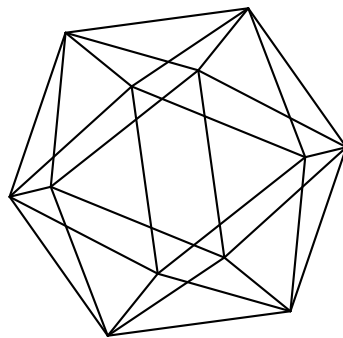# Max-Planck-Institut für Mathematik Bonn

Elliptic beta integrals and solvable models of statistical mechanics

by

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ABSTRACT. The univariate elliptic beta integral was discovered by the author in 2000. Recently Bazhanov and Sergeev have interpreted it as a star-triangle relation (STR). This important observation is discussed in more detail in connection to author's previous work on the elliptic modular double and supersymmetric dualities. We describe also a new Faddeev-Volkov type solution of STR, connections with the star-star relation, and higher-dimensional analogues of such relations. In this picture, Seiberg dualities are described by symmetries of the elliptic hypergeometric integrals (interpreted as superconformal indices) which, in turn, represent STR and Kramers-Wannier type duality transformations for elementary partition functions in solvable models of statistical mechanics.

CONTENTS

1. The simplest elliptic hypergeometric integrals

In the present paper we discuss relations between a new class of special functions, called elliptic hypergeometric functions, and solvable models of statistical mechanics. We describe the most complicated known integrable systems defined on $2d$ (two-dimensional) lattices representing continuous spin generalizations of the well known Ising model and its various extensions. Actually, these novel integrable models correspond to some discretized $2d$ quantum field theories. Also we indicate

connections with the $4d$ supersymmetric field theories, where elliptic hypergeometric integrals have found recently the major application. We start from a brief technical introduction to the needed results on special functions and discuss the physical systems they apply to in the following chapters.

General theory of elliptic hypergeometric integrals was formulated in [**S1, S3, S5**]. We skip the structural definition of these integrals and refer for the corresponding details to a reasonably short survey given in [**S9**].

Let us denote

$$(z;q)_\infty = \prod_{k=0}^\infty (1 - zq^k), \quad |q| < 1, \quad z \in \mathbb{C},$$

the standard infinite q-product and

$$\Gamma(z;p,q) = \prod_{i,j=0}^\infty \frac{1 - z^{-1}p^{i+1}q^{j+1}}{1 - zp^i q^j}, \quad |p|,|q| < 1, \quad z \in \mathbb{C}^*,$$

the standard elliptic gamma function. Below we use the conventions

$$\Gamma(a,b;p,q) := \Gamma(a;p,q)\Gamma(b;p,q), \quad \Gamma(az^{\pm 1};p,q) := \Gamma(az;p,q)\Gamma(az^{-1};p,q),$$

$$\Gamma(az^{\pm 1}y^{\pm 1};p,q) := \Gamma(azy;p,q)\Gamma(az^{-1}y;p,q)\Gamma(azy^{-1};p,q)\Gamma(az^{-1}y^{-1};p,q).$$

One has the symmetry $\Gamma(z;p,q) = \Gamma(z;q,p)$ and the inversion formula

$$\Gamma\left(a,\frac{pq}{a};p,q\right) = 1, \quad \text{or} \quad \Gamma(a,a^{-1};p,q) = \frac{1}{\theta(a;p)\theta(a^{-1};q)},$$

which follows from the difference equations

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q), \qquad \Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q),$$

where

$$\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty$$

is a theta function. The standard odd Jacobi theta function has the form [**WW**]

$$\theta_1(u|\tau) = -\theta_{11}(u) = -\sum_{k \in \mathbb{Z}} e^{\pi i \tau (k+1/2)^2} e^{2\pi i (k+1/2)(u+1/2)}$$

$$= i p^{1/8} e^{-\pi i u}(p;p)_\infty \theta(e^{2\pi i u};p),$$

where we denoted $p = e^{2\pi i \tau}$.

The univariate elliptic beta integral [**S1**] forms a cornerstone of a new powerful class of exactly computable integrals. It is described by the following explicit formula

$$(1.1) \qquad \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^6 \Gamma(t_j z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)} \frac{dz}{iz} = \prod_{1 \le j < k \le 6} \Gamma(t_j t_k;p,q),$$

where $\mathbb{T}$ is the unit circle with positive orientation,

$$\kappa = \frac{(p;p)_\infty (q;q)_\infty}{4\pi},$$

and six complex parameters t_j, $j = 1,\ldots,6$, satisfy the inequalities $|t_j| < 1$ and the balancing condition

$$(1.2) \qquad \prod_{j=1}^6 t_j = pq.$$

We use the word "integral" in two meanings. When referred to the exactly computable cases, like (1.1) or the standard Euler beta integral lying on its bottom, it means either the function defined by the left-hand side or, more often, the whole identity. In other cases it means an integral representation for a function of interest or a class of functions with common structure.

As shown in [**S3**], the left-hand side of relation (1.1) serves as the orthogonality measure for the most general known family of biorthogonal functions with the properties characteristic to classical orthogonal polynomials (Chebyshev, Hermite, Laguerre, Jacobi, ... , Askey-Wilson polynomials). In the same paper the elliptic beta integral has been generalized to the following function

$$(1.3) \qquad V(t_1, \ldots, t_8; p, q) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^{8} \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{\mathrm{i}z},$$

where $|t_j| < 1$ and $\prod_{j=1}^{8} t_j = (pq)^2$. This is a natural elliptic analogue of the Gauss hypergeometric function since its features generalize most of the special function properties of the $_2F_1$-series [**S5**, **S9**]. For $t_j t_k = pq$, $j \neq k$, V-function reduces to the elliptic beta integral and, for this reason, it can be called the elliptic beta integral of a higher order.

In [**S4**], the author has introduced the following universal integral transformation for functions analytical in the vicinity of the unit circle $\mathbb{T}$:

$$(1.4) \qquad g(w; t) = \kappa \int_{\mathbb{T}} \Delta(t; w, z; p, q) f(z; t) \frac{dz}{\mathrm{i}z},$$

where the kernel

$$(1.5) \qquad \Delta(t; w, z; p, q) := \Delta(t; w, z) = \Gamma(tw^{\pm 1} z^{\pm 1}; p, q), \quad |t| < 1,$$

is a particular product of four elliptic gamma functions. In [**SW**], it was shown that this integral transformation obeys the key property making it very similar to the Fourier transformation. Namely, its inverse is obtained essentially by the reflection $t \to t^{-1}$.

An explicit example of the pair of functions $g(w; t)$ and $f(z; t)$ in (1.4) can be easily found from the elliptic beta integral. Indeed, let us denote $t_5 = tw$ and $t_6 = tw^{-1}$ (so that $t^2 \prod_{j=1}^{4} t_j = pq$). Then,

$$(1.6) \qquad f(z; t) = \frac{\prod_{j=1}^{4} \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)},$$

$$(1.7) \qquad g(w; t) = \Gamma(t^2; p, q) \prod_{1 \leq i < j \leq 4} \Gamma(t_i t_j; p, q) \prod_{j=1}^{4} \Gamma(tt_j w^{\pm 1}; p, q),$$

where $|tw^{\pm 1}|, |t_j| < 1$.

Because of the permutational symmetry, any of the original variables t_j can be associated with the distinguished parameter t. After fixing $t_1 = sy, t_2 = sy^{-1}$ and $t_3 = rx, t_4 = rx^{-1}$, one can rewrite the elliptic beta integral in the form

$$\int_{\mathbb{T}} \varphi(z) \Delta(r; x, z) \Delta(s; y, z) \Delta(t; w, z) \frac{dz}{\mathrm{i}z}$$

$$(1.8) \qquad = \chi(r, s, t) \Delta(rs; x, y) \Delta(rt; x, w) \Delta(st; y, w),$$

where $rst = \pm\sqrt{pq}$ and

$$\varphi(z) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi\Gamma(z^{\pm 2}; p, q)} = \frac{1}{4\pi}(p;p)_\infty (q;q)_\infty \theta(z^2; p)\theta(z^{-2}; q),$$

(1.9) $$\chi(r, s, t) = \Gamma(r^2, s^2, t^2; p, q).$$

A key application of definition (1.4) consists in the construction of a tree of identities for multiple elliptic hypergeometric integrals with many parameters [**S4**]. Using one of the corresponding symmetry transformations, the following relation has been derived in [**S8**]

(1.10) $$\phi(x; c, d|\xi; s) = \kappa \int_{\mathbb{T}} R(c, d, a, b; x, w|s)\phi(w; a, b|\xi; s)\frac{dw}{iw},$$

where the "basis vector" ϕ has the form

(1.11) $$\phi(w; a, b|\xi; s) = \Gamma(sa\xi^{\pm 1}, sb\xi^{\pm 1}, \sqrt{\frac{pq}{ab}}w^{\pm 1}\xi^{\pm 1}; p, q),$$

and the "rotation" integral operator kernel is

$$R(c, d, a, b; x, w|s) = \frac{1}{\Gamma(\frac{pq}{ab}, \frac{ab}{pq}, w^{\pm 2}; p, q)}$$

$$\times V\left(sc, sd, \sqrt{\frac{pq}{cd}}x, \sqrt{\frac{pq}{cd}}x^{-1}, \frac{pq}{as}, \frac{pq}{bs}, \sqrt{\frac{ab}{pq}}w, \sqrt{\frac{ab}{pq}}w^{-1}; p, q\right).$$

The function ϕ is a generalization of the kernel $\Delta(t; x, z)$, since for $ab = pq/s^2$ one has the reduction

$$\phi(w; a, \frac{pq}{as^2}|\xi; s) = \Delta(s; w, \xi).$$

Using the Δ-kernel, relation (1.10) was rewritten also in [**S8**] in a more compact form

$$\Delta(\alpha; x, \xi)\Delta(\beta; y, \xi) = \kappa \int_{\mathbb{T}} r(\alpha, \beta, \gamma, \delta; x, y; t, w)\Delta(\gamma; t, \xi)\Delta(\delta; w, \xi)\frac{dw}{iw},$$

$$r(\alpha, \beta, \gamma, \delta; x, y; t, w) = \frac{1}{\Gamma(\delta^{\pm 2}, w^{\pm 2}; p, q)}V\left(\alpha x^{\pm 1}, \beta y^{\pm 1}, \frac{pq}{\gamma}t^{\pm 1}, \frac{w^{\pm 1}}{\delta}\right),$$

where $\alpha\beta = \gamma\delta$ and

$$V\left(\alpha x^{\pm 1}, \beta y^{\pm 1}, \frac{pq}{\gamma}t^{\pm 1}, \frac{w^{\pm 1}}{\delta}\right) = \kappa \int_{\mathbb{T}} \frac{\Delta(\alpha; x, z)\Delta(\beta; y, z)\Delta(\frac{pq}{\gamma}; t, z)\Delta(\frac{1}{\delta}; w, z)}{\Gamma(z^{\pm 2}; p, q)}\frac{dz}{iz}.$$

Here we use the condensed notation for parameters of the V-function: $V(\ldots\alpha x^{\pm 1}\ldots) = V(\ldots\alpha x, \alpha x^{-1}\ldots)$.

The function ϕ emerges also in the context of the Sklyanin algebra [**Sk**] (the algebra of the Yang-Baxter equation solutions),

$$S_\alpha S_\beta - S_\beta S_\alpha = i(S_0 S_\gamma + S_\gamma S_0),$$

(1.12) $$S_0 S_\alpha - S_\alpha S_0 = i\frac{J_\beta - J_\alpha}{J_\gamma}(S_\beta S_\gamma + S_\gamma S_\beta),$$

where J_α are the structure constants and (α, β, γ) is any cyclic permutation of $(1, 2, 3)$. Namely, one has to consider the generalized eigenvalue problems $A\phi = \lambda B\phi$, where A and B are linear combinations of four generators S_a, $a = 0, 1, 2, 3$,

and λ is a spectral parameter. The function ϕ is defined uniquely up to multiplication by a constant with the help of two such equations using a pair of Sklyanin algebras forming an elliptic modular double [**S8**]. This algebra represents an elliptic extension of the Faddeev modular double [**F**], but there are actually two different modular doubles at the elliptic level which obey different sets of involutions.

Relevance of the Sklyanin algebra in this setting was noticed first by Rains [**R1**]. For special quantized values of the parameters, the ϕ-function reduces to the intertwining vectors of Takebe [**T**], which were used by Rosengren in [**Ros**] for the derivation of a discrete spin version of relation (1.10). In our case both Casimir operators of the algebra (1.12), $K_0 = \sum_{a=0}^{3} S_a^2$ and $K_2 = \sum_{\alpha=1}^{3} J_\alpha S_\alpha^2$, take continuous values, i.e. we deal with the continuous spin representations related to the integral operator form of the Yang-Baxter equation.

The following scalar product has been introduced in [**S8**]

$$(1.13) \qquad \langle \chi, \psi \rangle = \kappa \int_{\mathbb{T}} \frac{\chi(z)\psi(z)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{\mathrm{i}z}.$$

It has been shown that both the V-function itself and the ϕ-vectors form biorthogonal systems of functions with respect to this measure. In particular, one has the relation

$$\kappa \int_{\mathbb{T}} \frac{\phi(e^{\mathrm{i}\varphi'}; \frac{pq}{c}, \frac{pq}{d} | \xi; s^{-1}) \phi(e^{\mathrm{i}\varphi}; c, d | \xi; s)}{\Gamma(\xi^{\pm 2}; p, q)} \frac{d\xi}{\mathrm{i}\xi}$$

$$(1.14) \qquad = \frac{2\pi}{(p; p)_\infty (q; q)_\infty} \Gamma\left(\frac{pq}{cd}, \frac{cd}{pq}, e^{\pm 2\mathrm{i}\varphi}; p, q\right) \sqrt{1 - v^2}\, \delta(v - v'),$$

where $v = \cos\varphi$, $v' = \cos\varphi'$, and $\delta(v)$ is the Dirac delta-function. (There is a missprint in formula (3.2) of [**S8**] which misses the first factor standing on the right-hand side of (1.14).) Positivity of the biorthogonality measure and of the ϕ-function corresponds to the unitarity of representations of the elliptic modular double. Setting $cd = pq/s^2$, we obtain

$$(2\kappa)^2 \int_{-1}^{1} \frac{\Delta(s^{-1}; e^{\mathrm{i}\varphi'}, e^{\mathrm{i}\chi}) \Delta(s; e^{\mathrm{i}\varphi}, e^{\mathrm{i}\chi})}{\Gamma(e^{\pm 2\mathrm{i}\chi}; p, q)} \frac{dX}{\sqrt{1 - X^2}}$$

$$(1.15) \qquad = \Gamma\left(s^2, s^{-2}, e^{\pm 2\mathrm{i}\varphi}; p, q\right) \sqrt{1 - v^2}\, \delta(v - v'),$$

where $X = \cos\chi$.

The tetrahedral symmetry transformation for V-function, discovered in [**S3**], can be rewritten in the following form:

$$V(\alpha x^{\pm 1}, \beta y^{\pm 1}, \gamma w^{\pm 1}, \delta z^{\pm 1}) = \Gamma(\alpha^2, \beta^2, \gamma^2, \delta^2; p, q) \Delta(\alpha\beta; x, y) \Delta(\gamma\delta; w, z)$$

$$(1.16) \qquad \times V(\sqrt{pq}\beta^{-1} x^{\pm 1}, \sqrt{pq}\alpha^{-1} y^{\pm 1}, \sqrt{pq}\delta^{-1} w^{\pm 1}, \sqrt{pq}\gamma^{-1} z^{\pm 1})$$

$$= \Delta(\alpha\gamma; x, w) \Delta(\alpha\delta; x, z) \Delta(\beta\gamma; y, w) \Delta(\beta\delta; y, z)$$

$$(1.17) \qquad \times V(\beta x^{\pm 1}, \alpha y^{\pm 1}, \delta w^{\pm 1}, \gamma z^{\pm 1})$$

$$(1.18) \qquad = \Gamma(\alpha^2, \beta^2, \gamma^2, \delta^2; p, q) \Delta(\alpha\beta; x, y) \Delta(\alpha\gamma; x, w) \Delta(\alpha\delta; x, z) \Delta(\beta\gamma; y, w)$$

$$\times \Delta(\beta\delta; y, z) \Delta(\gamma\delta; w, z) V(\sqrt{pq}\alpha^{-1} x^{\pm 1}, \sqrt{pq}\beta^{-1} y^{\pm 1}, \sqrt{pq}\gamma^{-1} w^{\pm 1}, \sqrt{pq}\delta^{-1} z^{\pm 1}),$$

where $\alpha\beta\gamma\delta = \pm pq$. The latter two transformations are obtained by repeated application of the first relation in combination with permutation of the parameters. The full symmetry group of the V-function is the Weyl group $W(E_7)$ for the exceptional root system E_7 [**R3**]. Therefore, there are $72 = \dim W(E_7)/S_8$ relations

similar to (1.16), (1.17), (1.18), we just picked up three of them by breaking the S_8 permutational symmetry and gathering the elliptic gamma functions into the Δ-blocks.

The outstanding physical application of the elliptic beta integral has been discovered by Dolan and Osborn [**DO**]. They have shown that the simplest superconformal (topological) indices of $\mathcal{N} = 1$ supersymmetric field theories coincide with known elliptic hypergeometric integrals. Exact computability or the Weyl group symmetry transformations of such integrals describe the Seiberg duality of $\mathcal{N} = 1$ theories [**Sb**], since they prove coincidence of the corresponding superconformal indices.

In this picture, the left-hand side of the univariate elliptic beta integral evaluation formula (1.1) describes the superconformal index of the supersymmetric quantum chromodynamics with $SU(2)$ gauge group and $SU(6)$ flavor group. This theory has one vector superfield (gauge fields) in the adjoint representation of $SU(2)$ and a set of chiral superfields (matter fields) in the fundamental representation of $SU(2) \times SU(6)$. The elementary particles representing these fields describe the spectrum of the theory in the high energy limit, where the coupling constant is vanishing due to the asymptotic freedom. In the deep infrared region the theory is strongly coupled, all colored particles confine, and one has the Wess-Zumino type model for mesonic fields lying in the 15-dimensional totally antisymmetric tensor representation of $SU(6)$. The superconformal index of the latter theory is described by the right-hand side expression of formula (1.1). This construction gives a group-theoretical interpretation of the elliptic beta integral. After renormalizing the parameters $t_k = (pq)^{1/6} y_k$, the balancing condition takes the form $\prod_{k=1}^{6} y_k = 1$, which is nothing else than the unitarity condition for the maximal torus variables of the group $SU(6)$. This is the simplest example of the Seiberg duality discovered in [**Sb**]. Further detailed investigation of such interrelations and their consequences can be found in [**SV1**], where many new elliptic beta integrals on root systems have been conjectured and many new supersymmetric dualities have been found.

The elliptic hypergeometric integrals emerge also in the context of the relativistic Calogero-Sutherland type models [**S7**]. However, the first non-trivial example of the elliptic hypergeometric functions was found from the exactly solvable models of statistical mechanics. Namely, in [**FT**] Frenkel and Turaev have shown that the Boltzmann weights (elliptic $6j$-symbols) of the RSOS models of Date et al [**DJKMO**], generalizing Baxter's eight-vertex model [**Bax1**], are determined by particular values of the terminating $_{12}V_{11}$ elliptic hypergeometric series (in modern notations of [**S9**]). The same series has been found by Zhedanov and the author [**SZ**] in a completely different setting, as a particular solution of the Lax pair equations for a classical discrete integrable system. In [**S2, S3**], a family of meromorphic functions obeying a novel two-index biorthogonality relation has been discovered. It was explicitly conjectured in [**S2**] that these functions determine a new family of solutions of the Yang-Baxter equation for discrete spin models. Since the $V(t_1, \ldots, t_8; p, q)$ function is an integral generalization of the latter functions, in [**S6**] it was conjectured that the V-function determines a solution of the Yang-Baxter equation. A simple connection of the terminating $_{12}V_{11}$-series and V-function with the Yang-Baxter equation for RSOS models was discussed in [**KS**]. Recently, Bazhanov and Sergeev [**BS**] have shown that the elliptic beta integral can be rewritten as a star-triangle relation (STR) which yields a new two-dimensional

solvable model of statistical mechanics. This is a new important application of integral (1.1) which is described in the next section. In this paper we show that the symmetry transformations for the V-function have similar interpretation as the star-star relations. Moreover, we conjecture that all known exact formulas for elliptic hypergeometric integrals describing the Seiberg duality transformations (at the level of superconformal indices) [**SV1**], in turn, represent STR and Kramers-Wannier type duality transformations [**KW, W**] for elementary partition functions in solvable models of statistical mechanics [**Bax2**].

2. The elliptic beta integral STR solution and star-star relation

In [**BS**], Bazhanov and Sergeev have interpreted the elliptic beta integral evaluation formula as a star-triangle relation which gave a new solution of this relation. In order to describe it, let us introduce the parameter η related to the bases p and q as

$$e^{-2\eta} = pq$$

and pass to the additive notation

$$z = e^{iu}, \quad x \to e^{ix}, \quad y \to e^{iy}, \quad w \to e^{iw}.$$

Introduce also the exponential form of the parameters

$$r = e^{-\alpha}, \quad s = e^{\alpha+\gamma-\eta}, \quad t = e^{-\gamma},$$

so that the balancing condition $r^2 s^2 t^2 = 1$ is satisfied automatically. Finally, denote

$$(2.1) \qquad W(\alpha; x, u) := \Delta(e^{\alpha-\eta}; e^{ix}, e^{iu}).$$

Then relation (1.8) can be rewritten as

$$\int_0^{2\pi} S(u; p, q) W(\eta - \alpha; x, u) W(\alpha + \gamma; y, u) W(\eta - \gamma; w, u) du$$

$$(2.2) \qquad = \chi(\alpha, \gamma; p, q) W(\alpha; y, w) W(\eta - \alpha - \gamma; x, w) W(\gamma; x, y),$$

where

$$(2.3) \qquad S(u; p, q) = \frac{(p; p)_\infty (q; q)_\infty}{4\pi} \theta(e^{2iu}; p) \theta(e^{-2iu}; q),$$

$$(2.4) \qquad \chi(\alpha, \gamma; p, q) = \Gamma(r^2, s^2, t^2; p, q).$$

As observed in [**BS**], equality (2.2) is nothing else than the star-triangle relation playing an important role for solvable models of statistical mechanics. It is symbolized by figure 1 given below, where the black vertex of the star-shaped figure on the left-hand side means the integration over u-variable with the weight $S(u)$, and W-weights are associated with the edges connecting the black vertex with white ones. On the right-hand side one has the product of three W-weights connecting only white vertices.

Suppose we have a two dimensional square lattice with spin variables $a, b, c, \ldots$ sitting at vertices. One associates the self-interaction energy $S(a)$ with each spin (vertex). For each horizontal bond connecting spins a and b the energy contribution is given by the Boltzmann weight $W_{fg}(a, b)$, and the energy contribution from each vertical bond connecting spins b and d is given by the weight $\overline{W}_{fg}(b, d)$. The variables f and g are called rapidities. Then, as described in detail by Baxter in
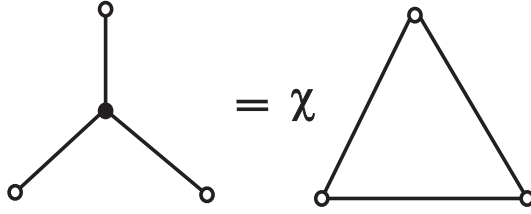
FIGURE 1. The star-triangle relation.

[**Bax3, Bax4**], the general STR for these quantities have the following functional equations form:

$$\sum_d S(d)\overline{W}_{fg}(d,b)W_{fh}(c,d)\overline{W}_{gh}(a,d) = R_{fgh}W_{fg}(c,a)\overline{W}_{fh}(a,b)W_{gh}(c,b),$$

$$(2.5)\sum_d S(d)\overline{W}_{fg}(b,d)W_{fh}(d,c)\overline{W}_{gh}(d,a) = R_{fgh}W_{fg}(a,c)\overline{W}_{fh}(b,a)W_{gh}(b,c).$$

The second equation is satisfied automatically if the Boltzmann weights are symmetric in spin variables

$$W_{fg}(a,b) = W_{fg}(b,a), \qquad \overline{W}_{fg}(a,b) = \overline{W}_{fg}(b,a).$$

Usually the normalization constants factorize, $R_{fgh} = r_{gh}r_{fg}/r_{fh}$. Then the weights satisfy the unitarity relation of the form

$$\sum_d S(d)\overline{W}_{fg}(a,d)\overline{W}_{gf}(d,b) = \frac{r_{fg}r_{gf}}{S(a)}\delta_{ab}$$

and the reflection equation $W_{fg}(a,b)W_{gf}(a,b) = 1$.

A subclass of solutions of (2.5) emerges from the weights depending only on differences of the rapidities,

$$(2.6) \qquad W_{fg}(a,b) = W(f-g;a,b), \quad \overline{W}_{fg}(a,b) = W(\eta-f+g;a,b),$$

where the parameter η is called the crossing parameter. Then the precise identification of equality (2.2) with (2.5) is reached after setting $\alpha = f-g$, $\gamma = g-h$ (so that $f-h = \alpha+\gamma$), equating $S(d)$ to $S(u;p,q)$ and R_{fgh} to $\chi(\alpha,\gamma;p,q)$ functions, and fixing appropriately the range of summation (integration) over the variable $d=u$. We call (2.1), (2.3), (2.4) the elliptic beta integral STR solution. As shown in [**BS**], it generalizes many known solvable models of statistical mechanics [**Bax2**]: the Ising model, Ashkin-Teller, chiral Potts, Fateev-Zamolodchikov Z_N-model, Kashiwara-Miwa and Faddeev-Volkov models. Moreover, as will be shown below, it comprises also a new Faddeev-Volkov type integrable system with continuous spins.

There is direct relation between spin systems on lattices of three types — the honeycomb, triangular, and rectangular lattices. Indeed, one can start from the honeycomb lattice, as depicted on the left-hand side of figure 2. Applying the star-triangle transformation to each black vertex one transforms the whole honeycomb lattice to the triangular one [**W**]. In a similar way, one can apply STR to each white vertex and obtain another triangular lattice having only black vertices. This is quite evident and does not require further explanations. However, further transformation of the triangular lattice to the square one is more tricky.

Consider the left-hand side of figure 3. Take the horizontal line in the middle of the drawn piece of the lattice. Pick up the triangles above and below it which
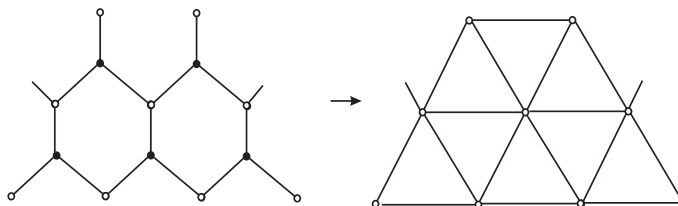
FIGURE 2. A honeycomb-triangluar lattice transformation induced by the star-triangle relation.

FIGURE 3. A triangluar-rectangular lattice transformation induced by the star-triangle relation.

intersect only at one point lying on this line (they are shown in bold lines). Apply to them the triangle-star relation replacing them by stars and continue this procedure up and down line-by-line of the resulting lattice. As a result, one obtains eventually the square lattice. Taking into account the nontrivial χ-multiplier in STR, one can thus connect partition functions of the square lattice model to the partition functions of two other types of models.
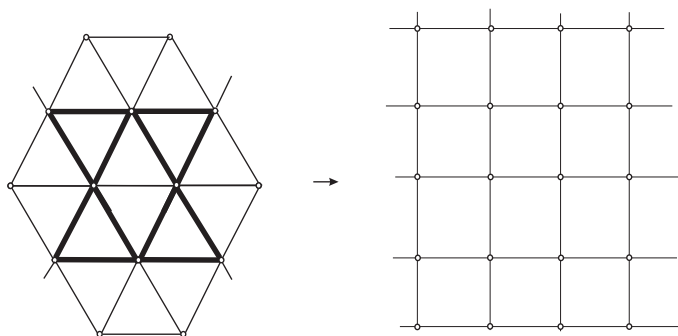
In [**BS**], the parameters x and u in (2.1) were considered as true spin variables. However, because of the $x \to -x$ and $u \to -u$ symmetries, the Boltzmann weights W and S depend on their trigonometric combinations. Therefore one can count as the true spin variables $U = \cos u$, $X = \cos x$, etc, with their values ranging from -1 to 1. The change of the variables in the measure is elementary

$$\int_{\mathbb{T}} f\left(\frac{1}{2}(z + z^{-1})\right)\frac{dz}{iz} = \int_{0}^{2\pi} f(\cos u)du = 2\int_{-1}^{1} f(U)\frac{dU}{\sqrt{1 - U^2}}.$$

The Boltzmann weight $W(\alpha; x, u)$ satisfies the reflection symmetry

$$W(\alpha; x, u)W(-\alpha; x, u) = 1,$$

following from the reflection equation for the elliptic gamma function. In terms of the spin variables $X = \cos x$ and $Y = \cos y$ the unitarity relation takes the form

$$\int_{-1}^{1} S(u; p, q) W(\eta - \alpha; x, u) W(\eta + \alpha; y, u) \frac{dU}{\sqrt{1 - U^2}}$$

(2.7)
$$= \frac{\Gamma(e^{2\alpha}, e^{-2\alpha}; p, q)}{S(x; p, q)} \sqrt{1 - X^2} \, \delta(X - Y).$$

This equality has been established by the author in [**S8**]. Note that positivity of the Boltzmann weights $S(u; p, q)$ and $W(\alpha; x, u)$ corresponds to the unitarity of the elliptic modular double representations [**S8**]. In particular, they are positive for $x, u \in [0, 2\pi]$, real α such that $|\sqrt{pq} e^{\alpha}| < 1$, and

$$1) \ \ p^* = p, \quad q^* = q, \qquad \text{or} \qquad 2) \ \ p^* = q.$$

At the level of superconformal indices, relations similar to (2.7) describe the Seiberg dualities for gauge field theories with equal number of colors and flavors and the chiral symmetry breaking [**SV2**].

Relation (2.2) is not changed if one replaces W and χ by

$$\widetilde{W}(\alpha; x, u) = \frac{W(\alpha; x, u)}{m(\alpha)},$$

(2.8)
$$\tilde{\chi}(\alpha, \gamma; p, q) = \frac{m(\alpha) m(\gamma) m(\eta - \alpha - \gamma)}{m(\eta - \alpha) m(\eta - \gamma) m(\alpha + \gamma)} \chi(\alpha, \gamma; p, q)$$

for arbitrary normalizing factor $m(\alpha)$.

The star-triangle relation is one of the three known forms of the Yang-Baxter equation. The second, probably the most popular form, is the vertex type relation symbolically written in terms of the R-matrices as

(2.9) $\mathbf{R}^{(12)}(\lambda) \mathbf{R}^{(13)}(\lambda + \mu) \mathbf{R}^{(23)}(\mu) = \mathbf{R}^{(23)}(\mu) \mathbf{R}^{(13)}(\lambda + \mu) \mathbf{R}^{(12)}(\lambda),$

where λ and μ are spectral parameters. The third type is referred to as the IRF (interaction around the face) Yang-Baxter equation. The star-star relation, which was discussed in detail in [**Bax3**], belongs to the latter type of equations and has the form

$$\sum_{g} S(g) W_1(a, g) W_2(b, g) W_3(c, g) W_4(d, g)$$

(2.10)
$$= R \frac{m(b, c) p(a, b)}{m(a, d) p(c, d)} \sum_{g} S(g) W'_1(a, g) W'_2(b, g) W'_3(c, g) W'_4(d, g),$$

where $W_j(a, b), W'_j(a, b), m(b, c), p(a, b)$ are two-spin Boltzmann weights and $S(g)$ is the spin self-interaction weight (it was omitted in formula (1.1) of [**Bax3**]). The left-hand side can be interpreted as an elementary partition function for a system of four spins a, b, c, d sitting in four square vertices connected by edges to the spin g sitting in the square center, and the summation is going over the values of the central spin, see figure 4 below. The right hand side has a similar interpretation of a statistical sum multiplied by the additional Boltzmann weights associated with opposite edges of the square (a, b, c, d). Formula (2.10) can be thought of as a generalized Kramers-Wannier duality transformation [**KW, W**].

Relation (2.10) should be compared with the V-function symmetry transformations written in the form (1.16), (1.17), and (1.18). Some of them coincide with
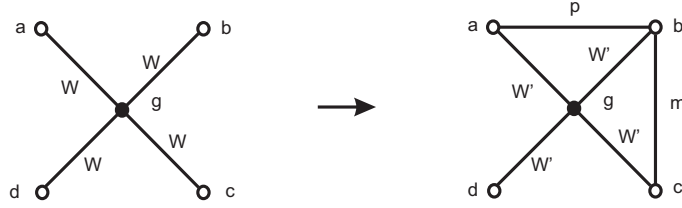
FIGURE 4. A star-star relation for the square lattice. Additional Boltzmann weights p and m are indicated by edges connecting corresponding vertices on the right-hand side.

(2.10) after appropriate identifications of the Boltzmann weights. For instance, equation (1.16) corresponds to the choice

$$W_1(a,g) = \Delta(\alpha; x, g), \qquad W_1'(a,g) = \Delta(\sqrt{pq}\beta^{-1}; x, g),$$
$$W_2(b,g) = \Delta(\beta; y, g), \qquad W_2'(b,g) = \Delta(\sqrt{pq}\alpha^{-1}; y, g),$$
$$W_3(c,g) = \Delta(\sqrt{pq}\gamma; w, g), \quad W_3'(c,g) = \Delta(\delta^{-1}; w, g),$$
$$W_4(d,g) = \Delta(\sqrt{pq}\delta; z, g), \quad W_4'(d,g) = \Delta(\gamma^{-1}; z, g),$$

where $\alpha\beta\gamma\delta = 1$, g is the integration variable for the V-function, and $S(g) = \kappa/\Gamma(g^{\pm 2}; p, q)$. Other factors have the form

$$R = \frac{\Gamma(\alpha^2, \beta^2; p, q)}{\Gamma(\gamma^{-2}, \delta^{-2}; p, q)}, \qquad m(b,c) = m(a,d) = 1,$$
(2.11)
$$p(a,b) = \Delta(\alpha\beta; x, y), \qquad p(c,d) = \Delta(\alpha\beta; w, z).$$

A similar interpretation is valid for relation (1.17). It corresponds to the choice

$$W_1(a,g) = \Delta(\alpha; x, g), \qquad W_1'(a,g) = \Delta(\beta; x, g),$$
$$W_2(b,g) = \Delta(\sqrt{pq}\gamma; w, g), \qquad W_2'(b,g) = \Delta(\sqrt{pq}\delta; w, g),$$
$$W_3(c,g) = \Delta(\beta; y, g), \qquad W_3'(c,g) = \Delta(\alpha; y, g),$$
$$W_4(d,g) = \Delta(\sqrt{pq}\delta; z, g), \qquad W_4'(d,g) = \Delta(\sqrt{pq}\gamma; z, g),$$

where, again, $\alpha\beta\gamma\delta = 1$ and $S(g) = \kappa/\Gamma(g^{\pm 2}; p, q)$. As to other factors, $R = 1$ and

$$m(b,c) = \Delta(\sqrt{pq}\beta\gamma; y, w), \qquad m(a,d) = \Delta(\sqrt{pq}\beta\gamma; x, z),$$
(2.12)
$$p(a,b) = \Delta(\sqrt{pq}\alpha\gamma; x, w), \qquad p(c,d) = \Delta(\sqrt{pq}\alpha\gamma; y, z).$$

There are three star-star relations for the Ising type models listed in [**Bax3**] as equations (2.16), (5.1), and (5.2). Our first option (2.11) corresponds to relation (5.2) in [**Bax3**]. Relations (2.16) and (5.2) in [**Bax3**] are obtained from each other by a reflection with respect to the lattice square diagonal (b, d). Our second option (2.12) corresponds to relation (5.1) in [**Bax3**] with nonconstant p- and m-weights. However, we have the third nontrivial form of the symmetry transformation for the V-function (1.18). It corresponds to a more complicated type of the star-star

relation

$$(2.13) \quad \sum_g S(g)W_1(a,g)W_2(b,g)W_3(c,g)W_4(d,g)$$

$$= R\frac{m(b,c)p(a,b)t(a,c)}{m(a,d)p(c,d)t(b,d)} \sum_g S(g)W_1'(a,g)W_2'(b,g)W_3'(c,g)W_4'(d,g),$$

where $t(a,c)$ is a new diagonal Boltzmann weight. Explicitly, we have

$$\begin{aligned}
W_1(a,g) &= \Delta(\alpha; x, g), & W_1'(a,g) &= \Delta(\sqrt{pq}\alpha^{-1}; x, g), \\
W_2(b,g) &= \Delta(\beta; y, g), & W_2'(b,g) &= \Delta(\sqrt{pq}\beta^{-1}; y, g), \\
W_3(c,g) &= \Delta(\sqrt{pq}\gamma; w, g), & W_3'(c,g) &= \Delta(\gamma^{-1}; w, g), \\
W_4(d,g) &= \Delta(\sqrt{pq}\delta; z, g), & W_4'(d,g) &= \Delta(\delta^{-1}; z, g),
\end{aligned}$$

where $\alpha\beta\gamma\delta = 1$. Other factors in (2.13) are $R = \Gamma(\alpha^2, \beta^2; p, q)/\Gamma(\gamma^{-2}, \delta^{-2}; p, q)$ and

$$\begin{aligned}
m(b,c) &= \Delta(\sqrt{pq}\beta\gamma; y, w), & m(a,d) &= \Delta(\sqrt{pq}\beta\gamma; x, z), \\
p(a,b) &= \Delta(\alpha\beta; x, y), & p(c,d) &= \Delta(\alpha\beta; w, z), \\
t(a,c) &= \Delta(\sqrt{pq}\alpha\gamma; x, w), & t(b,d) &= \Delta(\sqrt{pq}\alpha\gamma; y, z).
\end{aligned}$$

Perhaps, this type of the star-star relation was not considered in the literature before. Note that all such relations represent symmetry groups of the partition functions. In the case of V-function this is $W(E_7)$, i.e. one has much bigger symmetry than that seen explicitly in the chosen spin system interpretation. We have described thus a new (elliptic hypergeometric) class of solutions of the star-star relation which should lead to new solvable models of statistical mechanics similar to the checkerboard Ising model. Known systems of such type were investigated in detail in [**BSt**]. A natural general conclusion from our consideration is that the symmetry of STR can be richer than a direct sum of symmetries of the Boltzmann weights and the lattice.

3. A hyperbolic beta integral STR solution

We describe now another solution of the star-triangle relation associated with the modified form of the elliptic beta integral when one of the bases p or q can lie on the unit circle [**DS2**]. It simplifies also consideration of the degeneration limits to q-beta integrals of the Mellin-Barnes type (hyperbolic beta integrals).

First we describe the modified elliptic gamma function introduced in [**S3**]. It is convenient to use additive notation and introduce three pairwise incommensurate quasiperiods ω_1, ω_2, ω_3 together with the definitions

$$\begin{aligned}
q &= e^{2\pi i\frac{\omega_1}{\omega_2}}, & p &= e^{2\pi i\frac{\omega_3}{\omega_2}}, & r &= e^{2\pi i\frac{\omega_3}{\omega_1}}, \\
\tilde{q} &= e^{-2\pi i\frac{\omega_2}{\omega_1}}, & \tilde{p} &= e^{-2\pi i\frac{\omega_2}{\omega_3}}, & \tilde{r} &= e^{-2\pi i\frac{\omega_1}{\omega_3}}.
\end{aligned}$$

(3.1)

Here $\tilde{q}, \tilde{p}$, and $\tilde{r}$ are particular ($\tau \to -1/\tau$) modular transformations of q, p, and r. Assume that $\operatorname{Im}(\omega_1/\omega_2), \operatorname{Im}(\omega_3/\omega_1), \operatorname{Im}(\omega_3/\omega_2) > 0$, or $|q|, |p|, |r| < 1$. Then the modified elliptic gamma function is constructed as a product of two elliptic gamma functions

$$\begin{aligned}
G(u; \omega_1, \omega_2, \omega_3) &= \Gamma(e^{2\pi i\frac{u}{\omega_2}}; p, q)\Gamma(re^{-2\pi i\frac{u}{\omega_1}}; r, \tilde{q}) \\
&= e^{-\frac{\pi i}{3}B_{3,3}(u;\omega)}\Gamma(e^{-2\pi i\frac{u}{\omega_3}}; \tilde{r}, \tilde{p}),
\end{aligned}$$

(3.2)

where $B_{3,3}(u;\omega)$ is the third diagonal Bernoulli polynomial (for the general definition of such polynomials, see Appendix A),

$$B_{3,3}\left(u + \sum_{n=1}^{3} \frac{\omega_n}{2};\omega\right) = \frac{u(u^2 - \frac{1}{4}\sum_{k=1}^{3}\omega_k^2)}{\omega_1\omega_2\omega_3}.$$

The $G(u;\omega)$-function satisfies the following system of three linear difference equations of the first order

$$G(u + \omega_1;\omega) = \theta(e^{2\pi i \frac{u}{\omega_2}};p)G(u;\omega),$$
$$G(u + \omega_2;\omega) = \theta(e^{2\pi i \frac{u}{\omega_1}};r)G(u;\omega),$$
$$G(u + \omega_3;\omega) = e^{-\pi i B_{2,2}(u;\omega)}G(u;\omega),$$

where $B_{2,2}(u;\omega)$ is the second diagonal Bernoulli polynomial,

$$B_{2,2}(u;\omega) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.$$

The second equality in (3.2) follows from the fact that both expressions for $G(u;\omega)$ satisfy the above set of equations and the normalization $G(\frac{1}{2}\sum_{k=1}^{3}\omega_k;\omega) = 1$.

It is easy to see that $G(u;\omega)$ is well defined for $|p|,|r| < 1$ and $|q| \leq 1$, the $|q| = 1$ case being permitted in difference from the $\Gamma(z;p,q)$-function. Evidently, we have the symmetry relation

$$G(u;\omega_1,\omega_2,\omega_3) = G(u;\omega_2,\omega_1,\omega_3)$$

and the reflection equation

$$G(a;\omega)G(b;\omega) = 1, \quad a + b = \sum_{k=1}^{3}\omega_k.$$

For $\mathrm{Im}(\omega_1/\omega_2) > 0$, we can take the limit $\omega_3 \to \infty$ in such a way that

$$\mathrm{Im}(\omega_3/\omega_1), \quad \mathrm{Im}(\omega_3/\omega_2) \to +\infty$$

and $p,r \to 0$. Then,

$$(3.3) \qquad \lim_{p,r\to 0} G(u;\omega) = \gamma(u;\omega_1,\omega_2) = \frac{(e^{2\pi i u/\omega_1}\tilde{q};\tilde{q})_\infty}{(e^{2\pi i u/\omega_2};q)_\infty}.$$

For $\mathrm{Re}(\omega_1),\mathrm{Re}(\omega_2) > 0$ and $0 < \mathrm{Re}(u) < \mathrm{Re}(\omega_1 + \omega_2)$ this γ-function has the following integral representation

$$(3.4) \qquad \gamma(u;\omega_1,\omega_2) = \exp\left(-\int_{\mathbb{R}+i0} \frac{e^{ux}}{(1 - e^{\omega_1 x})(1 - e^{\omega_2 x})}\frac{dx}{x}\right),$$

which shows that $\gamma(u;\omega_1,\omega_2)$ is a meromorphic function of u even for $\omega_1/\omega_2 > 0$, when $|q| = 1$ and the infinite product representation (3.3) is not valid any more. The inversion relation for this function has the form

$$\gamma(u;\omega_1,\omega_2)\gamma(\omega_1 + \omega_2 - u;\omega_1,\omega_2) = e^{\pi i B_{2,2}(u;\omega)}.$$

For more details on this function see Appendix A.

Let $\mathrm{Im}(\omega_1/\omega_2) \geq 0$ and $\mathrm{Im}(\omega_3/\omega_1),\mathrm{Im}(\omega_3/\omega_2) > 0$, and let six complex parameters g_k, $k = 1,\ldots,6$, satisfy the constraints $\mathrm{Im}(g_k/\omega_3) < 0$ and

$$(3.5) \qquad \sum_{k=1}^{6} g_k = \omega_1 + \omega_2 + \omega_3.$$

Then [**DS2**],

$$(3.6) \qquad \int_{-\omega_3/2}^{\omega_3/2} \frac{\prod_{k=1}^{6} G(g_k \pm u; \omega)}{G(\pm 2u; \omega)} du = \tilde{\kappa} \prod_{1 \le k < l \le 6} G(g_k + g_l; \omega),$$

where the integration goes along the straight line segment connecting points $-\omega_3/2$ and $\omega_3/2$, and

$$(3.7) \qquad \tilde{\kappa} = \frac{-2\omega_2(\tilde{q}; \tilde{q})_\infty}{(q; q)_\infty (p; p)_\infty (r; r)_\infty}.$$

Here and below we use the shorthand notation

$$G(a \pm b; \omega) := G(a + b, a - b; \omega) := G(a + b; \omega)G(a - b; \omega).$$

The proof of equality (3.6) is rather simple. It is necessary to substitute in it the second form of $G(u; \omega)$-function (3.2), check that all exponential factors cancel and, after a change of notation, the formula reduces to the standard elliptic beta integral.

Let us introduce the crossing parameter

$$\eta = -\frac{1}{2} \sum_{k=1}^{3} \omega_k$$

and denote

$$(3.8) \qquad g_{1,2} = -\alpha \pm x, \quad g_{3,4} = \alpha + \gamma - \eta \pm y, \quad g_{5,6} = -\gamma \pm w,$$

so that the balancing condition (3.5) is satisfied automatically. Introduce also the modified Boltzmann weight, or the kernel for the modified form of the integral transformation (1.4),

$$W'(\alpha; x, u) = G(\alpha - \eta \pm x \pm u; \omega).$$

Then relation (3.6) can be rewritten as

$$\int_{-\omega_3/2}^{\omega_3/2} \phi(u; \omega) W'(\eta - \alpha; x, u) W'(\alpha + \gamma; y, u) W'(\eta - \gamma; w, u) du$$

$$(3.9) \qquad = \chi(\alpha, \gamma; \omega) W'(\alpha; y, w) W'(\eta - \alpha - \gamma; x, w) W'(\gamma; x, y),$$

where

$$\phi(u; \omega) = \frac{1}{\tilde{\kappa} G(\pm 2u; \omega)} = \frac{1}{\tilde{\kappa}} e^{-\pi i B_{2,2}(2u; \omega_1, \omega_2)} \theta(e^{-4\pi i u/\omega_2}; p) \theta(e^{-4\pi i u/\omega_1}; r),$$

$$(3.10) \quad \chi(\alpha, \gamma; \omega) = G(-2\alpha, -2\gamma, 2\alpha + 2\gamma - 2\eta; \omega).$$

Substituting the second form of the modified elliptic gamma function, we find

$$W'(\alpha; x, u) = \exp\left(-\frac{4\pi i}{3}\left(B_{3,3}(\alpha - \eta; \omega) + \frac{3\alpha(x^2 + u^2)}{\omega_1 \omega_2 \omega_3}\right)\right)$$

$$(3.11) \qquad \qquad \times \Delta\left(e^{2\pi i(\eta - \alpha)}; \frac{x}{\omega_3}, \frac{u}{\omega_3}; \tilde{p}, \tilde{r}\right).$$

We see that this Boltzmann weight is obtained from (2.1) after a reparametrization of variables and multiplication by an exponential of a quadratic polynomial of the spin variables. This means that there exists a nontrivial symmetry transformation of the star-triangle relation modifying its solutions in the described way.

The distinguished property of the modified elliptic beta integral is that it is well defined for $|q| = 1$. Therefore the limit $\omega_3 \to \infty$ leads to q-beta integrals well defined in this regime as well. Let $\text{Re}(\omega_1)$, $\text{Re}(\omega_2) > 0$. Then, for $\omega_3 \to +i\infty$, one has $p, r \to 0$ and $G(u; \omega)$ goes to $\gamma(u; \omega_1, \omega_2)$-function. Let us substitute $g_6 = \sum_{k=1}^{3} \omega_k - A$ in formula (3.6), where $A = \sum_{k=1}^{5} g_k$, and apply the inversion formula to the corresponding modified elliptic gamma function. Then the formal limit $\omega_3 \to +i\infty$ reduces this integration formula to

$$(3.12) \qquad \int_{-i\infty}^{+i\infty} \frac{\prod_{j=1}^{5} \gamma(g_k \pm u; \omega)}{\gamma(\pm 2u, A \pm u; \omega)} du = -2\omega_2 \frac{(\tilde{q}; \tilde{q})_\infty}{(q; q)_\infty} \frac{\prod_{1 \leq j < k \leq 5} \gamma(g_j + g_k; \omega)}{\prod_{k=1}^{5} \gamma(A - g_k; \omega)},$$

where the integration contour is the straight line for $\text{Re}(g_k) > 0$ or the Mellin-Barnes type contour, if these restrictions for parameters are violated. Let us remind also that

$$\frac{(\tilde{q}; \tilde{q})_\infty}{(q; q)_\infty} = \sqrt{-i\frac{\omega_1}{\omega_2}} e^{\frac{\pi i}{12}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)},$$

where $\sqrt{-i} = e^{-\pi i/4}$ since for $\omega_1/\omega_2 = ia$, $a > 0$, the square root should be positive.

Let us introduce parameter g_6 anew (it should not be confused with the previous variable g_6 which we have eliminated) using the condition

$$(3.13) \qquad \sum_{k=1}^{6} g_k = \omega_1 + \omega_2$$

(note the difference with (3.5)). Now we can apply the inversion formula to γ-functions to move some of them from the denominator of the integral kernel to its numerator. It is convenient here to define the hyperbolic gamma function $\gamma^{(2)}(u)$:

$$(3.14) \qquad \gamma^{(2)}(u; \omega) = e^{-\frac{\pi i}{2} B_{2,2}(u; \omega)} \gamma(u; \omega).$$

Then, after the change of the integration variable $u = iz$, the integral (3.12) takes the compact form

$$(3.15) \qquad \int_{-\infty}^{\infty} \frac{\prod_{j=1}^{6} \gamma^{(2)}(g_k \pm iz; \omega)}{\gamma^{(2)}(\pm 2iz; \omega)} dz = 2\sqrt{\omega_1 \omega_2} \prod_{1 \leq j < k \leq 6} \gamma^{(2)}(g_j + g_k; \omega).$$

Validity of the described limit $\omega_3 \to \infty$ at the level of integrals was rigorously justified in [**R2**] using a slightly different notation. Integral (3.15) was proven first (using a different approach) by Stokman [**St**] who called it the hyperbolic beta integral. We followed the formal limiting procedure suggested in [**DS2**].

Similar to (3.8), let us fix the parameters as

$$(3.16) \qquad g_{1,2} = -\alpha \pm ix, \quad g_{3,4} = \alpha + \gamma - \eta \pm iy, \quad g_{5,6} = -\gamma \pm iw$$

with the crossing parameter $\eta = -(\omega_1 + \omega_2)/2$. Then formula (3.15) can be rewritten as a star-triangle relation

$$\int_{-\infty}^{\infty} S(z) W(\eta - \alpha; x, z) W(\alpha + \gamma; y, z) W(\eta - \gamma; w, z) dz$$
$$(3.17) \qquad = \chi(\alpha, \gamma) W(\alpha; y, w) W(\eta - \alpha - \gamma; x, w) W(\gamma; x, y),$$

where

$$W(\alpha; x, z) = \gamma^{(2)}(\alpha - \eta \pm ix \pm iz; \omega),$$

$$S(z) = \frac{1}{2\sqrt{\omega_1 \omega_2}\gamma^{(2)}(\pm 2iz; \omega)} = \frac{2\sinh\frac{2\pi z}{\omega_1}\sinh\frac{2\pi z}{\omega_2}}{\sqrt{\omega_1 \omega_2}},$$

(3.18) $$\chi(\alpha, \gamma) = \gamma^{(2)}(-2\alpha, -2\gamma, 2\alpha + 2\gamma - 2\eta; \omega).$$

These Boltzmann weights are positive for real x, z, α, $\eta < \alpha < -\eta$, $\eta < 0$, and either real $\omega_{1,2}$ or $\omega_1^* = \omega_2$.

Consider a particular reduction of integration formula (3.15). For this we replace parameters $g_j \to g_j + i\mu$, $j = 1, 2, 3$, and $g_j \to g_j - i\mu$, $j = 4, 5, 6$. Since the integrand is symmetric in z we can rewrite the left-hand side as

$$2\int_0^\infty \frac{\prod_{j=1}^3 \gamma^{(2)}(g_j + i\mu \pm iz, g_{j+3} - i\mu \pm iz; \omega)}{\gamma^{(2)}(\pm 2iz; \omega)} dz$$

$$= 2\int_{-\mu}^\infty \prod_{j=1}^3 \gamma^{(2)}(g_j - iz, g_{j+3} + iz; \omega)\rho_1(z)\rho_2(z)dz,$$

where

$$\rho_1(z) = \frac{e^{-2\pi(z+\mu)(\omega_1^{-1} + \omega_2^{-1})}}{\gamma^{(2)}(\pm 2i(z+\mu); \omega)} \underset{\mu \to +\infty}{\to} 1$$

and

$$\rho_2(z) = e^{2\pi(z+\mu)(\omega_1^{-1} + \omega_2^{-1})}\prod_{j=1}^3 \gamma^{(2)}(g_j + 2i\mu + iz, g_{j+3} - 2i\mu - iz; \omega)$$

$$\underset{\mu \to +\infty}{\to} e^{\frac{\pi}{\omega_1 \omega_2}\left(-2\mu(\omega_1 + \omega_2) + \frac{i}{2}\sum_{j=1}^3 \left(g_{j+3}^2 - g_j^2 + (g_j - g_{j+3})(\omega_1 + \omega_2)\right)\right)}(1 + o(1)).$$

On the right-hand side we find

$$2\sqrt{\omega_1 \omega_2}\prod_{j=1}^3\prod_{k=4}^6 \gamma^{(2)}(g_j + g_k; \omega)\rho_3(g),$$

where

$$\rho_3(g) = \prod_{1 \le j < k \le 3} \gamma^{(2)}(g_j + g_k + 2i\mu, g_{j+3} + g_{k+3} - 2i\mu; \omega)$$

$$\underset{\mu \to +\infty}{\to} e^{\frac{\pi i}{2}\sum_{1 \le j < k \le 3}\left(B_{2,2}(g_{j+3} + g_{k+3} - 2i\mu) - B_{2,2}(g_j + g_k + 2i\mu)\right)}(1 + o(1)).$$

One can check that the leading asymptotics of $\rho_3(g)$ coincides with that of the $\rho_2(z)$-function. Taking the limit $\mu \to +\infty$, which is uniform, one comes to the following exact integration formula [**B**]

(3.19) $$\int_{-\infty}^\infty \prod_{j=1}^3 \gamma^{(2)}(g_j - iz, g_{j+3} + iz; \omega)dz = \sqrt{\omega_1 \omega_2}\prod_{j=1}^3\prod_{k=4}^6 \gamma^{(2)}(g_j + g_k; \omega),$$

where $\sum_{k=1}^6 g_k = \omega_1 + \omega_2$.

Let us change notation for the integral parameters

$$g_1 = -\alpha + \mathrm{i}x, \qquad g_2 = \alpha + \gamma - \eta + \mathrm{i}y, \qquad g_3 = -\gamma + \mathrm{i}w,$$
$$g_4 = -\alpha - \mathrm{i}x, \qquad g_5 = \alpha + \gamma - \eta - \mathrm{i}y, \qquad g_6 = -\gamma - \mathrm{i}w,$$

where $\eta = -(\omega_1 + \omega_2)/2$ is a crossing parameter; the balancing condition $\sum_{k=1}^{6} g_k = -2\eta$ is satisfied then automatically. Now we can rewrite equality (3.19) as the star-triangle relation

$$\int_{-\infty}^{\infty} W(\eta - \alpha; x, z) W(\alpha + \gamma; y, z) W(\eta - \gamma; w, z) \frac{dz}{\sqrt{\omega_1 \omega_2}}$$

(3.20)
$$= \chi\, W(\alpha; y, w) W(\eta - \alpha - \gamma; x, w) W(\gamma; x, y),$$

where the Boltzmann weight is defined as

$$W(\alpha; x, z) = \gamma^{(2)}(\alpha - \eta \pm \mathrm{i}(x - z); \omega)$$

and the normalization constant is

(3.21)
$$\chi = \gamma^{(2)}(-2\alpha, -2\gamma, 2\alpha + 2\gamma - 2\eta; \omega).$$

Note that $W(\alpha; x, y) = W(\alpha; y, x)$ and $W(\alpha; x, y) > 0$ in the same domain of parameters as before. Denoting $\omega = b$, $\omega_2 = b^{-1}$, and $\alpha = -(b + b^{-1})\theta/(2\pi)$ one can see that $W(\alpha; x, z)$ coincides with the Boltzmann weight of the Faddeev-Volkov model [**FV, VF**] denoted as $W_\theta(x - z)$ in [**BMS**] (our η differs by sign from the definition chosen in [**BMS**]) up to some normalization factor F_θ.

We thus see that the Faddeev-Volkov model solution of the star-triangle relation [**VF**] is a particular case of our hyperbolic beta integral STR solution (3.18). The fact that the left-hand side of STR for the Faddeev-Volkov model represents a particular limiting case of the elliptic beta integral was known to the author already in 2008. After seeing [**BMS**] and understanding this fact, the author was interested whether a similar interpretation exists for the elliptic beta integral itself. However, this idea was not developed further, partially because the origin of the normalizing factor F_θ given in [**BMS**] was not understood at that time. Fortunately, Bazhanov and Sergeev have independently answered this question in [**BS**].

4. Partition functions

The partition function of a homogeneous two dimensional discrete spin system on the square lattice with the Boltzmann weights $W(\alpha; u_i, u_j)$ (2.1) and $S(u_j)$ (2.3) has the form

$$Z = \int \prod_{(ij)} W(\alpha; u_i, u_j) \prod_{(kl)} W(\eta - \alpha; u_k, u_l) \prod_m S(u_m) du_m,$$

where the first product is taken over the horizontal edges (ij), the second product goes over all vertical edges (k, l), and the third product (in m) is taken over all internal vertices of the lattice. Let us take the elliptic beta integral STR solution of [**BS**] and consider the contribution to Z coming from a particular vertex u surrounded by the vertices u_1, u_2, u_3, u_4:

$$\int_{0}^{2\pi} S(u) W(\alpha; u_1, u) W(\alpha; u, u_3) W(\eta - \alpha; u_2, u) W(\eta - \alpha; u, u_4) du.$$

Substituting explicit expressions for the weights, one can easily see that this integral is equal to the elliptic hypergeometric function $V(t_1, \ldots, t_8; p, q)$ described above (1.3) with the following restricted set of parameters

$$\{t_1, t_2, t_3, t_4\} = \{e^{\alpha - \eta} e^{2\pi i u_1}, e^{\alpha - \eta} e^{-2\pi i u_1}, e^{\alpha - \eta} e^{2\pi i u_3}, e^{\alpha - \eta} e^{-2\pi i u_3}\},$$

$$\{t_5, t_6, t_7, t_8\} = \{e^{-\alpha} e^{2\pi i u_2}, e^{-\alpha} e^{-2\pi i u_2}, e^{-\alpha} e^{2\pi i u_4}, e^{-\alpha} e^{-2\pi i u_4}\}.$$

In total, there are 5 independent parameters, instead of 7 for generic V-function (in addition to the bases p and q). Therefore we conclude that the full partition function Z is given by an elliptic hypergeometric integral constructed as a tower of intertwined (restricted) elliptic analogues of the Gauss hypergeometric function similar to the Bailey tree for integrals [**S4**].

According to the general reflection method used in [**BS**], the leading asymptotics of the partition function for two-dimensional $N \times M$ lattice when its size goes to infinity, $N, M \to \infty$, has the form

$$Z \underset{N, M \to \infty}{=} m(\alpha)^{NM},$$

where $m(\alpha)$ is the normalizing factor for Boltzmann weights which guarantees that on the right-hand side of STR the $\tilde{\chi}$-multiplier (2.8) is equal to unity, $\tilde{\chi} = 1$. This condition is satisfied if

$$(4.1) \qquad \frac{m(\alpha)}{m(\eta - \alpha)} \Gamma(e^{-2\alpha}; p, q) = 1, \quad \text{or} \quad m(\alpha + \eta) = \Gamma(e^{2\alpha}; p, q) m(-\alpha).$$

Let us introduce the function
$$(4.2)$$
$$M(x; p, q, t) = \exp\left(\sum_{n \in \mathbb{Z}/\{0\}} \frac{(\sqrt{pqt}x)^n}{n(1 - p^n)(1 - q^n)(1 + t^n)} \right) = \frac{\Gamma(xt\sqrt{pqt}; p, q, t^2)}{\Gamma(x\sqrt{pqt}; p, q, t^2)},$$

where

$$\Gamma(z; p, q, t) = \prod_{j,k,l=0}^{\infty} (1 - zt^j p^k q^l)(1 - z^{-1} t^{j+1} p^{k+1} q^{l+1}), \quad |p|, |q|, |t| < 1,$$

is the second order elliptic gamma function satisfying the t-difference equation

$$\Gamma(tz; p, q, t) = \Gamma(z; p, q)\Gamma(z; p, q, t).$$

The reflection equation $\Gamma(z^{-1}; p, q, t) = \Gamma(pqtz; p, q, t)$ leads to the equality

$$M(x^{-1}; p, q, t) M(x; p, q, t) = 1.$$

It is easy also to check validity of the functional equation

$$M(x; p, q, t) M(t^{-1}x; p, q, t) = \Gamma\left(x\sqrt{\frac{pq}{t}}; p, q \right),$$

which is equivalent to (4.1) after fixing $t = pq$ and $x = e^{2\alpha}$. Therefore we find the needed normalizing function

$$(4.3) \qquad m(\alpha) = M(e^{2\alpha}; p, q, pq), \quad m(\alpha)m(-\alpha) = 1.$$

The function $-\log m(\alpha)$ defines thus the free energy per edge of the integrable lattice model under consideration.

Now we discuss the partition function for the general hyperbolic beta integral solution of the star-triangle relation (3.18). The needed normalization constant $m(\alpha)$ is found from the equation

$$(4.4) \qquad \frac{m(\alpha)}{m(\eta - \alpha)} \gamma^{(2)}(-2\alpha; \omega) = 1, \quad \text{or} \quad m(\alpha/2 + \eta) = \gamma^{(2)}(\alpha; \omega) m(-\alpha/2),$$

where $\eta = -(\omega_1 + \omega_2)/2$.

Let us define the function

$$\mu(u; \omega_1, \omega_2, \omega_3) = \frac{\gamma^{(3)}(u + \frac{1}{2}\sum_{k=1}^{3}\omega_k + \omega_3; \omega_1, \omega_2, 2\omega_3)}{\gamma^{(3)}(u + \frac{1}{2}\sum_{k=1}^{3}\omega_k; \omega_1, \omega_2, 2\omega_3)},$$

where $\gamma^{(3)}$-function is the hyperbolic gamma function of the third order defined in Appendix B. Using the integral representation for it, we can write

$$(4.5) \qquad \mu(u; \omega) = \exp\left(-\frac{\pi i a}{6} - \int_{\mathbb{R}+i0} \frac{e^{vx}}{(e^{\omega_1 x} - 1)(e^{\omega_2 x} - 1)(e^{\omega_3 x} + 1)} \frac{dx}{x}\right),$$

where $v = u + \sum_{k=1}^{3}\omega_k/2$ and

$$a = B_{3,3}(v + \omega_3; \omega_1, \omega_2, 2\omega_3) - B_{3,3}(v; \omega_1, \omega_2, 2\omega_3)$$
$$= \frac{3}{2\omega_1\omega_2}\left(u^2 - \frac{\omega_1^2 + \omega_2^2 + 3\omega_3^2}{12}\right)$$

For a special choice of the third quasiperiod variable $\omega_3 = \omega_1 + \omega_2$, this function appeared for the first time in [LZ].

Using the reflection equation

$$\gamma^{(3)}\left(\sum_{k=1}^{3}\omega_k - u; \omega_1, \omega_2, \omega_3\right) = \gamma^{(3)}(u; \omega_1, \omega_2, \omega_3)$$

and the difference equation

$$\gamma^{(3)}(u + \omega_3; \omega_1, \omega_2, \omega_3) = \gamma^{(2)}(u; \omega_1, \omega_2)\gamma^{(3)}(u; \omega_1, \omega_2, \omega_3),$$

one can easily check that $\mu(u; \omega)\mu(-u; \omega) = 1$ and

$$\mu(u; \omega)\mu(u - \omega_3; \omega) = \gamma^{(2)}\left(u + \frac{1}{2}(\omega_1 + \omega_2 - \omega_3); \omega_1, \omega_2\right).$$

The latter relation coincides with equation (4.4) for $u = 2\alpha$ and $\omega_3 = \omega_1 + \omega_2$. Therefore we find the free energy per edge as $-\log m(\alpha)$, where

$$(4.6) \qquad m(\alpha) = \mu(2\alpha; \omega_1, \omega_2, \omega_1 + \omega_2).$$

By construction this function satisfies also the reflection equation $m(\alpha)m(-\alpha) = 1$. Denoting $\omega_1 = b, \omega_2 = b^{-1}$ and substituting the infinite product representation of the $\gamma^{(3)}$-function given in Appendix B, we find the expression

$$m(\alpha) = \exp\left(-\pi i \alpha^2 - \frac{\pi i}{24}(1 - 2(b + b^{-1})^2)\right)$$
$$(4.7) \qquad \times \frac{(\tilde{q}e^{2\pi i u/b}; \tilde{q}^2)_\infty}{(qe^{2\pi i u b}; q^2)_\infty} \prod_{j,k=0}^{\infty} \frac{1 + e^{\pi i u/(b + b^{-1})}\tilde{p}^{j+1}\tilde{q}^{2k}}{1 - e^{\pi i u/(b + b^{-1})}\tilde{p}^{j+1}\tilde{q}^{2k}},$$

where it is assumed that $|q| < 1$, $q = e^{2\pi i b^2}$, $\tilde{q} = e^{-2\pi i/b^2}$, and $\tilde{p} = e^{-\pi i/(1 + b^2)}$.

We turn now to the Faddeev-Volkov model solution of STR (3.20). In this case we have no self-interaction of the spins sitting in lattice vertices, and the Boltzmann

weights attached to edges are simplified. But the partition function asymptotics is the same as in the previous case, since evidently the normalizing constant $m(\alpha)$ is found from the same equation (4.4). The free energy per edge for this model was computed already by Bazhanov, Mangazeev, and Sergeev in [**BMS**], where the Boltzmann weights normalizing factor was denoted as F_θ. Comparing this constant with our $m(\alpha)$, we see that they coincide for $\alpha = -(b + b^{-1})\theta/(2\pi)$, as necessary. However, our infinite product representation of $m(\alpha)$ in (4.7) differs drastically from that given in [**BMS**] (which was the source of author's old time confusion).

5. Conclusion

After the discovery of elliptic hypergeometric integrals, for a long time the author was drawing attention of experts (including the second author of [**BS**]) in two-dimensional conformal field theory and solvable models of statistical mechanics for a potential emergence of such functions in these fields. The connection between the elliptic beta integral and the star-triangle relation found in [**BS**] and the star-star relation described above confirms this expectation. However, the nature appeared to be much richer than it was imagined in [**S2, S6, S9**]. As mentioned already, the Dolan-Osborn discovery of a stunningly unexpected coincidence of elliptic hypergeometric integrals with superconformal (topological) indices in four dimensional supersymmetric gauge theories strongly pushed forward the development of the theory and raised many interesting open questions [**DO, SV1**]. The interpretation of exact computability of the elliptic beta integrals as the confinement phenomenon in quantum field theory is a new type of conceptual perception of the exact mathematical formulas.

As to the models considered in this paper, we have described a generalization of the Faddeev-Volkov solution of STR [**VF**] with the continuous spin variables taking values on the real line, which was not considered in [**BS**]. It has some nontrivial self-interaction energy for each vertex and a more complicated form of the Boltzmann weights for edges, though the free energy per edge appears to be the same as in the Faddeev-Volkov model. In [**VF**] the Yang-Baxter equation was proved using the quantum pentagonal relation [**FKV**]. It would be interesting to interpret in a similar way the model we have described here. Some time ago the author has tried to find an elliptic analogue of the pentagon relation in analogy with the constructions described in [**V**], but could not do it yet. Clearly the elliptic beta integral gives already an analytic form of that wanted operator relation, but it is hard to formulate it in terms of the commutation relations of some explicit operators.

In [**FV**], Faddeev and Volkov have considered a lattice Virasoro algebra and described an integrable model in the discrete $2d$ space-time (it was discussed also in detail in [**FKV**]). The elliptic beta integral yields more general solutions of STR than that of [**VF**], and it is natural to ask for explicit realization of the corresponding models similar to [**FV**]. During the work on [**SV1**], G. Vartanov and the author have suggested that there should exist some elliptic deformation of the primary fields $V_\alpha(z)$ built from free $2d$ bosonic fields (in the spirit similar to the situation discussed in [**SWy**]) such that the three point correlation function would be given by the elliptic beta integral and the four point function would be described by the V-function satisfying the elliptic hypergeometric equation [**S9**] (so that the

tetrahedral symmetries of the V-function would describe the s-t channels duality). Unfortunately, such hypothetical results are not conceivable at the present moment.

From the point of view of superconformal indices the partition function associated with the elliptic beta integral solution of STR looks as a superconformal index for a particular $SU(2)$-quiver gauge theory on a two dimensional lattice. Recently there was a great deal of activity on interrelations between four-dimensional super-Yang-Mills theories and two-dimensional field theories, see, e.g., [**AGT, CNV, GPRR, NS, SWy**]. In this framework, the elliptic hypergeometric integrals describing superconformal indices of $\mathcal{N} = 2$ quiver gauge theories have been interpreted by Gadde et al in [**GPRR**] as correlation functions of some $2d$ topological quantum field theories.

Therefore it is natural to expect that superconformal indices of all four dimensionanl CFTs are related to discretizations of $2d$ CFT models and other integrable systems. A connection of the Yang-Baxter moves with the Seiberg duality has been briefly discussed in [**HV**]. In this context, superconformal indices of all quiver gauge theories should correspond to full partition functions of some spin systems. In view of the abundance of supersymmetric dualities and rich structure of the corresponding superconformal indices (twisted partition functions) [**SV1**], the author considers the present moment only as a beginning of uncovering new two-dimensional and higher-dimensional integrable models hidden behind the elliptic hypergeometric functions.

For instance, the elliptic Selberg integral defined on the BC_n root system reads [**DS1, S9**]:

$$\kappa_n \int_{\mathbb{T}^n} \prod_{1 \le j < k \le n} \frac{\Gamma(tz_j^{\pm 1}z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1}z_k^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{m=1}^6 \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{\mathrm{i} z_j}$$

$$(5.1) \qquad\qquad = \prod_{j=1}^n \left(\frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \prod_{1 \le m < s \le 6} \Gamma(t^{j-1} t_m t_s; p, q) \right),$$

where $|t|, |t_m| < 1$, $t^{2n-2} \prod_{m=1}^6 t_m = pq$, and

$$\kappa_n = \frac{(p; p)_\infty^n (q; q)_\infty^n}{(4\pi)^n n!}.$$

After some work, this formula can be given the STR type shape

$$\int_{[0, 2\pi]^n} S(\mathbf{u}; t, p, q) W(\eta - \alpha; x, \mathbf{u}) W(\alpha + \gamma; y, \mathbf{u}) W(\eta - \gamma; w, \mathbf{u})[d\mathbf{u}],$$

$$(5.2) \qquad = W_t(\alpha; y, w) W_t(\eta - \alpha - \gamma; w, x) W_t(\gamma; x, y),$$

where we denoted

$$[d\mathbf{u}] = \kappa_n \prod_{j=1}^n \frac{\Gamma(t; p, q) du_j}{\Gamma(t^j; p, q)},$$

and the crossing parameter η is defined as

$$e^{-2\eta} = pq t^{n-1}.$$

The Boltzmann weights have the form

$$(5.3) \qquad S(\mathbf{u}; t, p, q) = \prod_{1 \le j < k \le n} \frac{\Gamma(te^{\pm \mathrm{i} u_j \pm \mathrm{i} u_k}; p, q)}{\Gamma(e^{\pm \mathrm{i} u_j \pm \mathrm{i} u_k}; p, q)} \prod_{j=1}^n \frac{1}{\Gamma(e^{\pm 2\mathrm{i} u_j}; p, q)}$$

and

$$(5.4) \qquad W(\alpha; x, \mathbf{u}) := \frac{1}{m(\alpha)} \prod_{j=1}^{n} \Gamma(\sqrt{pq}\, e^{\alpha} e^{\pm ix} e^{\pm iu_j}; p, q),$$

$$(5.5) \qquad W_t(\alpha; x, y) := \frac{1}{m(\alpha)} \prod_{j=1}^{n} \Gamma(\sqrt{pq}\, e^{\alpha} t^{j-\frac{n+1}{2}} e^{\pm ix} e^{\pm iy}; p, q),$$

and satisfy the reflection relations

$$W(\alpha; x, \mathbf{u})W(-\alpha; x, \mathbf{u}) = 1, \qquad W_t(\alpha; x, y)W_t(-\alpha; x, y) = 1.$$

The normalization constant $m(\alpha)$ for $n > 1$ has a substantially more complicated form than that for $n = 1$. To describe it we introduce the function

$$M(x; p, q, t, s) = \frac{\Gamma(xts^2, xt^{1-n}s; p, q, t, s^2)}{\Gamma(xts, xt^{1-n}s^2; p, q, t, s^2)},$$

a particular ratio of four elliptic gamma functions of the *third order*. More precisely, one has

$$\Gamma(z; p, q, t, s) := \prod_{i,j,k,l=0}^{\infty} \frac{1 - z^{-1}p^{i+1}q^{j+1}t^{k+1}s^{l+1}}{1 - zp^i q^j t^k s^l}$$

for $z \in \mathbb{C}^*$, $|p|, |q|, |t|, |s| < 1$, with the reflection equation $\Gamma(z, pqtsz^{-1}; p, q, t, s) = 1$ and the difference equation $\Gamma(sz; p, q, t, s) = \Gamma(z; p, q, t)\Gamma(z; p, q, t, s)$. Then,

$$(5.6) \qquad m(\alpha) = M(e^{2\alpha}; p, q, t, pqt^{n-1}),$$

with the standard reflection relation $m(\alpha)m(-\alpha) = 1$.

Let us discuss a physical meaning of the obtained model. Consider a honeycomb lattice on the plane with two types of vertices – black and white with two adjacent vertices always being of different color, see the left-hand side of figure 2. Into each white vertex we put an independent single component continuous spin x. Into each black vertex we put n independent spins u_j, $j = 1, \ldots, n$, or one n-dimensional spin with n continuous components. These "spins" are quite different from those of the Ising model where they take only the values $+1$ and -1 (i.e., they represent the fields and not the compact spins). In different words, one associates to each black vertex the $SP(2n)$-group space related to the root system BC_n. We associate with each black vertex the self-interaction Boltzmann weight (5.3) with an additional interaction between "spin" components in the internal space. To each bond connecting "black-and-white" vertices we attach the Boltzmann weight (5.4). Then, on the left-hand side of (5.2) we have the partition function of an elementary cell with the black vertex in the center and the integral taken over the u_j-spin values. If we apply this star-triangle relation to each black vertex we come to a different spin system associated with the plain triangular lattice having only the white vertices with the bond Boltzmann weights described by the function (5.5), see the right-hand side of figure 2. Such a transformation of lattices looks quite similar to a transformation of the honeycomb-triangular Ising systems considered in [**W**]. Perhaps there exists also another STR type duality transformation involving only the white vertices (with some self-interaction) allowing for a transition to yet another triangular lattice system. In addition to this uncertainty, it remains also unclear the free energy per edge of which model is described by the function (5.6).

Positivity of the Boltzmann weights of this model can be analyzed along the lines of elliptic modular double involutions discussed in [**S8**]. In particular, these

weights are clearly positive for $x, u_j \in [0, 2\pi]$, real t, α and $\rho := |\sqrt{pq}e^{\alpha}| < 1$, $\rho < |t|^{\frac{n-1}{2}} < \rho^{-1}$, with either $p^* = p, q^* = q$ or $p^* = q$. For $n > 1$ the crossing parameter η looks like an arbitrary free variable, not related to other parameters of the system, but, in fact, it is essentially equivalent to the coupling constant t for u_j-spins. If $t = 1$, relation (5.2) reduces to n-th power of the standard STR.

Define the BC_n-root system generalization of the V-function:

$$(5.7) \qquad I(t_1, \dots, t_8; t; p, q) = \prod_{1 \le j < k \le 8} \Gamma(t_j t_k; p, q, t)$$

$$\times \kappa_n \int_{\mathbb{T}^n} \prod_{1 \le j < k \le n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^{n} \frac{\prod_{k=1}^{8} \Gamma(t_k z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{i z_j},$$

where parameters $t, t_1, \dots, t_8 \in \mathbb{C}$, satisfy $|t|, |t_j| < 1$, and $t^{2n-2} \prod_{j=1}^{8} t_j = p^2 q^2$ constraints. As shown by Rains [R3], this function obeys the same $W(E_7)$ Weyl group of symmetries as in the $n = 1$ case. The key transformation has the form

$$(5.8) \qquad I(t_1, \dots, t_8; t; p, q) = I(s_1, \dots, s_8; t; p, q),$$

where

$$\begin{cases} s_j = \rho^{-1} t_j, & j = 1, 2, 3, 4 \\ s_j = \rho t_j, & j = 5, 6, 7, 8 \end{cases} ; \quad \rho = \sqrt{\frac{t_1 t_2 t_3 t_4}{pq t^{1-n}}} = \sqrt{\frac{pq t^{1-n}}{t_5 t_6 t_7 t_8}}, \quad |t|, |t_j|, |s_j| < 1.$$

Introduce variables x_j by relation $t^{(n-1)/4} t_j = (pq)^{1/4} e^{2\pi i x_j}$, so that the balancing condition becomes $\sum_{j=1}^{8} x_j = 0$. Then (5.8) describes the invariance of the integral I with respect to the Weyl reflection

$$x \to S_v(x) = x - \frac{2\langle x, v \rangle}{\langle v, v \rangle} v, \quad x, v \in \mathbb{R}^8,$$

where $\langle x, v \rangle = \sum_{k=1}^{8} x_k v_k$ is the scalar product and the vector v has components $v_k = 1/2$, $k = 1, 2, 3, 4$, and $v_k = -1/2$, $k = 5, 6, 7, 8$. Together with the group S_8 permuting the parameters x_j, this transformation generates full exceptional reflection group $W(E_7)$.

Equality (5.8) can be rewritten in the star-star relation form

$$\int_{[0, 2\pi]^n} S(\mathbf{u}; t, p, q) W(\eta - \alpha; x, \mathbf{u}) W(\eta - \beta; y, \mathbf{u}) W(\gamma; w, \mathbf{u}) W(\delta; z, \mathbf{u})[d\mathbf{u}]$$

$$(5.9) \qquad = R \frac{P_t(\alpha + \beta; x, y)}{P_t(\alpha + \beta; w, z)} \int_{[0, 2\pi]^n} S(\mathbf{u}; t, p, q)$$

$$\times W(\beta; x, \mathbf{u}) W(\alpha; y, \mathbf{u}) W(\eta - \delta; w, \mathbf{u}) W(\eta - \gamma; z, \mathbf{u})[d\mathbf{u}],$$

where $\alpha + \beta = \gamma + \delta$ and

$$R = \prod_{l=0}^{n-1} \frac{\Gamma(t^{-l} e^{-2\alpha}, t^{-l} e^{-2\beta}; p, q)}{\Gamma(t^{-l} e^{-2\gamma}, t^{-l} e^{-2\delta}; p, q)}, \quad P_t(\alpha; x, y) = \prod_{l=0}^{n-1} \Gamma(t^{-l} e^{-\alpha} e^{\pm i x \pm i y}; p, q).$$

In complete analogy with $n = 1$ case (1.16), (1.17), (1.18), one can obtain two other differently looking star-star relations for $n > 1$ by an iterative application of this formula after permutations of parameters.

Using the matrix integral representations for elliptic hypergeometric integrals, in [SV1] relation (5.8) was shown to describe a new electric-magnetic duality between two four-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theories with

the gauge group $G = SP(2n)$. Namely, the electric theory has the flavor group $SU(8) \times U(1)$; it contains the vector superfield in the adjoint representation of G, one chiral scalar multiplet in the fundamental representations of G and $SU(8)$, and the field described by the antisymmetric tensor of the second rank of G. The magnetic theory has the flavor group $SU(4)_l \times SU(4)_r \times U(1)_B \times U(1)$ and a similar set of quantum fields, as well as $2n$ additional gauge invariant mesonic fields — the antisymmetric tensors of $SU(4)$-flavor subgroups. The elliptic Selberg integral (5.1) describes the confinement phenomenon in the $SP(2n)$ super-Yang-Mills gauge theory with 6 chiral superfields in the fundamental and 1 chiral superfield in the antisymmetric representations of $SP(2n)$, respectively, — its dual magnetic phase contains only a peculiar set of mesonic fields without local gauge symmetry.

From the present paper point of view relations (5.2) and (5.9) should have an appropriate physical interpretation in the context of discrete integrable models for $n > 1$ similar to the $n = 1$ case. We described already one possible honeycomb lattice model that can be associated with the elliptic Selberg integral. The system lying behind relation (5.9) resembles the checkerboard Ising model with the continuous spins. As an elementary cell one has a square with four white vertices (with the single component spins $x, y, \ldots$ sitting in them) and one black vertex in the center (with the n-component spin $\mathbf{u}$ sitting in it and the integral taken over its values), see figure 4. There are again three differently looking star-star relations for $n > 1$ obtained by repeated application of the same formula (5.9) in conjugation with permutation of parameters, quite similar to the $n = 1$ case. Equality (5.2) can be considered then as their reduction to STR.

We did not discuss in this paper an important physical question of the existence of phase transitions in the described models and the spectrum of scaling exponents. For clarifying this point it is necessary to single out the temperature like variable associated with one of the parameters p or q [**BS**] and investigate the behavior of the partition functions per edge (defined by $m(\alpha)$'s) when the temperature varies from large to small values. Since many known systems with nontrivial phase transitions are represented by the limiting cases of the elliptic beta integral STR solutions, there are nontrivial critical phenomena. However, their classification requires separate analysis and lies beyond the scope of the present work.

In [**SV1**] a large number of proven and conjectural evaluation formulas for elliptic beta integrals on root systems and their nontrivial symmetry transformation analogues for higher order integrals has been listed. Actually, it was conjectured that there exist infinitely many such integrals, and for each of them one can expect suitable application in the context of solvable models of statistical mechanics and other types of integrable systems.

Appendix A. The modified q-gamma function

The function $\gamma(u; \omega_1, \omega_2)$ (3.3) or its various transformed versions are referred to in different papers as the double sine function [**KLS, PT**], the non-compact

quantum dilogarithm [**F, FKV, PT, V, BS**], the hyperbolic gamma function [**Ru, B**], or the modified q-gamma function [**S9**].

The functional equations satisfied by $\gamma(u; \omega)$ have the form

$$(A.1) \qquad \frac{\gamma(u + \omega_1; \omega)}{\gamma(u; \omega)} = 1 - e^{2\pi i \frac{u}{\omega_2}}, \qquad \frac{\gamma(u + \omega_2; \omega)}{\gamma(u; \omega)} = 1 - e^{2\pi i \frac{u}{\omega_1}}.$$

Using a modular transformation for theta functions one can derive another representation for $\gamma(u; \omega_1, \omega_2)$ complementary to (3.3):

$$(A.2) \qquad \gamma(u; \omega_1, \omega_2) = e^{\pi i B_{2,2}(u;\omega)} \frac{(e^{-2\pi i u/\omega_2} q; q)_\infty}{(e^{-2\pi i u/\omega_1}; \tilde{q})_\infty}.$$

The non-compact quantum dilogarithm [**F**] in the notation of [**BMS**] (in [**FKV**] it was denoted as $e_b(z)$) has the form

$$\varphi(z) = \exp\left(\frac{1}{4} \int_{\mathbb{R}+i0} \frac{e^{-2izw}}{\sinh(wb)\sinh(w/b)}\right) \frac{dw}{w}$$
$$= \exp\left(\int_{\mathbb{R}+i0} \frac{e^{wu}}{(e^{wb} - 1)(e^{w/b} - 1)}\right) \frac{dw}{w},$$

where

$$u = \frac{1}{2}(b + b^{-1}) - iz.$$

For $q = e^{2\pi i b^2}$, $\tilde{q} = e^{-2\pi i/b^2}$ and $\text{Im}(b^2) > 0$, one can write

$$\varphi(z) = \frac{(e^{2\pi i b u}; q)_\infty}{(e^{2\pi i u/b} \tilde{q}; \tilde{q})_\infty} = \frac{(-q^{1/2} e^{2\pi b z}; q)_\infty}{(-\tilde{q}^{1/2} e^{2\pi z/b}; \tilde{q})_\infty}.$$

Therefore,

$$\varphi(z) = \gamma\left(\frac{1}{2}(b + b^{-1}) - iz; b, b^{-1}\right)^{-1}.$$

The $S_b(u)$ function used in [**PT**] coincides with

$$\gamma^{(2)}(u; \omega_1, \omega_2) = e^{-\frac{\pi i}{2} B_{2,2}(u;\omega_1,\omega_2)} \gamma(u; \omega_1, \omega_2)$$

for $\omega_1 = b$ and $\omega_2 = b^{-1}$, and another similar function of [**PT**] is

$$G_b(u) = e^{-\frac{\pi i}{12}(3 + b^2 + b^{-2})} \gamma(u; b, b^{-1}).$$

The $\gamma(z)$-function used in [**V**] coincides with the infinite products ratio on the right-hand side of (A.2) for $z = \omega_2 u$ and $\tau = \omega_1/\omega_2$.

For $\text{Re}(\omega_1), \text{Re}(\omega_2) > 0$ and $0 < \text{Re}(u) < \text{Re}(\omega_1 + \omega_2)$ the function $\gamma^{(2)}(u; \omega_1, \omega_2)$ has the following integral representation

$$(A.3) \qquad \gamma^{(2)}(u; \omega_1, \omega_2) = \exp\left(-\text{PV} \int_{\mathbb{R}} \frac{e^{ux}}{(1 - e^{\omega_1 x})(1 - e^{\omega_2 x})} \frac{dx}{x}\right),$$

where the principal value of the integral means $\text{PV}\int_{\mathbb{R}} = 2^{-1}(\int_{\mathbb{R}+i0} + \int_{\mathbb{R}-i0})$. Using the fact that $\text{PV}\int_{\mathbb{R}} dx/x^k = 0$ for $k > 1$, one can write

$$\gamma^{(2)}(u; \omega_1, \omega_2) = \exp\left(-\int_0^\infty \left(\frac{\sinh(2u - \omega_1 - \omega_2)x}{2\sinh(\omega_1 x)\sinh(\omega_2 x)} - \frac{2u - \omega_1 - \omega_2}{2\omega_1\omega_2 x}\right)\right) \frac{dx}{x}.$$

Comparing this expression with the hyperbolic gamma function $G_h(z; \omega)$ defined in [**Ru**], one can see that

$$G_h(z; \omega) = \gamma^{(2)}\left(\frac{1}{2}(\omega_1 + \omega_2) - iz; \omega\right), \quad \text{Re}(\omega_1), \text{Re}(\omega_2) > 0.$$

Changing in (A.3) the sign of the integration variable $x \to -x$ and simultaneously $\omega_k \to -\omega_k$, $u \to -u$, we find that

$$(A.4) \qquad \gamma^{(2)}(u; \omega_1, \omega_2) = \exp\left(+PV \int_{\mathbb{R}} \frac{e^{ux}}{(1 - e^{\omega_1 x})(1 - e^{\omega_2 x})} \frac{dx}{x} \right),$$

where $\mathrm{Re}(\omega_1), \mathrm{Re}(\omega_2) < 0$ and $\mathrm{Re}(\omega_1 + \omega_2) < \mathrm{Re}(u) < 0$.

The double sine function is defined as $S_2(u; \omega) = 1/\gamma^{(2)}(u; \omega)$ and its properties were described in detail in the Appendix of [**KLS**]. For the $\gamma^{(2)}$-function we have

$$\gamma^{(2)}(u; \omega_1, \omega_2)^* = \gamma^{(2)}(u^*; \omega_1^*, \omega_2^*), \quad \gamma^{(2)}\left(\frac{\omega_1 + \omega_2}{2} \pm u; \omega_1, \omega_2\right) = 1,$$

and $\gamma^{(2)}(au; a\omega_1, a\omega_2) = \gamma^{(2)}(u; \omega_1, \omega_2)$ for arbitrary complex $a \neq 0$. After such a rescaling in (A.4) with $a = 2\pi i$ one gets the definition of the hyperbolic gamma function given in [**R2**].

The asymptotics we are interested in for $\mathrm{Im}(\omega_1/\omega_2) > 0$ have the form

$$\lim_{u \to \infty} e^{\frac{\pi i}{2} B_{2,2}(u; \omega)} \gamma^{(2)}(u; \omega) = 1, \qquad \text{for } \arg \omega_1 < \arg u < \arg \omega_2 + \pi,$$

$$\lim_{u \to \infty} e^{-\frac{\pi i}{2} B_{2,2}(u; \omega)} \gamma^{(2)}(u; \omega) = 1, \qquad \text{for } \arg \omega_1 - \pi < \arg u < \arg \omega_2.$$

Appendix B. General multiple gamma functions

Barnes introduced a multiple zeta function as the following m-fold series [**Bar**]

$$\zeta_m(s, u; \omega) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{1}{(u + \Omega)^s}, \qquad \Omega = n_1 \omega_1 + \ldots + n_m \omega_m,$$

where $s, u \in \mathbb{C}$. This series converges for $\mathrm{Re}(s) > m$ under the condition that all ω_j lie in one half-plane defined by a line passing through zero. Because of the latter requirement, the sequences $n_1 \omega_1 + \ldots + n_m \omega_m$ do not have accumulation points on the finite plane for any $n_j \to +\infty$. It is convenient to assume for definiteness that $\mathrm{Re}(\omega_j) > 0$.

The function $\zeta_m(s, u; \omega)$ satisfies equations

$$(B.1) \qquad \zeta_m(s, u + \omega_j; \omega) - \zeta_m(s, u; \omega) = -\zeta_{m-1}(s, u; \omega(j)), \quad j = 1, \ldots, m,$$

where $\omega(j) = (\omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_m)$ and $\zeta_0(s, u; \omega) = u^{-s}$. The Barnes multiple gamma function is defined by the equality

$$\Gamma_m(u; \omega) = \exp(\partial \zeta_m(s, u; \omega)/\partial s)\big|_{s=0}.$$

It satisfies finite difference equations

$$(B.2) \qquad \Gamma_m(u + \omega_j; \omega) = \frac{1}{\Gamma_{m-1}(u; \omega(j))} \Gamma_m(u; \omega), \qquad j = 1, \ldots, m,$$

where $\Gamma_0(u; \omega) := u^{-1}$.

The multiple sine-function is defined as

$$S_m(u; \omega) = \frac{\Gamma_m(\sum_{k=1}^{m} \omega_k - u; \omega)^{(-1)^m}}{\Gamma_m(u; \omega)}.$$

It is more convenient to work with the hyperbolic gamma function

$$\gamma^{(m)}(u; \omega) = S_m(u; \omega)^{(-1)^{m-1}}$$

satisfying the equations

$$\gamma^{(m)}(u+\omega_j;\omega) = \gamma^{(m-1)}(u;\omega(j))\,\gamma^{(m)}(u;\omega), \qquad j=1,\ldots,m.$$

Note that the elliptic gamma function can be written as a special combination of four Barnes gamma functions of the third order [S9], and similar relations are valid for higher order elliptic gamma functions used in the present paper.

One can derive the integral representation [N]

$$\gamma^{(m)}(u;\omega) = \exp\left(-\mathrm{PV}\int_{\mathbb{R}}\frac{e^{ux}}{\prod_{k=1}^{m}(e^{\omega_k x}-1)}\frac{dx}{x}\right)$$

$$= \exp\left(-\frac{\pi\mathrm{i}}{m!}B_{m,m}(u;\omega) - \int_{\mathbb{R}+\mathrm{i}0}\frac{e^{ux}}{\prod_{k=1}^{m}(e^{\omega_k x}-1)}\frac{dx}{x}\right)$$

$$= \exp\left(\frac{\pi\mathrm{i}}{m!}B_{m,m}(u;\omega) - \int_{\mathbb{R}-\mathrm{i}0}\frac{e^{ux}}{\prod_{k=1}^{m}(e^{\omega_k x}-1)}\frac{dx}{x}\right),$$

where $\mathrm{Re}(\omega_k) > 0$ and $0 < \mathrm{Re}(u) < \mathrm{Re}(\sum_{k=1}^{m}\omega_k)$ and $B_{m,m}$ are multiple Bernoulli polynomials defined by the generating function

(B.3)
$$\frac{x^m e^{xu}}{\prod_{k=1}^{m}(e^{\omega_k x}-1)} = \sum_{n=0}^{\infty}B_{m,n}(u;\omega_1,\ldots,\omega_m)\frac{x^n}{n!}.$$

Infinite product representations for these functions have been derived in [N]. In particular, for $|p|, |q| < 1$ and $|r| > 1$ we have

$$\gamma^{(3)}(u;\omega) = e^{-\frac{\pi\mathrm{i}}{6}B_{3,3}(u;\omega)}\prod_{j,k=0}^{\infty}\frac{(1-e^{2\pi\mathrm{i}u/\omega_1}\tilde{q}^{j+1}r^{-(k+1)})(1-e^{2\pi\mathrm{i}u/\omega_2}p^j q^k)}{1-e^{2\pi\mathrm{i}u/\omega_3}\tilde{p}^{j+1}r^{-k}},$$

which is used in the main text after the reduction $\omega_3 = 2(\omega_1+\omega_2)$ (or $p = q^2$, $r = \tilde{q}^{-2}$, $\tilde{p} = e^{-\pi\mathrm{i}\omega_2/(\omega_1+\omega_2)}$).

The functions $m(\alpha)$ (4.3), (4.6), and (5.6) defining the free energy per edge as described in the main part of the paper are related to particular cases of the Lerch type generalization of the Barnes zeta-function:

$$\zeta_m(s,u;\beta;\omega) = \sum_{n_1,\ldots,n_m=0}^{\infty}\frac{\prod_{k=1}^{m}\beta_k^{n_k}}{(u+\Omega)^s}, \qquad \Omega = n_1\omega_1+\ldots+n_m\omega_m,$$

converging for all $|\beta_k| < 1$, or $\mathrm{Re}(s) > m$ and $|\beta_k| = 1$ (provided the same constraints on ω_j are valid as in the plain Barnes case). The univariate case, i.e. the proper Lerch zeta-function, is described, e.g., in [WW].

The function $\zeta_m(s,u;\beta;\omega)$ satisfies the following set of finite difference equations
(B.4)
$$\beta_j\zeta_m(s,u+\omega_j;\beta;\omega) - \zeta_m(s,u;\beta;\omega) = -\zeta_{m-1}(s,u;\beta(j);\omega(j)), \quad j=1,\ldots,m,$$

where $\omega(j) = (\omega_1,\ldots,\omega_{j-1},\omega_{j+1},\ldots,\omega_m)$, $\beta(j) = (\beta_1,\ldots,\beta_{j-1},\beta_{j+1},\ldots,\beta_m)$, and $\zeta_0(s,u;\beta;\omega) = u^{-s}$.

Similar to the Barnes case, one can easily derive the integral representations

$$\zeta_m(s,u;\beta;\omega) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-ut}}{\prod_{k=1}^{m}(1-\beta_k e^{-\omega_k t})}dt$$

$$= \frac{\mathrm{i}\Gamma(1-s)}{2\pi}\int_{C_H}\frac{(-t)^{s-1}e^{-ut}}{\prod_{k=1}^{m}(1-\beta_k e^{-\omega_k t})}dt,$$

where C_H is the Hankel contour encircling the half-line $[0,\infty)$ counterclockwise, and using them analytically continue ζ_m-function in s and β_k to different regions of parameters. The β_k-deformation of the Barnes multiple gamma function defined as $\Gamma_m(u;\beta;\omega) = \exp(\partial\zeta_m(s,u;\beta;\omega)/\partial s)\big|_{s=0}$ satisfies the finite difference equations

$$(B.5) \quad \Gamma_m(u+\omega_j;\beta;\omega)^{\beta_j} = \frac{1}{\Gamma_{m-1}(u;\beta(j);\omega(j))}\Gamma_m(u;\beta;\omega), \qquad j=1,\ldots,m,$$

where $\Gamma_0(u;\beta;\omega) := u^{-1}$.

When β_k are primitive roots of unity, $\beta_k^{n_k}=1$, $n_k=2,3,\ldots$, it is possible to rewrite $\zeta_m(s,u;\beta;\omega)$ as linear combinations of the standard Barnes zeta functions. It follows from the simple identity

$$\frac{1}{1-\beta_k z} = \frac{\prod_{l=0,2,\ldots,n_k-1}(1-\beta_k^l z)}{1-z^{n_k}}.$$

This allows expressing the functions like (4.5) as linear combinations of the standard Barnes gamma functions, which was used in the construction of infinite product representations of the functions $m(\alpha)$ (4.3), (4.6), and (5.6). In particular, function (4.5) is emerging from the $m=3$ case with the choice $\beta_1=\beta_2=1$, $\beta_3=-1$.

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