A proof of the cyclic sum conjecture for multiple zeta values

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Abstract

We shall give a proof of M. Hoffman's cyclic sum conjecture for multiple zeta values. As a consequence, we obtain another proof of the (original) sum formula without using generating functions.

1 The Cyclic Sum Formula

For *admissible indices* $(k_1, k_2, \ldots, k_n)$, i.e., for integers $k_1, \ldots, k_{n-1} \geq 1$ and $k_n \geq 2$, multiple zeta values are defined by

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{0<a_1<a_2<\cdots<a_n} \frac{1}{{a_1}^{k_1} {a_2}^{k_2} \cdots {a_n}^{k_n}}.$$

Let n and k be integers with $0 < n \leq k$, and $S(k,n)$ the set of compositions (ordered permutations) of k of length n, i.e.,

$$S(k,n) = \{(k_1, k_2, \ldots, k_n) \mid k_1 + k_2 + \cdots + k_n = k,\ k_i \geq 1 \text{ for any } i\}.$$

We say two elements of $S(k,n)$ are cyclically equivalent if they are cyclic permutations of each other. Let $\Pi(k,n)$ be the set of cyclic equivalence classes of $S(k,n)$. The aim of this paper is to prove the following theorem which was conjectured by M. Hoffman in [4] (Please notice that we have —frivolously— changed the roles of k and n, compared to Hoffman's original notation).

Theorem 1 *For all $\alpha \in \Pi(k,n)$ with $n < k$,*

$$\sum_{(k_1,k_2,\ldots,k_n)\in\alpha} \zeta(k_1, k_2, \ldots, k_{n-1}, k_n + 1) = \sum_{(k_1,k_2,\ldots,k_n)\in\alpha} \sum_{i=0}^{k_n-2} \zeta(i+1, k_1, \ldots, k_{n-1}, k_n - i)$$

where the inner sum on the right-hand side is treated as 0 whenever $k_n = 1$.

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Proof For any positive integers $n,\ k_1,\ldots,k_n$ with $k_1+\cdots+k_n>n$ (i.e., at least one of the k_i's is >1), let $T(k_1,k_2,\ldots,k_n)$ be defined as the convergent series

$$T(k_1,k_2,\ldots,k_n)=\sum_{0\le a_0<a_1<a_2<\cdots<a_n}\frac{1}{a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}(a_n-a_0)}.$$

(For convergence, cf. Lemma 1.)

The theorem is readily proved when we apply the following Key Lemma for all cyclic permutations of $(k_1,k_2,\ldots,k_n)$ and add them up. Q.E.D.

Key Lemma *For any positive integers $n,\ k_1,\ldots,k_n$ with $k_i>1$ for some i, we have*

$$\begin{aligned}&T(k_1,k_2,\ldots,k_n)-T(k_n,k_1,k_2,\ldots,k_{n-1})\\&\quad=\zeta(k_1,k_2,\ldots,k_{n-1},k_n+1)-\sum_{i=0}^{k_n-2}\zeta(i+1,k_1,\ldots,k_{n-1},k_n-i),\end{aligned}$$

where the sum on the right is understood to be 0 if $k_n=1$.

Proof For any integers $r\ge 2$ and $i\ge 0$, we have

$$\begin{aligned}&\sum_{0<a_0<a_1<\cdots<a_n}\frac{1}{a_0^{i}a_1^{k_1}a_2^{k_2}\cdots a_n^{r}(a_n-a_0)}\\=&\sum_{0<a_0<a_1<\cdots<a_n}\frac{1}{a_0^{i+1}a_1^{k_1}a_2^{k_2}\cdots a_{n-1}^{k_{n-1}}a_n^{r-1}}\Big(\frac{1}{a_n-a_0}-\frac{1}{a_n}\Big)\\=&\sum_{0<a_0<a_1<\cdots<a_n}\frac{1}{a_0^{i+1}a_1^{k_1}a_2^{k_2}\cdots a_{n-1}^{k_{n-1}}a_n^{r-1}(a_n-a_0)}-\zeta(i+1,k_1,k_2,\ldots,k_{n-1},r).\end{aligned}$$

Putting $r=k_n-i$ and adding up the above equation for $i=0,1,\ldots,k_n-2$, we obtain

$$\begin{aligned}&\sum_{0<a_0<\cdots<a_n}\frac{1}{a_1^{k_1}a_2^{k_2}\cdots a_n^{k_n}(a_n-a_0)}\\=&\sum_{0<a_0<\cdots<a_n}\frac{1}{a_0^{k_n-1}a_1^{k_1}a_2^{k_2}\cdots a_{n-1}^{k_{n-1}}a_n(a_n-a_0)}-\sum_{i=0}^{k_n-2}\zeta(i+1,k_1,\ldots,k_{n-1},k_n-i).\end{aligned}$$

The left-hand side of this equation is equal to

$$T(k_1,k_2,\ldots,k_n)-\zeta(k_1,k_2,\ldots,k_{n-1},k_n+1)$$

and the first sum on the right-hand side can be written as

$$
\begin{aligned}
& \sum_{0<a_0<a_1<\cdots<a_n} \frac{1}{a_0^{k_n} a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-1}}} \Big(\frac{1}{a_n - a_0} - \frac{1}{a_n} \Big) \\
= & \sum_{0<a_0<a_1<\cdots<a_{n-1}} \frac{1}{a_0^{k_n} a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-1}}} \sum_{a_n = a_{n-1}+1}^{\infty} \Big(\frac{1}{a_n - a_0} - \frac{1}{a_n} \Big) \\
= & \sum_{0<a_0<a_1<\cdots<a_{n-1}} \frac{1}{a_0^{k_n} a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-1}}} \sum_{j=0}^{a_0-1} \frac{1}{a_{n-1} - j} \\
= & \sum_{0\le j<a_0<a_1<\cdots<a_{n-1}} \frac{1}{a_0^{k_n} a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-1}} (a_{n-1} - j)} \\
= & \; T(k_n, k_1, k_2, \ldots, k_{n-1}).
\end{aligned}
$$

Q.E.D.

Lemma 1 *For $k_1 + k_2 + \cdots + k_n > n > 0$, $T(k_1, k_2, \ldots, k_n)$ converges.*

Proof To see the convergence of $T(k_1, \ldots, k_n)$ with $k_1 + k_2 + \cdots + k_n > n > 0$, it is enough to check the convergence of $T(k_1, \ldots, k_n)$ with $k_1 + k_2 + \cdots + k_n = n + 1$, because the other $T(k_1, \ldots, k_n)$ can be bounded by one of these values, i.e., from the condition $k_1 + \cdots + k_n > n$ we have at least one entry k_i which is greater than 1, and for this i we have

$$T(\underbrace{1, \ldots, 1}_{i-1}, 2, \underbrace{1, \ldots, 1}_{n-i}) > T(k_1, \ldots, k_n).$$

So we need only check convergence for all permutations of $(1, 1, \ldots, 1, 2)$. First, we treat $T(1, 1, \ldots, 1, 2)$ as follows:

$$
\begin{aligned}
T(1, 1, \ldots, 1, 2) &= \sum_{0\le a_0<a_1<a_2<\cdots<a_n} \frac{1}{a_1 a_2 \cdots a_{n-1} a_n^2 (a_n - a_0)} \\
&\le \zeta(1, 1, \ldots, 1, 3) + \sum_{\substack{0<a_1<\cdots<a_n, \\ 0<t<a_n}} \frac{1}{t a_1 a_2 \cdots a_{n-1} a_n^2} \\
&= \zeta(1, 1, \ldots, 1, 3) + n\zeta(\underbrace{1, \ldots, 1}_{n}, 2) + \sum_{i=1}^{n-1} \zeta(\underbrace{1, \ldots, 1}_{i-1}, 2, \underbrace{1, \ldots, 1}_{n-i-1}, 2)
\end{aligned}
$$

Thus, we get the convergence of $T(1, 1, \ldots, 1, 2)$. From this fact and the formula in Key Lemma, we can get the convergence of $T(1, 2, 1, 1, \ldots, 1)$, $T(1, 1, 2, 1, 1, \ldots, 1)$, $\ldots$, $T(1, 1, \ldots, 1, 2, 1)$ by induction. Q.E.D.

It is worth pointing out that Theorem 1 provides another proof of the sum formula:

Theorem 2 (Sum Formula [2],[9]) *For any integers $0 < n \le k$, we have*

$$\sum_{(k_1,k_2,\ldots,k_n)\in S(k,n)} \zeta(k_1,k_2,\ldots,k_{n-1},k_n+1) = \zeta(k+1).$$

Proof We add up the equation in Theorem 1 for all cyclic equivalence classes of $S(k,n)$ to obtain

$$\sum_{(k_1,k_2,\ldots,k_n)\in S(k,n)} \zeta(k_1,k_2,\ldots,k_{n-1},k_n+1)$$

$$= \sum_{(k_1,k_2,\ldots,k_n)\in S(k,n)} \sum_{i=0}^{k_n-2} \zeta(i+1,k_1,\ldots,k_{n-1},k_n-i).$$

Now, we use the following lemma:

Lemma 2 *For any integers k and n with $0 < n < k$,*

$$\sum_{(k_0,k_1,\ldots,k_n)\in S(k,n+1)} \zeta(k_0,k_1,\ldots,k_{n-1},k_n+1)$$

$$= \sum_{(k_1,k_2\ldots,k_{n-1},t)\in S(k,n)} \sum_{i=0}^{t-2} \zeta(i+1,k_1,\ldots,k_{n-1},t-i)$$

Proof The left-hand side is a sum of $\binom{k-1}{n}$ terms and the right-hand side has $\left(\sum_{t=2}^{k-n+1}(t-1)\binom{k-t-1}{n-2}\right)$ terms, and by using the formula $\sum_{i=1}^{n} i\binom{n+s-i-1}{n-i} = \binom{n+s}{s+1}$, we can see that these two numbers are the same.

Next, for any index $(k_0,k_1,\ldots,k_n) \in S(k,n+1)$, we have $(k_1,k_2,\ldots,k_{n-1},k_0+k_n) \in S(k,n)$ and so on the right-hand side of the equation in Lemma 2, the terms $\sum_{i=0}^{k_0+k_n-2} \zeta(i+1,k_1,\ldots,k_{n-1},k_0+k_n-i)$ occur. If we fix $i = k_0 - 1$, then the condition $0 \le i = k_0 - 1 \le k_0 + k_n - 2$ is satisfied, so on the right-hand side, the term $\zeta(k_0,k_1,\ldots,k_n)$ occurs. Thus, the entries of both sides are the same and each entry appears exactly once, so the equation is valid. Q.E.D.

By using the lemma, the above equation gives the following equation for any integers $0 < n < k$,

$$\sum_{(k_1,k_2,\ldots,k_n)\in S(k,n)} \zeta(k_1,k_2,\ldots,k_{n-1},k_n+1)$$

$$= \sum_{(k_0,k_1,\ldots,k_n)\in S(k,n+1)} \zeta(k_0,k_1,\ldots,k_{n-1},k_n+1).$$

Thus, the theorem follows by induction on n. Q.E.D.

2 Duality and the Cyclic Sum Formula

In this section we apply the duality formula ([8]) to the right-hand side of the equation of Theorem 1, and we will see a kind of symmetry of the cyclic sum formula.

First, we review the definition of the dual index and the duality formula, and define the dual cyclic equivalence class.

For any integers $s \geq 1$ and $a_1, b_1, a_2, b_2, \ldots, a_s, b_s \geq 1$, we define two admissible indices k and k' to be "dual" to each other if

$$\mathsf{k} = (\underbrace{1, \ldots, 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, \ldots, 1}_{a_2 - 1}, b_2 + 1, \ldots, \underbrace{1, \ldots, 1}_{a_s - 1}, b_s + 1)$$

and

$$\mathsf{k}' = (\underbrace{1, \ldots, 1}_{b_s - 1}, a_s + 1, \underbrace{1, \ldots, 1}_{b_{s-1} - 1}, a_{s-1} + 1, \ldots, \underbrace{1, \ldots, 1}_{b_1 - 1}, a_1 + 1).$$

Then, the duality formula is the following:

Theorem 3 (Duality (cf.[8])) *For any admissible index* $\mathsf{k} = (k_1, k_2, \ldots, k_n)$ *and its dual* $\mathsf{k}' = (h_1, h_2, \ldots, h_m)$*,*

$$\zeta(k_1, k_2, \ldots, k_n) = \zeta(h_1, h_2, \ldots, h_m).$$

For any $0 < n < k$ and any $\alpha \in \Pi(k, n)$, it is easy to see that the dual indices of all admissible indices in α are cyclic permutations of each other. Thus, we postulate that $\beta \in \Pi(k, k-n)$ is the "dual" of $\alpha \in \Pi(k, n)$, if the dual index of an admissible index in α is in β.

We can now restate the cyclic sum formula:

Theorem 4 *For any* $0 < n < k$*,* $\alpha \in \Pi(k, n)$ *and its dual* $\beta \in \Pi(k, k-n)$*,*

$$\sum_{(k_1, \ldots, k_n) \in \alpha} \zeta(k_1, \ldots, k_{n-1}, k_n + 1) = \sum_{(h_1, \ldots, h_{k-n}) \in \beta} \zeta(h_1, \ldots, h_{k-n-1}, h_{k-n} + 1).$$

Proof If we take an admissible index $(k_1, \ldots, k_n) = (\underbrace{1, \ldots, 1}_{a_1 - 1}, b_1 + 1, \ldots, \underbrace{1, \ldots, 1}_{a_s - 1}, b_s + 1) \in \alpha$, then the right-hand side of the equation in Theorem 2 is

$$\frac{\#\alpha}{n} \sum_{(<a_1, b_1>, \ldots, <a_s, b_s>)} \sum_{i=0}^{b_s - 1} \zeta(i+1, \underbrace{1, \ldots, 1}_{a_1 - 1}, b_1+1, \ldots, \underbrace{1, \ldots, 1}_{a_{s-1} - 1}, b_{s-1}+1, \underbrace{1, \ldots, 1}_{a_s - 1}, b_s+1-i)$$

where the sum runs through all simultaneous cyclic permutations of those s pairs. The above expression is equal to

$$\frac{\#\alpha}{n} \sum_{(<a_1, b_1>, \ldots, <a_s, b_s>)} \Big(\zeta(1, \underbrace{1, \ldots, 1}_{a_1 - 1}, b_1 + 1, \ldots, \underbrace{1, \ldots, 1}_{a_s - 1}, b_s + 1)$$

$$+\sum_{i=1}^{b_s-1}\zeta\big(i+1,\underbrace{1,\ldots,1}_{a_1-1},b_1+1,\ldots,\underbrace{1,\ldots,1}_{a_{s-1}-1},b_{s-1}+1,\underbrace{1,\ldots,1}_{a_s-1},b_s+1-i)\Big).$$

Now we use the duality formula to obtain

$$\frac{\#\alpha}{n}\sum_{(<a_1,b_1>,\ldots,<a_s,b_s>)}\Big(\zeta(\underbrace{1,\ldots,1}_{b_s-1},a_s+1,\ldots,\underbrace{1,\ldots,1}_{b_2-1},a_2+1,\underbrace{1,\ldots,1}_{b_1-1},a_1+2)$$

$$+\sum_{i=1}^{b_s-1}\zeta(\underbrace{1,\ldots,1}_{b_s-1-i},a_s+1,\underbrace{1,\ldots,1}_{b_{s-1}-1},a_{s-1}+1,\ldots,\underbrace{1,\ldots,1}_{b_1-1},a_1+1,\underbrace{1,\ldots,1}_{i-1},2)\Big),$$

and from the definition, the dual of $(k_1,\ldots,k_n)$ is $(h_1,\ldots,h_{k-n})=(\underbrace{1,\ldots,1}_{b_s-1},a_s+1,$ $\ldots,\underbrace{1,\ldots,1}_{b_1-1},a_1+1)$, thus the above expression is equal to

$$\sum_{(h_1,h_2,\ldots,h_{k-n})\in\beta}\zeta(h_1,h_2,\ldots,h_{k-n-1},h_{k-n}+1).$$

Q.E.D.

We can also prove Theorem 2 by using Theorem 4 and the duality theorem, i.e., if we add up the equation of Theorem 4 for all entries of $\Pi(k,n)$, then we get the equation between the sum of all zeta values of weight $k+1$ and depth n and the sum of all zeta values of weight $k+1$ and depth $k-n$. If we apply the duality theorem to the right-hand side of the equation thus obtained, then the latter becomes the sum of all zeta values of weight $k+1$ and depth $n+1$, so it is the same formula as in the last part of the proof of Theorem 2, and we get the sum formula again.

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