset $L \subseteq \{1, \dots, n\}$, we denote by $\delta_{L,r}$ the convex hull in $\mathbb{R}^n$ of $\{re_i : i \in L\}$, where $e_1, \dots, e_n$ denotes the canonical basis in $\mathbb{R}^n$.

If $I$ is an ideal of $\mathcal{O}_n$, $I \neq 0$, then we denote by $\mathrm{ord}(I)$ the maximum of those integers $s \geqslant 1$ such that $I \subseteq m^s$.

**Lemma 3.9.** *Let $I_1, \dots, I_n$ be monomial ideals in $\mathcal{O}_n$ such that $I_1 + \cdots + I_n$ has finite colength. Let us consider, for a given integer $r \geqslant 1$, the ideal $Q_r = (I_1 + m^r) \cdots (I_n + m^r)$. Then, there exists an integer $r_0 \geqslant 1$ such that for all $r \geqslant r_0$ the following hold:*

1. *every compact face of $\Gamma_+(I_1 \cdots I_n)$ is a compact face of $Q_r$;*
2. *let $\Delta$ be a face of $\Gamma_+(Q_r)$ not intersecting $\Gamma(I_1 \cdots I_n)$, let us write $\Delta$ as $\Delta = \Delta_1 + \cdots + \Delta_n$, where $\Delta_i$ is a face of $I_i + m^r$, for all $i = 1, \dots, n$, and let $S = \{i : \Delta_i \cap \Gamma(I_i) \neq \emptyset\}$; then $S \neq \emptyset$ and there exists some $L \subsetneq \{1, \dots, n\}$ such that*
$$\Delta = \sum_{i \in S} \Delta_i + \big(n - |S|\big)\delta_{L,r},$$
*and $\Delta_i$ is a face of $\Gamma(I_i^L)$ if $I_i^L \neq 0$.*

*Proof.* Let us define, for a given integer $j \in \{1, \dots, n\}$, the ideal

$$L_j = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} I_{i_1} \cdots I_{i_j}.$$

Since the ideal $L_1$ has finite colength, then there exists an integer $r_0 \geqslant 1$ such that $m^{r_0} \subseteq L_1$. Then, for any integer $r \geqslant r_0$, we observe that $Q_r$ is expressed as

$$Q_r = L_n + m^r L_{n-1} + \cdots + m^{r(n-1)} L_1. \tag{16}$$

Relation (16) shows that we can increase the integer $r$ in order to have that any compact face of $L_n$ is a compact face of $Q_r$. Then item (1) holds.

If $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ then we define $v_0 = \min_i v_i$. We also define $L(v) = \{i : v_i = v_0\}$. For any vector $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$ and any $r \geqslant 1$ we have

$$\ell(v, m^r) = rv_0 \quad \text{and} \quad \Delta(v, m^r) = \delta_{L(v),r}. \tag{17}$$

Let us suppose that $r > \operatorname{ord}(I_i^L)$, for all $i = 1, \dots, n$ and all $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$. Let $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$ and let $j \in \{1, \dots, n\}$ such that $I_j^{L(v)} \neq 0$. Then

$$\ell(v, I_j) \leqslant \ell(v, I_j^{L(v)}) = \operatorname{ord}(I_j^{L(v)}) v_0 < rv_0 = \ell(v, m^r).$$

In particular, there exists an integer $r_1 \geqslant r_0$ such that for all $r \geqslant r_1$ we have

$$\Delta(v, I_j + m^r) \cap \Delta(v, I_j) \neq \emptyset, \tag{18}$$

for all vector $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$ and all $j$ such that $I_j^{L(v)} \neq 0$.

Let us consider an integer $r_2 \geqslant r_1$ such that each compact face of $\Gamma_+(I_i)$ is a compact face of $\Gamma_+(I_i + m^r)$, for all $i = 1, \dots, n$ and all $r \geqslant r_2$. Then the number of compact faces of $\Gamma_+(I_i + m^r)$ does not depend on $r$, if $r \geqslant r_2$, for all $i = 1, \dots, n$. In particular, there exists an integer $r_3 \geqslant r_2$ such that the number of compact faces of $\Gamma_+(Q_r)$ does not depend on $r$ if $r \geqslant r_3$.

For each face $\Delta$ of $\Gamma_+(Q_{r_3})$, let us choose a vector $v_\Delta$ such that $\Delta = \Delta(v_\Delta, Q_{r_3})$. Let us consider the decomposition $\Delta = \Delta_1 + \cdots + \Delta_n$, where $\Delta_i = \Delta(v_\Delta, I_i + m^{r_3})$, for all $i = 1, \dots, n$.

Let us suppose that $\Delta$ is face of $\Gamma_+(Q_{r_3})$ such that $\Delta \cap \Gamma(I_1 \cdots I_n) = \emptyset$. Then the set $S = \{i : \Delta_i \cap \Gamma(I_i) \neq \emptyset\}$ is non-empty, by (16). Moreover, if $L$ denotes the set $L(v_\Delta)$ we have that $\{j : I_j^L \neq 0\} \subseteq S$, by (18). In particular, if $i \notin S$ then $I_i^L = 0$ and $\Delta_i = \delta_{L,r_3}$, by (17).

We remark that, for a given $i \in \{1, \dots, n\}$, any face of $\Gamma_+(I_i + m^r)$, for $r \geqslant r_2$, is determined by its intersection with $\Gamma_+(I_i)$ and its intersection with the family of the coordinate axis. Then the vector $v_\Delta$ is integrated in a natural way in a family of vectors $v_\Delta^r$, for $r \geqslant r_3$, satisfying

$$\begin{aligned}
\Delta(v_\Delta^r, I_i + m^r) \cap \Gamma(I_i) &= \Delta_i \cap \Gamma(I_i), \;\; \text{for all } i \in S \\
\Delta(v_\Delta^r, I_i + m^r) \cap \Gamma(m^r) &= \delta_{L,r}, \;\; \text{for all } i \notin S \\
L(v_\Delta^r) &= L.
\end{aligned}$$

Then we can consider an integer $r_\Delta \geqslant r_3$ such that if $j \in S$ verifies that $I_j^L \neq 0$, then

$$\Delta(v_\Delta^r, I_j + m^r) \subseteq \Delta(v_\Delta^r, I_j) \cap \mathbb{R}^n_L,$$

for all $r \geqslant r_\Delta$. Hence, if $r \geqslant r_\Delta$, the face $\Delta$ is written as

$$\Delta = \sum_{i \in S} \Delta_i + (n - |S|)\delta_{L,r},$$

where $\Delta_j$ is a face of $\Gamma(I_j^L)$ for all $j \in S$ such that $I_j^L \neq 0$. □

**Theorem 3.10.** *Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$. Suppose that $\sigma(I_1, \dots, I_n) < \infty$. Let $g_1, \dots, g_n \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \dots, n$. Then the following conditions are equivalent:*

(1) *the ideal $\langle g_1, \dots, g_n \rangle$ has finite colength and $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$;*
(2) *$g \in \mathcal{R}(I_1, \dots, I_n)$.*

*Proof.* Let $g$ denote the map $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ given by $g = (g_1, \dots, g_n)$. For a given $r \geqslant 1$ we define the ideals

$$\begin{aligned} P_r &= g_1(I_2 + m^r) \cdots (I_n + m^r) + \cdots + g_n(I_1 + m^r) \cdots (I_{n-1} + m^r) \\ Q_r &= (I_1 + m^r) \cdots (I_n + m^r). \end{aligned}$$

Let us see that (1) implies (2). By Nakayama's Lemma we can suppose that $g_i$ is a polynomial, for all $i = 1, \dots, n$. By Proposition 2.5, $(g_1, \dots, g_n)$ is a $\sigma$-joint reduction of $(I_1, \dots, I_n)$. In particular, it is a joint reduction of $(I_1, \dots, I_n)$, by Proposition 2.8. Therefore $g$ is non-degenerate with respect to $(I_1, \dots, I_n)$, by Proposition 3.7.

Let $r_0$ be an integer such that $P_r$ is a joint reduction of $Q_r$, for all $r \geqslant r_0$. Let us fix a subset $L \subsetneq \{1, \dots, n\}$, $L \neq \emptyset$, and an integer $r \geqslant r_0$. Since reductions are stable under ring morphisms, we have that $P_r^L$ is a reduction of $Q_r^L$. Therefore the map $g^L$ is non-degenerate with respect to $(I_1 + m^r)^L, \dots, (I_n + m^r)^L$, by Proposition 3.7. Let us remark that $(I_i + m^r)^L \neq 0$, for all $i = 1, \dots, n$.

Let $C = \{i : I_i^L \neq 0\}$. The condition $\sigma(I_1, \dots, I_n) < \infty$ implies that $I_1 + \cdots + I_n$ has finite colength and. Therefore $C \neq \emptyset$. Without loss of generality we can suppose that $C = \{1, \dots, s\}$, for some $1 \leqslant s \leqslant n$. We have to see that $(g_1^L, \dots, g_s^L) : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^s, 0)$ is non-degenerate with respect to $I_1^L, \dots, I_s^L$.

Since $g_i$ is a polynomial, for all $i = 1, \dots, n$, let us assume that

$$\operatorname{supp}(g_i) \cap \Gamma_+(m^r) = \emptyset, \text{ for all } i = 1, \dots, n. \tag{19}$$

Let $H = (I_1^L + m_L^r) \cdots (I_s^L + m_L^r)$. Then $Q_r^L = H m_L^{r(n-s)}$. In particular, we have

$$\Gamma_+(Q_r^L) = \Gamma_+(H) + \Gamma_+(m_L^{r(n-s)}). \tag{20}$$

By Lemma 3.9 (1) we can suppose that $r_0$ is big enough in order to have that each compact face of $I_1^L \cdots I_s^L$ is a compact face of $H$. This fact together with (20) implies that if $v$ is a vector of $(\mathbb{R}_+ \smallsetminus \{0\})^q$, where $q = |L|$, then the set

$$\Delta(v, I_1^L \cdots I_s^L) + \Delta(v, m_L^{r(n-s)}) \tag{21}$$

is a compact face of $\Gamma_+(Q_r^L)$.

By hypothesis the map $g^L$ is non-degenerate with respect to $(I_1+m^r)^L,\dots,(I_n+m^r)^L$. Then $g^L$ verifies the $(K_v)$ condition with respect to these ideals (see Definition 3.1). Therefore, writing down this condition and considering (19) and (21), we have

$$\big\{x\in\mathbb{C}^n_L : (g_1^L)_{\Delta_1}(x)=\cdots=(g_s^L)_{\Delta_s}(x)=0\big\}\subseteq\big\{x\in\mathbb{C}^n_L : \prod_{i\in L}x_i=0\big\},$$

where $\Delta_i=\Delta(v,I_i^L)$, for all $i=1,\dots,s$. This shows that the map $(g_1^L,\dots,g_s^L):(\mathbb{C}^n_L,0)\to(\mathbb{C}^s,0)$ is non-degenerate with respect to $I_1^L,\dots,I_s^L$. Then $g\in\mathcal{R}(I_1,\dots,I_n)$, since we started from an arbitrary $L\subsetneq\{1,\dots,n\}$.

Let us see that (2) implies (1). Let us suppose that $g\in\mathcal{R}(I_1,\dots,I_n)$. By Proposition 2.5 and Proposition 3.7, item (1) holds if and only if there exists an integer $r_0$ such that $g$ is non-degenerate with respect to $I_1+m^r,\dots,I_n+m^r$, for all $r\geqslant r_0$.

Let $r_0$ be an integer such that items (1) and (2) of Lemma 3.9 hold for all $r\geqslant r_0$. Let us fix an integer $r\geqslant r_0$ and let us fix a compact face $\Delta$ of $\Gamma_+(Q_r)$. Let us write $\Delta$ as $\Delta=\Delta_1+\cdots+\Delta_n$, where $\Delta_i$ is a face of $\Gamma_+(I_i+m^r)$, for all $i=1,\dots,n$. We have to see that

$$\big\{x\in\mathbb{C}^n : (g_1)_{\Delta_1}(x)=\cdots=(g_n)_{\Delta_n}(x)=0\big\}\subseteq\{x\in\mathbb{C}^n : x_1\cdots x_n=0\}. \tag{22}$$

Let $\Delta'=\Delta\cap\Gamma_+(I_1\cdots I_n)$ and let $\Delta_i'=\Delta_i\cap\Gamma_+(I_i)$, for all $i=1,\dots,n$. If $\Delta'\neq\emptyset$, then $\Delta'=\Delta_1'+\cdots+\Delta_n'$ and $(g_i)_{\Delta_i}=(g_i)_{\Delta_i'}$, for all $i=1,\dots,n$. Thus inclusion (22) holds, since $g$ is non-degenerate with respect to $I_1,\dots,I_n$ by hypothesis.

Let us suppose that $\Delta'=\emptyset$. By Lemma 3.9, there exists a subset $L\subsetneq\{1,\dots,n\}$ such that, if $S=\{i:\Delta_i'\neq\emptyset\}$ and $C_L=\{i:I_i^L\neq 0\}$, then $C_L\subseteq S$ and $\Delta$ is written as

$$\Delta=\sum_{i\in S}\Delta_i+(n-|S|)\delta_{L,r}.$$

Let us suppose that $C_L=\{i_1,\dots,i_s\}$, for some $1\leqslant i_1<\cdots<i_s\leqslant n$, $s\leqslant t$, where $t=|S|$. Therefore we have

$$\Delta=\Delta^1+\Delta^2,$$

where $\Delta^1$ is a face of $m^{r(n-t)}I_{i_1}^L\cdots I_{i_s}^L$ and $\Delta^2=\sum_{i\in S\smallsetminus C_L}\Delta_i$.

Then we observe that the set of common zeros of $(g_1)_{\Delta_1},\dots,(g_n)_{\Delta_n}$ is contained in the set of common zeros of $(g_{i_1})_{\Delta_{i_1}'},\dots,(g_{i_s})_{\Delta_{i_s}'}$.

Since $\Delta_i'$ is a face of $I_i^L$, for all $i\in C_L$, then $(g_i)_{\Delta_i'}=(g_i^L)_{\Delta_i'}$, for all $i\in C_L$. Then the inclusion (22) follows, since the map $(g_{i_1}^L,\dots,g_{i_s}^L):(\mathbb{C}^n_L,0)\to(\mathbb{C}^s,0)$ is non-degenerate with respect to $I_{i_1}^L,\dots,I_{i_s}^L$, by hypothesis. □

Let us suppose that $I_1,\dots,I_n$ are ideals of finite colength of $\mathcal{O}_n$. Then Rees showed in [17] that the mixed multiplicity $e(I_1,\dots,I_n)$ can be computed in terms of Samuel mutiplicities via the following formula:

$$e(I_1,\dots,I_n)=\frac{1}{n!}\sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\emptyset}}(-1)^{n-|J|}e\Big(\prod_{j\in J}I_j\Big).$$

If we assume that $I_i$ is a monomial ideal for all $i = 1, \dots, n$, then $e(\prod_{j\in J} I_j)$ can be computed effectively using [3], for all $J \subseteq \{1, \dots, n\}$, $J \neq \emptyset$. That is, we can apply [3, Theorem 5.1] to deduce that if $f_J$ denotes the polynomial given by the sum of all $x^k$ such that $k$ is a vertex of $\Gamma_+(\prod_{j\in J} I_j)$, for all non-empty $J \subseteq \{1, \dots, n\}$, then

$$e(I_1, \dots, I_n) = \frac{1}{n!} \sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\emptyset}} (-1)^{n-|J|} \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle x_1 \frac{\partial f_J}{\partial x_1}, \dots, x_n \frac{\partial f_J}{\partial x_n}\rangle}.$$

Thus we have an effective method to compute the mixed multiplicity $e(I_1, \dots, I_n)$ when $I_i$ are monomial ideals of finite colength of $\mathcal{O}_n$. Let us suppose now that some of these ideals do not have finite colength but still $\sigma(I_1, \dots, I_n) < \infty$. Then, by the above discussion, the effective computation of $\sigma(I_1, \dots, I_n)$ reduces to compute some $r \geqslant 1$ such that $\sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)$. If $g = (g_1, \dots, g_n) \in \mathcal{R}(I_1, \dots, I_n)$, then we found in the proof of Theorem 3.10 that $g$ is non-degenerate with respect to $I_1 + m^r, \dots, I_n + m^r$, when $r$ is an integer such that $\Gamma_+(Q_r)$ satisfy conditions (1) and (2) of Lemma 3.9. Hence $e(g_1, \dots, g_n) = e(I_1 + m^r, \dots, I_n + m^r)$ and therefore $\sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)$. Obviously, the problem of finding an integer $r$ satisfying these conditions is easy when $n = 2$, and needs a more careful analysis in higher dimensions.

To end the paper we show a result about the computation of the monomials which are integral over the ideal generated by the components of a given map of $\mathcal{R}(I_1, \dots, I_n)$.

**Proposition 3.11.** *Let $I_1, \dots, I_n$ monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. Let $g = (g_1, \dots, g_n) \in \mathcal{R}(I_1, \dots, I_n)$. Then*

$$I_1 \cap \cdots \cap I_n \subseteq \overline{\langle g_1, \dots, g_n\rangle}.$$

*Proof.* Let $J$ be the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. Let $x^k$ be a monomial in $\mathcal{O}_n$. By Rees' multiplicity theorem we know that $x^k \in \overline{J}$ if and only if $e(J) = e(J, x^k)$ (see [9, p. 222]).

By a result of Northcott-Rees (see [9, p. 166] or [14]), we can consider general $\mathbb{C}$-linear combinations $h_1, \dots, h_n$ of $g_1, \dots, g_n, x^k$ such that the ideal $H$ generated by $h_1, \dots, h_n$ is a reduction of $J + \langle x^k\rangle$. Then $e(H) = e(J, x^k)$. Therefore, let $A$ be a squared matrix of size $n$ with entries in $\mathbb{C}$ and let $B$ be a row matrix with $n$ columns with entries in $\mathbb{C}$ such that

$$\begin{bmatrix} g_1 & \cdots & g_n & x^k \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix}. \tag{23}$$

Since the coefficients of $A$ are generic, we can suppose that $A$ is invertible. In particular, multiplying both sides of (23) by $A^{-1}$, we obtain:

$$\begin{bmatrix} g_1 & \cdots & g_n & x^k \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \\ BA^{-1} \end{bmatrix} = \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix} A^{-1}, \tag{24}$$

where $\mathbf{I}_n$ denotes the identity matrix of size $n$. We observe that the entries of the left hand side of (24) are of the form $g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k$, for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Relation (24) implies that $H = \langle g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k\rangle$. Then, we obtain that

$$e(J) \geqslant e(J, x^k) = e(H) = e(g_1 + \alpha_1 x^k, \dots, g_n + \alpha_n x^k), \tag{25}$$

for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. If $x^k \in I_1 \cap \dots \cap I_n$, then $e(g_1+\alpha_1 x^k, \dots, g_n+\alpha_n x^k) \geqslant \sigma(I_1, \dots, I_n)$, by Lemma 2.2. But by Theorem 3.10, the equality $e(J) = \sigma(I_1, \dots, I_n)$ holds, since we assume that $g \in \mathcal{R}(I_1, \dots, I_n)$. Then (25) implies that $e(J) = e(J, x^k)$ and hence $x^k \in \overline{J}$. □

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## References

[1] C. Bivià-Ausina, *The analytic spread of monomial ideals*, Comm. Algebra **31**, no. 7 (2003), 3487–3496.

[2] C. Bivià-Ausina, *The integral closure of modules, Buchsbaum-Rim multiplicities and Newton polyhedra*, J. London Math. Soc. (2) **69** (2004), 407–427.

[3] C. Bivià-Ausina, *Non-degenerate ideals in formal power series rings*, Rocky Mountain J. Math. **34**, 2 (2004), 495–511.

[4] C. Bivià-Ausina, T. Fukui and M.J. Saia, *Newton graded algebras and the codimension of non-degenerate ideals*, Math. Proc. Cambridge Philos. Soc. **133** (2002), 55–75.

[5] A. Corso, C. Huneke and W. Vasconcelos, *On the integral closure of ideals*, Manuscripta Math. **95** (1998), 331–347.

[6] D. Delfino, A. Taylor, W.V. Vasconcelos, R.H. Villarreal and N. Weininger, *Monomial ideals and the computation of multiplicities*, in Commutative Ring Theory and its Applications, Lecture Notes in Pure and Appl. Math. 231, Marcel Dekker, New York, 2003, 87–106.

[7] J.F. Gately, *Unique factorization of *-products of one-fibered monomial ideals*, Comm. Algebra **28** (7) (2000), 3137–3153.

[8] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2005). `http://www.singular.uni-kl.de`.

[9] C. Huneke and I. Swanson, Integral Closure of Ideals, Rings, and Modules. London Math. Soc. Lecture Note Series 336 (2006), Cambridge University Press.

[10] C. Huneke and N. V. Trung, *On the core of ideals*, Compos. Math. **141**, No. 1 (2005), 1–18.

[11] D. Katz and J. Verma, *Extended Rees algebras and mixed multiplicities*, Math. Z. **202** (1989), no. 1, 111–128.

[12] A.G. Khovanskii, *Newton polyhedra and toroidal varieties*, Funct. Anal. Appl. **11** (1978), 289–296.

[13] A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1–31.

[14] D.G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158.

[15] C. Polini, B. Ulrich and M.A. Vitulli, *The core of zero-dimensional monomial ideals*, Adv. Math. **211**, No. 1 (2007), 72–93.

[16] D. Rees, *$\mathfrak{a}$-transforms of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. **57** (1961), 8–17.

[17] D. Rees, *Generalizations of reductions and mixed multiplicities*, London Math. Soc. (2) **29** (1984), 397–414.

[18] D. Rees and J. Sally, *General elements and joint reductions*, Mich. Math. J. **35**, No. 2 (1988), 241–254.

[19] M. Soares, *Bounding Poincaré-Hopf indices and Milnor numbers*, Math. Nachr. **278**, No. 6 (2005), 703–711.

[20] I. Swanson, *Mixed multiplicities, joint reductions and quasi-unmixed local rings*, J. London Math. Soc. (2) **48**, No. 1 (1993), 1–14.

[21] B. Teissier, *Cycles évanescents, sections planes et conditions of Whitney*, Singularités à Cargèse, Astérisque, no. 7–8 (1973), 285–362.
[22] B. Teissier, *Monômes, volumes et multiplicités*, Introduction à la théorie des singularités, II, Travaux en Cours, 37, Hermann, 1988, 127–141.
[23] B. Teissier, *Monomial ideals, binomial ideals, polynomial ideals*, Math. Sci. Res. Inst. Publ. **51** (2004), 211-246.
[24] N.V. Trung, *Positivity of mixed multiplicities*, Math. Ann. **319** (2001), 33-63.
[25] N.V. Trung and J. Verma, *Mixed multiplicities of ideals versus mixed volumnes of polytopes*, to appear in Trans. Amer. Math. Soc.
[26] W. Vasconcelos, Integral closure. Rees algebras, multiplicities, algorithms, Springer Monogr. Math., 2005.
[27] J. Verma, *Joint reductions of complete ideals*, Nagoya Math. J. **118** (1990), 155-163.
[28] D.Q. Viêt, *Sequences determining mixed multiplicities and reductions of ideals*, Comm. Algebra **31** (2003), no. 10, 5047–5069.

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