Max-Planck-Institut für Mathematik
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Fabio Perroni
De-Qi Zhang

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Fabio Perroni
De-Qi Zhang

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
University of Bayreuth
Germany

Department of Mathematics
National University of Singapore
Singapore

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PSEUDO-AUTOMORPHISMS OF POSITIVE ENTROPY ON THE BLOWUPS OF PRODUCTS OF PROJECTIVE SPACES

FABIO PERRONI AND DE-QI ZHANG

ABSTRACT. We use a concise method to construct pseudo-automorphisms f_n of the first dynamical degree $d_1(f_n) > 1$ on the blowups of the projective n-space for all $n \geq 2$ and more generally on the blowups of products of projective spaces. These f_n, for $n = 3$ have positive entropy, and for $n \geq 4$ seem to be the first examples of pseudo-automorphisms with $d_1(f_n) > 1$ on rational varieties of higher dimensions.

1. INTRODUCTION

We work over the field $\mathbb{C}$ of complex numbers.

A birational map $f : X \dashrightarrow X'$ of varieties is a *pseudo-isomorphism* if it is an isomorphism outside codimension-two closed subsets of X and X'. If we assume further $X = X'$, then f is called a *pseudo-automorphism*. By the minimal model program (which we will not use at all), a variety of dimension ≥ 3 may have more than two minimal models, but all of them are pseudo-isomorphic to each other. In dimension two, every pseudo-automorphism of a normal projective surface is an automorphism, and all the minimal models of a given surface are isomorphic to each other.

The main result of the paper is the following:

Theorem 1.1. *Let $w = w_{p,q,r}$ be the Coxeter element* (*unique up to conjugation*) *of the Weyl group $W(T_{p,q,r})$* (cf. 2.1). *Suppose that $r \geq 3$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Then there exist a blowup $X = X_{p,q,r}$ of $(\mathbb{P}^{r-1})^{p-1} = \mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1}$ at $q + r$ points P_i lying on a cuspidal curve $C \subset (\mathbb{P}^{r-1})^{p-1}$ of multi-degree r and a pseudo-automorphism $f_w : X \dashrightarrow X$ such that $(f_w)^*|H^2(X, \mathbb{Z})$ equals w. In particular, the first dynamical degree $d_1(f_w)$ of f_w is equal to the spectral radius $\rho(w)$ of w and larger than* 1 (*cf.* [4]).

Here C is the cuspidal curve of arithmetic genus one embedded in $(\mathbb{P}^{r-1})^{p-1}$ by the product map $\Phi_{|D_1|} \times \cdots \times \Phi_{|D_{p-1}|}$ for some Cartier divisors D_i of degree r on C. For instance, when $p = 2$, we can take $C = \{(1, z, z^2, ..., z^{r-2}, z^r)\}$ in affine coordinates.

When $p = 2$ and $n = q + r$, we can take $w = (12 \cdots n) r_{I,1}$, where the permutation is on the part e_j of the standard basis of the hyperbolic lattice $\Lambda_n = h_1\mathbb{Z} + \sum_{j=1}^{n} e_j\mathbb{Z}$

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(naturally identified with $H^2(X,\mathbb{Z})$) and $r_{I,1}$ is the reflection corresponding to the root $\alpha_{I,1} = h_1 - \sum_{j=1}^{r} e_j$ (cf. 2.1).

As a consequence of Theorem 1.1 and Corollary 4.10 late on, we have:

Corollary 1.2. (1) *When $\{p,q,r\} = \{2,3,7\}$ (as unordered sets) and $r \geq 3$, f_w is a pseudo-automorphism of the blowup of $(\mathbb{P}^{r-1})^{p-1}$ at $q+r$ points and $d_1(f_w) = 1.17628\ldots$ is the Lehmer number of the Lehmer polynomial $x^{10}+x^9-(x^7+x^6+x^5+x^4+x^3)+x+1$.*

(2) *When $\{p,q,r\} = \{2,4,5\}$ (as unordered sets) and $r \geq 3$, f_w is a pseudo-automorphism of the blowup of $(\mathbb{P}^{r-1})^{p-1}$ at $q+r$ points and $d_1(f_w) = 1.28064\ldots$ is the largest root of the Salem polynomial $x^8 - x^5 - x^4 - x^3 + 1$.*

(3) *When $\{p,q,r\} = \{3,3,4\}$ (as unordered sets), f_w is a pseudo-automorphism of the blowup of $(\mathbb{P}^{r-1})^{p-1}$ at $q+r$ points and $d_1(f_w) = 1.40127\ldots$ is the largest root of the Salem polynomial $x^6 - x^4 - x^3 - x^2 + 1$.*

(4) *If $(p,q,r) = (2,q,4)$ and $q \geq 5$, the topological entropy $h(f_w) = \log d_1(f_w) > 0$.*

The three types of $T_{p,q,r}$ in (1), (2) and (3) above are the only T-shaped minimal hyperbolic Coxeter diagrams (cf. [9, Table 5]). The three Salem numbers above are the smallest Salem numbers of degrees 10, 8 and 6, respectively. Hence one also realizes the Lehmer number as $d_1(f_w)$ of the pseudo-automorphism of X (a 10-point blowup of $\mathbb{P}^6$).

We remark that $h(f_w) = \log 1.28064\ldots$ is the smallest known topological entropy (> 0) of a pseudo-automorphism on a rational threefold which is not of product type. In [2], the authors have constructed a pseudo-automorphism f on the blowup of $\mathbb{P}^3$ at 2 points and 13 curves with $h(f) = \log 1.28064\ldots$. Our construction is different from theirs; for instance, f is induced by a quadratic birational map on $\mathbb{P}^3$, while the f_w in Corollary 1.2 (4) all come from cubic maps; see the end of Section 4 for more details.

When $(p,q,r) = (2,7,3)$, f_w is an automorphism of the blow-up of the projective plane at 10 points. This automorphism coincides with the one constructed in [1, Appendix] and [10, Theorem 1.1].

When $(p,q,r) = (2,6,4)$, w (or its power) seems to have been geometrically realized early by Coble and Cossec-Dolgachev (cf. [5, p. 39]).

The structure of the paper is the following. In Section 2 we first recall the definition of the Weyl group $W(p,q,r)$ and of Coxeter elements. We then introduce marked cubic curves, we define an action of $W(p,q,r)$ on the markings and we study some properties of this action. In Section 3 we state Theorem 3.1 which will be used in the proof of Theorem 1.1. In Section 4 we prove Theorems 3.1 and 1.1 in the case $p=2$ and we also study some aspects of the geometry of $X_{2,q,4}$ and of the pseudo-automorphism f_w. In the last Section 5 we complete the proof of Theorems 1.1 and 3.1 for all $p \geq 2$.

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2. Preliminaries

2.1. Weyl groups and roots (cf. [8])

Let $p \geq 2$, $q \geq 2$ and $r \geq 3$ be integers. Let $n := p+q+r-2$. We now define the *root system* L_n of type $T_{p,q,r}$. Let

$$\Lambda = \Lambda_n = \mathbb{Z}h_1 + \cdots + \mathbb{Z}h_{p-1} + \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{q+r}$$

be the lattice of rank $n+1$ with basis

$$h_1, h_2, \dots, h_{p-1}, e_1, \dots, e_{q+r}.$$

Late on in Section 4, we treat the case $p = 2$ and set $e_0 = h_1$. The following equations define an *inner product* on Λ (cf. [12, §3]):

$$\begin{aligned} h_i^2 = h_i \cdot h_i &= r-2 \ (1 \leq i < p), \\ h_i \cdot h_j &= r-1 \ (i \neq j), \ \ h_i \cdot e_j = 0, \\ e_i^2 = e_i \cdot e_i &= -1 \ (1 \leq i \leq q+r), \ e_i \cdot e_j = 0 \ (i \neq j). \end{aligned}$$

Set

$$\kappa := r \sum_{i=1}^{p-1} h_i - ((p-1)(r-1)-1) \sum_{j=1}^{q+r} e_j.$$

We will see that κ corresponds to the anti-canonical divisor of some blowup X of $(\mathbb{P}^{r-1})^{p-1}$ at $q+r$ points, and Λ_n is isomorphic to $H^2(X, \mathbb{Z})$. The *root system* (of type $T_{p,q,r}$) is

$$L_n := \kappa^{\perp} \cap \Lambda_n = \{\alpha \in \Lambda_n \,|\, \alpha \cdot \kappa = 0\}.$$

The *simple roots*:

$$\beta_1 = -h_1 + h_2, \ \beta_2 = -h_2 + h_3, \ \dots, \beta_{p-2} = -h_{p-2} + h_{p-1},$$

$$\alpha_0 = h_1 - \sum_{i=1}^{r} e_i, \ \ \alpha_1 = e_1 - e_2, \ \ \alpha_2 = e_2 - e_3, \ \ \dots, \ \ \alpha_{q+r-1} = e_{q+r-1} - e_{q+r}$$

form a basis of L_n. The corresponding Dynkin diagram is shown in Figure 1.

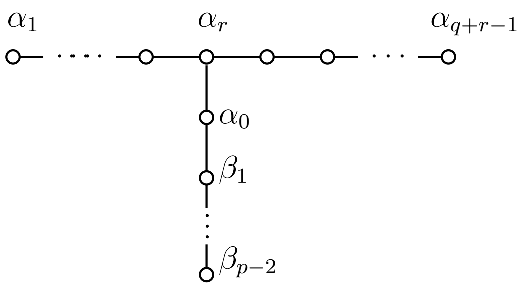

Figure 1.

Any $\alpha \in L_n$ with $\alpha^2 = -2$ determines the *reflection* $r_\alpha \in O(L_n)$ given by:

$$r_\alpha(x) = x + (x \cdot \alpha)\alpha.$$

For distinct $i, j \geq 1$, $r_{e_i - e_j}$ (resp. $r_{h_i - h_j}$) is the *transposition* interchanging the basis elements e_i and e_j (resp. h_i and h_j) while fixing the other e_k's and h_ℓ's. For any $1 \leq k < p$ and subset $I \subseteq \{1, 2, \ldots, n\}$ with $|\, I\, | = r$, we define the 'root'

$$\alpha_{I,k} = h_k - \sum_{i \in I} e_i \in L_n$$

and the reflection (called *a Cremona involution*):

$$r_{I,k} := r_{\alpha_{I,k}} .$$

Its action on Λ is given as follows:

$$\begin{aligned} r_{I,k}(h_k) &= h_k + (h_k \cdot \alpha_{I,k})\alpha_{I,k} = (r-1)h_k - (r-2)\sum_{i \in I} e_i, \\ r_{I,k}(h_i) &= h_i + (h_i \cdot \alpha_{I,k})\alpha_{I,k} = (r-1)h_k + h_i - (r-1)\sum_{j \in I} e_j \ \ (i \neq k), \\ r_{I,k}(e_i) &= e_i + \alpha_{I,k} \ \ (i \in I), \\ r_{I,k}(e_j) &= e_j \ \ (j \notin I). \end{aligned}$$

The *Weyl group*

$$W := W(p, q, r) = W(T_{p,q,r}) \subset O(L_n) \subset O(\Lambda_n)$$

is the subgroup of $O(L_n)$ generated by the reflections

$$r_{\beta_i}\, (1 \leq i \leq p-2),\ r_{\alpha_j}\ (0 \leq j < q + r)\, .$$

Elements in the set below are called (real) *roots*

$$\Delta_n := \{w(\beta_i),\, w(\alpha_j)\,|\, w \in W,\, 1 \leq i \leq p-2,\, 0 \leq j < q+r\}.$$

Definition 2.2. A Coxeter element w of W is the product $w = \prod_{i=1}^{n} r_{\gamma_i}$ where $\{\gamma_i\}_{i=1}^{n} = \{\beta_i\}_{i=1}^{p-2} \cup \{\alpha_j\}_{j=0}^{q+r-1}$ as sets. When $p = 2$, choose $(\gamma_1, \dots, \gamma_n) = (\alpha_1, \dots, \alpha_{q+r-1}, \alpha_0)$, we get $w = (12\dots n)r_{I,1}$ with $I = \{1, 2, \dots, r\}$, the product of a permutation (on $e_1, \dots, e_n$) and a Cremona involution. This Coxeter element will be also denoted by $w_{2,q,r}$.

Remark 2.3. Coxeter elements are conjugate to each other, since the Dynkin diagram $T_{p,q,r}$ is a tree (cf. [8, §3.16, §8.14]).

2.4. Marked cuspidal curves

Let

$$C = \{YZ^2 = X^3\} \subset \mathbb{P}^2$$

be the plane *cuspidal curve* (of arithmetic genus 1). Consider the subset

$$\Lambda_C \subset (\mathrm{Pic}^r(C))^{p-1} \times C^{q+r}, \text{ or equivalently } \Lambda_C \subset (\mathrm{Pic}^r(C))^{p-1} \times (\mathrm{Pic}^1(C))^{q+r}, \; r \geq 3$$

consisting of $(n+1)$-tuples

$$(D; c) := (D_1, \dots, D_{p-1}; c_1, \dots, c_{q+r})$$

with c_i contained in the smooth locus $C \setminus \{(0, 1, 0)\}$ of C.

Given $(D; c) \in \Lambda_C$, define a *marking* on C

$$\rho = \rho_{(D;c)} : \Lambda \to \mathrm{Pic}(C)$$

by setting

$$\rho(h_i) = D_i, \;\; \rho(e_j) = [c_j].$$

Here a marking is a group homomorphism $\rho\colon \Lambda \to \mathrm{Pic}(C)$ such that $\rho(h_i) \in \mathrm{Pic}^r(C)$ and $\rho(e_j) = [p_j]$, with $p_j \in C \setminus \{(0, 1, 0)\}$.

Remark 2.5. The $(n+1)$-tuple $(D; c) \in \Lambda_C$ and the marking $\rho = \rho_{(D;c)}$ on C determine each other uniquely.

As observed in [10, Proposition 4.1, Theorem 4.3], since $\mathrm{Aut}(C)$ acts transitively on

$$\mathrm{Pic}^0(C) \cong \mathbb{C}$$

and for any $u \in \Lambda$

$$\deg(\rho(u)) = \frac{1}{((r-1)(p-1)-1)}(\kappa \cdot u),$$

we have:

Lemma 2.6. *ρ is determined, up to isomorphism, by its restriction*

$$\rho_0 : \mathrm{Ker}(\deg \circ \rho) = L_n \to \mathrm{Pic}^0(C).$$

Here two markings ρ and ρ' are *isomorphic* if there is an $f \in \mathrm{Aut}(C)$ such that $f^* \circ \rho = \rho'$.

Set

$$U_C := \{(D; c) \in \Lambda_C \mid \rho_{(D;c)}(\alpha) \neq 0, \, \forall\, \alpha \in \, \Delta_n\}.$$

As observed in [11, Example 3], applying the defining condition of U_C to the roots $\alpha = e_i - e_j$, and $\alpha_{I,k}$ with $|I| = r$, we have:

Remark 2.7. If $(D; c) \in U_C$, then $c_i \neq c_j$ $(i \neq j)$, and $\sum_{i \in I} c_i \notin |D_k|$ $(\forall I, \, |\, I\,| = r, \forall k = 1, \ldots, p-1)$, i.e., no r points of $P(k)_i := \Phi_{|D_k|}(c_i) \in \mathbb{P}^{r-1}$, for k fixed, are contained in a hyperplane of $\mathbb{P}^{r-1}$. Here $\Phi_{|D_k|} \colon C \to \mathbb{P}^{r-1}$ is the embedding determined by D_k (cf. Lemma 4.1).

Definition 2.8. Using markings, there is an action of W on Λ_C. It is defined by the formula (cf. Remark 2.5):

$$\rho_{w(D;c)} := \rho_{(D;c)} \circ w \, .$$

Thus W acts on U_C because $w\Delta_n = \Delta_n$. Namely, we have:

Lemma 2.9. *If $w \in W$ and $(D; c) \in U_C$ then $w(D; c) \in U_C$.*

2.10. The correspondence between vectors of $\Lambda_n \otimes \mathbb{C}$ and markings on C

Let $v = \sum_{i=1}^{p-1} \xi_i h_i + \sum_{j=1}^{q+r} \eta_j e_j \in \Lambda_n \otimes \mathbb{C}$. We will define an $(n+1)$-tuple $(D^v; c^v)$ in the following way. Let $p(t) = (t, t^3, 1) \in C$ be a parametrization. Define t_j, c_j^v and D_i^v $(1 \leq i < p)$, by

$$\begin{aligned} r\, t_0 &= v \cdot h_1 = (r-2)\xi_1 + (r-1) \sum_{i=2}^{p-1} \xi_i, \\ t_j &= v \cdot e_j = -\eta_j \; (1 \leq j \leq q+r), \\ c_j^v &= p(t_j - t_0) \in C, \\ D_i^v &= [rp(0) + \, p(\xi_1) - p(\xi_i)] \in \mathrm{Pic}^r(C). \end{aligned} \tag{1}$$

In this way we get the $(n+1)$-*tuple*:

$$(D^v; c^v) := (D_1^v, \ldots, D_{p-1}^v; \, c_1^v, \ldots, c_{q+r}^v) \; \in \; (\mathrm{Pic}^r(C))^{p-1} \, \times \, C^{q+r}.$$

Then $(D^v; c^v)$ determines a marking ρ^v on C by setting $\rho^v(h_i) = D_i^v$, $\rho^v(e_j) = [c_j^v]$.

Lemma 2.11. *The restriction $\rho_0^v : L_n \to \mathrm{Pic}_0(C) \cong \mathbb{C}$ of ρ^v satisfies:*

$$\rho_0^v(u) = (u \cdot v)[p(1) - p(0)].$$

Hence for a root $\alpha \in \Delta_n$, we have $\rho^v(\alpha) = 0$ if and only if $\alpha \cdot v = 0$. In particular, the $(n+1)$-tuple $(D^v; c^v) \in U_C$ if and only if $0 \notin \Delta_n \cdot v$.

Proof. Direct computations show that the formula above is true for the elements β_i ($1 \leq i \leq p-2$), α_j ($0 \leq j < q+r$) as defined in 2.1. This proves the result since these elements form a basis of L_n. □

Remark 2.12. Conversely, for any $(n+1)$-tuple $(D; c)$, we can use the equations in 2.10 to define a vector v such that $(D; c) = (D^v, c^v)$.

Lemma 2.13. *For any $w \in W$, we have $\rho^v \circ w^{-1} = \rho^{w(v)}$, and $w^{-1}(D^v, c^v) = (D^{w(v)}; c^{w(v)})$.*

Proof. The first part follows from Lemma 2.11 and Remark 2.6, since $w \in O(\Lambda_n)$. The second follows from the first and Definition 2.8 (cf. Remark 2.5). □

Lemma 2.14. (*cf.* [10, Corollary 7.7]) *Let $u, v \in \Lambda_n \otimes \mathbb{C}$ with $\Delta_n \cdot u \not\ni 0 \notin \Delta_n \cdot v$. Then*

$$u = av + b\kappa \iff (D^u; c^u) \cong (D^v; c^v)\,.$$

Proof. The $(n+1)$-tuples are determined by their markings on C or equivalently by their restrictions on L_n (cf. Remarks 2.5 and 2.6), while the latter is determined by the inner product on $L_n = \Lambda \cap \kappa^{\perp}$ (cf. Lemma 2.11). The lemma follows since $\mathrm{Aut}(C)$ acts on $\mathrm{Pic}^0(C)$ by scalar multiplication. □

2.15. Let $w \in W$ with *spectral radius* $\rho(w) > 1$. When $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, the root system L_n is hyperbolic ($\kappa^2 < 0$). Hence $\rho(w)$ is a Salem number and $\det(xI - w) = S(x) \cdot C(x)$, where $S(x)$ is a Salem polynomial (cf. [9, Proposition 7.1]). We say that $\lambda \in \mathbb{C}$ is a *leading eigenvalue* if $S(\lambda) = 0$. So $\rho(w)$ is a leading eigenvalue. We say that $v \in L_n \otimes \mathbb{C}$ is a *leading eigenvector* if $w(v) = \lambda v$ with λ a leading eigenvalue.

Proposition 2.16. *Let $r \geq 3$. Let $v \in L_n \otimes \mathbb{C} = (\Lambda \otimes \mathbb{C}) \cap \kappa^{\perp}$ be an eigenvector of some $w \in W$ with eigenvalue λ. Then $0 \notin \Delta_n \cdot v$, i.e. $(D^v, c^v) \in U_C$ in the sense of Lemma* 2.11*, if either one of the following two conditions is satisfied.*

1. *w is a Coxeter element and v is a leading eigenvector.*
2. *λ is not a root of unity; and w has no periodic roots, i.e., no positive power of w fixes a root in Δ_n.*

Proof. The results follow from the calculation in [10, Theorems 2.6 and 2.7], as our diagram is bipartite; see also [9, Discussions before Theorem 1.3 and after Theorem 3.1]. Indeed, in (1), the root system L_n is hyperbolic of signature $(1, n-1)$. □

Remark 2.17. (1) happens exactly when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ (cf. [9, Table 5]).

3. Main Theorem

The following result will be used to prove Theorem 1.1. The proof is contained in the next sections.

Theorem 3.1. *Let w be an element of the Weyl group $W = W(p, q, r)$, with $r \geq 3$. Let $v \in L_n \otimes \mathbb{C} = (\Lambda \otimes \mathbb{C}) \cap \kappa^{\perp}$ be an eigenvector of w with $w(v) = \lambda v$. Assume that $0 \notin \Delta_n \cdot v$, i.e., $(D^v, c^v) \in U_C$ in the sense of Lemma 2.11. Then there exist a blowup $X = X_{p,q,r} \to (\mathbb{P}^{r-1})^{p-1} = \mathbb{P}^{r-1} \times \cdots \times \mathbb{P}^{r-1}$ at $q + r$ points P_i lying on the cuspidal curve $\Phi_{|D^v|}(C) \subset (\mathbb{P}^{r-1})^{p-1}$ of multi-degree r and a pseudo-automorphism $f_w : X \dashrightarrow X$ such that $(f_w)^*|H^2(X, \mathbb{Z})$ equals w. When $|\lambda| > 1$, λ equals $|\lambda|$, the spectral radius $\rho(w)$ of w and also the first dynamical degree $d_1(f_w)$ of f_w.*

4. Proof of Theorems when $p = 2$

We will frequently use the following result.

Lemma 4.1. *Let C be the cuspidal curve of arithmetic genus 1 and let D be a Cartier divisor on C of degree r.*

1. *If $r = 1$, then there is a unique smooth point P of C such that $P \sim D$ (linear equivalence).*
2. *If $r = \deg(D) \geq 3$, then the complete linear system $|D|$ is base point free and defines an embedding $\Phi_{|D|} : C \to \mathbb{P}^{r-1}$.*

Proof. By the Riemann-Roch theorem (true for all projective curves as in Hartshorne's book, Ch IV, Ex 1.9) and Serre duality for Cohen-Macaulay projective variety, we have $h^0(C, \mathcal{O}_C(D)) = r$. The result follows. Indeed, the second part of (1) is worked out in Hartshorne's book, Ch II, Example 6.11.4. $\square$

We now prove Theorem 3.1 when $p = 2$. In the definition of the lattice Λ_n and L_n, we set $p = 2$ and $e_0 = h_1$. Let $(D; c) \in U_C$ and consider the embedding

$$\Phi_{|D|} : C \to \mathbb{P}^{r-1}$$

given by the base-point free complete linear system $|D|$. Set $P_i := \Phi_{|D|}(c_i)$. Let

$$\pi_{(D;c)} : X = X_{(D;c)} \to \mathbb{P}^{r-1}$$

be the blowup of the n points P_i with $E_i = \pi_{(D;c)}^{-1}(P_i)$. For any $w \in W$, set $(D'; c') := w(D; c)$ and define similarly $\Phi_{|D'|}$, P'_i, $\pi_{(D';c')} : X' = X_{(D';c')} \to \mathbb{P}^{r-1}$, E'_i.

The result below should be well known but we work it out since we need to extend it to the case $p > 2$ in Section 5. Our statement also incorporates the marking on the curve C embedded in $\mathbb{P}^{r-1}$.

Proposition 4.2. *Let $p = 2$. Let $w \in W$ and $(D; c) \in U_C$. Define $(D'; c') := w(D; c)$. Consider the blowups*

$$\pi \colon X_{(D;c)} \to \mathbb{P}^{r-1}\,, \quad \pi' \colon X_{(D';c')} \to \mathbb{P}^{r-1}$$

at the points $P_i = \Phi_{|D|}(c_i)$ (resp. $P'_i = \Phi_{|D'|}(c'_i)$).

Then there exists a pseudo-isomorphism $f_w \colon X_{(D;c)} \dashrightarrow X_{(D';c')}$ such that

$$f_w^* \colon H^2(X_{(D';c')}, \mathbb{Z}) \to H^2(X_{(D;c)}, \mathbb{Z})$$

coincides with w after the identifications $[E'_j] = e_j = [E_j]$, $j \geq 1$, $\pi'^*[H] = e_0 = \pi^*[H]$. Here H is the hyperplane of $\mathbb{P}^{r-1}$ and E_i (resp. E'_i) is the exceptional divisor over P_i (resp. P'_i).

Proof. Since W is generated by the transpositions $r_{e_i - e_j}$ and the Cremona involution $r_{I,1}$, we need to prove the result only when w is one of them.

Our proof is top down: first construct a pseudo-isomorphism $X = X_{(D;c)} \dashrightarrow X'$ and then show that X' equals $X_{(D';c')}$, the blowup of $\mathbb{P}^{r-1}$ at the n points $\Phi_{|D'|}(c'_i)$.

Consider first the case $w = r_{I,1}$ with $I = \{1, 2, \dots, r\}$. Let $X_P = X_{P_1,\dots,P_r} \to \mathbb{P}^{r-1}$ be the blowup of the r points P_i ($1 \leq i \leq r$). Since these r points P_i span the whole space (cf. Remark 2.7), we can take the standard Cremona involution $\Psi_P = \Psi_{P_1,\dots,P_r} : \mathbb{P}^{r-1} \dashrightarrow \mathbb{P}^{r-1}$. Ψ_P is given by the linear system $|\mathcal{O}_{\mathbb{P}^{r-1}}(r-1) - (r-2)\sum_{i=1}^r P_i|$. A basis of this linear system is: $\sum_{j \neq i} H_j$, $i \in \{1, \dots, r\}$, where H_i is the hyperplane passing through $r-1$ points $\{P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_r\}$. The base locus of the linear system (the place where Ψ_P is not defined) is the union of $H_i \cap H_j$ ($1 \leq i < j \leq r$). Using new coordinate system so that $P_1 = [1 : 0 : \cdots : 0]$, …, $P_r = [0 : \cdots : 0 : 1]$, our Ψ_P is given by

$$\Psi_P : [X_1 : \cdots : X_r] \ \to \ [\frac{1}{X_1} : \cdots : \frac{1}{X_r}].$$

Let $E_i \subset X_P$ be the inverse image of P_i and $E_0 \subset X_P$ the total transform of a hyperplane of $\mathbb{P}^{r-1}$. Then it is known that Ψ_P lifts to an involutive pseudo-automorphism $\widetilde{\Psi_P} : X_P \to X_P$ exchanging E_i with the proper transform $H'_i \subset X_P$ of H_i (cf. [6]). This means that

$$\widetilde{\Psi_P}^* E_i = H'_i \sim E_0 - \sum_{j \neq i} E_j \quad \text{(linear equivalence)}$$

Denote by $e_i = [E_i] \in H^2(X_P, \mathbb{Z})$. Then

$$\widetilde{\Psi_P}^* e_i = [\widetilde{\Psi_P}^* E_i] = [E_0 - \sum_{j \neq i} E_j] = e_0 - \sum_{j \neq i} e_j = e_i + (e_0 - \sum_{i=1}^r e_i) = w(e_i).$$

By the definition of the Cremona involution in terms of the linear system,

$$\widetilde{\Psi_P}^* E_0 = (r-1)E_0 - (r-2)\sum_{i=1}^r E_i$$

and hence $\widetilde{\Psi_P}^* e_0 = w(e_0)$.

The blowup $X_P \to \mathbb{P}^{r-1}$ is centered at r smooth points $P_i = \Phi_{|D|}(c_i)$, and hence gives an isomorphism between the proper transform $C_X \subset X_P$ of C and C. Since $C = \Phi_{|D|}(C) \subset \mathbb{P}^{r-1}$ is a non-degenerate curve, it is not contained in any hyperplane H_i. Hence C_X is not contained in H'_i. Now $\deg(H'_i|C_X) = \deg(E_0 - \sum_{j\neq i} E_j)|C_X) = \deg(\mathcal{O}_{\mathbb{P}^{r-1}}(1)|C) - (r-1) = 1$ since $C = \Phi_{|D|}(C)$ is a curve of degree r in $\mathbb{P}^{r-1}$. Thus C_X meets H'_i only at one point and transversally. Since the Cremona involution $\Psi_P : \mathbb{P}^{r-1} \dashrightarrow \mathbb{P}^{r-1}$ blows up r smooth points P_i on C and collapses H'_i to a point called P'_i in the codomain $\mathbb{P}^{r-1}$, it maps $C \subset \mathbb{P}^{r-1}$ isomorphically to a curve C' in the codomain $\mathbb{P}^{r-1}$. As sets, we have $\{P'_i\} = \{P_i\}$. This C' is also the isomorphic image of $C_X \subset X_P$ via the map $X_P \overset{\widetilde{\Psi_P}}{\dashrightarrow} X_P \to \mathbb{P}^{r-1}$. This isomorphism of curves factors as $C_X \overset{\widetilde{\Psi_P}}{\to} C'_X \to C'$.

Let us calculate the very ample divisor $D' = \mathcal{O}_{\mathbb{P}^{r-1}}(1)|C'$ giving rise to the embedding $C' \subset \mathbb{P}^{r-1}$. By the above identification $C_X = C'_X = C'$ and further the identification $C_X = \Phi_{|D|}(C) = C$, we have

$$D' = E_0|C'_X = \widetilde{\Psi_P}^* E_0|C_X = ((r-1)E_0 - (r-2)\sum_{i=1}^{r} E_i)|C_X = (r-1)D - (r-2)\sum_{i=1}^{r} c_i = w(D)$$

(cf. Definition 2.8). Let $c'_i \in C'$ be the preimage of the point $P'_i \in \mathbb{P}^{r-1}$ via the embedding $\Phi_{|D'|} : C' \to \mathbb{P}^{r-1}$. Under the same identification $C = \Phi_{|D|}(C) = C_X = C'_X = C'$, we have (cf. Lemma 4.1):

$$C = C' \ni c'_i = P'_i = H'_i \,|\, C_X \sim (E_0 - \sum_{j\neq i} E_j)|C_X = D - \sum_{j\neq i} c_j = w(c_i).$$

For $r+1 \le j \le n$, the point P_j is not contained in the indeterminacy set: the union of $H_i \cap H_j$, otherwise, the r points $P_1, \dots, P_{i-1}, P_j, P_{i+1}, \dots, P_r$ are contained in the hyperplane H_i, contradicting Remark 2.7. Let Q_j $(r+1 \le j \le n)$ be the Ψ_P-image of P_j. For $1 \le i \le r$, set $Q_i = P_i$. Let $\pi_{(D;c)} : X = X_{(D;c)} \to \mathbb{P}^{r-1}$ be the blowup of the n points P_i, $E_0 \subset X$ the pullback of the hyperplane of $\mathbb{P}^{r-1}$, $E_i = \pi_{(D;c)}^{-1}(P_i)$ $(i \ge 1)$, and e_i $(i \ge 0)$ the cohomology class of E_i in $H^2(X, \mathbb{Z})$. Let $\pi' : X' \to \mathbb{P}^{r-1}$ be the blowup of the n points Q_i, $E'_0 \subset X'$ the pullback of the hyperplane of $\mathbb{P}^{r-1}$, $E'_i = (\pi')^{-1}(Q_i)$ $(i \ge 1)$, and e'_i $(i \ge 0)$ the cohomology class of E'_i in $H^2(X', \mathbb{Z})$. Then $\widetilde{\Psi_P}$ lifts to a pseudo-isomorphism $f_w : X \to X'$. Identify $H^2(X_P, \mathbb{Z})$ with its embedded image (via pullback) in $H^2(X, \mathbb{Z})$. By the calculation above and the construction, we have $f_w^* e'_i = w(e_i)$ for all $i \le r$ and $f_w^*(E'_j) = E_j$ $(j > r)$ (so $f_w^*(e'_j) = e_j = w(e_j)$), if we identify $H^2(X', \mathbb{Z}) = H^2(X, \mathbb{Z})$ by letting $e_i = e'_i$ $(i \ge 0)$; thus $f_w^* = w$.

By the argument above, if we set $(D'; c') = w(D; c)$, then the above $\pi' : X' \to \mathbb{P}^{r-1}$ is just the blowup of n points $P'_i = \Phi_{|D'|}(c'_i)$ on the curve $C' = \Phi_{|D'|}(C') \subset \mathbb{P}^{r-1}$, i.e., it is $\pi_{(D';c')}$. This proves Proposition 4.2 when w is a Cremona involution.

Next, consider the case where $w = r_{e_a - e_b}$ is a transposition of the basis elements e_a and e_b and fixing the others. Take an automorphism σ of $\mathbb{P}^{r-1}$ interchanging two points P_a and P_b. Let $C' = \sigma(C) \subset \sigma(\mathbb{P}^{r-1}) = \mathbb{P}^{r-1}$. Set $P'_a = P_a$, $P'_b = P_b$ and $P'_j = \sigma(P_j)$ $(j \neq a, b)$. Let $X' \to \mathbb{P}^{r-1}$ be the blowup of the n points P'_i with E'_i the inverse of P'_i. Then σ lifts to an isomorphism $f_w : X \to X'$. We see that $f_w^* = w$ if we identify $H^2(X', \mathbb{Z}) = H^2(X, \mathbb{Z})$ by letting $[E'_i] = e_i = [E_i]$ as above. Define $(D'; c')$ so that $D' = D$, $c'_a = c_a, c'_b = c_b$ and $c'_j = \sigma(c_j)$ $(j \neq a, b)$. Using the identification $C = C_X = C'_X = C'$ as above, we obtain $(D'; c') = w(D; c)$. This implies Proposition 4.2 as in the previous case. □

4.3. Proof of Theorem 3.1 when $p = 2$

Given v as in Theorem 3.1, we define $(D^v; c^v)$ as in 2.10 (cf. Lemma 2.11). Set $(D; c) = (D^v; c^v)$. Then we get the pseudo-isomorphism $f_w : X = X_{(D;c)} \to X' = X_{(D';c')}$ as in Proposition 4.2 with $(D'; c') = w(D; c)$ and $f_w^* = w$ on $H^2(X', \mathbb{Z})$ identified with $H^2(X, \mathbb{Z})$ by letting $[E'_i] = [E_i]$ and $[(\pi')^* H'] = [\pi^* H]$. By Lemmas 2.13 and 2.14,

$$(D'; c') = w(D^v; c^v) = (D^{w^{-1}(v)}; c^{w^{-1}(v)}) = (D^{\lambda^{-1}v}; c^{\lambda^{-1}v}) = (D^v; c^v) = (D; c)$$

(up to the action of $\mathrm{Aut}(C)$). Thus we get an isomorphism between $\pi_{(D';c')} : X_{(D';c')} \to \mathbb{P}^{r-1}$ in Proposition 4.2 and $\pi_{(D;c)} : X = X_{(D;c)} \to \mathbb{P}^{r-1}$ so that f_w is a pseudo-automorphism. This proves Theorem 3.1. Indeed, for the final part (when $|\lambda| > 1$), the Coxeter system is hyperbolic, so λ is the largest root of a Salem polynomial and also the spectral radius $\rho(w)$ of w (cf. [9, Proposition 7.1]). Thus $d_1(f_w) = \rho(f_w^* | H^2(X, \mathbb{Z})) = \rho(w) = \lambda$ by the definition of $d_1(f_w)$ (cf. [4]).

4.4. Proof of Theorem 1.1 when $p = 2$

Theorem 1.1 (1) follows from Proposition 2.16, Theorem 3.1 and its proof, by taking λ in Proposition 2.16 to be the spectral radius of w; see also 2.15 and Remark 2.17.

4.5. Concrete construction of f_w on $X_{2,q,r}$ as in Theorem 1.1

We first construct a pseudo-automorphism f such that $f_* = w$ where $w = (12 \cdots n) r_{I,1}$ is a Coxeter element of the root system L_n of type $T_{2,n-r,r}$ (cf. Definition 2.2). Then $f_w = f^{-1}$ meets the requirement. To do so, take an eigenvector v of w such that $w(v) = \lambda v$ and λ is the spectral radius of $w \in O(L_n)$ (which turns out to be $d_1(f_w)$, since $f_w^* = w$).

Define the $(n+1)$-tuple $(D; c) = (D^v; c^v)$ as in 2.10. Let $P_i = \Phi_{|D|}(c_i) \in \Phi_{|D|}(C) \subset \mathbb{P}^{r-1}$. Choose a new coordinate system of $\mathbb{P}^{r-1}$ such that $P_1 = [1 : 0 : \cdots : 0], \ldots, P_r = [0 : \cdots : 0 : 1]$. Consider the standard Cremona involution:

$$\gamma : \mathbb{P}^{r-1} \to \mathbb{P}^{r-1},\ [X_1 : \cdots : X_r] \mapsto [\frac{1}{X_1} : \cdots : \frac{1}{X_r}].$$

Let $\pi = \pi_{(D;c)} : X = X_{(D;c)} \to \mathbb{P}^{r-1}$ be the blowup at the n points P_i and let $E_i = \pi^{-1}(P_i)$ and $E_0 \subset X$ the total transform of a hyperplane of $\mathbb{P}^{r-1}$. Then by the proof of Proposition 4.2 and Theorem 3.1, there is a projective automorphism g of $\mathbb{P}^{r-1}$ such that $g \circ \gamma$ lifts to a pseudo-isomorphism $f = f_{(12\cdots n)} \circ f_{r_{I,1}} : X \to X'$ where $f_{(12\cdots n)}$ is the lifting of g and so is an isomorphism. Moreover, $f_* = (f_{(12\cdots n)})_*(f_{r_{I,1}})_* = w$ on $H^2(X, \mathbb{Z})$ identified with $H^2(X', \mathbb{Z})$ by letting $[E_i] = e_i = [E_i']$. Recall that $X' = X_{(D';c')}$ is the blowup at n points $P_i' = \Phi_{|D'|}(c_i') \in \mathbb{P}^{r-1}$ and since v is an eigenvector we have

$$X' = X_{(D';c')} = X_{w(D^v;c^v)} = X_{(D^{w^{-1}(v)};c^{w^{-1}(v)})} = X_{(D^{\lambda^{-1}v};c^{\lambda^{-1}v})} = X_{(D^v;c^v)} = X$$

(up to isomorphism). Now the identification $(D'; c') = w(D; c)$ with $(D; c)$ and the fact that $w(c_i) = c_{i+1} \pmod n$ for $i > r$ force $(g \circ \gamma)(P_i) = P_{i+1} \pmod n$. Conversely, if we can find g as above then we can forget about the eigenvector v or so, and straightaway say that $(g \circ \gamma)^{-1}$ lifts to a pseudo-automorphism f_w on the blowup $X \to \mathbb{P}^{r-1}$ at the n points P_i which satisfies the conclusion of Theorem 1.1.

Remark 4.6. When $p = 2$, our f_w in Theorem 1.1 lifts to an isomorphism. Indeed, by the construction in 4.5, it is enough to lift $f = f_{(12\cdots n)} \circ f_{r_{I,1}} : X \dashrightarrow X$ to an isomorphism. By [6, VI, Lemma 1], there is a further blowup $\sigma : X_1 \to X$ and a blowup $X_2 \to \mathbb{P}^{r-1}$ such that $f_{r_{I,1}}$ lifts to an isomorphism $f_1 : X_1 \to X_2$. We can take a corresponding blowup $X_3 \to X$ of the images of the centers lying below the exceptional divisors on X_2 to lift the isomorphism $f_{(12\cdots n)}$ to an isomorphism $f_2 : X_2 \to X_3$. Now the isomorphism $f_3 = f_2 \circ f_1 : X_1 \to X_3$ (resp. $f_4 = f_3^{-1}$) is a lifting of f (resp. $f^{-1} = f_w$).

4.7. On the geometry of $X_{2,q,4}$ with $q \geq 5$

In this subsection, we prove the following:

Proposition 4.8. *Let $w = w_{2,q,4}$ ($q \geq 5$). Let f_w be the pseudo-automorphism of $X := X_{2,q,4}$ in Theorem 1.1 and $C_X \subset X$ the proper transform of $C_D := \Phi_{|D|}(C) \subset \mathbb{P}^3$. Then:*

1. *f_w stabilizes the cuspidal curve C_X and permutes members F_t' of the rational pencil $|-K_X/2|$ each of which is a strict transform of an irreducible quadric hypersurface $F_t \subset \mathbb{P}^3$ with $F_t' \cap F_{t'}' = C_X$ ($t \neq t'$). Moreover, all the quadrics F_t, except two: F_i ($i = 1, 2$), are smooth.*
2. *f_w stabilizes the blowup F_1' of the quadric cone F_1 whose vertex P is the cusp of C_D. When $q = 5$, every effective divisor E with the class $[E]$ fixed by f_w^* is a union of members in $|-K_X/2|$.*
3. *$S := F_1'$ (the $(q+4)$-point blowup of the quadric F_1) is disjoint from the indeterminacy of f_w. The restriction $f_S := f_w \,|\, S$ is a well defined automorphism of S with $d_1(f_S) = d_1(f_w) > 1$.*

Proof. By the proof, $f_w(C_X) = C_X$ holds in Theorem 1.1 for any (p, q, r). Since C_D has arithmetic genus 1 and degree 4, it is contained in a linear system $|\mathcal{I}(2)|$ of quadrics of dimension ≥ 1. This follows from the long cohomology sequence associated to

$$0 \to \mathcal{I}(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{C_D}(2) \to 0\,.$$

Alternatively, we may assume that $C_D = \{(1, z, z^2, z^4)\}$ in new coordinates. By a direct calculation, $|\mathcal{I}(2)|$ is a pencil spanned by $F_1 := \{X_2^2 = X_1X_3\}$ and $F_2 := \{X_3^2 = X_1X_4\}$, every member $F_t \neq F_i$ $(i = 1, 2)$ is smooth, and $(\mathrm{Sing}(F_i)) \cap (C_D \setminus \mathrm{Sing}(C_D)) = \emptyset$.

Let $\pi : X \to \mathbb{P}^3$ be the blowup at the $q + 4$ points P_i as in Theorem 1.1 with $E_i = \pi^{-1}(P_i)$ and $E_0 \subset X$ the total transform of a hyperplane of $\mathbb{P}^3$. For $F \in |\mathcal{I}(2)|$, the proper transform F' of F satisfies $F' \sim 2E_0 - \sum_{i=1}^{q+4} E_i$ (linear equivalence), so $-K_X = -(\pi^* K_{\mathbb{P}^3} + 2\sum_{i=1}^{q+4} E_i) \sim 2F'$. Since $-K_X$ is preserved by f_w, we have $2(f_w^* F' - F') \sim 0$, so $f_w^* F' - F' \sim 0$, because the rational manifold X is simply connected and hence cohomologous divisors are just linear equivalent divisors.

F', or equivalent $F = \pi(F')$, is irreducible. Otherwise, $F = L_1 \cup L_2$ with two hyperplanes L_i. Since all $(q+4)$ points $P_i \in C_D$ belong to F, we may assume that L_1 contains 5 of P_i. This contradicts Remark 2.7 (cf. Proposition 2.16). For two distinct such F, say F_t, $F_{t'}$, the intersection $F_t \cap F_{t'}$ includes C_D and hence equals C_D by comparing the degree. This proves (1).

If E is a divisor whose class $[E]$ is fixed by f_w^* (e.g., $E = aF'$), then either $\dim |E| \leq 0$ or $|E|$ is composed of a pencil, otherwise, f_w would descend to a surface or threefold automorphism of the first dynamical degree equal to 1 via a fibration with general fiber a curve or a point, contradicting the fact that $d_1(f_w) > 1$ (cf. [3]). In particular, $|aF'|$ $(a > 0)$ is composed of a pencil (necessarily parametrized by a curve $B \cong \mathbb{P}^1$ because the irregularity $q(B) \leq q(X) = 0$) stabilized by f_w. The induced action of f_w on $B \cong \mathbb{P}^1$ has at least one fixed point. Namely, at least one $F'_0 \in |F'|$ is f_w-stable.

When $q = 5$, the characteristic polynomial of $f_w^*|H^2(X, \mathbb{Z})$ has the form $\phi_8(x)(x+1)(x-1)$ (cf. [9, Table 5]), where $x-1$ corresponds to the f_w-invariant class $\kappa = [-K_X] = 2[F']$. If E is an integral divisor with $f_w^*[E] = [E]$ then $bE \sim aF'$ for some coprime integers a, b. Since $[F'] \cdot [F'] = (\kappa)^2/4 = 4 - q = -1$, we get $b = \pm 1$. In particular, every effective divisor E with $[E]$ fixed by f_w^* is a member of the pencil $|aF'|$ and hence equal to a union of $F'_t \in |F'|$ by the Stein factorization.

As in 4.6 or [6], $f_w : X \dashrightarrow X$ (and $f_{r_{I,1}}$) is well-defined outside the proper transforms $H'_{ij} := H'_i \cap H'_j$ of the lines $H_{ij} := H_i \cap H_j$ $(1 \leq i < j \leq 4)$ (cf. the notation of Proposition 4.2). Our $F'_1 \in |F'|$ has the characteristic property as being the only singular member in

$|F'|$ whose singular point $\pi^{-1}(P)$ is the cusp of C_X (P being the vertex of F_1). Since

$$H'_i.H'_j.F'_t = (E_0 - \sum_{i\neq\ell=1}^{4} E_\ell).(E_0 - \sum_{j\neq\ell=1}^{4} E_\ell).(2E_0 - \sum_{\ell=1}^{q+4} E_\ell) = 2(E_0^3) - \sum_{i,j\neq\ell=1}^{4} (E_\ell)^3 = 2-2 = 0$$

H'_{ij} is either contained in F'_t or disjoint from F'_t. If H'_{ij} is contained in F'_1, then the line H_{ij} is contained in the cone F_1 and passes through its vertex P, and H_i intersects the non-degenerate curve C_D at its cusp P and three points P_j ($j \neq i, 1 \leq j \leq 4$), hence $4 = \deg(C_D) = C_D.H_i \geq 2+3$, a contradiction. Thus no H'_{ij} intersects F'_1. Our f_w is well defined at $S := F'_1$, and $f_w(F'_1) \in |F'|$ satisfies the same characteristic property as F'_1 and hence equals F'_1. The isomorphism $f_4 : X_1 \to X_3$ in 4.6 (lifting f_w) is just f_w around S and hence $f_S = f_w|S$ is an automorphism of S. This proves (2).

Using the lifting f_4 of f_w, we have $f_S^*(L_{f_w}|S) = d_1(f_w)L_{f_w}|S$, where L_{f_w} is the eigenvector of $f_w^*|H^2(X,\mathbb{Z}) = w$ corresponding to the eigenvalue $d_1(f_w) = \rho(w)$. To prove (3), we only need to show $L_{f_w}|F'_t \neq 0$. To do so, write $L_{f_w} = v = \sum_{i=0}^{q+4} v_i e_i$ as in 2.10. Then

$$(L_{f_w}|F'_t).(E_i|F'_t) = L_{f_w}.E_i.F'_t = (\sum_{j=0}^{q+4} v_j E_j).E_i.(2E_0 - \sum_{j=1}^{q+4} E_j) = -v_i(E_i)^3 = -v_i\,.$$

Since e_0 is not an eigenvector of $w_{2,q,4}$, it follows that $L_{f_w}|F'_t \neq 0$. □

Remark 4.9. The Salem number 1.28064... is also realized in [10] as $d_1(f_{13})$ of an automorphism f_{13} on the blowup X_{13} of $\mathbb{P}^2$ at 13 points on a cubic curve. The map f_S in Proposition 4.8 (3) with $q = 5$ is not the descent of f_{13} because the characteristic polynomial of $f_{13}^*|H^2(X_{13},\mathbb{Z})$ is of the form $\phi_8(x)(x^4+1)(x^2-1)$. Since $W(2,5,4)$ can be embedded in $W(2,7,3)$, our Proposition 4.8 (3) with $q = 5$ is compatible with [10].

As a consequence of Theorem 1.1 and Proposition 4.8, we have:

Corollary 4.10. *Let $f_w\colon X_{2,q,4} \dashrightarrow X_{2,q,4}$ be as in Theorem* 1.1 *with $q \geq 5$ and let $S = F'_1$ be as in Proposition* 4.8. *Then the topological entropy $h(f_w)$ of f_w satisfies:*

$$h(f_w) = h(f_S) = \log d_1(f_S) = \log d_1(f_w) > 0.$$

Proof. By the Poincaré duality and noting that $d_1(f_w)$ is a Salem number, we have $d_1(f_w) = d_2(f_w)$. Thus $\log d_1(f_w) \geq h(f_w) \geq h(f_S) = \log d_1(f_S) = \log d_1(f_w)$ (cf. [4], [7], [13], and taking an equivariant resolution of S), and we are done. □

5. Proof of Theorems for all $p \geq 2$

We now prove Theorem 3.1 for $p \geq 3$. Let $w \in W$. Let $(D;c) \in U_C$. Denote by $(D';c') = w(D;c)$. Consider the embedding:

$$\Phi_{(D;c)} : C \to (\mathbb{P}^{r-1})^{p-1},\ \ (x \mapsto (\Phi_{|D_1|}(x), \ldots, \Phi_{|D_{p-1}|}(x))).$$

Set $P_j = (\Phi_{|D_1|}(c_j), \dots, \Phi_{|D_{p-1}|}(c_j))$. Let

$$\pi_{(D;c)} : X = X_{(D;c)} \to (\mathbb{P}^{r-1})^{p-1}$$

be the blowup at the $q+r$ points P_j with $E_j = \pi^{-1}_{(D;c)}(P_j)$. Similarly, we define $\Phi_{(D';c')}$, P'_j, $\pi_{(D';c')} : X' = X_{(D';c')} \to (\mathbb{P}^{r-1})^{p-1}$, E'_j.

For the result below, see [12, Theorem 1]. Our statement also incorporates the marking on the curve C embedded in $(\mathbb{P}^{r-1})^{p-1}$.

Proposition 5.1. *Suppose that $w \in W$ and $(D;c) \in U_C$. Then there is a pseudo-isomorphism $f_w : X \to X' = X_{(D',c')}$ such that $f_w^* = w$ if we identify $H^2(X,\mathbb{Z}) = \sum_{i=1}^{p-1} \mathbb{Z}h_i + \sum_{j=1}^{q+r} \mathbb{Z}e_j = H^2(X',\mathbb{Z})$ by letting $[E_j] = e_j = [E'_j]$ $(j \geq 1)$ and $[\pi^*_{(D;c)}\mathcal{O}_{\mathbb{P}_i^{r-1}}(1)] = h_i = [\pi^*_{(D';c')}\mathcal{O}_{\mathbb{P}_i^{r-1}}(1)]$ where $\mathbb{P}_i^{r-1}$ is the i-th factor of the product $(\mathbb{P}^{r-1})^{p-1}$.*

Proof. The proof is similar to Proposition 4.2. Since the Weyl group is generated by the reflections $r_{h_i-h_j}$ (resp. $r_{e_i-e_j}$) corresponding to the exchange of the factors $\mathbb{P}_i^{r-1}$ and $\mathbb{P}_j^{r-1}$ (resp. P_i and P_j of the blowup), and the Cremona involution r_{α_0} with $\alpha_0 = h_1 - \sum_{i=1}^r e_i$, we have only to consider the case $w = r_{\alpha_0}$. This w is realized by the lifting $f_w : X \to X'$ of the following standard (geometric) Cremona involution (cf. [12, Lemma in §3]):

$$\begin{aligned}\Psi\ :\ &(\mathbb{P}^{r-1})^{p-1} \ \to\ (\mathbb{P}^{r-1})^{p-1},\\ &([X_1 : \cdots : X_r], [Y_1 : \cdots : Y_r], \dots, [Z_1 : \cdots : Z_r]) \ \mapsto\\ &([\frac{1}{X_1} : \cdots : \frac{1}{X_r}], [\frac{Y_1}{X_1} : \cdots : \frac{Y_r}{X_r}], \dots, [\frac{Z_1}{X_1} : \cdots : \frac{Z_r}{X_r}]).\end{aligned}$$

Here, with new coordinates, we may assume that $P_1, \dots, P_r$ are images of the standard vertices $[1 : 0 : \cdots : 0], \dots, [0 : \cdots : 0 : 1]$ in $\mathbb{P}^{r-1}$ via the diagonal embedding $\mathbb{P}^{r-1} \to (\mathbb{P}^{r-1})^{p-1}$ $(P \mapsto (P, \dots, P))$, and $X \to (\mathbb{P}^{r-1})^{p-1}$ is the blowup of $q+r$ points P_i and $X' \to (\mathbb{P}^{r-1})^{p-1}$ is the blowup of $Q_1 := P_1, \dots, Q_r := P_r$ and $Q_j := \Psi(P_j)$ $(r < j \leq q+r)$. By the form of the map,

$$f_w^* h_1 = (r-1)h_1 - (r-2)\sum_{i=1}^{r}[E_i] = w(h_1)$$

if we identify $H^2(X,\mathbb{Z}) = H^2(X',\mathbb{Z})$ (here and below) by letting $h_i = h'_i$, $[E_j] = e_j = [E'_j]$. Here and below $E_i \subset X$ (resp. $E'_i \subset X'$) is the inverse of P_i (resp. Q_i), h_i (resp. h'_i) is the (cohomology class of) total transform of the hyperplane $\mathcal{O}_{\mathbb{P}_i^{r-1}}(1)$ of the i-th factor of the domain (resp. codomain) of Ψ. From the form of Ψ, we have also

$$f_w^* h_i = (r-1)h_1 - (r-2+1)\sum_{i=1}^{r}[E_i] + h_i = w(h_i)\ (1 \leq i < p)$$

where the h_i's in the middle of the display and the extra 1 in $r-2+1$ correspond to the numerators $Y_1, \dots, Z_r$ in the defining rational functions of Ψ. As observed in [12, Lemma in §3], using the affine coordinates $((x_2,\dots,x_r),(y_2,\dots,y_r),\dots,(z_2,\dots,z_r))$ of $(\mathbb{P}^{r-1})^{p-1}$ around the point P_1 (the diagonal image of the point $[1:0:\cdots:0]\in\mathbb{P}^{r-1}$), the map $X\xrightarrow{f_w}X'\to(\mathbb{P}^{r-1})^{p-1}$ takes the following form around E_1:

$$\begin{aligned}E_1 \ni\ &((x_2,\dots,x_r),(y_2,\dots,y_r),\dots,(z_2,\dots,z_r)\\ &\mapsto([0:\frac{1}{x_2}:\cdots:\frac{1}{x_r}],[1:\frac{y_2}{x_2}:\cdots:\frac{y_r}{x_r}],\dots,[1:\frac{z_2}{x_2}:\cdots:\frac{z_r}{x_r}]).\end{aligned}$$

Hence for the hyperplane $H_{1i}\subset(\mathbb{P}^{r-1})^{p-1}$ defined by $X_i=0$, its proper transform $H'_{1i}\subset X'$ satisfies (when $i=1$) $f_w^*H'_{1i}=E_i$. Since Ψ is an involution and by a similar observation, for all $1\le i\le r$, we have (noting that $[H_{1i}]=h_1$):

$$[f_w^*E'_i]=[H'_{1i}]=h_1-\sum_{i\neq j=1}^{r}[E_j]=w(e_i)$$

if we identify $H^2(X,\mathbb{Z})=H^2(X',\mathbb{Z})$ as above. The equality $f_w^*e'_j=e_j$ $(r<j\le q+r)$ is by the definition of Q_j. Thus we have $f_w^*=w$ on $H^2(X',\mathbb{Z})$ (identified with $H^2(X,\mathbb{Z})$).

To check that $X'\to(\mathbb{P}^{r-1})^{p-1}$ is just the blowup of points P'_i determined by the $(n+1)$-tuple $w(D;c)$, we can argue as in Proposition 4.2. Indeed, let $C_X\subset X$ be the proper transform of $C=\Phi_{(D;c)}(C)\subset(\mathbb{P}^{r-1})^{p-1}$ (which is isomorphic to C since we blow up only smooth points on C). Then for $1\le i\le r$, we have $\deg(H'_{1i}|C_X)=\deg(H_{1i}|\Phi_{(D;c)}(C))-\deg(\sum_{i\neq j=1}^{r}E_j)|C_X=r-(r-1)=1$. Hence C_X meets H'_{1i} at only one point and transversally. So the map $X\xrightarrow{f_w}X'\to(\mathbb{P}^{r-1})^{p-1}$ collapses H'_{1i} to a smooth point Q_i on the image C' of C which is contained in the codomain $(\mathbb{P}^{r-1})^{p-1}$ of the Cremona involution Ψ. With the identification $C'=C_X=\Phi_{(D;c)}(C)=C$, we have

$$[\mathcal{O}_{\mathbb{P}_i^{r-1}}(1)|C']=((r-1)h_1-(r-1)\sum_{i=1}^{r}[E_i]+h_i)|C_X=w(h_i)|C=w(D)_i=D'_i$$

which is a degree $r\ge 3$ (very ample) divisor and embeds C' onto C'_i ($\subset\mathbb{P}_i^{r-1}$, the i-th factor of the codomain of Ψ). With the identification $C'=C_X=\Phi_{(D;c)}(C)=C=C_1:=\Phi_{|D_1|}(C)$ ($\subset\mathbb{P}_1^{r-1}$, the first factor of the domain of Ψ), the point $Q_i\in C'$ is given by

$$[H'_{1i}|C_X]=[H_{1i}|C_1]-\sum_{i\neq j=1}^{r}E_j|C_X=D_1-\sum_{i\neq j=1}^{r}c_j=w(c_i)\in C.$$

Hence Q_i is one of P'_i $(1\le i\le r)$ defined before Proposition 5.1. By the construction, $Q_j=\Psi(P_j)=P'_j$ for $r<j\le q+r$. Thus $X'=X_{(D';c')}$. This proves Proposition 5.1. □

5.2. Proof of Theorems 3.1 and 1.1

The same argument for $p=2$ now works for all $p\ge 2$, but with Proposition 4.2 replaced by Proposition 5.1.

5.3. On the geometry of $X_{3,q,3}$ with $q\ge 4$

Proposition 5.4. *Let* $X_{3,q,3}$ $(q \geq 4)$ *and* f_w *be as in Theorem* 1.1. *Then* $X_{3,q,3}$ *is the blowup of* $\mathbb{P}^2 \times \mathbb{P}^2$ *at* $q+3$ *points, and* f_w *permutes members of the linear system* $|-K_X/3|$ *of dimension* ≥ 2. *When* $q = 4$, *every divisor* E *with class* $[E]$ *fixed by* f_w^* *satisfies* $E \sim a(-K_X/3)$ (*linear equivalence*) *for some* $a \in \mathbb{Z}$.

The proof is similar to that of Proposition 4.8 and is left to the reader.

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Department of Mathematics, University of Bayreuth, Germany
E-mail address: `fabio.perroni@uni-bayreuth.de`

Department of Mathematics, National University of Singapore, Singapore
E-mail address: `matzdq@nus.edu.sg`