A generalisation of the Morse inequalities

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1 Introduction

In [1] Arnold calls a Legendrian embedding $f\colon L \to J^1M$, into the 1-jet bundle, that can be obtained from the zero section by a smooth deformation through Legendrian embeddings a quasi-function. Intersections of f with $M_0 \times \mathbb{R}$, where M_0 denotes the zero section of T^*M, are called critical points of f and these are said to be nondegenerate if the intersection is transverse. It can be shown that every such quasi-function is of the form $f = \psi|_{M_0\times\{0\}}$, where $\psi = \psi^1$ for some contact isotopy $(\psi^t)_{0\leq t\leq 1}$ of J^1M. Under the assumption that M is closed (that is, compact and boundaryless) and a Riemannian metric on M has been chosen, we show that to each nondegenerate critical point $c \in L$ of f we may naturally assign a number

$$\mu(c, \psi^t) \in \frac{n}{2} + \mathbb{Z}$$

as the Maslov index of the path $\gamma\colon t \mapsto \psi^t((\psi^1)^{-1}f(c))$. Assuming that all critical points are nondegenerate we denote by p_k the number of critical points $c \in L$ such that $\mu(c, \psi^t) = k$ and by b_k the kth Betti number of M (with coefficients in any field). We prove the following:

Theorem 1.1 *Let $f\colon L \to J^1M$ be a quasi-function on a closed Riemannian manifold M of dimension n having only nondegenerate critical points. Then*

$$p_{\frac{n}{2}-k} - p_{\frac{n}{2}-k+1} + \cdots \geq b_k - b_{k-1} + \cdots + (-1)^k b_0$$

for every $k \in \mathbb{Z}$.

Theorem 1.1 is a refinement of a theorem by Chekanov announced by Arnold in [1] (see also Chekanov [5]). This bounds the number of critical points, in the nondegenerate case, below by the sum of the Betti numbers. A proof of Chekanov's Theorem is also given by Chaperon [4] and is based on a result by Théret [16] concerning the existence of a generating function. Our proof is independent of their work.

A method of generating contact isotopies is by using a time-dependent Hamiltonian $H\colon [0,1]\times J^1M \to \mathbb{R}$. Indeed all contact isotopies may be obtained in this way. A special instance is given by the following: given a Morse function $f\colon M \to \mathbb{R}$, let $H(x, y, z) = -f(x)$ define a time-independent Hamiltonian. The critical points of f are in one-to-one correspondence with the critical points of the associated quasi-function. In particular the Morse index $\mathrm{ind}_H(x_0)$ of a critical point $x_0 \in M$ of f is related to the Maslov index by

$$\mu(x_0, \psi_H^t) = \frac{n}{2} - \mathrm{ind}_H(x_0)$$

and thus in this special case Theorem 1.1 reduces to the classical Morse inequalities.

Our methods also refine a related theorem in the symplectic setting proved by Hofer [9] and Laudenbach and Sikorav [11]. This pertains to finding a lower bound for the number of points of intersection of the zero section of the cotangent bundle with its image under a time-dependent Hamiltonian symplectomorphism. An outline of our approach follows.

Let ψ^t be a contact isotopy of J^1M with Hamiltonian $H\colon [0,1]\times J^1M\to\mathbb{R}$. When $M=\mathbb{R}^n$ we may associate to H an action functional as follows. Consider the path space

$$\mathcal{P}=\left\{c=(x,y)\colon [0,1]\to T^*\mathbb{R}^n \,\middle|\, y(0)=0\right\},$$

and, given $c\in\mathcal{P}$, let $z=z_c\colon [0,1]\to\mathbb{R}$ be the unique solution of

$$\dot z=\langle y,\dot x\rangle - H(t,x,y,z)\qquad z(0)=0$$

(assuming such a solution exists). Define the action functional $\Phi_H\colon \mathcal{P}\to\mathbb{R}$ by

$$\Phi_H(c)=\int_0^1\Big(\langle y,\dot x\rangle - H(t,x,y,z)\Big)\,dt.$$

A calculation shows that the fibre critical points of Φ_H with respect to the fibration $\mathcal{P}\to\mathbb{R}^n : (x,y)\mapsto x(1)$ generate the Legendrian submanifold $\psi^1(L_0)$.

We consider discretisations of the form $\Phi_H^N\colon \mathcal{P}^N\to\mathbb{R}$ for $N\in\mathbb{N}$ which also have this property. Here

$$\mathcal{P}^N=\Big\{c^N=(x_0,\dots,x_N,y_1,\dots,y_N)\in\mathbb{R}^{(2N+1)n}\,\Big|\, x_i,\,y_j\in\mathbb{R}^n\Big\},$$

which is fibred over $\mathbb{R}^n$ by $\mathcal{P}^N\to\mathbb{R}^n : c^N\mapsto x_N$, can be thought of as a space of discrete paths in $\mathbb{R}^{2k}$, and Φ_H^N is the discrete action functional which is given by

$$\Phi_H^N(c^N)=\sum_{j=1}^N\Big(\langle y_j,x_j-x_{j-1}\rangle - V_{j-1}(x_{j-1},y_j,z_{j-1})\Big).$$

The functions $V_j\colon \mathbb{R}^{2n+1}\to\mathbb{R}$ are constructed by using the contact isotopy ψ^t, and the z_j are given by the iterative rule

$$z_0=0,\qquad z_j=\langle y_j,x_j-x_{j-1}\rangle - V_{j-1}(x_{j-1},y_j,z_{j-1})+z_{j-1},\quad j>0.$$

We show that such discretisations always exist for N sufficiently large and H having compact support.

A path $c\in\mathcal{P}$ is a critical point of Φ_H if and only if (c,z_c) is a trajectory of ψ^t starting in $L_0\times\{0\}$, where L_0 denotes the zero section of $T^*\mathbb{R}^n$, and ending in $L_0\times\mathbb{R}$. Similarly, a discrete path $c^N\in\mathcal{P}^N$ is a critical point of Φ^N if and only if $(c^N,\{z_j\})$ corresponds to a "broken trajectory" of ψ^t with $(x_0,y_0,z_0)\in L_0\times\{0\}$ and $(x_N,y_N,z_N)\in L_0\times\mathbb{R}$. Furthermore given such a path c^N, critical for Φ^N, and such that $\psi^1(L_0\times\{0\})$ intersects $L_0\times\mathbb{R}$ transversally at (x_N,y_N,z_N), $d^2\Phi_H^N(c^N)$ *will* be nondegenerate. We show that, given a critical point $c\in\mathcal{P}$ with the same property, for sufficiently large N the signature of the second variation of the discretisation is given by:

Proposition 1.2

$$\operatorname{sign} d^2\Phi_H^N(c^N)=2\mu(c^N,\psi^t).$$

This generalises a theorem of Robbin and Salamon in [13] for quadratic Hamiltonians. Also, it strengthens the theorem of Viterbo [17] which states that, in the symplectic case, the difference of the signature of the second variation of Φ^N at two critical points is independent of N, but at the cost of having to take N sufficiently large.

The action functional and its discretisation are extended to the 1-jet bundle of a closed manifold M as follows. First, M is embedded in $\mathbb{R}^k$ for a suitable k. Next, a lift $\widetilde{H}\colon J^1\mathbb{R}^k \to \mathbb{R}$ of H is chosen such that $\widetilde{H}|_{J^1\mathbb{R}^k|_M} = H \circ \rho$ where $\rho\colon J^1\mathbb{R}^k|_M \to J^1M$ is the natural projection map. Finally, the appropriate action functional is defined to be $\Phi_{\widetilde{H}}$ restricted to the space of paths having $x(1) \in M$, and similarly for the discretisation. Theorem 1.1 follows from generalising Proposition 1.2 and appealing to stabilised Morse theory (the Conley index).

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2 Contact structures

In this section we recall some basic notions of contact structures. A *contact structure* on a $2n+1$-dimensional manifold Q is a maximally nonintegrable field of hyperplanes $\xi \subset TQ$. If we assume that ξ is transversally orientable, that is ξ is given by the kernel of some 1-form α, then the contact condition can be stated as

$$\alpha \wedge (d\alpha)^n \neq 0. \tag{1}$$

In this case α is said to be a *contact form* for the contact structure ξ. An important equivalent formulation of the contact condition (1) is that $d\alpha$ is a nondegenerate 2-form on ξ.

Given a contact manifold (Q, ξ) and an integral submanifold $L \subset Q$ note that, for each $p \in Q$, the tangent space T_pL is an isotropic subspace of the symplectic vector space $(\xi_p, d\alpha_p)$, that is, $d\alpha_p$ vanishes on T_pL. This implies, in particular, that the dimension of L can be at most n. In such a case L is said to be *Legendrian*.

A diffeomorphism $\psi\colon Q \to Q$ which preserves the oriented field of hyperplanes ξ is called a *contactomorphism*, that is, equivalently, if

$$\psi^*\alpha = e^g\alpha$$

for some function $g\colon Q \to \mathbb{R}$.

Note that if $\psi\colon Q \to Q$ is a contactomorphism, then, by a direct calculation, its derivative restricts to give a linear conformal symplectomorphism

$$d\psi(p)|_{\xi_p}\colon (\xi_p, d\alpha_p) \to (\xi_{\psi(p)}, d\alpha_{\psi(p)}),$$

for every $p \in Q$.

A smooth family $(\psi^t)_{0\leq t\leq 1}$ of diffeomorphisms $\psi^t\colon Q \to Q$ with $\psi^0 = \mathrm{id}$ is called an *isotopy* of Q. When each ψ^t is a contactomorphism the family ψ^t is called a *contact isotopy*.

A *contact vector field* is defined to be a vector field $X\colon Q \to TQ$ which satisfies

$$\mathcal{L}_X\alpha = h\alpha$$

for some function $h\colon Q \to \mathbb{R}$.

The *contact Hamiltonian* associated to a contact vector field $X\colon Q \to TQ$ is defined to be the function $H\colon M \to \mathbb{R}$ given by

$$H = -\iota(X)\alpha. \tag{2}$$

It is easy to check that every function $H\colon Q\to\mathbb{R}$ is the contact Hamiltonian of a unique contact vector field $X=X_H\colon Q\to TQ$.

1-jet bundles

The standard contact structure on the 1-jet bundle $J^1M=T^*M\times\mathbb{R}$ of an n-dimensional manifold M is given as kernel of the 1-form

$$\alpha = dz - \lambda_{\mathrm{can}}.$$

Here z denotes the standard coordinate on $\mathbb{R}$ and λ_{can} the canonical 1-form on T^*M which is given by the defining property $\sigma^*\lambda_{\mathrm{can}}=\sigma$ for every 1-form $\sigma\colon M\to T^*M$.

Let ψ^t be a contact isotopy of J^1M and $X\colon[0,1]\times J^1M\to TJ^1M$ be the associated time-dependent contact vector field defined by

$$\frac{d}{dt}\psi^t(p)=X(t,\psi^t(p)).$$

Associated to X is the time-dependent Hamiltonian $H\colon[0,1]\times J^1M\to\mathbb{R}$ given by (2). We express this in local coordinates.

About any point $(c,z)\in J^1M$ one can construct a distinguished set of local coordinates $x_1,\dots,x_n,y_1,\dots,y_n,z$ in which α takes the form

$$\alpha = dz-\sum_i y_i\,dx_i.$$

In these coordinates it is easy to check that X is given by Hamilton's equations:

$$\dot x_i=\frac{\partial H}{\partial y_i} \tag{3a}$$

$$\dot y_i=-\frac{\partial H}{\partial x_i}-y_i\frac{\partial H}{\partial z} \tag{3b}$$

$$\dot z=\langle y,\dot x\rangle - H \tag{3c}$$

and thus ψ^t is the time-t map of the flow associated to Hamilton's equations for H.

3 The variational principle

In this section we describe an abstract method for generating all solutions of Hamilton's equations subject to $c(0)\in M_0$, the zero section of T^*M, and $z(0)=0$. Let $H\colon[0,1]\times J^1M\to\mathbb{R}$ be a time-dependent Hamiltonian. To ensure that (4) below has a solution $z\colon[0,1]\to\mathbb{R}$ for every smooth path $c\colon[0,1]\to T^*M$ assume that ∂_zH is bounded. Let $\pi\colon\mathcal{P}_M\to M$ be the fibre bundle where

$$\mathcal{P}_M=\{c\colon[0,1]\to T^*M\mid c(0)\in M_0\}$$

is a space of smooth paths and π is the projection $c\mapsto\pi_{T^*M}c(1)\in M$. To each path $c\in\mathcal{P}_M$ associate a function $z=z_c\colon[0,1]\to\mathbb{R}$ given as the unique solution of

$$\dot z(t)=\lambda_{\mathrm{can}}(\dot c(t))-H(t,c(t),z(t)),\qquad z(0)=0. \tag{4}$$

Now define the *action functional* $\Phi_H\colon\mathcal{P}_M\to\mathbb{R}$ by

$$\Phi_H(c)=z(1)=\int_0^1\Big(\lambda_{\mathrm{can}}(\dot c)-H(t,c,z)\Big)\,dt.$$

A point $c \in \mathcal{P}_M$ is called a *fiber critical point* of Φ_H if the differential of Φ_H disappears along $T_c(\mathcal{P}_M)_{\pi(c)}$, the tangent space at c of the fibre over $\pi(c)$.

Proposition 3.1 *There is a one-to-one correspondence between the set of fibre critical points of Φ_H and the solutions of Hamilton's equations subject to $c(0) \in M_0$, $z(0) = 0$. Furthermore, $c \in \mathcal{P}_M$ is fibre critical if and only if*

$$\iota(\dot{c})d\lambda_{\text{can}} + dH_{t,z}(c) + \partial_z H(t, c, z)\lambda_{\text{can}} = 0 \tag{5}$$

Proof. Let $(-\varepsilon, \varepsilon) \to \mathcal{P}_M : s \mapsto c_s$ be a parametrised family of paths and abbreviate $c_0 = c$. Denote by $\gamma : [0, 1] \to c^*T(T^*M)$ the associated variational field:

$$\gamma(t) = \left.\frac{\partial}{\partial s}\right|_0 c_s(t).$$

Define $z_s : [0, 1] \to \mathbb{R}$ for each s by (4) and denote the derivative with respect to s at $s = 0$ by ζ. Now ζ evaluated at $t \in [0, 1]$ is given by the expression

$$\begin{aligned}
\zeta(t) &= \left.\frac{\partial}{\partial s}\right|_0 \int_0^t c_s^*(\lambda_{\text{can}} - H_{z_s}\, d\tau) \\
&= \int_0^t c^*\mathcal{L}_\gamma(\lambda_{\text{can}} - H_z\, d\tau) - \partial_z H\zeta\, d\tau \\
&= \int_0^t \Big(\iota(\gamma)(-\iota(\dot{c})d\lambda_{\text{can}} - dH_{\tau,z}) - \partial_z H\zeta\Big)\, d\tau + \lambda_{\text{can}}(\gamma(t))
\end{aligned}$$

where the third equality follows from Cartan's identity for the Lie derivative. Thus the 1-form α applied to the vector field (γ, ζ) satisfies

$$\alpha(\gamma(t), \zeta(t)) = \int_0^t \Big(\iota(\gamma)(-\iota(\dot{c})d\lambda_{\text{can}} - dH_{\tau,z} - \partial_z H\lambda_{\text{can}}) - \partial_z H\alpha(\gamma, \zeta)\Big)\, d\tau$$
$$\alpha(\gamma(0), \zeta(0)) = 0.$$

This we can solve explicitly. Evaluating at $t = 1$ we find

$$\alpha(\gamma(1), \zeta(1)) = \int_0^1 e^{\int_t^1 -\partial_z H\, d\tau}\iota(\gamma)(-\iota(\dot{c})d\lambda_{\text{can}} - dH_{t,z} - \partial_z H\lambda_{\text{can}})\, dt.$$

Thus it follows that $c \in \mathcal{P}_M$ is fibre critical if and only if (5) holds which, in local coordinates, is just equations (3a, b). □

Note that at a fibre critical point c the differential of Φ_H is given by

$$d\Phi_H(c)(\gamma) = \lambda_{\text{can}}(\gamma(1)).$$

It thus follows that c is a critical point only if $c(1) \in M_0$.

4 Variational families

The notion of a variational family formalises the variational principle that was considered in the previous section. In this section we consider some basic properties of variational families in general. We start by briefly recalling some ideas from the theory of symplectic reduction.

Suppose that (P, ω) is a symplectic manifold and that $W \subset P$ is a coisotropic submanifold. This means that T_pW^ω is contained in T_pW for each $p \in W$ where T_pW^ω denotes the symplectic complement of T_pW. Then the subspaces T_pW^ω determine an isotropic distribution TW^ω in TP which by the closedness of ω is integrable. It follows from Frobenius' theorem that W is foliated by isotropic leaves. If we assume that the quotient space $P' = W/\sim$ is a manifold, where the equivalence relation is given by: $p_0 \sim p_1$ if p_0 and p_1 lie in the same leaf, then we note that the P' is naturally a symplectic manifold.

We next recall that if L is a Lagrangian submanifold of P which intersects W cleanly, that is, $L \cap W$ is a submanifold of P and $T_p(L \cap W) = T_pL \cap T_pW$ for all $p \in L \cap W$, then the image of L in P' is also Lagrangian, but may only be immersed.

To pass from Lagrangian immersions to Legendrian immersions we use the process of contactisation. This applies to all symplectic manifolds (P, ω) where the symplectic form is given as the differential of a 1-form $-\lambda$. Such manifolds are said to be *exact.* A *contactisation* of an exact symplectic manifold (P, ω) is a contact manifold $(P \times \mathbb{R}, dz - \lambda)$ where λ is a 1-form satisfying the relation $\omega = -d\lambda$. Here z denotes the coordinate on $\mathbb{R}$. Given an exact symplectic manifold (P, ω), a *contactisation* of a Lagrangian immersion $f\colon L \to P$ is a Legendrian immersion $\tilde{f}\colon L \to P \times \mathbb{R}$ lifting f. If the Lagrangian immersion $f\colon L \to P$ is *exact*, that is, $f^*\lambda = dS$ for some function $S\colon L \to \mathbb{R}$, then the lift $L \to P \times \mathbb{R}$: $c \mapsto (f(c), S(c))$ is Legendrian. Conversely it is easy to see that if f admits a lift then it must be exact. Note however that the contactisation of an exact Lagrangian immersion is not unique since the function S is unique only up to the addition of a locally constant function.

We now give the definition of a *variational family.* This is a pair consisting of a fibre bundle $\pi\colon E \to X$ and a smooth function $\Phi\colon E \to \mathbb{R}$. The variational is called *transversal* if the graph of $d\Phi$ in T^*E intersects the conormal bundle of the fibres

$$N_E = \left\{(c, \eta) \in T^*E \,\middle|\, \eta \in (\ker d\pi(c))^\perp\right\}$$

transversally. For a transversal variational family (E, Φ) the set of fibre critical points is given by

$$\mathcal{C}_{E,\Phi} = \left\{c \in E \,\middle|\, d\Phi(c) \in (\ker d\pi(c))^\perp\right\}$$

which by transversality is a manifold. To each fibre critical point we can uniquely associate a covector $v^* \in T^*_{\pi(c)}X$ by the relation $v^* \circ d\pi(c) = d\Phi(c)$. This gives rise to a map $\iota_\Phi : \mathcal{C}_{E,\Phi} \to J^1X$ given by mapping $c \in \mathcal{C}_{E,\Phi}$ to the 1-jet $(\pi(c), v^*, \Phi(c))$.

The following proposition is due to Hörmander [10]. We include here a proof for the sake of completeness.

Proposition 4.1 *Given a transversal variational family (E, Φ), the map $\iota_\Phi\colon \mathcal{C}_{E,\Phi} \to J^1M$ is a Legendrian immersion.*

Proof. Note that the graph of $d\Phi$ is a Lagrangian submanifold of T^*E and N_E is a coisotropic submanifold. Note also that the quotient $N_E/\sim$ can be identified with T^*X. It follows from the above discussion that the map $\iota'_\Phi\colon \mathcal{C}_{E,\Phi} \to T^*X$, given by composing ι_Φ with the projection $J^1X \to T^*X$, is a Lagrangian immersion. Moreover the pullback of λ_{can} by ι'_Φ is given by

$${\iota'_\Phi}^*\lambda_{\text{can}} = v^* \circ d(\pi|_{\mathcal{C}_{E,\Phi}}) = d(\Phi|_{\mathcal{C}_{E,\Phi}})$$

and so ι'_Φ is an exact immersion. Contactising, it follows that ι_Φ is a Legendrian immersion. $\square$

Denote by L_Φ the image of ι_Φ and by X_0 the zero section of T^*X. Clearly, a point $c \in \mathcal{C}_{E,\Phi}$ is a critical point of Φ if and only if $\iota_\Phi(c)$ is in $L_\Phi \cap X_0 \times \mathbb{R}$. Moreover, when E is a finite dimensional vector bundle we have:

Lemma 4.2 *A point $c \in \operatorname{crit}\Phi$ is a nondegenerate critical point of Φ if and only if L_Φ and $X_0 \times \mathbb{R}$ intersect transversally at $\iota_\Phi(c)$.*

The proof of this uses the following elementary fact:

Lemma 4.3 *Let (V, ω) be a symplectic vector space and $N \subset V$ a coisotropic subspace. Then given two Lagrangian subspaces Λ_0 and Λ_1 satisfying*

$$\Lambda_0 \subset N, \qquad \Lambda_1 \cap N^\omega = \{0\},$$

Λ_0 is transverse to Λ_1 if and only if, in the quotient, the reduced spaces $\overline{\Lambda}_0$ and $\overline{\Lambda}_1$ are transversal.

Proof of Lemma 4.2. Apply Lemma 4.3 in case $V = T_p(T^*E)$, $N = T_pN_E$, $\Lambda_0 = T_pE_0$ and $\Lambda_1 = T_p(\operatorname{graph}(d\Phi))$ where $p = (c, d\Phi(c))$ and E_0 denotes the zero section of T^*E. □

Example 4.4 (Action) Let $H\colon [0,1] \times J^1M \to \mathbb{R}$ be a time-dependent Hamiltonian and $\Phi_H\colon \mathcal{P}_M \to \mathbb{R}$ the action functional given in Section 3. By Proposition 3.1 the pair $(\mathcal{P}_M, \Phi_H)$ is a variational family for the Legendrian submanifold $\psi^1(M_0 \times \{0\})$, where $\psi^t = \psi^t_H$ denotes the contact isotopy generated by H. Note that we do not say anything about transversality.

Restricted variational families

Let M be a submanifold of X and let $L \to J^1X\colon c \mapsto (f(c), S(c))$ be a Legendrian immersion transversal to $J^1X|_M$. We extend the notion of symplectic reduction to such Legendrian immersions as follows.

Evidently $f\colon L \to T^*X$ is an exact Lagrangian immersion. Define $f'\colon L' \to T^*M$, where $L' = f^{-1}(T^*X|_M)$, by $f' = r \circ f \circ i$. Here $i\colon L' \hookrightarrow L$ is the inclusion and $r\colon T^*X|_M \to T^*M$ is the natural projection. By transversality L' is a manifold of the same dimension as M. Using $r^*\lambda_{\mathrm{can}} = j^*\lambda_{\mathrm{can}}$ where $j\colon T^*X|_M \hookrightarrow T^*X$ is the inclusion, it is now easy to see that f' is again an exact Lagrangian immersion with $f'^*\lambda_{\mathrm{can}} = d(S \circ i)$. The lift $L' \to J^1M\colon c \mapsto (f'(c), S \circ i(c))$ we refer to as the *reduction* of (f, S).

Suppose that $(\pi\colon E \to X,\ \Phi\colon E \to \mathbb{R})$ is a transversal variational family and denote by

$$(\pi_M = \pi|_{E_M}\colon E_M \to M,\ \Phi_M = \Phi|_{E_M}\colon E_M \to \mathbb{R}),$$

where $E_M = E|_M$, the variational family obtained by restricting to M. In the case of a finite dimensional vector bundle these are related by:

Lemma 4.5 *Given a transversal variational family (E, Φ), the restricted variational family (E_M, Φ_M) is transversal if and only if L_Φ is transverse to $J^1X|_M$. Moreover, if both variational families are transversal then L_{Φ_M} is the reduction of L_Φ.*

Proof. The first part of the lemma follows easily by choosing local coordinates on X in which M is linear and the second part follows by a direct calculation. □

Corollary 4.6 *If both* (E, Φ) *and* (E_M, Φ_M) *are transversal then* $c \in E_M$ *is a nondegenerate critical point of* Φ_M *if and only if* L_Φ *and* $TM^\perp \times \mathbb{R}$*, where* $TM^\perp \subset T^*X$ *denotes the conormal bundle of* M*, intersect transversally at* $\iota_\Phi(c)$*.*

Proof. This follows immediately from Lemmas 4.2, 4.3 and 4.5. □

5 Continuous-time theory

In this section we assume that M is embedded in $\mathbb{R}^k$ and that $\psi^t : J^1M \to J^1M$ is a given contact isotopy. We construct a variational family $(\mathcal{P}_{\mathbb{R}^k}, \Phi_H)$ on $\mathbb{R}^k$ whose restriction to M, $(\mathcal{P}, \Phi)$, generates the Legendrian submanifold $\psi^1(M_0 \times \{0\}) \subset J^1M$, and consider the second variation of the function Φ.

5.1 Construction of variational family

Suppose that $H\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ is a time-dependent Hamiltonian with the property that for every $x \in M$

$$H(t,x,y,z) = H(t,x,y',z) \quad \text{whenever} \quad y - y' \perp T_xM.$$

Denoting by $\rho\colon J^1\mathbb{R}^k|_M \to J^1M$ the natural projection $(x,y,z) \mapsto (x, y|_{T_xM}, z)$, it follows that H descends to a time-dependent Hamiltonian $\overline{H}\colon [0,1] \times J^1M \to \mathbb{R}$ given by

$$H|_{J^1\mathbb{R}^k|_M} = \overline{H} \circ \rho. \tag{6}$$

We insist that H_t is C^1-bounded for every $t \in [0,1]$. This ensures that the Hamiltonian flows of H and $\overline{H}$ define global diffeomorphisms for all $t \in [0,1]$.

The next lemma shows how the Hamiltonian flow of H is related to that of $\overline{H}$ (*cf.* [2]).

Lemma 5.1 *Let* H *be a time-dependent Hamiltonian on* $J^1\mathbb{R}^k$ *and denote by* φ^t *the associated contact isotopy. Then the following are equivalent:*

(i) H *satisfies* (6) *for some function* $\overline{H}\colon [0,1] \times J^1M \to \mathbb{R}$*,*

(ii) φ^t *leaves* $J^1\mathbb{R}^k|_M$ *invariant,*

(iii) φ^t *descends to a contact isotopy on* J^1M*.*

Proof. (i) ⇒ (ii). Let $X = (\dot{x}, \dot{y}, \dot{z})$ denote the vector field generated by H. It is sufficient to show that $X_t(p) \in T_p(J^1\mathbb{R}^k|_M)$ for all $p \in J^1\mathbb{R}^k|_M$. To see this note that, for any $p = (x,y,z) \in J^1\mathbb{R}^k|_M$,

$$\begin{aligned} \dot{x}_t(p) &= \partial_y H_t(x,y,z) \\ &= \partial_y H_t(x, \Pi(x)y, z) \\ &= \Pi(x)\partial_y H_t(x,y,z). \end{aligned}$$

Here T^*M has been identified with TM, using the induced inner product, and $\Pi(x) \in \mathbb{R}^{k \times k}$ denotes the orthogonal projection onto T_xM. It follows $\dot{x}_t(p) \in T_xM$ and hence $X_t(p) \in T_p(J^1\mathbb{R}^k|_M)$ as required.

(ii) $\Rightarrow$ (iii). We first claim that if a contactomorphism φ preserves $J^1\mathbb{R}^k|_M$ then it has the property that

$$\rho(\varphi(p)) = \rho(\varphi(q)) \quad \text{whenever} \quad \rho(p) = \rho(q). \tag{7}$$

To see this note that the subspace $N_p = T_p(J^1\mathbb{R}^k|_M) \cap \xi_p$ is a coisotropic subspace of the symplectic vector space $(\xi_p, d\alpha_p)$, for each $p \in J^1\mathbb{R}^k|_M$, and that the symplectic complement is given simply by $N_p^\omega = T_p(\rho^{-1}(\rho(p)))$. Also note that $d\varphi(p)$ restricts to a map from N_p to $N_{\varphi(p)}$. Property (7) now follows from the fact that the fibers of ρ are connected.

Consequently, we may define $\psi^t : J^1M \to J^1M$ by $\rho \circ \varphi^t \circ j = \psi^t \circ \rho$, where j denotes the inclusion $J^1\mathbb{R}^k|_M \hookrightarrow J^1\mathbb{R}^k$. Clearly ψ^t is a diffeomorphism. That ψ^t is a contactomorphism follows from a direct calculation using the identity $j^*\alpha = \rho^*\alpha$.

(iii) $\Rightarrow$ (i). Since φ^t descends to a contact isotopy ψ^t on J^1M it is clear that φ^t preserves $J^1\mathbb{R}^k|_M$. Also it follows that the contact vector fields X, $\overline{X}$ generated by the isotopies φ^t, ψ^t respectively are related by $\rho_*X = \overline{X}$. These satisfy $\iota(X)\alpha = -H$, $\iota(\overline{X})\alpha = -\overline{H}$, where $\overline{H}$ is the Hamiltonian function generating ψ^t. A direct calculation now shows that H and $\overline{H}$ are related by (6). This completes the proof of the lemma. $\Box$

Remark 5.2 It follows from Lemma 5.1 that, when H satisfies (6), there is a commutative diagram

$$\begin{array}{ccc} J^1\mathbb{R}^k|_M & \xrightarrow{\varphi^t} & J^1\mathbb{R}^k|_M \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ J^1M & \xrightarrow{\psi^t} & J^1M \end{array}$$

but note that φ^t is not uniquely determined by the restriction of H to $J^1\mathbb{R}^k|_M$.

We now return to the main problem and assume, from now on, that our contact isotopy ψ^t is generated by $\overline{H}$. We define the variational family $(\mathcal{P}, \Phi)$ as follows.

Let $\mathcal{P} = \mathcal{P}_{\mathbb{R}^k,M}$ be the space of paths

$$\mathcal{P} = \{c = (x,y) \colon [0,1] \to T^*\mathbb{R}^k \mid y(0) = 0,\, x(1) \in M\}$$

in $T^*\mathbb{R}^k$, and let π be the projection $\mathcal{P} \to M$ given by $c = (x,y) \mapsto x(1)$. Define the action functional $\Phi = \Phi_{H,M} \colon \mathcal{P} \to \mathbb{R}$ by

$$\Phi(c) = \int_0^1 \Big(\langle y(t), \dot{x}(t)\rangle - H(t, x(t), y(t), z(t))\Big)\, dt$$

where $z \colon [0,1] \to \mathbb{R}$ is the unique solution of the initial value problem

$$\dot{z}(t) = \langle y(t), \dot{x}(t)\rangle - H(t, x(t), y(t), z(t)), \qquad z(0) = 0. \tag{8}$$

Lemma 5.1 now implies the following:

Proposition 5.3 *$(\mathcal{P}, \Phi)$ is a variational family for the Legendrian submanifold $\psi^1(M_0 \times \{0\}) \subset J^1M$.*

In particular, $c \in \mathcal{P}$ is a critical point of Φ if and only if (c, z) is a solution of Hamilton's equations and $c(1)$ is in $TM^\perp$, the conormal bundle of M.

Remark 5.4 The variational families $(\mathcal{P}_M, \Phi_{\overline{H}})$, of Example 4.4, and $(\mathcal{P}, \Phi)$, considered above, both generate the Legendrian submanifold $\psi^1(M_0 \times \{0\}) \subset J^1M$. These are related as follows: let $c \in \mathcal{P}$ be a fibre critical point of Φ, then $\overline{c} \in \mathcal{P}_M$, given by composing c with the projection $T^*\mathbb{R}^k|_M \to T^*M$, is a fibre critical point of $\Phi_{\overline{H}}$. Conversely any fibre critical point $\overline{c}$ of $\Phi_{\overline{H}}$ lifts uniquely to a fibre critical point c of Φ by using the contact isotopy φ^t.

5.2 The second variation

From the proof of Proposition 3.1 the differential of Φ is given by

$$\begin{aligned} d\Phi(c)(\gamma) = &\int_0^1 e^{\int_t^1 -\partial_z H\, d\tau}\Big(\big\langle \eta, \dot{x} - \partial_y H\big\rangle + \big\langle \dot{y} + \partial_x H + y\partial_z H, \xi\big\rangle\Big)\, dt \\ &+ \langle y(1), \xi(1)\rangle \end{aligned}$$

where $\gamma \in T_c\mathcal{P} = \big\{\gamma = (\xi, \eta) \colon [0,1] \to \mathbb{R}^{2k} \,\big|\, \eta(0) = 0,\, \xi(1) \in T_{x(1)}M\big\}$. In order to deal with the boundary term $\langle y(1), \xi(1)\rangle$ we now restrict Φ to the space

$$\mathcal{E} = \mathcal{E}_{\mathbb{R}^k, M} = \big\{c \in \mathcal{P} \,|\, c(1) \in TM^\perp\big\}$$

of paths satisfying Lagrangian boundary conditions. Before giving a formula for the second variation of Φ at a critical point we introduce some notation.

Let $c \in \mathcal{P}$ be a critical point of Φ and denote by $z \colon [0,1] \to \mathbb{R}$ the corresponding solution of (8). Define, for reasons that will become apparent later, the matrix valued functions $G_{\xi\xi}, G_{\xi\eta}, G_{\eta\xi}, G_{\eta\eta} \colon [0,1] \to \mathbb{R}^{k\times k}$ and $G_\zeta \colon [0,1] \to \mathbb{R}$ by

$$\begin{aligned} G_{\xi\xi} &= \partial^2_{xx}H + \partial^2_{xz}Hy^T + y\partial^2_{zx}H + \partial^2_{zz}Hyy^T \\ G_{\xi\eta} &= \partial^2_{xy}H + y\partial^2_{zy}H \\ G_{\eta\xi} &= \partial^2_{yx}H + \partial^2_{yz}Hy^T \\ G_{\eta\eta} &= \partial^2_{yy}H \\ G_\zeta &= \partial_z H. \end{aligned} \tag{9}$$

Here the derivatives of H are evaluated at $(t, c(t), z(t)) \in [0,1] \times J^1\mathbb{R}^k$. Note that $G_{\xi\xi}(t)$ and $G_{\eta\eta}(t)$ are symmetric and that $G_{\xi\eta}(t) = G_{\eta\xi}(t)^T$. Construct from these the symmetric matrix valued function $S = S_G \colon [0,1] \to \mathbb{R}^{2k\times 2k}$ by

$$S(t) = \begin{pmatrix} G_{\xi\xi}(t) & G_{\xi\eta}(t) + \frac{1}{2}G_\zeta(t)\mathbb{1} \\ G_{\eta\xi}(t) + \frac{1}{2}G_\zeta(t)\mathbb{1} & G_{\eta\eta}(t) \end{pmatrix}. \tag{10}$$

Also, denote by J_0 the $2k \times 2k$ matrix

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

and by $\mathcal{C}_c = \mathcal{C}_{c,M}$ the tangent space of $\mathcal{E}$ at c:

$$\mathcal{C}_c = \big\{\gamma \colon [0,1] \to \mathbb{R}^{2k} \,\big|\, \gamma(0) \in \Lambda_0,\, \gamma(1) \in \Lambda_1\big\}.$$

Here $\Lambda_0 = \mathbb{R}^k \times \{0\}$ and $\Lambda_1 = T_{c(1)}(TM^\perp)$ are Lagrangian subspaces of $\mathbb{R}^{2k}$.

Proposition 5.5 *The second differential of Φ at a critical point c is of the form*

$$d^2\Phi(c)(\gamma_1,\gamma_2) = \langle A(c)\gamma_1,\gamma_2\rangle_{L^2}$$

where $\gamma_1,\gamma_2 \in \mathcal{C}_c$ and the second variation, $A(c)$, is given by

$$A(c)\gamma = e^{\int_t^1 -G_\zeta\,d\tau}\left(J_0\dot\gamma - S\gamma + \tfrac{1}{2}G_\zeta J_0\gamma\right). \tag{11}$$

Proof. Fix a two parameter family of curves

$$c_{s_1,s_2} = (x_{s_1,s_2}, y_{s_1,s_2}) \in \mathcal{E}, \qquad s_1,\ s_2 \in (-\varepsilon,\varepsilon)$$

with $c_{0,0} = c$ and denote the derivatives by

$$\xi_{i,s_1,s_2} = \frac{\partial}{\partial s_i}x_{s_1,s_2}, \quad \eta_{i,s_1,s_2} = \frac{\partial}{\partial s_i}y_{s_1,s_2}, \quad \zeta_{i,s_1,s_2} = \frac{\partial}{\partial s_i}z_{s_1,s_2}.$$

Here, of course, z_{s_1,s_2} is given by solving (8) with x,y,z replaced by $x_{s_1,s_2}, y_{s_1,s_2}, z_{s_1,s_2}$. Abbreviate $\xi_i = \xi_{i,0,0}$, $\eta_i = \eta_{i,0,0}$, $\zeta_i = \zeta_{i,0,0}$ and set $\gamma_i = (\xi_i,\eta_i)$, $\mu_i = (\xi_i,\eta_i,\zeta_i)$. We now proceed with the proof of the proposition.

Differentiate the identity

$$\dot z_{s_1,s_2} = \left\langle y_{s_1,s_2}, \dot x_{s_1,s_2}\right\rangle - H(\cdot, x_{s_1,s_2}, y_{s_1,s_2}, z_{s_1,s_2})$$

with respect to s_2 and s_1 successively and then evaluate at $s_1 = s_2 = 0$ to obtain

$$\begin{aligned}\partial_{s_1}\dot\zeta_2 = {} & \left\langle \partial_{s_1}\eta_2, \dot x\right\rangle + \left\langle \eta_2, \dot\xi_1\right\rangle + \left\langle \eta_1, \dot\xi_2\right\rangle + \left\langle y, \partial_{s_1}\dot\xi_2\right\rangle \\ & - d^2H_t(\mu_1,\mu_2) - \langle \partial_x H, \partial_{s_1}\xi_2\rangle - \langle \partial_y H, \partial_{s_1}\eta_2\rangle - \partial_z H \partial_{s_1}\zeta_2.\end{aligned}$$

Using the fact that c is critical rewrite this as

$$\dot\zeta = \left\langle \eta_2, \dot\xi_1\right\rangle + \left\langle \eta_1, \dot\xi_2\right\rangle + \frac{d}{dt}\langle y, \partial_{s_1}\xi_2\rangle + \langle y\partial_z H, \partial_{s_1}\xi_2\rangle - \partial_z H\zeta - d^2H_t(\mu_1,\mu_2)$$

where $\zeta = \partial_{s_1}\zeta_2$ or more concisely, by using the substitution $\lambda = \zeta - \langle y, \partial_{s_1}\xi_2\rangle$, as

$$\frac{d}{dt}\lambda = \left\langle \eta_2, \dot\xi_1\right\rangle + \left\langle \eta_1, \dot\xi_2\right\rangle - \partial_z H\lambda - d^2H_t(\mu_1,\mu_2).$$

Since $\lambda(0) = 0$ the solution to this ODE evaluated at $t = 1$ is

$$\lambda(1) = \int_0^1 e^{\int_t^1 -G_\zeta\,d\tau}\Big(\left\langle \eta_2, \dot\xi_1\right\rangle + \left\langle \eta_1, \dot\xi_2\right\rangle - d^2H_t(\mu_1,\mu_2)\Big)\,dt.$$

Also since $0 = \partial_{s_1}\langle y(1), \xi_2(1)\rangle = \langle \eta_1(1), \xi_2(1)\rangle + \langle y(1), \partial_{s_1}\xi_2(1)\rangle$ it follows that $\lambda(1) = \zeta(1) + \langle \eta_1(1), \xi_2(1)\rangle$. Now partially integrate the second summand on the right hand side to obtain

$$\zeta(1) = \int_0^1 e^{\int_t^1 -G_\zeta\,d\tau}\Big(\left\langle \eta_2, \dot\xi_1\right\rangle - \left\langle \dot\eta_1, \xi_2\right\rangle - G_\zeta\langle \eta_1, \xi_2\rangle - d^2H_t(\mu_1,\mu_2)\Big)\,dt.$$

Since c is critical observe, by Proposition 3.1, that

$$\alpha(\mu_i(t)) = \zeta_i(t) - \langle y(t), \xi_i(t)\rangle = 0$$

for all $t \in [0,1]$ thus, in particular,

$$d^2H_t(\mu_1,\mu_2) = \langle G_{\xi\xi}\xi_1, \xi_2\rangle + \langle G_{\xi\eta}\eta_1, \xi_2\rangle + \langle G_{\eta\xi}\xi_1, \eta_2\rangle + \langle G_{\eta\eta}\eta_1, \eta_2\rangle.$$

The proposition now follows. $\square$

Remark 5.6 $A(c)$ extends to a self-adjoint operator on $L^2([0,1],\mathbb{R}^{2k})$ with dense domain

$$W^{1,2}_{c,M} = \left\{\gamma \in W^{1,2}([0,1];\mathbb{R}^{2k}) \,\middle|\, \gamma(0) \in \Lambda_0,\, \gamma(1) \in \Lambda_1\right\}.$$

In addition, $A(c)$ is injective if and only if $\Psi(1)\Lambda_0$ is transverse to Λ_1 where $\Psi(t) = \Psi_{c,H}(t) \in \mathrm{Sp}(2k)$ is given by

$$\dot{\Psi} = -J_0 S\Psi, \qquad \Psi(0) = \mathbb{1}.$$

In such a case we will say that c is a *nondegenerate* critical point of Φ.

Remark 5.7 The contact isotopy φ^t, generated by H, linearised along the path $a = (c,z)\colon [0,1] \to J^1\mathbb{R}^k$ and restricted to contact planes, gives rise to the path of linear conformal symplectomorphisms

$$d\varphi^t(a(0))|_{\xi_{a(0)}} : (\xi_{a(0)}, d\alpha_{a(0)}) \to (\xi_{a(t)}, d\alpha_{a(t)}).$$

This path may be naturally identified with the path $e^{\int_0^t -\frac{1}{2}G_\zeta\, d\tau}\Psi_{c,H}$, where $\Psi_{c,H}$ is given in Remark 5.6, via the canonical symplectomorphisms

$$(\xi_p, d\alpha_p) \to (\mathbb{R}^{2k}, \omega_0)\colon (\xi, \eta, \langle y, \xi\rangle) \mapsto (\xi, \eta)$$

where $p = (x, y, z)$.

Remark 5.8 Identify T^*M with TM using the induced inner product. The tangent space to $TM^\perp$ at $p = (x,y)$ is given by

$$T_p(TM^\perp) = \left\{(\xi,\eta) \in \mathbb{R}^{2k} : \xi \in T_xM,\, (d\Pi(x)\xi)y + \Pi(x)\eta = 0\right\}$$

where $\Pi(x) \in \mathbb{R}^{k\times k}$ denotes the orthogonal projection onto T_xM. Note that defining $I\!I_p\colon T_xM \to \mathbb{R}$ by $I\!I_p(\xi) = \frac{1}{2}\langle (d\Pi(x)\xi)y, \xi\rangle$ gives a quadratic form on T_xM, the *second fundamental form* of M at x along the normal vector y

Remark 5.9 Let $G\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ be the time-dependent Hamiltonian given by

$$G(t,\xi,\eta,\zeta) = \tfrac{1}{2}\langle G_{\xi\xi}(t)\xi,\xi\rangle + \langle G_{\eta\xi}(t)\xi,\eta\rangle + \tfrac{1}{2}\langle G_{\eta\eta}(t)\eta,\eta\rangle + G_\zeta(t)\zeta.$$

We call such Hamiltonians, where $G_{\xi\xi}(t), G_{\eta\xi}(t), G_{\eta\eta}(t) \in \mathbb{R}^{k\times k}$ with $G_{\xi\xi}(t)$, $G_{\eta\eta}(t)$ symmetric and $G_\zeta(t) \in \mathbb{R}$, *quadratic*, and define $G_{\xi\eta}(t) = G_{\eta\xi}(t)^T$. Define the functional $\Phi_c = \Phi_{c,H,M}\colon \mathcal{C}_c \to \mathbb{R}$ by

$$\Phi_c(\gamma) = \int_0^1 \left(\langle \eta, \dot{\xi}\rangle - G(t,\xi,\eta,\zeta)\right) dt + I\!I_{c(1)}(\xi(1)),$$

where $\zeta\colon [0,1] \to \mathbb{R}$ is the unique solution of the initial value problem

$$\dot{\zeta}(t) = \langle \eta(t), \dot{\xi}(t)\rangle - G(t,\xi(t),\eta(t),\zeta(t)), \qquad \zeta(0) = 0. \tag{12}$$

Notice that if c is a nondegenerate critical point of Φ, then 0 is the unique critical point of Φ_c, also nondegenerate, and in this case the above proposition shows that the second variation of Φ at c agrees with the second variation of Φ_c.

Remark 5.10 A quadratic Hamiltonian $G\colon [0,1]\times J^1\mathbb{R}^k \to \mathbb{R}$ as above generates the contact isotopy

$$\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} = e^{\int_0^t -\frac{1}{2}G_\zeta\, d\tau}\Psi_G(t)\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix}$$
$$\zeta(t) = e^{\int_0^t -G_\zeta\, d\tau}\zeta_0 + \tfrac{1}{2}\int_0^t e^{\int_s^t -G_\zeta\, d\tau}(\langle G_{\eta\eta}\eta,\eta\rangle - \langle G_{\xi\xi}\xi,\xi\rangle)\, ds$$

where $\Psi_G\colon [0,1]\to \mathrm{Sp}(2k)$ is given by

$$\dot{\Psi}_G = -J_0 S_G \Psi_G, \qquad \Psi_G(0) = \mathbb{1}. \tag{13}$$

Conversely, given any smooth family of symplectic matrices $\Psi\colon [0,1]\to \mathrm{Sp}(2k)$ and any smooth positive function $a\colon [0,1]\to \mathbb{R}_{>0}$ there exists a contact isotopy of the form

$$\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} = a(t)\Psi(t)\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix}$$
$$\zeta(t) = a(t)^2\zeta_0 + Q(t,\xi_0,\eta_0)$$

where $\Psi = \Psi_G$ for some quadratic Hamiltonian G and $Q(t,\cdot,\cdot)$ is quadratic. We call such contact isotopies *pseudo-linear*.

6 Discrete variational families

In this section we construct a family of finite dimensional variational families $(\mathcal{P}^N,\Phi^N)$ for our Legendrian submanifold $\psi^1(M_0\times\{0\})\subset J^1M$. This is done in two stages. The first stage is to consider the case of a contactomorphism C^1-close to the identity. The second stage is the general case. Alternative constructions are given by Chekanov [5] and Théret [16]. The corresponding construction in the case of cotangent bundles was first carried out by Laudenbach and Sikorav [11].

6.1 Generating functions of type V

We generalise the notion of a generating function of type V, as defined in [12], to the contact setting. Let $\varphi\colon J^1\mathbb{R}^k\to J^1\mathbb{R}^k$ be a contactomorphism and denote it as follows:

$$(x_0,y_0,z_0)\mapsto (x_1,y_1,z_1) = (u(x_0,y_0,z_0), v(x_0,y_0,z_0), w(x_0,y_0,z_0)).$$

A *generating function of type V* for φ is a function $V\colon \mathbb{R}^{2k+1}\to\mathbb{R}$ such that

$$\begin{aligned} x_1 - x_0 &= \frac{\partial V}{\partial y}(x_0,y_1,z_0)\\ y_1 - y_0 &= -\frac{\partial V}{\partial x}(x_0,y_1,z_0) - y_0\frac{\partial V}{\partial z}(x_0,y_1,z_0)\\ z_1 - z_0 &= \langle y_1, x_1 - x_0\rangle - V(x_0,y_1,z_0)\end{aligned} \tag{14}$$

if and only if $(x_1,y_1,z_1) = \varphi(x_0,y_0,z_0)$. In this case, observe that φ has compact support (that is, φ is equal to the identity outside a compact set) if and only if V has.

Proposition 6.1 (i) *Every contactomorphism $\varphi\colon J^1\mathbb{R}^k\to J^1\mathbb{R}^k$ which is sufficiently close to the identity in the C^1-topology admits a unique generating function of type V.*

(ii) *For each smooth function* $V\colon \mathbb{R}^{2k+1} \to \mathbb{R}$ *having compact support and sufficiently small first and second derivatives there exists a unique contactomorphism* $\varphi\colon J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ *such that* V *is a generating function of type* V *for* φ

Proof. For (i), the assumption on φ ensures that the map

$$\mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1} : (x_0, y_0, z_0) \mapsto (x_0, v(x_0, y_0, z_0), z_0)$$

has a global inverse. Thus there exists a map $f\colon \mathbb{R}^{2k+1} \to \mathbb{R}^k$ such that

$$y_0 = f(x_0, v(x_0, y_0, z_0), z_0).$$

Also, since $\varphi\colon (x_0, y_0, z_0) \mapsto (x_1, y_1, z_1)$ is a contactomorphism, we have

$$dz_1 - \sum y_{1i}\, dx_{1i} = e^g \left(dz_0 - \sum y_{0i}\, dx_{0i}\right) \tag{15}$$

for some function $g\colon \mathbb{R}^{2k+1} \to \mathbb{R}$. Thus, using the notation set up earlier, the chain rule gives

$$\begin{aligned} dz_1 - \sum y_{1i}\, dx_{1i} = \sum \left(\partial_{x_{0i}} w - \langle y_1, \partial_{x_{0i}} u\rangle\right) dx_{0i} \\ + \sum \left(\partial_{y_{0i}} w - \langle y_1, \partial_{y_{0i}} u\rangle\right) dy_{0i} + \left(\partial_{z_0} w - \langle y_1, \partial_{z_0} u\rangle\right) dz_0. \end{aligned}$$

Comparing with (15) we find that

$$\partial_{y_0} w - (\partial_{y_0} u)^T y_1 = 0, \tag{16}$$

and

$$\partial_{x_0} w - (\partial_{x_0} u)^T y_1 = -\left(\partial_{z_0} w - \langle y_1, \partial_{z_0} u\rangle\right) y_0. \tag{17}$$

We claim that $V\colon \mathbb{R}^{2n+1} \to \mathbb{R}$ defined by

$$\begin{aligned} V(x_0, y_1, z_0) &:= \langle y_1, x_1 - x_0\rangle - z_1 + z_0 \\ &= \langle y_1, u(x_0, f(x_0, y_1, z_0), z_0) - x_0\rangle - w(x_0, f(x_0, y_1, z_0), z_0) + z_0 \end{aligned} \tag{18}$$

satisfies (14).

This claim is proved by a straightforward calculation, differentiating (18) and using the identities (16) and (17). The uniqueness of generating functions of type V is clear.

For (ii), let $V\colon \mathbb{R}^{2k+1} \to \mathbb{R}$ be a smooth function satisfying the hypotheses. We construct φ as follows. Let $\mu\colon \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$ be the map defined by

$$(x_0, y_1, z_0) \mapsto \Big(x_0, (1 - \partial_{z_0} V(x_0, y_1, z_0))^{-1} \left(y_1 + \partial_{x_0} V(x_0, y_1, z_0)\right), z_0\Big),$$

and $\nu\colon \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$ the map defined by

$$(x_0, y_1, z_0) \mapsto \left(x_0 + \partial_{y_1} V, y_1, z_0 + \langle y_1, \partial_{y_1} V\rangle - V\right).$$

The hypotheses on V ensure that μ and ν are diffeomorphisms. We define φ to be the composition $\nu \circ \mu^{-1}$. That φ is a contactomorphism is seen directly as follows. Expand the formula for the contact 1-form $\alpha = dz_1 - \sum y_{1i}\, dx_{1i}$. After cancellations this becomes

$$dz_1 - \sum y_{1i}\, dx_{1i} = (1 - \partial_{z_0} V)\, dz_0 - \sum (\partial_{x_{0i}} V + y_{1i})\, dx_{0i}.$$

From the definition of the function y_0 we have the equality

$$\partial_{x_0} V + y_1 = (1 - \partial_{z_0} V)\, y_0$$

hence φ is a contactomorphism as required. □

6.2 The pseudo-linear case

When φ is the time-t_1 map of a contact isotopy generated by a quadratic Hamiltonian G the pair (ξ_1, η_1) is given in terms of (ξ_0, η_0) by

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = a\Psi \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} \tag{19}$$

where $a > 0$ and

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2k)$$

are as given in Remark 5.10. In this case the condition of φ being C^1-close to the identity is replaced by requirement that

$$\det(D) \neq 0 \tag{20}$$

which holds for $t_1 > 0$ sufficiently small. Again, we say a generating function $\mathbb{R}^{2k+1} \to \mathbb{R}$: $(\xi, \eta, \zeta) \mapsto W(\xi, \eta, \zeta)$ is *quadratic* if it is quadratic in ξ and η and linear in ζ. A simple modification of the proof of Proposition 6.1 now gives:

Proposition 6.2 *Every contactomorphism $\varphi\colon J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ generated by a quadratic Hamiltonian with (20) holding admits a unique quadratic generating function of type V.*

Conversely, for each function $W = W(\xi, \eta, \zeta)$ on $\mathbb{R}^{2k+1}$, quadratic in ξ and η and linear in ζ, satisfying

$$1 - \partial_\zeta W \neq 0, \qquad \det(\mathbb{1} + \partial^2_{\xi\eta} W) \neq 0, \tag{21}$$

there exists a unique pseudo-linear contactomorphism $\varphi\colon J^1\mathbb{R}^k \to J^1\mathbb{R}^k$ such that W is a generating function of type V for φ.

We now give an explicit formula for the quadratic generating function of the pseudo-linear contactomorphism φ above. Observe from (19) that when condition (20) holds

$$\begin{aligned} \xi_1 - \xi_0 &= (aA - \mathbb{1})\xi_0 + aB\eta_0 \\ &= (aA - \mathbb{1})\xi_0 + BD^{-1}\eta_1 - aBD^{-1}C\xi_0 \\ &= (aA - aBD^{-1}C - \mathbb{1})\xi_0 + BD^{-1}\eta_1 \\ &= \big(a(D^{-1})^T - \mathbb{1}\big)\xi_0 + BD^{-1}\eta_1. \end{aligned}$$

Here the last equality follows from the fact that $\Psi \in \mathrm{Sp}(2k)$. Also note that

$$\begin{aligned} \eta_1 - \eta_0 &= \eta_1 - a^{-1}D^{-1}\eta_1 + D^{-1}C\xi_0 \\ &= D^{-1}C\xi_0 + (\mathbb{1} - aD^{-1})\eta_1 + (a - a^{-1})D^{-1}\eta_1 \\ &= D^{-1}C\xi_0 + (\mathbb{1} - aD^{-1})\eta_1 + (a - a^{-1})D^{-1}(aC\xi_0 + aD\eta_0) \\ &= a^2D^{-1}C\xi_0 + (\mathbb{1} - aD^{-1})\eta_1 + (a^2 - 1)\eta_0. \end{aligned}$$

It follows that the generating function of type V for φ is given by

$$\begin{aligned} W(\xi_0, \eta_1, \zeta_0) = &-\tfrac{1}{2}\langle a^2D^{-1}C\xi_0, \xi_0\rangle + \big\langle\big(a(D^{-1})^T - \mathbb{1}\big)\xi_0, \eta_1\big\rangle \\ &+ \tfrac{1}{2}\langle BD^{-1}\eta_1, \eta_1\rangle + (1 - a^2)\zeta_0. \end{aligned} \tag{22}$$

6.3 Discrete-time variational theory

We consider the general case. Recall that M is a closed manifold embedded $\mathbb{R}^k$, and $H\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ satisfies (6) for some function $\overline{H}\colon [0,1] \times J^1M \to \mathbb{R}$. It follows that H will *not* have compact support. Hence we multiply by a smooth cutoff function and assume, by abuse of notation, that H is equal to zero outside a large compact set containing $\bigcup_{t\in[0,1]} \varphi^t(M)$, where φ^t denotes the contact isotopy generated by H, and M the set $M \times \{0\} \times \{0\} \subset J^1\mathbb{R}^k$.

We now construct a family of finite dimensional variational families generating the Legendrian submanifold $\psi^1(M_0 \times \{0\})$, by discretising time. Pick an integer N and define

$$\varphi_j^{j+1} = \varphi^{\frac{j+1}{N}} \circ \left(\varphi^{\frac{j}{N}}\right)^{-1}$$

for $j = 0, \dots, N-1$. Then

$$\varphi^1 = \varphi_{N-1}^{N} \circ \varphi_{N-2}^{N-1} \circ \cdots \circ \varphi_0^1$$

and for N sufficiently large each φ_j^{j+1} satisfies the hypotheses of Proposition 6.1 (i). Hence for each j there exists a function $V_j\colon \mathbb{R}^{2k+1} \to \mathbb{R}$ such that

$$\begin{aligned} x_{j+1} - x_j &= \frac{\partial V_j}{\partial y} \\ y_{j+1} - y_j &= -\frac{\partial V_j}{\partial x} - y_j\frac{\partial V_j}{\partial z} \\ z_{j+1} - z_j &= \langle y_{j+1}, x_{j+1} - x_j\rangle - V_j \end{aligned} \tag{23}$$

if and only if $(x_{j+1}, y_{j+1}, z_{j+1}) = \varphi_j^{j+1}(x_j, y_j, z_j)$.

Now define, by analogy with the continuous-time case, the space

$$\mathcal{P}^N = \mathcal{P}^N_{\mathbb{R}^k,M} = \left\{c = (x_0, \dots, x_N, y_1, \dots, y_N) \in \mathbb{R}^{(2N+1)k} \,\middle|\, x_N \in M\right\}$$

of discrete paths in $\mathbb{R}^{2k}$, and to each $c \in \mathcal{P}^N$ associate a sequence $(z_0, \dots, z_N)$ given by the iterative rule

$$z_0 = 0, \qquad z_j = \langle y_j, x_j - x_{j-1}\rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1}) + z_{j-1}, \quad j > 0. \tag{24}$$

Let $\pi\colon \mathcal{P}^N \to M$ denote the projection $c \mapsto x_N$. The *discrete action functional* $\Phi^N = \Phi^N_{H,M}\colon \mathcal{P}^N \to \mathbb{R}$ can now be defined by

$$\Phi^N(c) = z_N = \sum_{j=1}^{N} \Big(\langle y_j, x_j - x_{j-1}\rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1})\Big).$$

Proposition 6.3 $(\mathcal{P}^N, \Phi^N)$ *is a variational family for the Legendrian submanifold* $\psi^1(M_0 \times \{0\}) \subset J^1M$.

This follows from:

Lemma 6.4 *There is a one-to-one correspondence between the set of fibre critical points of* Φ^N *and solutions of* (23) *(for* $j = 0, \dots, N-1$*) with* $y_0 = z_0 = 0$, $x_N \in M$.

Proof. The partial derivatives of Φ^N are

$$\frac{\partial \Phi^N}{\partial x_j} = (1-\partial_z V_{N-1})\cdots(1-\partial_z V_{j+1})\,(y_j - y_{j+1} - \partial_x V_j - y_j \partial_z V_j)$$
$$\frac{\partial \Phi^N}{\partial y_{j+1}} = (1-\partial_z V_{N-1})\cdots(1-\partial_z V_{j+1})\,(x_{j+1} - x_j - \partial_y V_j)$$

for $j = 0, \ldots, N-1$ (on defining $y_0 = 0$). Since $1-\partial_z V_l$ is nonzero for each l, this proves the lemma. □

As before, $c \in \mathcal{P}^N$ is a critical point of Φ^N if and only if (c, z) is a solution of Hamilton's discrete equations (23) and $c_N = (x_N, y_N) \in TM^{\perp}$. In particular, c is a critical point of Φ^N if and only if it is obtained by sampling a path $c \in \operatorname{crit}\Phi$.

Remark 6.5 In applications it will be useful to note the following. Let $x = x_N$ and $\xi = (x_0, \ldots, x_{N-1}, y_1, \ldots, y_N)$ denote the base and fibre coordinates respectively of $\mathcal{P}^N = M \times \mathbb{R}^{2m}$ where $m = Nk$. Then Φ^N can be written in the form

$$\Phi^N(x,\xi) = \tfrac{1}{2}\langle P\xi, \xi\rangle + W(x,\xi).$$

where $P \in \mathbb{R}^{2m\times 2m}$ is a nondegenerate symmetric matrix of signature zero and W is given by

$$W(x,\xi) = \langle x, y_N\rangle - \sum_{j=1}^{N} V_{j-1}(x_{j-1}, y_j, z_{j-1})$$

Also, the gradient of W with respect to ξ, $\partial_\xi W$, is bounded.

Remark 6.6 Suppose $L \subset J^1M$ is a Legendrian submanifold given by a variational family $(E = M \times \mathbb{R}^l, S)$. Then $\psi^1(L)$ is also given by a variational family. This is seen as follows.

Choose any function $\widetilde{S}\colon \widetilde{E} \to \mathbb{R}$, where $\widetilde{E} = \mathbb{R}^k \times \mathbb{R}^l$, satisfying $\widetilde{S}|_E = S$, and define the space

$$\widetilde{\mathcal{P}}^N = \left\{(\xi, c) \in \mathbb{R}^l \times \mathbb{R}^{(2N+1)k} \;\middle|\; c \in \mathcal{P}^N\right\}$$

of augmented discrete paths. Now define the generalised discrete action functional $\widetilde{\Phi}^N\colon \widetilde{\mathcal{P}}^N \to \mathbb{R}$ by

$$\widetilde{\Phi}^N(\xi, c) = z_N = \sum_{j=1}^{N}\Big(\langle y_j, x_j - x_{j-1}\rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1})\Big) + z_0$$

where

$$z_0 = \widetilde{S}(x_0, \xi), \qquad z_j = \langle y_j, x_j - x_{j-1}\rangle - V_{j-1}(x_{j-1}, y_j, z_{j-1}) + z_{j-1}, \quad j > 0.$$

It is now easy to check that $(\widetilde{\mathcal{P}}^N, \widetilde{\Phi}^N)$ is a variational family for $\psi^1(L)$.

Remark 6.7 An alternative and more explicit way of constructing a discrete action functional is by setting

$$V_j(x_j, y_{j+1}, z_j) = H(t_j, x_j, y_{j+1}, z_j)(t_{j+1} - t_j)$$

where $t_j = j/N$ and N is sufficiently large so that the hypotheses of Proposition 6.1 (ii) hold for each V_j. A disadvantage of this method is that it does not generate the original Legendrian submanifold but one that is close to it. This is the approach used by Robbin and Salamon in the symplectic setting [13].

6.4 The second variation

We start with some notation. Let $c \in \mathcal{P}^N$ be a critical point of Φ^N and $\{z_j\}$ the associated sequence given by the iterative rule (24). For $j = 0, \dots, N-1$ define $W_{j,\xi\xi}, W_{j,\xi\eta}, W_{j,\eta\xi}, W_{j,\eta\eta} \in \mathbb{R}^{k\times k}$ and $W_{j,\zeta} \in \mathbb{R}$ by

$$\begin{aligned} W_{j,\xi\xi} &= \partial^2_{xx}V_j + \partial^2_{xz}V_j y_j^T + y_j\partial^2_{zx}V_j + \partial^2_{zz}V_j y_j y_j^T \\ W_{j,\xi\eta} &= \partial^2_{xy}V_j + y_j\partial^2_{zy}V_j \\ W_{j,\eta\xi} &= \partial^2_{yx}V_j + \partial^2_{yz}V_j y_j^T \\ W_{j,\eta\eta} &= \partial^2_{yy}V_j \\ W_{j,\zeta} &= \partial_z V_j \end{aligned}$$

where V_j is evaluated at (x_j, y_{j+1}, z_j). Note that $W_{j,\xi\xi}, W_{j,\eta\eta}$ are symmetric and $W_{j,\xi\eta} = W^T_{j,\eta\xi}$. Abbreviate $\alpha^l_j = (1-\partial_z V_l)\cdots(1-\partial_z V_j)$ for $l \geq j$, setting it to 1 otherwise. Denote by $W^N_c = W^N_{c,M}$ the tangent space of $\mathcal{P}^N$ at c:

$$W^N_c = \Big\{\gamma = (\xi_0, \dots, \xi_N, \eta_1, \dots, \eta_N) \in \mathbb{R}^{(2N+1)k} \,\big|\, \xi_N \in T_{x_N}M\Big\}.$$

We now give a formula for the second variation of Φ^N at c. As before, $\Pi(x) \in \mathbb{R}^{k\times k}$ denotes the orthogonal projection onto T_xM.

Proposition 6.8 *The second derivative of Φ^N at a critical point c is of the form*

$$d^2\Phi^N(c)(\gamma_1,\gamma_2) = \langle A^N(c)\gamma_1, \gamma_2\rangle$$

where $\gamma_1, \gamma_2 \in W^N_c$ and the second variation, $A^N(c)\colon W^N_c \to W^N_c\colon (\xi,\eta) \mapsto (u,v)$, is given by the expressions[1]

$$\begin{aligned} u_j &= \alpha^{N-1}_{j+1}(\eta_j - \eta_{j+1} - W_{j,\xi\xi}\xi_j - W_{j,\xi\eta}\eta_{j+1} - W_{j,\zeta}\eta_j) \\ v_{j+1} &= \alpha^{N-1}_{j+1}(\xi_{j+1} - \xi_j - W_{j,\eta\xi}\xi_j - W_{j,\eta\eta}\eta_{j+1}) \\ u_N &= (d\Pi(x_N)\xi_N)y_N + \Pi(x_N)\eta_N. \end{aligned}$$

Proof. Differentiate the formulae in the proof of Lemma 6.4. □

Remark 6.9 For $j = 0, \dots, N-1$ let $W_j\colon \mathbb{R}^{2k+1} \to \mathbb{R}$ be the function defined by

$$W_j(\xi_j, \eta_{j+1}, \zeta_j) = \tfrac{1}{2}\langle W_{j,\xi\xi}\xi_j, \xi_j\rangle + \langle W_{j,\eta\xi}\xi_j, \eta_{j+1}\rangle + \tfrac{1}{2}\langle W_{j,\eta\eta}\eta_{j+1}, \eta_{j+1}\rangle + W_{j,\zeta}\zeta_j.$$

Using these we may associate to the critical point c the function $\Phi^N_c = \Phi^N_{c,H,M}\colon W^N_c \to \mathbb{R}$ defined by

$$\Phi^N_c(\gamma) = \sum_{j=1}^N \Big(\langle\eta_j, \xi_j - \xi_{j-1}\rangle - W_{j-1}(\xi_{j-1}, \eta_j, \zeta_{j-1})\Big) + I\!I_{c_N}(\xi_N)$$

where ζ_j is given by the iterative rule

$$\zeta_0 = 0, \qquad \zeta_j = \langle\eta_j, \xi_j - \xi_{j-1}\rangle - W_{j-1}(\xi_{j-1}, \eta_j, \zeta_{j-1}) + \zeta_{j-1}, \quad j > 0 \tag{25}$$

and where $I\!I_{c_N}$ denotes the second fundamental form. As in the continuous-time case, if c is a nondegenerate critical point of Φ^N, then 0 is the unique critical point of Φ^N_c, also nondegenerate, and in this case Proposition 6.8 implies that the second variation of Φ^N at c agrees with the second variation of Φ^N_c.

[1] Define $\eta_0 = 0$.

Recall that in the continuous-time case given a critical point c of Φ we can construct a functional Φ_c whose second variation agrees with the second variation of Φ at c (see Remark 5.9). The following proposition shows how Φ_c^N and Φ_c are related.

Proposition 6.10 *Φ_c^N is the discretisation of Φ_c.*

Proof. Let $(c, z)\colon [0, 1] \to J^1M$ be a solution of Hamilton's equations beginning in $L_0 \times \{0\}$ and $(c^N, \{z_j\})$ the corresponding solution of Hamilton's discrete equations. Note that, by Remark 5.7, linearising Hamilton's equations for H along (c, z) and restricting to contact planes, which are canonically identified with $\mathbb{R}^{2k}$, yields the equations

$$\begin{aligned}\dot{\xi} &= G_{\eta\xi}\xi + G_{\eta\eta}\eta \\ \dot{\eta} &= -G_{\xi\xi}\xi - G_{\xi\eta}\eta - G_{\zeta}\eta.\end{aligned}$$

Similarly, linearising Hamilton's discrete equations along $(c^N, \{z_j\})$ and restricting to contact planes yields the equations

$$\begin{aligned}\xi_{j+1} - \xi_j &= W_{j,\eta\xi}\xi_j + W_{j,\eta\eta}\eta_{j+1} \\ \eta_{j+1} - \eta_j &= -W_{j,\xi\xi}\xi_j - W_{j,\xi\eta}\eta_{j+1} - W_{j,\zeta}\eta_j.\end{aligned}$$

It follows that the W_j define a family of generating functions of type V for the pseudo-linear contact isotopy generated by G and the result now follows. □

Corollary 6.11 *$(\mathcal{P}^N, \Phi^N)$ is always transversal as a variational family.*

Proof. We need to show that, given $\gamma = (\xi_0, \ldots, \xi_N, \eta_1, \ldots, \eta_N) \in W_c^N$ satisfying $\xi_N = 0$, if $d^2\Phi^N(c)(\gamma, \gamma')$ is 0 for all $\gamma' \in W_c^N$ then γ is identically 0. This is seen as follows. Suppose γ satisfies the hypotheses of the statement we are seeking to show, then

$$\begin{aligned}\xi_{j+1} - \xi_j &= W_{j,\eta\xi}\xi_j + W_{j,\eta\eta}\eta_{j+1} \\ \eta_{j+1} - \eta_j &= -W_{j,\xi\xi}\xi_j - W_{j,\xi\eta}\eta_{j+1} - W_{j,\zeta}\eta_j\end{aligned}$$

for $j = 1, \ldots, N-1$, and $\Pi(x_N)\eta_N = 0$. Denoting by A_j the conformal symplectic matrix associated to W_j, it follows that

$$\begin{pmatrix}\xi_{j+1} \\ \eta_{j+1}\end{pmatrix} = A_j \begin{pmatrix}\xi_j \\ \eta_j\end{pmatrix}$$

for $j = 1, \ldots, N-1$, and that (ξ_N, η_N) is in the symplectic complement of the coisotropic subspace $N_{c_N} = T_{c_N}(T^*\mathbb{R}^k|_M)$. Since φ^t preserves $J^1\mathbb{R}^k|_M$, we have $A_jN_{c_j} = N_{c_{j+1}}$. It follows that $(\xi_0, 0)$ is in the symplectic complement of N_{c_0} which implies that $\xi_0 = 0$ and hence γ is identically 0, as required. □

Corollary 6.12 *The second variation of Φ^N at a critical point c is nondegenerate if and only if $\Psi_{c,H}\Lambda_0$ is transverse to $\Lambda_1 = T_{c_N}(TM^\perp)$.*

Proof. This follows immediately from Corollaries 6.11 and 4.6 and Remark 5.7. □

Remark 6.13 If the variational family (E, S), given in Remark 6.6, is transversal, then the variational family $(\widetilde{\mathcal{P}}^N, \widetilde{\Phi}^N)$ will also be transversal. This follows by arguing as in the proof of Corollary 6.11 and using Lemma 4.5.

7 The Maslov index

In this section we define the Maslov index for the points of intersection of $\psi^1(M_0 \times \{0\})$ and $M_0 \times \mathbb{R}$, assuming that the intersection is transversal. We begin by summarising the properties of the Maslov index for various spaces of paths that we will require. Proofs of these may be found in [14].

Let $\Lambda\colon [a,b] \to \mathcal{L}(k)$ be a path of Lagrangian subspaces and let $Z = (X,Y)\colon [a,b] \to \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{2k})$ denote a choice of a frame for Λ. This means that $Z(t)\colon \mathbb{R}^k \to \mathbb{R}^{2k}$ is an injective linear map such that

$$\Lambda(t) = \operatorname{Im} Z(t), \qquad X(t)^T Y(t) = Y(t)^T X(t)$$

for all t. Also, let $V \in \mathcal{L}(k)$ be a fixed Lagrangian subspace. A *crossing* for the path Λ is a number $t \in [a,b]$ such that $\Lambda(t)$ intersects V nontrivially. At a crossing t the *crossing form* is defined to be the quadratic form

$$\Gamma(\Lambda, V, t)\colon \Lambda(t) \cap V \to \mathbb{R}$$

given by

$$v \mapsto \langle X(t)u, \dot{Y}(t)u \rangle - \langle Y(t)u, \dot{X}(t)u \rangle$$

where $v = Z(t)u$. A crossing t is said to be *regular* if $\Gamma(\Lambda, V, t)$ is nonsingular. When the path Λ has only regular crossings the *Maslov index*, $\mu(\Lambda, V)$, is defined by

$$\mu(\Lambda, V) = \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, V, a) + \sum_{a<t<b} \operatorname{sign} \Gamma(\Lambda, V, t) + \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, V, b)$$

where the sum is taken over all crossings t. By perturbing, keeping endpoints fixed, this definition extends to give a well defined Maslov index for all paths. The integers k_a and k_b are defined by

$$k_a = \dim \Lambda(a) \cap V, \qquad k_b = \dim \Lambda(b) \cap V.$$

We now state some properties of the Maslov index.

(Integrality) The integers μ, k_a and k_b are related by

$$\mu + \tfrac{1}{2}(k_a - k_b) \in \mathbb{Z}.$$

(Product) Under the natural identification of $\mathcal{L}(k) \times \mathcal{L}(k')$ as a submanifold of $\mathcal{L}(k+k')$

$$\mu(\Lambda \oplus \Lambda', V \oplus V') = \mu(\Lambda, V) + \mu(\Lambda', V').$$

(Localisation) For the path Λ given by the frame $t \mapsto (\mathbb{1}, A(t))$ and V the subspace $\mathbb{R}^k \times \{0\}$ the Maslov index is given by

$$\mu(\Lambda, V) = \tfrac{1}{2} \operatorname{sign} A(b) - \tfrac{1}{2} \operatorname{sign} A(a).$$

(Zero) If $\Lambda(t) \cap V$ has constant dimension for all t then $\mu(\Lambda, V) = 0$.

For a pair of Lagrangian paths $\Lambda, \Lambda'\colon [a,b] \to \mathcal{L}(k)$ the *relative crossing form* $\Gamma(\Lambda, \Lambda', t)$ on $\Lambda(t) \cap \Lambda(t)'$ is defined by

$$\Gamma(\Lambda, \Lambda', t) = \Gamma(\Lambda, \Lambda'(t), t) - \Gamma(\Lambda(t), \Lambda', t).$$

When the pair Λ, Λ' have only regular crossings the *relative Maslov index*, $\mu(\Lambda, \Lambda')$, is defined by

$$\mu(\Lambda, \Lambda') = \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, \Lambda', a) + \sum_{a<t<b} \operatorname{sign} \Gamma(\Lambda, \Lambda', t) + \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, \Lambda', b).$$

The relative Maslov index has the following property.

(Naturality) For a Lagrangian pair Λ, Λ' and a symplectic path Ψ

$$\mu(\Psi\Lambda, \Psi\Lambda') = \mu(\Lambda, \Lambda').$$

Finally, for a path of symplectic matrices $\Psi\colon [a,b] \to \mathrm{Sp}(2k)$ the *Maslov index*, $\mu(\Psi)$, is defined by[2]

$$\mu(\Psi) = \mu(\Psi V, V)$$

where $V = \mathbb{R}^k \times \{0\}$.

The following property of the Maslov index for symplectic paths will be used.

(Homotopy) Two symplectic paths Ψ, Ψ' with $\Psi(a) = \Psi'(a)$ and $\Psi(b), \Psi'(b) \in \mathrm{Sp}_0(2k)$ are homotopic within this class if and only if they have the same Maslov index, where $\mathrm{Sp}_0(2k)$ denote the set of matrices $\Psi \in \mathrm{Sp}(2k)$ such that ΨV transverse to V.

We now define the Maslov index for the points of intersection of $\psi^1(M_0 \times \{0\})$ and $M_0 \times \mathbb{R}$, or equivalently, the critical points of $\Phi_{\overline{H}}$.

Let $\overline{c} \in \mathcal{P}_M$ be a critical point of $\Phi_{\overline{H}}$ and denote by $c = (x, y) \in \mathcal{P}$ its unique lift to a critical point of Φ (see Remark 5.4). Induced by c is the path of symplectic matrices

$$\Psi_{c,H} \colon [0,1] \to \mathrm{Sp}(2k)$$

(see Remarks 5.6 and 5.7) and this reduces, via symplectic reduction of the coisotropic distribution $T(T^*\mathbb{R}^k|_M)$, to the path of symplectic linear transformations

$$\Psi_{\overline{c},\overline{H}}(t) \colon (T_{\overline{c}(0)}(T^*M), -d\lambda_{\mathrm{can}\,\overline{c}(0)}) \longrightarrow (T_{\overline{c}(t)}(T^*M), -d\lambda_{\mathrm{can}\,\overline{c}(t)}).$$

Now define the *Maslov index*, $\mu(\overline{c}, \overline{H})$, by

$$\mu(\overline{c}, \overline{H}) = \mu(\Psi_{\overline{c},\overline{H}} \overline{\Lambda}_0, \overline{\Lambda}_1) \tag{26}$$

where $\overline{\Lambda}_0 = T_{x(0)}M$ and $\overline{\Lambda}_1$ is the path obtained by reducing

$$\Lambda_1(t) = \left\{ (\xi, \eta) \in \mathbb{R}^{2k} : \xi \in T_{x(t)}M,\ \Pi(x(t))(d\Pi(x(t))\xi)y(t) + \Pi(x(t))\eta = 0 \right\}. \tag{27}$$

Here the right hand side of (26) is defined by choosing a unitary trivialisation of the vector bundle $\overline{c}^*T(T^*M)$. By the naturality property this is independent of the trivialisation chosen. By the integrality property

$$\mu(\overline{c}, \overline{H}) \in \frac{n}{2} + \mathbb{Z}.$$

If we assume that the embedding $M \hookrightarrow \mathbb{R}^k$ is an isometry, then this definition is independent of the choice of embedding. Indeed, the induced Riemannian structure on T^*M,

[2]Note that this definition differs from the one given in [14] since here the Maslov index is defined with respect to the horizontal and there with respect to the vertical.

via the embedding $M \hookrightarrow \mathbb{R}^k$, induces a splitting of $T_p(T^*M)$ into a horizontal and a vertical space:

$$T_p(T^*M) = H_p \oplus V_p,$$

given by

$$\begin{aligned} H_p &= \left\{(\xi, (d\Pi(x)\xi)y) \in \mathbb{R}^{2k} : \xi \in T_x M\right\} \\ V_p &= \left\{(0,\eta) \in \mathbb{R}^{2k} : \eta \in T_x M\right\} \end{aligned}$$

where $p = (x, y)$. In this notation $\overline{\Lambda}_1(t)$ is just the horizontal subspace $H_{\overline{c}(t)}$.

Lemma 7.1 *Let $\overline{c} \in \mathcal{P}_M$ and $c \in \mathcal{P}$ be critical point of $\Phi_{\overline{H}}$ and Φ respectively and related as above. Then there is an equality*

$$\mu(\overline{c}, \overline{H}) = \mu(\Psi_{c,H}\Lambda_0, \Lambda_1(1)) + \tfrac{1}{2}\operatorname{sign} I\!I_{c(1)}$$

where $\Lambda_1 \colon [0,1] \to \mathcal{L}(k)$ is given by (27).

The proof of this lemma uses the following fact about symplectic reductions.

Lemma 7.2 *Let $\Lambda_0, \Lambda_1 \colon [0,1] \to \mathcal{L}(k)$ be a pair of Lagrangian paths and $N \subset \mathbb{R}^{2k}$ a coisotropic subspace such that*

$$\Lambda_0(t) \cap N^\omega = \{0\}, \qquad \Lambda_1(t) \subset N.$$

Then the Maslov index of the pair (Λ_0, Λ_1) agrees with the Maslov index of the reduced pair $(\overline{\Lambda}_0, \overline{\Lambda}_1)$:

$$\mu(\Lambda_0, \Lambda_1) = \mu(\overline{\Lambda}_0, \overline{\Lambda}_1).$$

Proof. By choosing a basis for the isotropic subspace N^ω and then extending it to a basis of the Lagrangian subspace $\Lambda_1(t)$ we can assume that

$$\begin{aligned} N &= \mathbb{R}^n \times \{0\} \times \mathbb{R}^n \times \mathbb{R}^{k-n} \\ \Lambda_1 &= \mathbb{R}^n \times \{0\} \times \{0\} \times \mathbb{R}^{k-n} \end{aligned}$$

for all t. Hence it is sufficient to check that the assertion of the proposition holds when only Λ_0 is allowed to vary and Λ_1 and N are as above. Since $\Lambda_0(t) \cap N^\omega = \{0\}$ it follows that we may choose a Lagrangian frame for Λ_0 of the form $Z = (X, Y) \colon [0,1] \to \mathcal{L}(\mathbb{R}^k, \mathbb{R}^{2k})$ where

$$X = \begin{pmatrix} \overline{X} & A \\ 0 & \mathbb{1} \end{pmatrix}, \qquad Y = \begin{pmatrix} \overline{Y} & B \\ C & D \end{pmatrix}$$

and thus a Lagrangian frame for $\overline{\Lambda}_0$ is given by $\overline{Z} = (\overline{X}, \overline{Y}) \colon [0,1] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{2n})$. We compute the crossing form for the pair (Λ_0, Λ_1).

A crossing for (Λ_0, Λ_1) is a number $t \in [0,1]$ such that there is a $u = (u', 0) \in \mathbb{R}^k$ such that $\overline{Y}(t)u' = 0$, that is, if and only if t is a crossing for $(\overline{\Lambda}_0, \overline{\Lambda}_1)$. At a crossing t the crossing form is given by

$$\begin{aligned} \Gamma(\Lambda_0, \Lambda_1, t)(v) &= \left\langle X(t)u, \dot{Y}(t)u\right\rangle - \left\langle Y(t)u, \dot{X}(t)u\right\rangle \\ &= \left\langle \overline{X}(t)u', \dot{\overline{Y}}(t)u'\right\rangle \\ &= \Gamma(\overline{\Lambda}_0, \overline{\Lambda}_1, t)(v') \end{aligned}$$

where $v = Z(t)u$, $v' = \overline{Z}(t)u'$. This proves the lemma. □

Proof of Lemma 7.1. In view of Lemma 7.2 it is sufficient to show that

$$\mu(\Psi_{c,H}\Lambda_0, \Lambda_1) = \mu(\Psi_{c,H}\Lambda_0, \Lambda_1(1)) + \tfrac{1}{2}\operatorname{sign} \mathit{II}_{c(1)}.$$

Let $\Phi\colon [0,1] \to \mathrm{Sp}(2k)$ be a path of symplectic matrices satisfying

$$\Lambda_1(t) = \Phi(t)\Lambda_1(1)$$

and abbreviate $\Psi = \Psi_{c,H}$. Then

$$\begin{aligned}
\mu(\Psi\Lambda_0, \Lambda_1) &= \mu(\Psi\Lambda_0, \Phi\Lambda_1(1)) \\
&= \mu(\Phi^{-1}\Psi\Lambda_0, \Lambda_1(1)) \\
&= \mu(\Phi(1)^{-1}\Psi\Lambda_0, \Lambda_1(1)) + \mu(\Phi^{-1}\Psi(0)\Lambda_0, \Lambda_1(1)) \\
&= \mu(\Psi\Lambda_0, \Lambda_1(1)) + \mu(\Lambda_0, \Lambda_1).
\end{aligned}$$

By multiplying by a path of unitary matrices if necessary we may assume that

$$\Lambda_1(t) = \left\{(\xi, 0, -B(t)\xi, \eta) \in \mathbb{R}^{2k} : \xi \in \mathbb{R}^n,\, \eta \in \mathbb{R}^{k-n}\right\}$$

where $B\colon [0,1] \to \mathbb{R}^{n\times n}$ is a path of symmetric matrices with $B(0) = 0$. Now by the product and localisation properties of the Maslov index

$$\mu(\Lambda_0, \Lambda_1) = \tfrac{1}{2}\operatorname{sign} B(1) = \tfrac{1}{2}\operatorname{sign} \mathit{II}_{c(1)}.$$

This proves the lemma. □

8 The signature identity

In this section we generalise the signature theorem of Robbin and Salamon in [13] to the case of nondegenerate critical points of the action functional on the 1-jet bundle of a closed manifold.

Theorem 8.1 *Let $c \in \mathcal{P}$ be a nondegenerate critical point of Φ, $\overline{c}$ its projection to $\mathcal{P}_M$ and $c^N \in \mathcal{P}^N$ the corresponding critical point of Φ^N. Then, for sufficiently large N, the signature of the second variation of Φ^N at c^N is given by*

$$\operatorname{sign} A^N(c^N) = 2\mu(\overline{c}, \overline{H}).$$

This strengthens the theorem of Viterbo [17], which states that, in the symplectic case, the difference of the signature of the second variation of Φ^N at two critical points is independent of N, but at the cost of having to take N sufficiently large.

Below we prove a result which implies this theorem.

Let $G\colon [0,1] \times J^1\mathbb{R}^k \to \mathbb{R}$ be a quadratic Hamiltonian and denote by $\Psi_G\colon [0,1] \to \mathrm{Sp}(2k)$ the associated path of symplectic matrices given by (13). Also, let $F \subset \mathbb{R}^k$ be a linear subspace and $B\colon F \to F$ a linear transformation such that $\langle B\xi, \xi'\rangle = \langle \xi, B\xi'\rangle$ for all ξ, $\xi' \in F$. Define the Lagrangian subspace

$$\Lambda_{F,B} = \left\{(\xi, \eta) \in \mathbb{R}^{2k} : \xi \in F,\, B\xi + \Pi_F\eta\right\}$$

Where $\Pi_F \in \mathbb{R}^{k\times k}$ denotes the orthogonal projection onto F. Now define the function $\Phi^N_{G,F,B}\colon W^N_F \to \mathbb{R}$ by

$$\Phi^N_{G,F,B}(\gamma) = \sum_{j=1}^{N} \Big(\langle \eta_j, \xi_j - \xi_{j-1} \rangle - W_{j-1}(\xi_{j-1}, \eta_j, \zeta_{j-1}) \Big) + \tfrac{1}{2}\langle B\xi_N, \xi_N \rangle$$

where

$$W^N_F = \Big\{ \gamma = (\xi_0, \ldots, \xi_N, \eta_1, \ldots, \eta_N) \in \mathbb{R}^{(2N+1)k} \,\big|\, \xi_N \in F \Big\}$$

and where the W_j are computed using (22), and the ζ_j using (25). Denote by $A^N_{G,F,B}$ the second variation of $\Phi^N_{G,F,B}$.

In view of Proposition 6.10 and Lemma 7.1, Theorem 8.1 now follows from:

Theorem 8.2 *Let G, F and B be as above and assume that $\Psi_G(1)\Lambda_0$ is transverse to $\Lambda_{F,B}$. Then, for N sufficiently large, the signature of the second variation of $\Phi^N_{G,F,B}$ is given by*

$$\operatorname{sign} A^N_{G,F,B} = 2\mu(\Psi_G\Lambda_0, \Lambda_{F,B}) + \operatorname{sign} B.$$

The proof we give is based on that given by Robbin and Salamon [13] in the case where $F = \mathbb{R}^k$, $B = 0$ and G is independent of z.

Proof. Without loss of generality assume that

$$\Lambda_{F,B} = \big\{ (\xi', 0, -B\xi', \eta'') \in \mathbb{R}^n \times \mathbb{R}^{k-n} \times \mathbb{R}^n \times \mathbb{R}^{k-n} = \mathbb{R}^{2k} : \xi' \in \mathbb{R}^n,\ \eta'' \in \mathbb{R}^{k-n} \big\}$$

where $\dim F = n$ and where $B \in \mathbb{R}^{n\times n}$ is symmetric.

The proof of the theorem now proceeds in three steps.

Step 1. The theorem holds in the case

$$G(t, \xi', \xi'', \eta', \eta'', \zeta) = \tfrac{1}{2}\langle G_{\xi'\xi'}(t)\xi', \xi' \rangle.$$

In this case the path of symplectic matrices associated to G is given by

$$\Psi_G(t) = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ C(t) & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}$$

where

$$C(t) = \int_0^t -G_{\xi'\xi'}(\tau)\, d\tau.$$

Thus, from the product and the localisation properties of the Maslov index,

$$\mu(\Psi_G\Lambda_0, \Lambda_{F,B}) = \tfrac{1}{2}\operatorname{sign}(C(1) + B) - \tfrac{1}{2}\operatorname{sign} B.$$

Also, from (22),

$$W_j(\xi_j, \eta_{j+1}, \zeta_j) = \tfrac{1}{2}\langle W_{j,\xi'\xi'}\xi'_j, \xi'_j \rangle,$$

and defining

$$C^j = \sum_{i=0}^{j-1} -W_{i,\xi'\xi'}$$

for $j = 1, \dots, N$, it is easy to see that $C^N = C(1)$. Hence the second variation is given by

$$\langle A^N_{G,F,B}\gamma, \gamma\rangle = 2\sum_{j=1}^{N}\langle \eta_j, \xi_j - \xi_{j-1}\rangle - \sum_{j=0}^{N-1}\langle W_{j,\xi'\xi'}\xi'_j, \xi'_j\rangle + \langle B\xi'_N, \xi'_N\rangle.$$

Now define the following system of coordinates on $W^N_F = \mathbb{R}^{2Nk+n}$:

$$\begin{aligned}
u'_j &= \xi'_j - \xi'_{j-1} \\
u''_j &= \xi''_j - \xi''_{j-1} \\
v'_j &= \eta'_j - \tfrac{1}{2}C^j(\xi'_j + \xi'_{j-1}) \\
v''_j &= \eta''_j \\
w &= \xi'_N
\end{aligned}$$

and denote $u = (u'_1, u''_1, \dots, u'_N, u''_N)$, $v = (v'_1, v''_1, \dots, v'_N, v''_N)$. In these coordinates the L^2-inner product of u and v is given by

$$\begin{aligned}
\langle u, v\rangle &= \sum_{j=1}^{N}\langle u'_j, v'_j\rangle + \sum_{j=1}^{N}\langle u''_j, v''_j\rangle \\
&= \sum_{j=1}^{N}\langle \xi'_j - \xi'_{j-1}, \eta'_j\rangle - \tfrac{1}{2}\sum_{j=1}^{N}\langle \xi'_j - \xi'_{j-1}, C^j(\xi'_j + \xi'_{j-1})\rangle \\
&\quad + \sum_{j=1}^{N}\langle \xi''_j - \xi''_{j-1}, \eta''_j\rangle \\
&= \sum_{j=1}^{N}\langle \xi_j - \xi_{j-1}, \eta_j\rangle + \tfrac{1}{2}\sum_{j=1}^{N}\langle C^j\xi'_{j-1}, \xi'_{j-1}\rangle - \tfrac{1}{2}\sum_{j=1}^{N}\langle C^j\xi'_j, \xi'_j\rangle \\
&= \sum_{j=1}^{N}\langle \xi_j - \xi_{j-1}, \eta_j\rangle + \tfrac{1}{2}\sum_{j=0}^{N-1}\langle C^{j+1}\xi'_j, \xi'_j\rangle - \tfrac{1}{2}\sum_{j=0}^{N}\langle C^j\xi'_j, \xi'_j\rangle \\
&= \sum_{j=1}^{N}\langle \xi_j - \xi_{j-1}, \eta_j\rangle + \tfrac{1}{2}\sum_{j=0}^{N-1}\langle (C^{j+1} - C^j)\xi'_j, \xi'_j\rangle - \tfrac{1}{2}\big\langle C^N\xi'_N, \xi'_N\big\rangle \\
&= \sum_{j=1}^{N}\langle \xi_j - \xi_{j-1}, \eta_j\rangle - \tfrac{1}{2}\sum_{j=0}^{N-1}\langle W_{j,\xi'\xi'}\xi'_j, \xi'_j\rangle - \tfrac{1}{2}\big\langle C^N\xi'_N, \xi'_N\big\rangle \\
&= \tfrac{1}{2}\big\langle A^N_{G,F}\gamma, \gamma\big\rangle - \tfrac{1}{2}\big\langle \big(C^N + B\big)w, w\big\rangle.
\end{aligned}$$

Thus the second variation satisfies

$$\tfrac{1}{2}\big\langle A^N_{G,F,B}\gamma, \gamma\big\rangle = \langle u, v\rangle + \tfrac{1}{2}\big\langle \big(C^N + B\big)w, w\big\rangle$$

and hence

$$\begin{aligned}
\operatorname{sign} A^N_{G,F,B} &= \operatorname{sign}\big(C^N + B\big) \\
&= \operatorname{sign}(C(1) + B) \\
&= 2\mu(\Psi_G\Lambda_0, \Lambda_{F,B}) + \operatorname{sign} B.
\end{aligned}$$

Step 2. The theorem holds for quadratic Hamiltonians independent of ζ.

Let G be a ζ-independent Hamiltonian with $\Psi_G(1)\Lambda_0$ transverse to $\Lambda_{F,B}$ and suppose, for a given value of k', that $G' \colon \mathbb{R}^{2k'} \to \mathbb{R}$ is any symplectic shear of the form

$$G'(\xi', \eta') = \tfrac{1}{2}\langle G'_{\xi'\xi'}\xi', \xi'\rangle$$

where $G'_{\xi'\xi'}$ is a nondegenerate symmetric matrix of signature zero. (It follows that k' is necessarily even.) From Step 1 it follows that

$$\operatorname{sign} A^N_{G', \mathbb{R}^{k'}, 0} = 2\mu(\Psi_{G'}\Lambda_0', \Lambda_0') = 0$$

where $\Lambda_0' = \mathbb{R}^{k'} \times \{0\}$. Now define $G_0 = G \oplus G' \colon [0,1] \times J^1\mathbb{R}^{k+k'} \to \mathbb{R}$ by

$$G_0(t, \xi, \xi', \eta, \eta', \zeta) = G(t, \xi, \eta, \zeta) + G'(\xi', \eta').$$

By additivity of the signature

$$\operatorname{sign} A^N_{G_0, F\oplus\mathbb{R}^{k'}, B\oplus 0} = \operatorname{sign} A^N_{G,F,B}$$

and by the product property of the Maslov index

$$\mu(\Psi_{G_0}\Lambda_0 \oplus \Lambda_0', \Lambda_{F,B} \oplus \Lambda_0') = \mu(\Psi_G\Lambda_0, \Lambda_{F,B}).$$

Hence it suffices to prove the theorem for G_0 with F replaced by $\widetilde{F} = F \oplus \mathbb{R}^{k'}$ and B by $\widetilde{B} = B \oplus 0$.

Now, for k' sufficiently large, let G_1 be a symplectic shear of the form considered in Step 1, with F replaced by $\widetilde{F}$, satisfying

$$\mu(\Psi_{G_1}\widetilde{\Lambda}_0, \Lambda_{\widetilde{F},\widetilde{B}}) = \mu(\Psi_{G_0}\widetilde{\Lambda}_0, \Lambda_{\widetilde{F},\widetilde{B}})$$

where $\widetilde{\Lambda}_0 = \Lambda_0 \oplus \Lambda_0'$. This exists by the localisation and integrality properties of the Maslov index. Also, choose a symplectic matrix $\Phi \in \mathrm{Sp}(2k+2k')$ such that $\Lambda_{\widetilde{F},\widetilde{B}} = \Phi\widetilde{\Lambda}_0$. Then

$$\mu(\Phi^{-1}\Psi_{G_1}\widetilde{\Lambda}_0, \widetilde{\Lambda}_0) = \mu(\Phi^{-1}\Psi_{G_0}\widetilde{\Lambda}_0, \widetilde{\Lambda}_0)$$

and the symplectic path $\Phi^{-1}\Psi_{G_i}$ satisfies

$$\Phi^{-1}\Psi_{G_i}(0) = \Phi^{-1}, \qquad \Phi^{-1}\Psi_{G_i}(1)\widetilde{\Lambda}_0 \cap \widetilde{\Lambda}_0 = \{0\}.$$

Thus by the homotopy property of the Maslov index there exists a homotopy of symplectic paths $\Phi^{-1}\Psi_\lambda \colon [0,1] \to \mathrm{Sp}(2k+2k')$ between $\Phi^{-1}\Psi_{G_0}$ and $\Phi^{-1}\Psi_{G_1}$ and within the same class. From this we can construct a quadratic Hamiltonian G_λ such that $\Psi_\lambda = \Psi_{G_\lambda}$.

Now choose N sufficiently large such that for each λ the function $\Phi^N_{G_\lambda,\widetilde{F},\widetilde{B}}$ is defined. Then

$$\begin{aligned}
\operatorname{sign} A^N_{G_0,\widetilde{F},\widetilde{B}} &= \operatorname{sign} A^N_{G_1,\widetilde{F},\widetilde{B}} \\
&= 2\mu(\Psi_{G_1}\widetilde{\Lambda}_0, \Lambda_{\widetilde{F},\widetilde{B}}) + \operatorname{sign} \widetilde{B} \\
&= 2\mu(\Psi_{G_0}\widetilde{\Lambda}_0, \Lambda_{\widetilde{F},\widetilde{B}}) + \operatorname{sign} \widetilde{B}.
\end{aligned}$$

Here the first equality follows from Corollary 6.12.

Step 3. The general case.

To the quadratic Hamiltonian

$$G(t,\xi,\eta,\zeta) = \tfrac{1}{2}\langle G_{\xi\xi}(t)\xi,\xi\rangle + \langle G_{\eta\xi}(t)\xi,\eta\rangle + \tfrac{1}{2}\langle G_{\eta\eta}(t)\eta,\eta\rangle + G_\zeta(t)\zeta$$

associate the quadratic ζ-independent Hamiltonian

$$G'(t,\xi,\eta,\zeta) = \tfrac{1}{2}\langle G_{\xi\xi}(t)\xi,\xi\rangle + \langle (G_{\eta\xi}(t) + \tfrac{1}{2}G_\zeta(t)\mathbb{1})\xi,\eta\rangle + \tfrac{1}{2}\langle G_{\eta\eta}(t)\eta,\eta\rangle.$$

Now consider the homotopy $G_\lambda = \lambda G' + (1-\lambda)G$ between G and G'. Notice that $\Psi_{G_\lambda} = \Psi_G$ as symplectic paths for all λ. Step 3 now follows from Corollary 6.12. This completes the proof of the theorem. $\square$

9 Proof of the generalised Morse inequalities

Recall from Remark 6.5 that the discrete action functional $\Phi^N : M \times \mathbb{R}^{2m} \to \mathbb{R}$ can be written in the form

$$\Phi^N(x,\xi) = \tfrac{1}{2}\langle P\xi,\xi\rangle + W(x,\xi)$$

where $P \in \mathbb{R}^{2m\times 2m}$ is a nondegenerate symmetric matrix of signature zero and where the gradient of W in the direction of the fibre, $\partial_\xi W$, is bounded. We prove the generalised Morse inequalities by studying the critical points of Φ^N. As $M \times \mathbb{R}^{2m}$ is noncompact stabilised Morse theory will be used; see Conley [6].

In order to study the critical points of Φ^N we consider the (negative) gradient flow. This is defined by the equations

$$\frac{d\xi}{ds} = -P\xi - \frac{\partial W}{\partial \xi}(x,\xi), \qquad \frac{dx}{ds} = -\frac{\partial W}{\partial x}(x,\xi).$$

Now induced by P there is a splitting

$$\mathbb{R}^{2m} = E^- \oplus E^+$$

into the negative and positive eigenspaces. It follows that, for some $\delta > 0$,

$$\langle P\xi^-,\xi^-\rangle \le -\delta|\xi^-|^2$$
$$\langle P\xi^+,\xi^+\rangle \ge \delta|\xi^+|^2$$

for $\xi^- \in E^-$, $\xi^+ \in E^+$. Thus we can find constants $\varepsilon > 0$ and $R > 0$ such that

$$\frac{d}{ds}|\xi^-|^2 \ge \varepsilon \quad \text{if} \quad |\xi^-| \ge R$$
$$\frac{d}{ds}|\xi^+|^2 \le -\varepsilon \quad \text{if} \quad |\xi^+| \ge R.$$

Indeed, for $|\xi^-| \ge R$ and R sufficiently large,

$$\frac{d}{ds}\frac{|\xi^-|^2}{2} = \left\langle \xi^-, -P\xi^- - \partial_\xi W\right\rangle \ge \delta|\xi^-|^2 - \sup|\partial_\xi W| \cdot |\xi^-| \ge \varepsilon,$$

and similarly for $|\xi^+|^2 \ge R$. Thus an isolating block, in the sense of Conley, for the compact invariant set Λ of all bounded orbits of the gradient flow, is given by

$$N = \left\{(x,\xi^- + \xi^+) : \xi^\pm \in E^\pm,\ |\xi^\pm| \le R\right\}$$

with exit set

$$L = \{(x, \xi^- + \xi^+) \in N : |\xi^-| = R\}$$

for R sufficiently large.

We use the following notation

$$\begin{aligned} b_k(\Lambda) &= \dim H_k(N, L) \\ b_k(M) &= \dim H_k(M) \\ c_k(\Lambda) &= \# \{c \in \Lambda \mid d\Phi^N(c) = 0, \, \mathrm{ind}_{\Phi^N}(c) = k\} \\ p_k(\Lambda) &= \# \{c \in \Lambda \mid d\Phi^N(c) = 0, \, \mu(c, \overline{H}) = k\} \end{aligned}$$

where $\mathrm{ind}_{\Phi^N}(c)$ denotes the *Morse index* of Φ^N at c and is defined to be the number of negative eigenvalues of the Hessian $d^2\Phi^N(c)$. Here the numbers $b_k(\Lambda)$ are known as the *Conley–Betti numbers*. These are related to the $c_k(\Lambda)$ by the Morse inequalities.

Theorem 9.1 (Morse inequalities) *For* $k = 0, \ldots, n + 2m$

$$c_k(\Lambda) - c_{k-1}(\Lambda) + \cdots \pm c_0(\Lambda) \geq b_k(\Lambda) - b_{k-1}(\Lambda) + \cdots \pm b_0(\Lambda)$$

with equality holding for $k = n + 2m$.

These inequalities are proved in [12].

The Conley–Betti numbers are related to the Betti numbers of M by the Thom isomorphism. Specifically, denote by E^m m-dimensional Euclidean space, by B^m and S^{m-1} the closed unit ball and unit sphere respectively in E^m, and abbreviate $E_0^m = E^m \setminus \{0\}$. Then by homotopy

$$(M \times B^m \times B^m, M \times \partial B^m \times B^m) \simeq (M \times B^m, M \times S^{m-1}) \simeq (M \times E^m, M \times E_0^m).$$

Thus

$$H_k(N, L) = H_k(M \times E^m, M \times E_0^m) = H_{k-m}(M)$$

where the last equality is the Thom isomorphism. In other words

$$b_k(\Lambda) = b_{k-m}(M). \tag{28}$$

Also by Theorem 8.1

$$c_k = p_{\frac{n}{2}+m-k}. \tag{29}$$

The proof of Theorem 1.1, the generalised Morse inequalities, is now completed by putting together (28), (29) and the Morse inequalities.

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