On the integral cohomology of wreath products

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Abstract. Under mild conditions on the space X, we describe the additive structure of the integral cohomology of the space $X^p \times_{C_p} EC_p$ in terms of the cohomology of X. We give weaker results for other similar spaces, and deduce various corollaries concerning the cohomology of finite groups.

0. Introduction.

Let S be a group with a fixed action on a finite set Ω. By the wreath product $G \wr S$ of a group G with S we mean a split extension with kernel G^Ω, quotient S, and with the S-action on G^Ω given by permuting the copies of G. Our main interest is the integral cohomology of finite groups of the form $G \wr S$. We work in greater generality however, because it is no more difficult to study the cohomology of spaces of the form $X^\Omega \times_S E$, where E is an S-free S-CW-complex, and X is a CW-complex of finite type. The mod-p cohomology of certain such spaces plays a crucial role in Steenrod's definition of the reduced power operations [29]. Building on work of Steenrod, Nakaoka described the cohomology of such spaces with coefficients in any field [23]. The point about working over a field is that then the cellular cochain complex for X is homotopy equivalent to the cohomology of X, viewed as a complex with trivial differential. If the integral cohomology of X is free, then a similar result holds in this case. Evens used this to study the cohomology of (the classifying space of) the Lie group $U(m) \wr \Sigma_n$ in the course of his work on Chern classes of induced representations [13]. The study of the integral cohomology in the case when $H^*(X)$ is not free is much harder. The pioneers in this case were Evens and Kahn, who made a partial study of the important special case of $X^p \times_{C_p} EC_p$. We complete the study of this case in Section 4 below, which could be viewed as both an extension of and a simplification of [15, section 4]. Many, but not all, of our results are corollaries of this work. Our paper has the following structure.

In Section 1 we give some algebraic background. Most of this material is well-known, although we have not seen Lemma 1.4 stated explicitly before, and we believe that Lemma 1.1 is original. This lemma, which compares spectral sequences coming from double complexes consisting of 'the same groups', but with 'different differentials', is the key to our extension of Evens-Kahn's results [15]. Theorem 2.1 is a statement of the result of Nakaoka mentioned above, which for interest's sake we have made more general than the original. A weak version of this theorem could be stated as: 'if $H^*(X)$ is free, then the Cartan-Leray spectral sequence for $H^*(X^\Omega \times_S E)$ collapses at the E_2-page'. A similarly weakened version of Theorem 2.2 would say that over the integers (or any PID), this spectral sequence collapses at the E_r-page, $r = 2 + |\Omega|$, without any condition on $H^*(X)$.

The Cartan-Leray spectral sequence may be obtained from a double complex. In Sections 3 and 4 we study the other spectral sequence associated with the same double complex. In Section 3 we define the elements $\alpha \wr 1$ and give upper and lower bounds on their orders. In Section 4 we describe the additive structure of the integral cohomology of $X^p \times_{C_p} EC_p$, extending work of Evens-Kahn [15]. One surprising corollary of this result is that for $p \geq 5$, a cyclic summand of $H^i(X)$ of order p^j gives rise to $(p-1)/2$ cyclic summands of $H^*(X^p \times_{C_p} EC_p)$ of order p^{j+1}, not just the summand that one would expect in degree pi. In Section 5 we use the results of Section 4 to determine the differentials and extension problems in the Cartan-Leray spectral sequence for $X^p \times_{C_p} EC_p$.

Quillen's detection lemma [25], which states that the mod-p cohomology of $X^p \times_{C_p} EC_p$ is detected by two maps from X^p and $X \times BC_p$, is a corollary of Nakaoka's results. In Section 6 we describe the kernel of the analogous map in integral cohomology. In Section 7 we sketch how the methods of Sections 4, 5, and 6 may be applied to describe the p-local cohomology of $X^p \times_{\Sigma_p} E\Sigma_p$. In fact this turns out to be considerably simpler than the case of of the cyclic group.

In Section 8 we review a conjecture of A. Adem and H.-W. Henn concerning the exponent of integral cohomology of finite groups. We show that if G does not afford a counterexample, then neither does $G \wr C_p$. For a slightly stronger conjecture we obtain the more general result that if G and S do not afford counterexamples then neither does $G \wr S$. We also recall an upper bound on the (eventual) exponent of integral cohomology of p-groups, based on generalized Frattini subgroups, that we gave in [21]. Section 9 describes an example showing that this bound is not always best possible. The only connection between this section and wreath products is that evidence gleaned from wreath products led the author to believe for some time that such examples could not exist.

In Section 10, we assume that the p-local cohomology of X is a finitely generated algebra, so that the variety of all ring homomorphisms from $H^*(X)$ to an algebraically closed field k of characteristic p is affine. In this case the variety for $H^*(X^p \times_{C_p} EC_p)$ is also affine, and may be described in terms of the variety for $H^*(X)$. (This is easily deduced from [23], but is first stated explicitly in [25].) We use the results of Sections 4 and 6 to give a similar description, for each i, of the subvariety corresponding to the annihilator in $H^*(X^p \times_{C_p} EC_p)$ of the element p^i. First we prove some general properties of such subvarieties. In Section 11 we apply the results of Section 10 to cohomology of finite groups. The subvarieties that we study were introduced in this context by Carlson [11]. Let G be a p-group, let $W(G)$ be the variety of ring homomorphisms from $H^*(G)$ to k, and let $W_i(G)$ be the subvariety corresponding to $\mathrm{Ann}(p^i)$. Each $W_i(G)$ is a covariant functor of G. We show that $W_i(G)$ is contained in the image of $W(\Phi_i(G))$, where $\Phi_i(G)$ is the generalized Frattini subgroup introduced in Section 8. Carlson asked in [11] if the image of $W(\Phi^i(G))$ is contained in $W_i(G)$, where $\Phi^i(G)$ is the ith iterated Frattini subgroup of G. We give examples where this does not hold. We show however that the image of $W(\Phi(G) \cap Z(G))$ is contained in $W_1(G)$. We close with the remark that in all known examples, each $W_i(G)$ has a nice description.

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1. Notation and algebraic preliminaries.

We hope that our notation is either defined where used or is evident. As a rough guide we mention the following points.

- p is a prime number throughout.
- C_n is a cyclic group of order n.
- Ω is the set $\{1, \ldots, l\}$, or occasionally $\{1, \ldots, p\}$.
- $\Sigma_l = \Sigma(\Omega)$, is the symmetric group on Ω.
- All spaces are taken to be CW-complexes, and the topology on a product is chosen so that the product of CW-complexes is a CW-complex in the natural way.
- For G a group, EG is a contractible G-free G-CW-complex.
- R is a commutative ring, often the integers $\mathbb{Z}$ or the p-local integers $\mathbb{Z}_{(p)}$. Usually $- \otimes -$ and $\mathrm{Hom}(-,-)$ should be taken over R. In Sections 10 and 11 however, R is a more general commutative $\mathbb{Z}_{(p)}$-algebra, and $- \otimes -$ stands for the tensor product over $\mathbb{Z}_{(p)}$.
- Usually S stands for a subgroup of $\Sigma(\Omega)$, although in Section 9 a sphere called S makes a brief appearance, and in Section 10 S is a commutative $\mathbb{Z}_{(p)}$-algebra.
- All chain complexes and cochain complexes are bounded below. Double cochain complexes will be denoted $(E_0^{*,*}, d, d')$, $E_0^{*,*}$, or similarly, where the subscript is intended to suggest that we will take spectral sequences. Double complexes (and pages of the associated spectral sequences) will be depicted with the first index running horizontally:

$$\begin{array}{ccccccc} & \vdots & & \vdots & \\ \cdots & E_0^{i,j+1} & \xrightarrow{d} & E_0^{i+1,j+1} & \cdots \\ & \uparrow{\scriptstyle d'} & & \uparrow{\scriptstyle d'} & \\ \cdots & E_0^{i,j} & \xrightarrow{d} & E_0^{i+1,j} & \cdots \\ & \vdots & & \vdots & \end{array}.$$

Following Cartan-Eilenberg [12], we call the associated spectral sequence in which $d_0 = d'$ the 'type I spectral sequence', and the spectral sequence in which $d_0 = d$ the 'type II spectral sequence'. Note that higher differentials in type II spectral sequences will point upwards and leftwards on our illustrations.

The following lemma seems to be new, and is very useful in our calculation (in Section 4) of $H^*(X^p \times_{C_p} EC_p)$.

Lemma 1.1. *Let $(E_0^{*,*}, d, d')$ be a double cochain complex of abelian groups, and let $E_r^{*,*}$ be the corresponding type II spectral sequence. For any integer n, make a second double cochain complex $(\tilde{E}_0^{*,*}, \tilde{d}, \tilde{d}')$, where $\tilde{E}_0^{i,j} = E_0^{i,j}$, $\tilde{d} = d$, and $\tilde{d}' = nd'$. Then any element x of $E_0^{*,*}$ that represents an element of $E_r^{*,*}$ also represents an element of $\tilde{E}_r^{*,*}$, and $\tilde{d}_r(x) = n^r d_r(x)$.*

Proof. By assumption, there exist $x = x_1, \ldots, x_r \in E_0^{*,*}$ such that $d(x_1) = 0$ and $d(x_{i+1}) + d'(x_i) = 0$ for $1 \le i < r$, and then by definition $d_r(x)$ is represented by $d'(x_r)$.

Now if we let $y_i = n^{i-1}x_i$, we see that $x = y_1$, $\tilde{d}(y_1) = 0$, $\tilde{d}(y_{i+1}) + \tilde{d}'(y_i) = 0$ for $1 \leq i < r$, and that $\tilde{d}_r(x) = \tilde{d}'(y_r) = n^r d'(x_r) = n^r d_r(x)$. ∎

Remark. There is no simple relation between the type I spectral sequences associated with the complexes $E_0^{*,*}$ and $\tilde{E}_0^{*,*}$.

Proposition 1.2. *If C^* is a cochain complex of R-modules, then $C^{*\otimes\Omega}$ may be given the structure of a cochain complex of $R\Sigma(\Omega)$-modules as follows. The action of the transposition $\sigma_i = (i, i+1) \in \Sigma(\Omega)$ is defined on a homogeneous element $c_1 \otimes \cdots \otimes c_l$ by*

$$\sigma_i(c_1 \otimes \cdots \otimes c_l) = (-1)^{\deg(c_i)\deg(c_{i+1})} c_1 \otimes \cdots \otimes c_{i+1} \otimes c_i \otimes \cdots \otimes c_l.$$

Proof. First check that the action of each σ_i commutes with the differential on $C^{*\otimes\Omega}$. Now recall that $\Sigma(\Omega)$ has the following presentation as a Coxeter group [9]:

$$\Sigma(\Omega) = \langle \sigma_1, \ldots, \sigma_{l-1} \mid \sigma_i^2, (\sigma_i\sigma_j)^2, (\sigma_k\sigma_{k+1})^3 \rangle,$$

where $1 \leq i, j < l$, $1 \leq k < l-1$, and $|i-j| > 1$. Now check that each of the relators in this presentation acts trivially on $C^{*\otimes\Omega}$. ∎

Remark. The above proof was suggested to the author by Warren Dicks. It seems to be easier than proofs in which one works out the sign for the action of every element of $\Sigma(\Omega)$ (see for example [5,14]). On the other hand, one needs to know the sign of the action of an arbitrary element in most applications.

If C^* is the cellular cochain complex on a finite type CW-complex X, it may be shown that the cellular action of $\Sigma(\Omega)$ on X^Ω (with the product cell structure) induces the above action on $C^*(X^\Omega) = C^*(X)^{\otimes\Omega}$.

Let S be a subgroup of $\Sigma(\Omega)$, and let W_* be a chain complex of free RS-modules. Much of our work consists of studying double cochain complexes of the form

$$E_0^{i,j} = \mathrm{Hom}_S(W_i, (C^{*\otimes\Omega})^j),$$

and their associated spectral sequences. In many cases, W_* will be either the cellular cochain complex of a free S-CW-complex E, or will be a free resolution of the trivial RS-module R. In either of these cases there is a homomorphism (of RS-complexes, where S has the diagonal action on $W_* \otimes W_*$):

$$\Delta : W_* \longrightarrow W_* \otimes W_*$$

which gives rise to an anticommutative ring structure on $H^*\mathrm{Hom}_S(W_*, R)$. In this case, the isomorphism $(C^{*\otimes\Omega}) \otimes R \cong (C^{*\otimes\Omega})$ gives rise to a graded $H^*\mathrm{Hom}_S(W_*, R)$-module structure on $H^*\mathrm{Tot}E_0^{*,*}$, and a bigraded action on each of the two spectral sequences from the E_2-page onwards. (In each case, $H^*\mathrm{Hom}_S(W_*, R)$ is graded in the i-direction.)

Lemma 1.3, which we shall state but not prove, is due to Steenrod [29]. It is used in Steenrod's definition of the reduced p-powers, as well as in Nakaoka's work on wreath products (see [23], or Theorem 3.1 below). For most of our purposes the easier Lemma 1.4 will suffice however. Lemma 1.4 is implicit in a remark in [13], but I know of no explicit statement of it in the literature.

Lemma 1.3. *If W_* is (as above) a complex of free RS-modules and $f, g : C'^* \to C^*$ are homotopic maps between cochain complexes of R-modules, then the maps* $\mathrm{Hom}_S(1, f^{\otimes\Omega})$ *and* $\mathrm{Hom}_S(1, g^{\otimes\Omega})$ *from* $\mathrm{Hom}_S(W_*, C'^{*\otimes\Omega})$ *to* $\mathrm{Hom}_S(W_*, C^{*\otimes\Omega})$ *are homotopic.* ■

The proof of Lemma 1.3 is slightly simpler if it is also assumed that W_* is acyclic, and this is the statement given in the recent books [5] and [14]. Even if one is only interested in the special case of group cohomology however, it is worth having the stronger version of Lemma 1.3 because this justifies the application of Nakaoka's argument to wreath products of the form $G \wr S'$, where S' has S as a quotient and acts on G^Ω via the action of S on Ω.

Lemma 1.4. *Let W_* be a complex of free RS-modules, and let $f : C'^* \to C^*$ be a homotopy equivalence between cochain complexes of R-modules. Define a double cochain complex $E_0^{*,*}$ by*

$$E_0^{i,j} = \mathrm{Hom}_S(W_i, (C^{*\otimes\Omega})^j),$$

and similarly for $E_0'^{i,j}$. Then the map

$$\mathrm{Hom}_S(1, f^{\otimes\Omega}) : E_0'^{*,*} \to E_0^{*,*}$$

of double cochain complexes induces an isomorphism on the E_1-pages of the associated type I spectral sequences, and hence an isomorphism between the homologies of the corresponding total complexes.

Proof. The map $f^{\otimes\Omega} : C'^{\otimes\Omega} \to C^{\otimes\Omega}$ is a homotopy equivalence of R-complexes, and is an RS-map, so induces an RS-module isomorphism between $H^j(C'^{\otimes\Omega})$ and $H^j(C^{\otimes\Omega})$. If W_i is free with basis I_i, then $E_1^{i,j}$ (resp. $E_1'^{i,j}$) is isomorphic to a product of copies of $H^j(C^{\otimes\Omega})$ (resp. $H^j(C'^{\otimes\Omega})$) indexed by I_i, and the map is an isomorphism as claimed. ■

The conclusion of Lemma 1.4 does not necessarily hold for $f : C'^* \to C^*$ that only induces an isomorphism on cohomology, but recall the following lemma, from for example [28, p167].

Lemma 1.5. *A map between R-projective cochain complexes is a homotopy equivalence if and only if it induces an isomorphism on cohomology.* ■

We end this section with some remarks concerning the case when R is a principal ideal domain (PID) and C^* is an R-free cochain complex such that each $H^i(C^*)$ is a finitely generated R-module. For each i, fix a splitting of $H^i(C^*)$ as a direct sum of cyclic R-modules. Thus for some indexing set A,

$$H^*(C) \cong \bigoplus_{a \in A} H(a),$$

where $H(a)$ is isomorphic to $R/(r(a))$ and is a summand in degree $i(a)$, for some $r(a) \in R$ and integer $i(a)$.

For $r \in R$ and $i \in \mathbb{Z}$, define a cochain complex $C(r,i)^*$ such that each $C(r,i)^j = 0$, except that $C(r,i)^i \cong R$ and if r is non-zero, then $C(r,i)^{i-1} \cong R$. Define the differential in $C(r,i)^*$ so that $H^i(C(r,i)^*) \cong R/(r)$. Define a cochain complex C'^* by

$$C'^* = \bigoplus_{a \in A} C(a)^*,$$

where $C(a)^* = C(r(a), i(a))$. It is easy to construct a map from C'^* to C^* inducing an isomorphism on cohomology, which is therefore a homotopy equivalence by Lemma 1.5.

Now consider the RS-module structure of the complex $(C'^{*\otimes\Omega})^*$. (The motivation for this is provided by Lemmata 1.3 and 1.4.) As a complex of R-modules, $(C'^{*\otimes\Omega})^*$ splits as a direct sum of pieces of the form

$$C(a_1, \dots, a_l) = C(a_1)^* \otimes \cdots \otimes C(a_l)^*.$$

Call such a summand a 'cube', and call the $C(a_j)^*$ that arise the cube's 'sides'. The action of S permutes the cubes, and the stabilizer S' of the cube $C(a_1, \dots, a_l)^*$ is equal to the stabilizer of $(a_1, \dots, a_l) \in A^\Omega$. Each R-module summand of $C(a_1, \dots, a_l)^*$ of the form $C(a_1)^{i(a_1)-\epsilon(1)} \otimes \cdots \otimes C(a_l)^{i(a_l)-\epsilon(l)}$, where $\epsilon(j) = 0$ or 1 and $\epsilon(j) = 0$ if $r(a_j) = 0$, is in fact an RS'-summand. Define Ω^- by

$$\Omega^- = \{j \in \Omega \mid i(a_j) - \epsilon(j) \text{ is odd } \},$$

and $\Omega^+ = \Omega \setminus \Omega^-$. Then S' is a subgroup of $\Sigma(\Omega^+) \times \Sigma(\Omega^-) \leq \Sigma(\Omega)$, and the action of S' on $C(a_1)^{i(a_1)-\epsilon(1)} \otimes \cdots \otimes C(a_l)^{i(a_l)-\epsilon(l)}$ is given by the sign action of $\Sigma(\Omega^-)$ tensored with the trivial action of $\Sigma(\Omega^+)$. (This may be checked using Proposition 1.2.)

2. Nakaoka's argument and generalizations.

Recall that S is a subgroup of the symmetric group on a finite set Ω, and let E be a free S-CW-complex of finite type. Note that we do not require E to be contractible. For any finite type CW-complex X we may define an action of S on a product of copies of X indexed by Ω, letting S permute the factors. Now S acts 'diagonally' on $X^\Omega \times E$, and we may form the quotient space $(X^\Omega \times E)/S$. Topologists usually insist that S should act on the right of X^Ω and on the left of E, and write $X^\Omega \times_S E$ for the quotient. Assume that E is connected. By considering various covering spaces of $X^\Omega \times_S E$ it is easy to see that in this case the fundamental group of $X^\Omega \times_S E$ is the wreath product $G \wr \pi_1(E/S)$, where G is the fundamental group of X. By this wreath product we mean a split extension with kernel G^Ω and quotient $\pi_1(E/S)$, where $\pi_1(E/S)$ acts via the action of its quotient S on Ω. In particular, if E is simply connected, then $\pi_1(X^\Omega \times_S E)$ is the wreath product $G \wr S$. From now on we shall assume that E is both connected and simply connected for clarity—it is easy to make the necessary changes to cover the general case.

For any field k, Nakaoka's argument determines $H^*(X^\Omega \times_S E; k)$ in terms of $H^*(X; k)$ and $H^*(E/S; N)$ for various signed permutation kS-modules N. The argument applies equally to the cohomology of $X^\Omega \times_S E$ with non-trivial coefficients coming from some $kG \wr S$-modules (in fact, those which are 'tensor-induced' from kG-modules) and gives some information about $H^*(X^\Omega \times_S E; R)$ for any commutative ring R. The following account includes both of these generalizations.

Let R be a commutative ring, and let Ω, S, E and X be as above, with $\pi_1(X) = G$. All chain complexes will be chain complexes of R-modules unless otherwise stated and all unmarked tensor products will be over R. Let W_* be the cellular R-chain complex for E, so that W_* is a chain complex of free RS-modules, and is a free resolution for R over RS in the case when E is acyclic. Let U_* be the cellular chain complex of the universal cover of X, so that U_* is a chain complex of finitely generated free RG-modules. If M is an RG-module, then RG^Ω acts on $M^{\otimes\Omega}$ by letting G^Ω act component-wise, and S acts on $M^{\otimes\Omega}$ by permuting the factors of each monomial $m_1 \otimes m_2 \otimes \ldots \otimes m_l$. Together these actions define an $RG \wr S$-module structure on $M^{\otimes\Omega}$. Similarly, the total complex of $U_*^{\otimes\Omega}$ becomes a complex of $RG \wr S$-modules, where the action of S is as described in Proposition 1.2. From this point of view the signs that arise are due to the fact that the action of S on the Ω-cube does not necessarily preserve its orientation. The complex $U_*^{\otimes\Omega}$ is the cellular chain complex of the universal cover of X^Ω (with the product CW structure), where the action of $G \wr S$ comes from the action of S on X^Ω. Similarly, the complex $U_*^{\otimes\Omega} \otimes W_*$, with the diagonal action of $G \wr S$, is eaily seen to be the cellular chain complex of the universal cover of $X^\Omega \times_S E$.

Define a double cochain complex $E_0^{i,j}$ by:

$$E_0^{i,j} = \mathrm{Hom}_{RG\wr S}((U_*^{\otimes\Omega})_j \otimes W_i, M^{\otimes\Omega}).$$

The associated total complex is the complex of $G \wr S$-equivariant cochains on the universal cover of $X^\Omega \times_S E$ with values in $M^{\otimes\Omega}$, and so the cohomology of this complex is $H^*(X^\Omega \times_S E; M^{\otimes\Omega})$. There are two spectral sequences associated to the double complex $E_0^{i,j}$. The type I spectral sequence has E_2-page as follows:

$$E_2^{i,j} = H^i(E/S; H^j(X^\Omega; M^{\otimes\Omega})).$$

This is a spectral sequence of Cartan-Leray type for the covering of $X^\Omega \times_S E$ by $X^\Omega \times E$. From the E_2-page onwards it is isomorphic to the Serre spectral sequence for the fibration

$$X^\Omega \longrightarrow X^\Omega \times_S E \longrightarrow E/S.$$

Each of the two spectral sequences admits a bigraded $H^*(E/S; R)$-module structure from the E_2-page onwards, as was shown in Section 1.

So far we have not used the assumption that X is of finite type. This is required to establish the second of the following isomorphisms of double cochain complexes:

$$\begin{aligned} E_0^{*,*} &= \mathrm{Hom}_{G\wr S}(U_*^{\otimes\Omega} \otimes W_*, M^{\otimes\Omega}) \\ &\cong \mathrm{Hom}_S(W_*, \mathrm{Hom}_{G^\Omega}(U_*^{\otimes\Omega}, M^{\otimes\Omega})) \\ &\cong \mathrm{Hom}_S(W_*, \mathrm{Hom}_G(U_*, M)^{\otimes\Omega}). \end{aligned}$$

Thus we are reduced to the study of the cohomology of a double complex of the type discussed in Section 1.

Remark. (The algebraic case.) The case of interest in group cohomology may be recovered as the case when X and E/S both have trivial higher homotopy groups (note that this is more general than requiring E to be contractible). Of course, in the algebraic case we can do without the space X altogether, and just take the complex U_* to be a finite type (R-free) RG-projective resolution for R. Thus the above argument applies to groups G of type $FP(\infty)$ over R, rather than just those groups G having a $K(G,1)$ of finite type.

The following theorem is due to Nakaoka, although we have deliberately made the statement more general than the original (which considered only trivial coefficients).

Theorem 2.1 ([23,13,20]). *As above, let X be a connected CW-complex of finite type with fundamental group G, and let M be an R-free RG-module such that $H^*(X;M)$ is R-projective. Let E be a (connected) free S-CW-complex. Then there is an isomorphism of graded R-modules as follows.*

$$H^*(X^{\otimes\Omega} \times_S E; M^{\otimes\Omega}) \cong H^*(E/S; H^*(X;M)^{\otimes\Omega})$$

The spectral sequence with coefficients in $M^{\otimes\Omega}$ for the fibration $X^\Omega \to X^\Omega \times_S E \to E/S$ collapses at the E_2-page, and the additive extensions in the reconstruction of the cohomology from the E_∞-page are all trivial.

Remarks. Note that the freeness conditions are automatically satisfied if R is a field. In the case when $M = R$, the isomorphism of the theorem is moreover an isomorphism of R-algebras, and for general M it may be shown to be an isomorphism of $H^*(X^\Omega \times_S E; R) \cong H^*(E/S; H^*(X;R)^{\otimes\Omega})$-modules.

Proof. Under the conditions of the theorem, both $C^* = \mathrm{Hom}_G(U_*, M)$ and $C'^* = H^*(C^*)$ are R-projective. Viewing C'^* as a cochain complex with trivial differential it is easy to construct a chain map from C'^* to C^* inducing the identity map on cohomology (see [23] or [15]), which is a homotopy equivalence by Lemma 1.5. Now by either Lemma 1.3 or Lemma 1.4 we see that $H^*(X^\Omega \times_S E; M^{\otimes\Omega})$ may be calculated as the cohomology of the total complex of $E_0'^{i,j} = \mathrm{Hom}_S(W_i, (C'^{*\otimes\Omega})^j)$. But this double complex has trivial j-differential, so splits as a direct sum of double complexes concentrated in constant j-degree (or 'rows'). ∎

Remarks. There are other proofs of special cases of Theorem 2.1. There is a topological proof due to Adem-Milgram in the special case when S is cyclic of order p, E is contractible, S acts freely transitively on Ω, and the coefficients form a field [3].

There is also a short proof of the trivial coefficient case arising in cohomology of finite groups, due to Benson and Evens ([5, vol. II, p.130 and 14, Thrm. 5.3.1]). This proof is as follows: If G is finite, and $R = k$ is a field, then one may take U_* to be a minimal resolution for k over kG. In this case, if M is a simple kG-module, for example the trivial module k, then the differential in $\mathrm{Hom}_G(U_*, M)$ is trivial and so the double complex for computing $H^*(G \wr S; M^{\otimes\Omega})$ has one of its differentials trivial, and the type I spectral sequence collapses at the E_2-page. It is not true however that if also W_* is a minimal resolution for k over kS then $U_*^{\otimes\Omega} \otimes W_*$ is a minimal resolution for k over $kG \wr S$, because in general this will have a larger growth rate than a minimal resolution.

If the condition on $H^*(X;M)$ is weakened, Nakaoka's argument may still be applied, but the conclusion is far weaker. For example, consider the following theorem.

Theorem 2.2. *Take notation and conditions as in the statement of Theorem 2.1, but replace the condition that $H^*(X;M)$ should be R-projective by the condition that $H^*(X;M)$ should have projective dimension at most one over R. Then from E_2 onwards, the spectral sequence for the fibration $X^\Omega \to X^\Omega \times_S E \to E/S$ with $M^{\otimes\Omega}$ coefficients is a direct sum of spectral sequences $E_r^{*,*} = \bigoplus_\alpha E_{r,\alpha}^{*,*}$, where each $E_{r,\alpha}^{i,j}$ has 'height' at most $|\Omega|$. There is a corresponding direct sum decomposition of $H^*(X^\Omega \times_S E; M^{\otimes\Omega})$.*

Remark. By the phrase '$E_{r,\alpha}^{*,*}$ has height at most $|\Omega|$' we mean that there exist integers $n(\alpha) \leq n'(\alpha)$ such that $E_{r,\alpha}^{i,j} = 0$ if $j < n(\alpha)$ or $j > n'(\alpha)$, and $n'(\alpha) - n(\alpha) \leq |\Omega|$. In particular this implies that $d_{|\Omega|+1}$ is the last possibly non-zero differential.

Proof. Let $P_i \rightarrowtail Q_i \twoheadrightarrow H^i(X;M)$ be a projective resolution over R for $H^i(X;M)$, and build a cochain complex C'^* with $C'^i = Q_i \oplus P_{i+1}$ and differential given by the following composite.

$$C'^i \twoheadrightarrow P_{i+1} \rightarrowtail Q_{i+1} \rightarrowtail C'^{i+1}$$

Then using the projectivity of P_i and Q_i it is easy to construct a chain map from C'^* to $\mathrm{Hom}_G(U_*, M)$ inducing an isomorphism on cohomology, which is an equivalence by Lemma 1.5. Now by either Lemma 1.3 or Lemma 1.4, the double complex $E_0'^{*,*} = \mathrm{Hom}_S(W_*, C'^{\otimes\Omega})$ may be used to compute $H^*(X^\Omega \times_S E; R)$. By construction C'^* splits as a direct sum of complexes of length at most one, and so $(C'^{\otimes\Omega})^*$ splits as a direct sum of complexes of length at most $|\Omega|$, and hence $E_0'^{*,*}$ splits as a direct sum of double complexes of height at most $|\Omega|$. The claimed properties of the spectral sequence now follow from Lemma 1.4. ∎

Remarks. There is no easy generalization of the statement given after Theorem 2.1 concerning the ring structure of $H^*(X^\Omega \times_S E; R)$. It is known, for example, that the ring structure of the integral cohomology of X does not suffice to determine that of $X \times X$ [24]. This could be considered as the case of $X^\Omega \times_S E$ when R is the integers, E is a point, and S is the trivial subgroup of the symmetric group on a set Ω of size two.

One may ask about similar results to the above for generalized cohomology theories. We believe that the following statement is implicit in [10, chap. IX]: 'Let $h^*(-)$ be a complex oriented generalized cohomology theory associated with an H_∞-ring spectrum, and assume that $h^*(X)$ is free over h^* with basis B concentrated in even degrees. Then $h^*(X^\Omega \times_S E)$ is isomorphic to a direct sum of $\tilde{h}^*(E/S')$'s, shifted in degree, where S' runs over the stabilizers of a set of orbit representatives for the action of S on B^Ω.' For specific choices of S and E stronger results are known—see [19] for some recent results in the case when $S = C_p$ and E is contractible.

3. The order of $\alpha \wr 1$.

Throughout this section, take R to be either $\mathbb{Z}$ or one of its localizations, and let W_* be the cellular chain complex of an S-free S-CW-complex E with augmentation $\epsilon : W_* \to R$, so that the homology of $\ker(\epsilon)$ is the reduced R-homology of E. For any X of finite type and any $\alpha \in H^{2i}(X;R)$, $\alpha \wr 1 \in H^{2il}(X^\Omega \times_S E; R)$ may be defined ([29,23,14]). We give bounds on the order of $\alpha \wr 1$ in terms of the order of α. In fact it costs no extra work to replace $C^*(X;R)$ by an arbitrary R-free cochain complex of finite type, so we do so. Much of the section generalizes to the case when R is any PID, if the statements concerning orders are replaced by statements about annihilators.

For C^* a finite type R-free cochain complex, and $c \in C^*$ of even degree, define $c \wr 1 \in \mathrm{Hom}_S(W_*, C^{*\otimes\Omega})$ by

$$c \wr 1(w) = \epsilon(w).c \otimes \cdots \otimes c,$$

for any $w \in W_*$. The same formula may be used for c of odd degree, but in this case it defines an element of $\mathrm{Hom}_S(W_*, C^{*\otimes\Omega} \otimes \widehat{R})$, where $\widehat{R}$ is the RS-module of R-rank one on which S acts via the sign representation of $\Sigma(\Omega)$. If c is a cocycle, then so is $c \wr 1$, and using Lemma 1.3 it may be shown that the cohomology class of $c \wr 1$ depends only on the cohomology class of c (this is the only place where we require Lemma 1.3 rather than Lemma 1.4). Thus if $\alpha \in H^*(C^*)$, we may define a unique element $\alpha \wr 1$ of $H^*\mathrm{TotHom}_S(W_*, C^{*\otimes\Omega})$.

If α does not generate a direct summand of $H^*(C^*)$, pick α' and an integer n such that α' generates a direct summand of $H^*(C^*)$ and $\alpha = n\alpha'$. Then (check on the level of cochains) $\alpha \wr 1 = (n\alpha') \wr 1 = n^l(\alpha \wr 1)$. Hence it suffices to consider the case when α generates a summand of $H^*(C^*)$.

Theorem 3.1. *Fix S, W_*, R, as above. Let C^* be any R-free finite type cochain complex, and let $\alpha \in H^*(C^*)$. Then the order of $\alpha \wr 1 \in H^*\mathrm{TotHom}_S(W_*, C^{*\otimes\Omega})$ depends only on the order $O(\alpha)$ of α, and the order of a cyclic summand of $H^*(C^*)$ containing α. If α has infinite order, then so does $\alpha \wr 1$. If α generates a finite direct summand of $H^*(C^*)$, then*

$$O(\alpha) \leq O(\alpha \wr 1) \leq l'.O(\alpha),$$

where l' is the h.c.f. of the lengths of the S-orbits in Ω.

Proof. The reduction to the case when α generates a direct summand of $H^*(C^*)$ is discussed above. If α does generate a direct summand of $H^*(C^*)$, pick a splitting of $H^*(C^*)$ into cyclic summands as at the end of Section 1:

$$H^*(C^*) = \bigoplus_{a \in A} H(a).$$

Choose the splitting in such a way that α generates one of the summands, say $H(a_0)$. Construct C'^* as in Section 1. Then $H^*(C(a_0)^*)$ maps isomorphically to the summand of $H^*(C^*)$ generated by α, and $C(a_0)^{*\otimes\Omega}$ is an RS-summand of the complex $C'^{*\otimes\Omega}$. Hence the image of $H^*\mathrm{TotHom}_S(W_*, C(a_0)^{*\otimes\Omega})$ in $H^*\mathrm{TotHom}_S(W_*, C^{*\otimes\Omega})$ is a direct summand. Moreover, if $\alpha' \in H^*(C(a_0)^*)$ maps to α, then $\alpha' \wr 1$ maps to $\alpha \wr 1$. Thus we see that if α generates a summand of $H^i(C^*)$ of order n, then $O(\alpha \wr 1) = O(\alpha' \wr 1)$, where α' generates $H^i(C(n,i)^*)$ (see Section 1 for the definition of $C(n,i)^*$). Let $D(n,i)^* = C(n,i)^{*\otimes\Omega}$. It may be checked that as complexes of RS-modules, for any i,

$$D(n,2i)^{*-2li} \cong D(n,0)^* \cong D(n,2i+1)^{*-2li-l} \otimes \widehat{R},$$

where $\widehat{R}$ is the sign representation of $S \leq \Sigma(\Omega)$. Hence the order of $\alpha' \wr 1$ does not depend on i.

Let E' be a set of points permuted freely, transitively by S, and let E'' be a contractible free S-CW-complex with one orbit of zero cells. Then there are S-equivariant maps $E' \to E \to E''$, and hence augmentation preserving RS-maps $W'_* \to W_* \to W''_*$. Thus it suffices to verify the lower bound in the case when $W_* = W'_*$, and the upper bound in the case when $W_* = W''_*$. The lower bound follows from the Kuñneth theorem, because

$$\mathrm{TotHom}_S(W'_*, D(n,0)^*) \cong D(n,0)^* \cong C(n,0)^{*\otimes\Omega}.$$

For the upper bound in the case when α generates a summand $R/(n)$ for $n \neq 0$, we consider the type II spectral sequence, $E_*^{*,*}$, for

$$E_0^{i,j} = \mathrm{Hom}_S(W_i'', D(n,0)^j).$$

Since we assumed that W_0'' is free of rank one, $E_0^{0,0}$ is isomorphic to R, and some generator for $E_0^{0,0}$ is a cocycle representing $\alpha' \wr 1$. Since $E_0^{i,j}$ is zero for $j > 0$, $E_\infty^{0,0}$ is a subgroup of $H^0(\mathrm{Tot}E_0^{*,*})$. Hence the order of $\alpha' \wr 1$ is equal to the order of $E_\infty^{0,0}$. Now as RS-module, $D(n,0)^{-1}$ is isomorphic to the permutation module with basis Ω. (See the analysis at the end of Section 1.) The coboundary

$$R\Omega = D(n,0)^{-1} \xrightarrow{d} D(n,0)^0 = R$$

satisfies $d(\omega) = n$ for each $\omega \in \Omega \subseteq R\Omega$. Now since W_*'' is acyclic, the E_1-page of the spectral sequence is isomorphic to the cohomology of the group S with coefficients in $D(n,0)^*$. More precisely,

$$E_1^{i,j} = H^i(S, D(n,0)^j).$$

In particular, $E_1^{0,j} = (D(n,0)^j)^S$, the S-fixed points in $D(n,0)^j$. Thus $E_1^{0,0} = R$, and $E_1^{0,-1}$ is the free R-module with basis the S-orbits in Ω. It follows that

$$E_2^{0,0} = E_1^{0,0}/\mathrm{Im}(d' : E_1^{0,-1} \to E_1^{0,0}) \cong R/(nl'),$$

where l' is the h.c.f. of the lengths of the S-orbits in Ω. This gives the required upper bound, since $E_\infty^{0,0}$ is a quotient of $E_2^{0,0}$. ∎

Remark. Ther is an easy argument using the transfer which gives the weaker upper bound $O(\alpha \wr 1) \leq |S|.O(\alpha)$.

The most interesting case of the construction of $\alpha \wr 1$ is of course the case in which W_* is acyclic, and we shall concentrate on that case from now on. Proposition 3.2 gives an easy case in which the lower bound given above is attained for W_* acyclic, and hence for all W_*.

Proposition 3.2. *With notation as in Theorem 3.1, if α generates a summand of order $O(\alpha) = n$ with n coprime to l', then $O(\alpha \wr 1) = O(\alpha)$.*

Proof. By assumption, each of α and $l'\alpha$ generates a summand of $H^*(C^*)$ of order n, so $O(\alpha \wr 1) = O(l'\alpha) \wr 1$ and $O(\alpha \wr 1)$ divides nl' by 3.1. But $(l'\alpha) \wr 1 = l'^l(\alpha \wr 1)$, and hence $O(\alpha \wr 1)$ is coprime to l'. ∎

In view of Proposition 3.2, it is reasonable to consider the problem of the order of $\alpha \wr 1$ one prime at a time. An easy transfer argument shows that in this case it suffices to consider the case when S is a p-group. In this case, and for W_* acyclic, I know of no case where the upper bound of Theorem 3.1 (or rather its p-part) is not attained. The only cases in which I have been able to prove this are stated below.

Theorem 3.3. *Let S be a p-group of order p^n, and let S permute Ω freely, transitively. Let W_* be acyclic and let α generate a summand of $H^*(C^*)$ of order p^r. Then $O(\alpha \wr 1) = p^{r+n}$ if either S is cyclic, or S is elementary abelian.*

Proof. By Theorem 3.1, it suffices to show this for some single choice of C^* and element α generating a summand of $H^*(C^*)$ of even degree. We choose C^* to be a cochain complex for computing the integral cohomology of some p-group H, and then use techniques from cohomology of finite groups, including the Evens norm map [14] and a theorem of J. F. Carlson [11] which we state below as Theorem 3.4.

Recall (from for example [14]) that if G is a finite group, with subgroup H, then a choice of transversal T to H in G gives rise to an injective homomorphism

$$\phi_T : G \longrightarrow H \wr \Sigma(G/H),$$

where $\Sigma(G/H)$ is the permutation group on the set G/H of cosets of H in G. Furthermore, if T' is another transversal, then ϕ_T and $\phi_{T'}$ differ only by an inner automorphism of $H \wr \Sigma(G/H)$. If H is normal in G then the map $G \to \Sigma(G/H)$ factors through G/H, and G/H acts freely, transitively on the cosets of H. If α is an element of $H^*(H)$ of even degree, then the Evens norm $\mathcal{N}_H^G(\alpha) \in H^*(G)$ is defined to be $\phi_T^*(\alpha \wr 1)$ (for W_* acyclic), and is independent of T. There is a double coset formula for the image of $\mathcal{N}_H^G(\alpha)$ in $H^*(H)$, which in the case when H is normal in G is:

$$\mathrm{Res}_H^G \mathcal{N}_H^G(\alpha) = \prod_{t \in T} c_t^*(\alpha), \tag{1}$$

where c_t^* is the map of $H^*(H)$ induced by conjugation by t. Our strategy for proving the theorem is to find a group G expressible as an extension with quotient S and kernel some suitable H, and some $\alpha \in H^*(H)$ generating a summand of order p^r such that $\mathcal{N}_H^G(\alpha)$ has order p^{n+r}.

In the case when $S = C_{p^n}$ is cyclic, we may take G to be cyclic of order p^{n+r}, so that H is cyclic of order p^r. Then

$$H^*(H) = \mathbb{Z}[\alpha]/(p^r\alpha), \qquad H^*(G) = \mathbb{Z}[\alpha']/(p^{n+r}\alpha'),$$

where α and α' have degree two, and $\gamma \in H^{2i}(G)$ generates $H^{2i}(G)$ if and only if $\mathrm{Res}_H^G(\gamma)$ generates $H^{2i}(H)$. But by (1), $\mathrm{Res}_H^G \mathcal{N}_H^G(\alpha) = \alpha^{p^n}$, so $\mathcal{N}_H^G(\alpha)$ has order p^{n+r} as required. The case when S is non-cyclic of order four may be proved similarly, taking G to the the generalized quaternion group of order 2^{r+2} expressed as a central extension with quotient S and cyclic kernel.

The case $S = (C_p)^n$ for $n \neq 1$ and $(p, n) \neq (2, 2)$ is more complicated, because here it seems to be impossible to choose G with H a central subgroup. For $p = 2$, let P be the dihedral group of order eight, and for odd p, let P be the (unique) non-abelian group of order p^3 and exponent p. In each case the centre of P is cyclic of order p. Now let G be the central product of n copies of P and a cyclic group of order p^r. The group G has the following presentation:

$$G = \langle A_1, \ldots, A_n, B_1, \ldots, B_n, C \mid C^{p^r},\ A_i^p,\ B_i^p,\ [A_i, A_j],$$

$$[B_i, B_j],\ [A_i, C],\ [B_i, C],\ [A_i, B_j]C^{-\delta(i,j)p^{r-1}}\rangle \tag{2}$$

where $\delta(i,j)$ is the Dirac δ-function. Let Z be the subgroup of G generated by C. Then Z is the centre of G, Z is cyclic of order p^r, and G/Z is elementary abelian of rank $2n$. Let H be the subgroup of G generated by C, and the A_i's. Then H is normal in G, and $H = Z \times H'$, where H' is the elementary abelian group of rank n generated by the A_i's. The quotient G/H is elementary abelian of rank n, so is isomorphic to S as required.

Since Z is a direct summand of H, the map $\mathrm{Res}^H_Z : H^*(H) \to H^*(Z)$ is surjective. Let $H^*(Z) = \mathbb{Z}[\gamma]/(p^r\gamma)$, and let $\alpha \in H^2(H)$ be such that $\mathrm{Res}^H_Z(\alpha) = \gamma$. The exponent of $H^*(H)$ is p^r (by the Künneth theorem), so any such α generates a summand of $H^*(H)$. Since Z is central in G, c_g^* acts trivially on $H^*(Z)$ for any $g \in G$. Applying the double coset formula given in (1), it follows that

$$\mathrm{Res}^G_Z \mathcal{N}^G_H(\alpha) = \prod_{t \in G/H} c_t^*(\gamma) = \gamma^{p^n}.$$

The claim will follow if we can prove that any element of $H^*(G)$ whose image generates $H^{2i}(Z)$ for some $i > 0$ has order p^{n+r}. This is done in Lemma 3.5. ∎

Theorem 3.4. *([11]) Let G be a finite group, and let $x_1, \ldots, x_m$ be elements of $H^*(G)$ such that $H^*(G)$ is a finite module for the subalgebra they generate. Then the order $|G|$ of G divides the product $O(x_1)\cdots O(x_m)$.* ∎

Lemma 3.5. *Let G be the group with presentation (2) as above, and Z the subgroup generated by C. Then any element of $H^*(G)$ whose image under Res^G_Z generates $H^{2i}(Z)$, for some $i > 0$, has order p^{n+r}.*

Proof. Let α_1 be an element as in the statement. We shall exhibit $\alpha_2, \ldots, \alpha_{n+1} \in H^*(G)$ of order p such that $H^*(G)$ is finite over the subalgebra generated by $\alpha_1, \ldots, \alpha_{n+1}$. Then by Theorem 3.4, α_1 must have order at least p^{n+r}. The subgroup H of G has index p^n, and its (positive degree) cohomology has exponent p^r, so a transfer argument gives the other inequality. Recall that work of Quillen implies that for any p-group K, $H^*(K)$ is finite over a subring H'^* if and only if for every maximal elementary abelian subgroup E of K, $H^*(E)$ is finite over its subring $\mathrm{Res}^K_E(H'^*)$ [26].

The maximal elementary abelian subgroups of G have p-rank $n+1$, and all contain the central subgroup of Z order p. (One way to see this is to note that the subgroup of G generated by the A_i's and B_i's contains all elements of G of order p, and this group is an extraspecial group of order p^{2n+1}, whose elementary abelian subgroups are discussed in for example [7].) The quotient G/Z is elementary abelian of rank $2n$, and if E is a maximal elementary abelian subgroup of G, the image of E in G/Z is elementary abelian of rank n.

Recall that if E is an elementary abelian group of rank m, then $H^2(E) \cong \mathrm{Hom}(E, \mathbb{Q}/\mathbb{Z})$ is elementary abelian of the same rank. Let H'^* be the subalgebra of $H^*(E)$ generated by H^2. Elements of $H^*(E)$ of positive degree have exponent p, and $H'^* \otimes \mathbb{F}_p$ is naturally isomorphic to the ring $\mathbb{F}_p[E]$ of polynomial functions on E viewed as an $\mathbb{F}_p$-vector space. Recall from for example [6, Chap. 8], that the ring of invariants in $\mathbb{F}_p[E]$ under the action of the full automorphism group of E is a polynomial algebra with generators $c_{m,0}, \ldots, c_{m,m-1}$, where $c_{m,i}$ has degree $2(p^m - p^i)$. (The $c_{m,j}$'s are known as the Dickson invariants.) Recall also that if E' is a subgroup of E of rank m', then the image of $c_{m,j}$ in $\mathbb{F}_p[E']$ is zero for $j < m - m'$ and is a power of $c_{m',j-m+m'}$ otherwise.

Now let $\gamma_1, \dots, \gamma_n$ be the elements of $H^*(G/Z)$ corresponding to the Dickson invariants $c_{2n,n}, \dots, c_{2n,2n-1}$ in the subalgebra generated by $H^2(G/Z)$. Let E be a maximal elementary abelian subgroup of G, and let Z' be the order p subgroup of Z. Then E/Z' is a subgroup of G/Z of rank n, and the properties of the Dickson invariants stated above imply that $H^*(E/Z')$ is finite over the subalgebra generated by $\mathrm{Res}(\gamma_1), \dots, \mathrm{Res}(\gamma_n)$. Let α_{i+1} be the image of γ_i in $H^*(G)$. Now $\mathrm{Res}^G_E(\alpha_2), \dots, \mathrm{Res}^G_E(\alpha_{n+1})$ freely generate a polynomial subalgebra of $H^*(E) \otimes \mathbb{F}_p$ and $\mathrm{Res}^G_{Z'}(\alpha_i) = 0$ for $i > 1$. If α_1 is an element of $H^*(G)$ with $\mathrm{Res}^G_{Z'}(\alpha_1) \neq 0$, as in the statement, then it follows that $\mathrm{Res}^G_E(\alpha_1), \dots, \mathrm{Res}^G_E(\alpha_{n+1})$ also freely generate a polynomial subalgebra of $H^*(E) \otimes \mathbb{F}_p$, and so $H^*(E)$ is finite over the subalgebra they generate. Thus $H^*(G)$ is finite over the algebra generated by $\alpha_1, \dots, \alpha_{n+1}$ by Quillen's theorem, and then α_1 must have order at least p^{n+r} by Carlson's Theorem 3.4. ∎

Remark. The proof of Lemma 3.5 is based on Carlson's method for computing the exponent of the cohomology of the extraspecial groups [11]. Carlson does not use Dickson invariants to construct elements analogous to the α_i's for $i > 1$, but gives an existence proof for such elements via algebraic geometry.

4. Integral cohomology for the cyclic group of order p.

In this section we concentrate on the study of the integral cohomology of $X^p \times_{C_p} E$, where C_p acts by freely permuting the factors of X^p and E is contractible. As above, the results apply more generally to the cohomology of $\mathrm{Hom}_{C_p}(W_*, C^{*\otimes p})$ for any finite type cochain complex C^* of free abelian groups. Evens and Kahn obtained partial results for this case [15] and we give few details for that part of our calculation which is a repeat of theirs. The main new idea here is the use of Lemma 1.1, which enables us to complete the calculation of $H^*(X^p \times_{C_p} E)$ and to give simpler proofs of some of the results in Evens and Kahn's paper [15].

As in Section 1, let C^* be a cochain complex of finitely generated free abelian groups, for example, the cellular cochain complex of a CW-complex of finite type. Let E be a contractible C_p-free C_p-CW-complex, and let W_* be the cellular chain complex for E. Recall from Lemma 1.4 that for finding the cohomology of $\mathrm{Hom}_{C_p}(W_*, C^{*\otimes p})$, the complex C^* may be replaced by any homotopy equivalent complex. As in Section 1, we replace C^* by a complex C'^* consisting of a direct sum of pieces of the form $C(n,i)^*$, in bijective correspondence with the summands of $H^i(C^*)$ (in some fixed splitting) isomorphic to $\mathbb{Z}/(n)$. (Recall that $C(n,i)^*$ is a complex which has at most two non-zero groups, each of rank one, and that $H^*(C(n,i)^*)$ is isomorphic to $\mathbb{Z}/(n)$ concentrated in degree i.)

As in Section 1, the complex of abelian groups, $C'^{*\otimes p}$ splits as a direct sum of 'cubes' of the form $C(n_1,i_1)^* \otimes \cdots \otimes C(n_p,i_p)^*$. The action of C_p permutes the cubes freely, except for those cubes whose sides all correspond to the same cyclic summand of $H^*(C^*)$. The action of C_p on these cubes is by a cyclic permutation of the p distinct axes. Considering the cube as embedded in $\mathbb{R}^p$, the matrix for the action is a permutation matrix of order p. More care must be taken for $p = 2$ than for p odd, because a 2×2 permutation matrix of order two has determinant -1.

The cubes permuted freely by C_p cause no problem. Indeed, if D^* is any cochain complex of abelian groups, then the Eckmann-Shapiro lemma [8,5] shows that

$$H^*\mathrm{TotHom}_{C_p}(W_*, D^* \otimes \mathbb{Z}C_p) \cong H^*(D^*).$$

Moreover, each of the two spectral sequences arising from viewing $\mathrm{Hom}_{C_p}(W_*, D^* \otimes \mathbb{Z}C_p)$ as a double cochain complex has $E_2^{i,0} \cong H^i(D^*)$ and $E_2^{i,j} = 0$ for $j \neq 0$. We can of course find the cohomology of $C(n_1, i_1) \otimes \cdots \otimes C(n_p, i_p)$ using the Kunneth theorem. If $i = i_1 + \cdots + i_p$ and exactly r of $n_1, \ldots, n_p$ are nonzero, then $H^{i-j}(C(n_1, i_1) \otimes \cdots C(n_p, i_p))$ is isomorphic to a sum of $\binom{r-1}{j}$ copies of $\mathbb{Z}/(n_1, \ldots, n_p)$.

Similarly, the cubes of the form $C(0,i)^{\otimes p}$ where C_p acts by permuting the factors are easy to handle. The cochain complex $C(0,i)^{\otimes p}$ consists of a single $\mathbb{Z}C_p$-module of $\mathbb{Z}$-rank one in degree pi. This is the trivial $\mathbb{Z}C_p$-module except when $p = 2$ and i is odd, in which case it is $\hat{\mathbb{Z}}$, the module on which a generator for C_2 acts as multiplication by -1. Thus each such cube contributes a summand to the spectral sequence of the form $H^*(C_p; \mathbb{Z})$ (resp. $H^*(C_2; \hat{\mathbb{Z}})$ if $p = 2$ and i is odd) concentrated in $E_2^{*,pi}$.

As in Section 3, let $D(n,i)^* = C(n,i)^{\otimes p}$ for $n \geq 0$ be a complex of $\mathbb{Z}C_p$-modules where C_p acts by permuting the factors (with a sign if $p = 2$ and i is odd). The only contributions to $H^*(X^p \times_{C_p} E)$ not accounted for by the above remarks come from summands of the double cochain complex of the form $\mathrm{Hom}_{C_p}(W_*, D(n,i))$, for $n > 1$. It is easy to see that the module $D(n,i)^j$ is the zero module unless $p(i-1) \leq j \leq pi$, and that $D(n,i)^j$ is $\mathbb{Z}C_p$-free of rank $1/p\binom{p}{j}$ if $p(i-1) < j < pi$. In the case when p is odd, $D(n,i)^{pi}$ and $D(n,i)^{p(i-1)}$ are the trivial $\mathbb{Z}C_p$-module $\mathbb{Z}$. When $p = 2$, $D(n,i)^{2i}$ is isomorphic to $\mathbb{Z}$ (resp. $\hat{\mathbb{Z}}$) and $D(n,i)^{2(i-1)}$ is isomorphic to $\hat{\mathbb{Z}}$ (resp. $\mathbb{Z}$) if i is even (resp. odd). To describe the differential in $D(n,i)^*$ completely we would have to choose an explicit generating set for each $D(n,i)^j$, and we shall not do this. All that we shall require in the sequel is that $D(1,i)^*$ is exact (because $C(1,i)^*$ is exact), and that if we view $C(1,i)^*$ and $C(n,i)^*$ as consisting of the same groups but with different maps, then the differential in $D(n,i)^*$ is n-times that in $D(1,i)^*$. Note that for p odd, each $D(n,i)$ is isomorphic to $D(n,0)$ shifted in degree by pi, and for $p = 2$, $D(n,2i)$ is isomorphic to $D(n,0)$ shifted by $4i$ and $D(n, 2i+1)$ is isomorphic to $D(n,1)$ shifted by $4i$.

Theorem 4.1. *For p an odd prime, let $E_0^{i,j} = \mathrm{Hom}_{C_p}(W_i, D(n,0)^j)$, and let $E_*^{*,*}$ be the corresponding type II spectral sequence. Then for $i > 0$, $E_2^{i,j}$ is zero except that $E_2^{2i,0}$ and $E_2^{2i,-p}$ are cyclic of order p. Define a function $g(j)$ as follows:*

$$g(j) = \begin{cases} 1/p((p-1)(-1)^j + \binom{p-1}{j}) & \text{for } 0 \leq j \leq p-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $E_2^{0,-j}$ is an extension of $\mathbb{Z}/p$ by $(\mathbb{Z}/n)^{g(j)}$ for $j = 0, 2, 4, \ldots, p-3$, nonsplit if p divides n, and for other j, $E_2^{0,-j}$ is isomorphic to $(\mathbb{Z}/n)^{g(j)}$. If p divides n then the spectral sequence collapses at E_2. If p does not divide n then the non-zero higher differentials are $d_3, d_5, \ldots, d_p$, and in this case $E_\infty^{i,j} = 0$ for $i > 0$ and $E_\infty^{0,-j} \cong (\mathbb{Z}/n)^{g(j)}$.

Remark. The reader may find it helpful to consult figures 1 and 2, which illustrate cases of the above statement.

Proof. For this spectral sequence $E_1^{i,j}$ is isomorphic to $H^i(C_p; D(n,0)^j)$. Since $D(n,0)^{-j}$ is the trivial module $\mathbb{Z}$ for $j=0$ or $j=p$, free of rank $1/p\binom{p}{j}$ for $0<j<p$ and zero for other j, we see that $E_1^{i,j}$ is trivial for $i>0$ except that $E_1^{2i,0} \cong E_1^{2i,-p} \cong \mathbb{Z}/p$. Also $E_1^{0,-j}$ is free abelian of rank $f(j)$, where f is defined as follows:

$$f(j) = \begin{cases} 1 & \text{for } j=0 \text{ or } j=p, \\ 1/p\binom{p}{j} & \text{for } 0<j<p, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the only non-zero groups on the E_1-page occur on the three line segments $j=0$ and $i \geq 0$, $j=-p$ and $i \geq 0$, $i=0$ and $-p \leq j \leq 0$. The shape of the E_1-page implies that $E_2^{i,j} = E_1^{i,j}$ for $i>0$, and that the only possibly non-zero differentials after d_1 are the following:

$$\begin{aligned} d_3 &: E_3^{2,-p} \to E_3^{0,3-p}, \\ d_5 &: E_5^{4,-p} \to E_5^{0,5-p}, \\ &\ \vdots \\ d_{p-2} &: E_{p-2}^{p-3,-p} \to E_{p-2}^{0,-2}, \\ \text{and} \quad d_p &: E_p^{p-1+2i,-p} \to E_p^{0,2i} \quad \text{for all } i \geq 0. \end{aligned}$$

To determine the groups $E_2^{0,j}$ and the higher differentials we shall first consider the case $n=1$ and then apply Lemma 1.1.

The complex $D(1,0)^*$ is exact, which implies that for $n=1$ the total complex of $E_0^{*,*}$ is exact, and hence $E_\infty^{i,j}=0$ for all i and j. We also know the isomorphism type of $E_2^{i,j}$ for all i and j except for the cases when $i=0$ and $-p \leq j \leq 0$. Since each of the possibly non-trivial groups on the E_2-page is involved in at most one possibly non-trivial higher differential, it follows that in the case $n=1$, all the possibly non-zero differentials listed above must be isomorphisms, and that all groups $E_2^{0,j}$ except those such that $E_r^{0,j}$ appears in the above list must be trivial. This completes the proof in the case $n=1$.

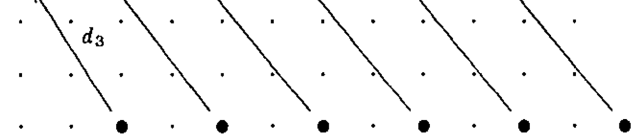

Fig. 1. The E_2-page and higher differentials for the spectral sequence of 4.1, when $p=7$ and $n=1$. '•' denotes a non-zero entry.

$\mathbb{Z}/7n$	-	$\mathbb{Z}/7$	-	$\mathbb{Z}/7$	-	$\mathbb{Z}/7$	$\cdots$
-	-	-	-	-	-	-	$\cdots$
$\mathbb{Z}/7n \oplus (\mathbb{Z}/n)^2$	-	-	-	-	-	-	$\cdots$
$(\mathbb{Z}/n)^2$	-	-	-	-	-	-	$\cdots$
$\mathbb{Z}/7n \oplus (\mathbb{Z}/n)^2$	-	-	-	-	-	-	$\cdots$
-	-	-	-	-	-	-	$\cdots$
$\mathbb{Z}/n$	-	-	-	-	-	-	$\cdots$
-	-	$\mathbb{Z}/7$	-	$\mathbb{Z}/7$	-	$\mathbb{Z}/7$	$\cdots$

Fig. 2. The E_2-page of the spectral sequence of 4.1 for $p = 7$.

Let Z^j and B^j stand for the cycles and boundaries respectively in $E_1^{0,-j}$ for the case $n = 1$. Then Z^j and B^j are free abelian of the same rank, and $Z^j = B^j$ except that Z^j/B^j has order p for $j = 0, 2, 4, \ldots, p-3$. Let $g'(j)$ stand for the rank of Z^j or B^j. By Lemma 1.1., the group of cycles for d_1 in $E_1^{0,-j}$ for general n is equal to Z^j, while the group of boundaries is equal to nB^j. It follows that for general n, $E_2^{0,-j}$ is the natural extension of Z^j/B^j by $B^j/nB^j \cong (\mathbb{Z}/n)^{g'(j)}$, which is nonsplit whenever possible. Thus to verify the claimed description of the E_2-page it suffices to show that $g'(j) = g(j)$. For this, note that the short exact sequence

$$0 \to Z^j \to E_1^{0,-j} \to B^{j-1} \to 0$$

implies that $g'(j) + g'(j-1) = f(j)$. Define polynomials $F(t)$, $G(t)$ in a formal variable t by

$$F(t) = \sum_j f(j)t^j, \qquad G(t) = \sum_j g'(j)t^j,$$

and note that

$$F(t) = \frac{p-1}{p}(1+t^p) + \frac{1}{p}(1+t)^p.$$

The relation given between f and g' implies that $(1+t)G(t) = F(t)$, from which it is easy to verify that $g' = g$.

It now remains only to check the given description of the higher differentials in the spectral sequence for general n. Once more we invoke Lemma 1.1. For $i > 0$, let $x \in E_0^{2i,-p}$ be an element representing a generator for $E_1^{2i,-p}$, and let $y \in E_0^{2i-1-p}$ (resp. $y \in E_0^{2i-p+1,0}$ if $2i > p$) be an element representing the image of x under d_{2i+1} (resp. d_p) in the spectral sequence for $n = 1$. If p divides n, then $n^j y$ will represent zero in E_2 for $j \geq 2$, and so in this case the higher differentials are trivial. If on the other hand p does not divide n then $n^j y$ will represent an element of order p in E_2 for $j \geq 1$, and so again the higher differentials are as claimed. ■

Theorem 4.1′. *For $p = 2$, let $E_0^{*,*}$ be the double cochain complex* $\mathrm{Hom}_{C_2}(W_*, D(n,0)^*)$. *Then in the corresponding type II spectral sequence, $E_2^{i,j} = 0$ except that $E_2^{2i+2,0} \cong E_2^{2i+1,-2} \cong \mathbb{Z}/2$ for all $i \geq 0$, and $E_2^{0,0} \cong \mathbb{Z}/(2n)$. If n is even the spectral sequence* *collapses at E_2. If n is odd, the spectral sequence collapses at E_3, and $E_3^{i,j} = 0$ except that $E_3^{0,0} \cong \mathbb{Z}/n$.*

Let $E'^{*,*}_0$ be the double cochain complex $\mathrm{Hom}_{C_2}(W_*, D(n,1)^*)$. Then in the corresponding type II spectral sequence $E_2^{i,j} = 0$ except that $E'^{2i+1,2}_2 \cong E'^{2i+2,0}_2 \cong \mathbb{Z}/2$ for all $i \geq 0$, and $E'^{0,1}_2 \cong \mathbb{Z}/n$. If n is even the spectral sequence collapses at E_2. If n is odd the spectral sequence collapses at E_3 and $E'^{i,j}_3 = 0$ except that $E'^{0,1}_3 \cong \mathbb{Z}/n$.

Proof. Similar to the proof of Theorem 4.1, and easier in spite of the extra complication introduced by the second action of C_2 on $\mathbb{Z}$. ∎

Theorem 4.2. *Let p be an odd prime and let W_* be an acyclic complex of free $\mathbb{Z}C_p$-modules. Let C^* be a cochain complex of finitely generated free abelian groups, and choose some splitting*

$$H^*(C^*) \cong \bigoplus_{a \in A} H(a),$$

where $H(a)$ is a summand of $H^{d(a)}(C^*)$, and is isomorphic to $\mathbb{Z}/(n(a))$, where (for simplicity and without loss of generality) each $n(a)$ is either 0, a power of p, or a positive integer coprime to p. Let C_p act on the set A^p by permuting the factors, so that elements of the form $(a_1, \ldots, a_p)$ are in free orbits provided that not all the a_i are equal, and elements of the form $(a, \ldots, a)$ are fixed. Then $H^*\mathrm{TotHom}_{C_p}(W_*, C^{*\otimes p})$ is a direct sum of the following summands.

a) *For each free C_p-orbit in A^p with orbit representative $(a_1, \ldots, a_p)$, and for each j, a direct sum of $\binom{r}{j-1}$ copies of $\mathbb{Z}/(n(a_1), \ldots, n(a_p))$ in degree $d(a_1) + \cdots + d(a_p) - j$. Here r is the number of a_i's such that $n(a_i)$ is non-zero.*

b) *For each a such that $n(a) = 0$, one copy of $\mathbb{Z}$ in degree $pd(a)$, and for each $i \geq 0$, one copy of $\mathbb{Z}/(p)$ in degree $pd(a) + 2 + 2i$.*

c) *For each a such that $n(a)$ is non-zero and coprime to p and for each j, a direct sum of $g(j)$ copies of $\mathbb{Z}/(na)$ in degree $pd(a) - j$, where $g(j)$ is as defined in the statement of Lemma 2.2.*

d) *For each a such that $n(a)$ is a power of p, for $j = 0, 2, \ldots, p-3$, a direct sum of $g(j) - 1$ copies of $\mathbb{Z}/(n(a))$ and one copy of $\mathbb{Z}/(pn(a))$ in degree $pd(a) - j$; for other j a sum of $g(j)$ copies of $\mathbb{Z}/(n(a))$ in degree $pd(a) - j$; and for each $i \geq 0$ one copy of $\mathbb{Z}/(p)$ in degree $pd(a) + 2 + 2i$ and one copy of $\mathbb{Z}/(p)$ in degree $p(d(a) - 1) + 2 + 2i$.*

Proof. The fact that we can split the cohomology as a direct sum of contributions of the types described, and the analyses of cases a) and b), were proved earlier. Cases c) and d) are descriptions of the cohomology of the total complex of $E_0^{*,*} = \mathrm{Hom}_{C_p}(W_*, D(n(\alpha), 0))$, and so follow from Theorem 4.1 except that in case d) we have to show that the extension with kernel $E_\infty^{0,-p+2i}$ and quotient $E_\infty^{2i,-p}$ representing H^{2i-p} is split for $1 \leq i \leq (p-1)/2$. For $i = 1$ this is obvious, because $g(p-2) = 0$ and hence $E_\infty^{0,2-p} = 0$. For other i we use (for the first time) the graded $H^*(C_p; \mathbb{Z})$-module structure of the spectral sequence, which is a filtration of the graded $H^*(C_p)$-module structure on $H^*\mathrm{Hom}_{C_p}(W_*, D(n,0)^*)$. It is easy to see that the product of a generator for $E_\infty^{2,-p}$ and a generator for $H^{2i}(C_p; \mathbb{Z})$ is a generator for $E_\infty^{2+2i,-p}$ and therefore that there is an element of $H^*(E_0^{*,*})$ of order p yielding a generator for $E_\infty^{2+2i,-p}$, and so the extension is split.

As an alternative we may solve the extension problem by calculating $H^*(\mathrm{Tot}E_0^{*,*} \otimes \mathbb{F}_p)$, which determines the number of cyclic summands of $H^*(\mathrm{Tot}E_0^{*,*})$ by the universal coefficient theorem. When p divides n, the differential in $D(n,0) \otimes \mathbb{F}_p$ is trivial, and so the type II spectral sequence for $H^*(\mathrm{Tot}E_0^{*,*} \otimes \mathbb{F}_p)$ collapses at the E_1-page, which makes this calculation easy. We leave the details to the interested reader. ∎

Theorem 4.2′. *Let $p = 2$, and let W_* be an acyclic complex of free $\mathbb{Z}C_2$-modules. Let C^* be a cochain complex of finitely generated free abelian groups and fix a splitting*

$$H^*(C^*) \cong \bigoplus_{a\in A} H(a),$$

as in the statement of Theorem 4.2. Then $H^*\mathrm{Hom}_{C_2}(W_*, C^{*\otimes 2})$ is a direct sum of the following summands.

a) For each $a \neq a' \in A$, one summand $\mathbb{Z}/(n(a), n(a'))$ in degree $d(a) + d(a')$, and if both $n(a)$ and $n(a')$ are nonzero, one summand $\mathbb{Z}/(n(a), n(a'))$ in degree $d(a) + d(a') - 1$.

b) For each a with $n(a) = 0$ and $d(a)$ even, one summand $\mathbb{Z}$ in degree $2d(a)$ and one summand $\mathbb{Z}/2$ in each degree $2d(a) + 2 + 2i$.

c) For each a with $n(a) = 0$ and $d(a)$ odd, one summand $\mathbb{Z}/2$ in each degree $2d(a)+1+2i$.

d) For each a with $n(a)$ odd and $d(a)$ even, one copy of $\mathbb{Z}/(n(a))$ in degree $2d(a)$.

e) For each a with $n(a)$ odd and $d(a)$ odd, one copy of $\mathbb{Z}/(n(a))$ in degree $2d(a) - 1$.

f) For each a with $n(a)$ a (strictly positive) power of 2 and $d(a)$ even, one copy of $\mathbb{Z}/(2n(a))$ in degree $2d(a)$ and one copy of $\mathbb{Z}/2$ in degree $2d(a) - 1$, and in each degree of the form $2d(a) + 1 + i$.

g) For each a with $n(a)$ a (strictly positive) power of 2 and $d(a)$ odd, one copy of $\mathbb{Z}/(n(a))$ in degree $2d(a) - 1$ and once copy of $\mathbb{Z}/2$ in each degree $2d(a) + i$.

Proof. This follows from the preamble together with Theorem 4.1′. Note that this is simpler than Theorem 4.2 in that there are no extension problems that need be resolved. ∎

5. The spectral sequence for the cyclic group of order p.

In the previous section we computed the integral cohomology of $X^p \times_{C_p} EC_p$ for any finite type CW-complex X, by replacing the double cochain complex forming the E_0-page of the Cartan-Leray spectral sequence with a direct sum of simpler complexes. In this section we solve the associated type I spectral sequences, and hence describe the differentials and extension problems in the Cartan-Leray spectral sequence for $X^p \times_{C_p} EC_p$.

For C^* a cochain complex of finitely generated free abelian groups, fix a splitting of $H^*(C^*)$ as a direct sum of cyclic groups,

$$H^*(C^*) = \bigoplus_{a\in A} H(a)$$

as in the statement of Theorem 4.2. As in the preamble to Section 4, choose a cochain complex C'^* splitting as a direct sum of subcomplexes $C'(a)^*$ (indexed by the same set A), and a homotopy equivalence f from C'^* to C^*, the cochains on X, such that the image of

$H^*(C'(a)^*)$ under f^* is $H(a) \subseteq H^*(C^*)$. As in Section 4, let W_* be the chain complex for EC_p. By Lemma 1.4, the double complexes

$$E_0^{*,*} = \mathrm{Hom}_{C_p}(W_*, ((C^*)^{\otimes p})^*) \quad \text{and} \quad E'^{*,*}_0 = \mathrm{Hom}_{C_p}(W_*, ((C'^*)^{\otimes p})^*)$$

give rise to isomorphic type I spectral sequences, from the E_1-page onwards. If C^* is the cellular cochain complex of a finite type CW-complex X, then the type I spectral sequence for $E_0^{*,*}$ is the Cartan-Leray spectral sequence for $X^p \times_{C_p} EC_p$.

We have already seen that $E'^{*,*}_0$ splits as a direct sum of subcomplexes indexed by the C_p-orbits in A^p. In the preamble to Section 4 we showed that the type I spectral sequences corresponding to free C_p-orbits in A^p have E_2-page concentrated in the column $E'^{0,*}_2$, so collapse at E_2 and give rise to no extension problems. We also showed that the trivial C_p-orbits of the form $(a, \ldots, a)$ such that $H(a)$ is infinite cyclic give rise to double complexes with a single nonzero row, and hence the type I spectral sequences for such orbits collapse at E_2 and give rise to no extension problems. Thus the Cartan-Leray spectral sequence for $X^p \times_{C_p} EC_p$ splits from E_2 onwards as a direct sum of the following:

a) Various pieces concentrated in $E_*^{0,*}$.
b) For each $a \in A$ such that $H(a) \subseteq H^i(X)$ is infinite cyclic, a piece concentrated in the row $E_*^{*,pi}$.
c) For each a such that $H(a) \subseteq H^i(X)$ is cyclic of order n, a copy of the type I spectral sequence for the double complex $\mathrm{Hom}_{C_p}(W_*, D(n,i)^*)$ defined in the introduction to Section 4.

Thus to solve the Cartan-Leray spectral sequence, it suffices to solve the type I spectral sequences for the double complexes of Theorems 4.1 and 4.1′.

Theorem 5.1. *Let p be an odd prime, and let $E_0^{*,*}$ be the double cochain complex of Theorem 4.1, and let $E_*^{*,*}$ be the corresponding type I spectral sequence. If p does not divide n, then $E_2^{i,j}$ is as follows, where the function g is as defined in Theorem 4.1:*

$$E_2^{i,j} = \begin{cases} (\mathbb{Z}/n)^{\oplus g(-j)} & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases}$$

In this case the spectral sequence clearly collapses and gives rise to no extension problems. If p divides n, the E_2-page is as follows:

$$E_2^{i,j} = \begin{cases} (\mathbb{Z}/n)^{\oplus g(-j)} \oplus \mathbb{Z}/p & \text{for } i = 0,\ j \text{ odd},\ 0 > j > 1-p, \\ (\mathbb{Z}/n)^{\oplus g(-j)} & \text{for } i = 0,\ j \text{ not as above}, \\ \mathbb{Z}/p & \text{for } i > 0,\ 0 \geq j \geq 1-p, \\ 0 & \text{otherwise}. \end{cases}$$

In this case the spectral sequence collapses at E_3, and the E_3-page is as follows:

$$E_3^{i,j} = \begin{cases} (\mathbb{Z}/n)^{\oplus g(-j)} & \text{if } i = 0, \\ \mathbb{Z}/p & \text{if either } j = 0,\ i > 0 \text{ and } i \text{ even}, \\ & \quad \text{or } j = 1-p \text{ and } i \text{ odd}, \\ & \quad \text{or } i = 1 \text{ and } j = -1, -3, \ldots, 2-p, \\ 0 & \text{otherwise}. \end{cases}$$

The only non trivial extensions in reassembling $H^*\mathrm{Tot}E_0^{*,*}$ from $E_\infty^{*,*}$ are that the extension with kernel $E_3^{1,-j}$ and quotient $E_3^{0,1-j}$ is non-split for $j = 1, 3, \ldots, p-2$.

Remark. Figure 3 illustrates the case of the above statement when $p = 7$ and n is a multiple of p. In the figure, circles indicate entries isomorphic to $\mathbb{Z}/7$, and squares indicate other non-zero entries. Entries in black remain non-zero in $E_3 = E_\infty$. Arrows represent non-zero d_2's, and double lines represent non-split extensions.

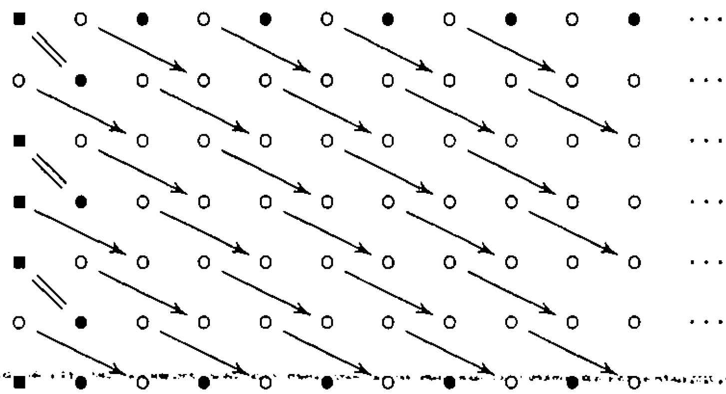

Fig. 3. The E_2-page of the spectral sequence of 5.1, for $p = 7$ and $p|n$.

Proof. Recall that $E_0^{i,j} = \mathrm{Hom}_{C_p}(W_i, D(n,0)^j)$, and so $E_1^{i,j} = \mathrm{Hom}_{C_p}(W_i, H^j(D(n,0)^*))$ because W_i is free, and then $E_2^{i,j} = H^i(C_p; H^j(D(n,0)^*))$. The complex $D(1,0)^*$ is exact, and if we let Z^j stand for the cycles in $D(1,0)^{-j}$, then for any n,

$$H^{-j}(D(n,0)^*) = Z^j/nZ^j.$$

Since Z^j is free abelian, the cochain complex $\mathrm{Hom}_{C_p}(W_*, Z^j)$ is also free abelian, and we have the following isomorphism.

$$\mathrm{Hom}_{C_p}(W_*, Z^j/nZ^j) \cong \mathrm{Hom}_{C_p}(W_*, Z^j)/n\mathrm{Hom}_{C_p}(W_*, Z^j)$$

Hence we may compute $E_2^{i,-j} = H^i(C_p; Z^j/nZ^j)$ by first computing $H^*(C_p; Z^j)$ and then applying a universal coefficient theorem.

Now $Z^0 = D(1,0)^0 = \mathbb{Z}$, $Z^j = 0$ unless $0 \le j \le p-1$, and for $0 < j \le p-1$ there is a short exact sequence of C_p-modules

$$0 \to Z^j \to F_j \to Z^{j-1} \to 0$$

where F_j is a free module of rank $\frac{1}{p}\binom{p}{j}$. Taking cohomology we obtain for $0 < j < p$ the following exact sequences.

$$0 \to H^0(C_p; Z^j) \to H^0(C_p; F_j) \to H^0(C_p; Z^{j-1}) \to H^1(C_p; Z^j) \to 0$$

$$0 \to H^{i+1}(C_p; Z^{j-1}) \to H^{i+2}(C_p; Z^j) \to 0$$

Using the second of these and the periodicity of the cohomology of C_p with arbitrary coefficients, it follows easily that for $i > 0$ and $0 \leq j \leq p-1$,

$$H^i(C_p; Z^j) = \begin{cases} \mathbb{Z}/p & \text{for } i+j \text{ even,} \\ 0 & \text{for } i+j \text{ odd.} \end{cases}$$

For each j, $H^0(C_p, Z^j)$ is a free abelian group, so we need only find its rank. From the first of the two exact sequences we obtain for $0 < j < p$ that

$$\mathrm{Rk}H^0(C_p; Z^j) + \mathrm{Rk}H^0(C_p; Z^{j-1}) = \mathrm{Rk}H^0(C_p; F_j) = \frac{1}{p}\binom{p}{j}.$$

Note also that $H^0(C_p; Z^0)$ has rank 1, and so the rank of $H^0(C_p; Z^j)$ satisfies the same recurrence relation as $g(j)$ defined in Lemma 2.2, so is equal to $g(j)$. Now the universal coefficient theorem tells us that

$$E_2^{i,j} \cong H^i(C_p; Z^j/nZ^j) \cong \mathrm{Tor}(H^{i+1}(C_p; Z^j), \mathbb{Z}/n) \oplus H^i(C_p; Z^j) \otimes \mathbb{Z}/n,$$

and we see that $E_2^{*,*}$ is as claimed.

It will turn out that the E_2-page, together with the cohomology of $E_0^{*,*}$ (which was calculated in the last section), suffices to determine the pattern of higher differentials. We claim that the only non-zero higher differential is

$$d_2 : E_2^{i,j} \to E_2^{i+2,j-1},$$

where $i \geq 0$, $0 \geq j > 1-p$ and $i+j$ is odd. Moreover, when $i = 0$, the kernel of d_2 is a direct summand of $E_2^{i,j}$. These claims imply those made in the statement concerning the E_3-page. Let

$$A_m = \bigoplus_{i+j=m} E_2^{i,j},$$

so that d_2 and the higher differentials give rise to maps from subquotients of A_m to subquotients of A_{m+1}. Then for $m > 0$, the order of A_m is p^p, while for $0 \geq m \geq 1-p$ the order of A_m is $n^{g(-m)}p^{2[(p-m)/2]}$ (here the square brackets indicate the greatest integer less than or equal to their contents). The order of $H^m\mathrm{Tot}(E_0^{*,*})$ is, by Theorem 4.1, equal to $n^{g(-m)}p$ for $m \geq 2-p$, and $n^{g(-m)}$ for $m \leq 1-p$. For $m = 1-p$, the orders of A_m and $H^m(E_0^{*,*})$ are equal, and so there are no non-zero higher differentials leaving A_{1-p}. (This is also easy to see directly, because A_{1-p} is concentrated in the bottom left corner of the E_2-page.) Now assume that m is even, and that we have shown that A_m consists of cycles for all the higher differentials. In this case, the universal cycles in A_{m+1} form a filtration of $H^{m+1}(E_0^{*,*})$, and in particular these two groups have the same order. The universal cycles in A_{m+2} modulo the image of A_{m+1} under all differentials forms a filtration of $H^{m+2} = H^{m+2}(E_0^{*,*})$. Taking the orders of various groups we obtain the following inequality,

$$|H^{m+2}| = |\text{cycles in } A_{m+2}|/|\text{image of } A_{m+1}| \leq |A_{m+2}|.|H^{m+1}|/|A_{m+1}|,$$

with equality if and only if A_{m+2} consists entirely of universal cycles. It is easy using the numbers given above to check that equality does hold, and so by induction A_{m+2} consists of universal cycles.

Now pick any strictly positive m. By the above inductive argument, all elements of A_{2m+2} are universal cycles, and it is clear that $E_2^{2m+2,0}$ cannot be hit by any differential. However, H^{2m+2} has order p, so the other $p-1$ summands of A_{2m+2} of order p must all be hit by some differential. The only way for this to happen is if

$$d_2 : E_2^{2m+1+i,-i} \to E_2^{2m+3+i,-i-1}$$

is an isomorphism for $0 \leq i < p-1$. The $H^*(C_p;\mathbb{Z})$-module structure of the spectral sequence allows us to deduce the claimed description of d_2 on $E_2^{i,j}$ for all $i > 0$. To see that d_2 on $E_2^{0,j}$ is non-zero for j odd and $0 > j > 1-p$, note that no higher differential can hit a generator for $E_2^{2,j-1}$, but this generator must be hit to ensure that the product over i of the orders of $E_\infty^{2+i,j-i-1}$ equals the order of $H^{2+j-1}(E_0^{*,*})$. To see that the kernel of d_2 on $E_2^{0,j}$ is isomorphic to $(\mathbb{Z}/n)^{g(-j)}$ for j odd, note that since $d_2 : E_2^{2,j} \to E_2^{4,j-1}$ is an isomorphism, the kernel of $d_2 : E_2^{0,j} \to E_2^{2,j-1}$ is equal to the kernel of the map from $H^0(C_p; Z^j/nZ^j) = E_2^{0,j}$ to $H^2(C_p; Z^j/nZ^j) = E_2^{2,j}$ induced by the cup product with a generator of $H^2(C_p;\mathbb{Z})$. This kernel contains the image of $H^0(C_p; Z^j)$, which is isomorphic to $(\mathbb{Z}/n)^{g(-j)}$, because $H^2(C_p; Z^j) = 0$ for odd j. Given the description of E_3 claimed in the statement, it is easy to see that there are no further non-zero differentials, and that the non-trivial extension problems must be as claimed. ∎

Theorem 5.1′. *Let $p = 2$, let $E_0^{*,*}$ and $E'^{*,*}_0$ be as in the statement of Theorem 4.1′, and let $E_*^{*,*}$, $E'^{*,*}_*$ be the corresponding type I spectral sequences. Then if n is odd,*

$$E_2^{i,j} = \begin{cases} \mathbb{Z}/n & \text{if } i = j = 0, \\ 0 & \text{otherwise} \end{cases}$$

$$E'^{i,j}_2 = \begin{cases} \mathbb{Z}/n & \text{if } i = 0,\ j = 1, \\ 0 & \text{otherwise} \end{cases}$$

and both spectral sequences of course collapse at E_2.

If n is even, then

$$E_2^{i,j} = \begin{cases} \mathbb{Z}/n & \text{if } i = j = 0, \\ \mathbb{Z}/2 & \text{if } j = -1 \text{ and } i \geq 0, \text{ or if } j = 0 \text{ and } i > 0, \\ 0 & \text{otherwise} \end{cases}$$

$$E'^{i,j}_2 = \begin{cases} \mathbb{Z}/n & \text{if } i = 0,\ j = 1, \\ \mathbb{Z}/2 & \text{if } j = 2 \text{ and } i \geq 0, \text{ or if } j = 1 \text{ and } i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Both spectral sequences collapse at E_3, and

$$E_3^{i,j} = \begin{cases} \mathbb{Z}/n & \text{if } i = j = 0, \\ \mathbb{Z}/2 & \text{if } (i,j) = (0,-1) \text{ or } (1,-1), \text{ or if } i > 0 \text{ and even and } j = 0 \text{ or } -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$E'^{i,j}_3 = \begin{cases} \mathbb{Z}/n & \text{if } i = 0 \text{ and } j = 1, \\ \mathbb{Z}/2 & \text{if } i \text{ is odd and } j = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

The only non-trivial extension in reassembling $H^*\mathrm{Tot}E_0^{*,*}$ and $H^*\mathrm{Tot}E_0'^{*,*}$ from the E_∞-pages is that the extension with kernel $E_3^{1,-1}$ and quotient $E_3^{0,0}$ is non-split.

Proof. Similar to, but far easier than, that of Theorem 5.1. ∎

6. A detection lemma.

Let X be a finite type CW-complex and let α and β be the maps:

$$\alpha : X^p \times EC_p \to X^p \times_{C_p} EC_p, \qquad \beta : X \times BC_p \to X^p \times_{C_p} EC_p.$$

Here α is the covering map, and β is induced by the map $\Delta : (x, e) \mapsto (x, \ldots, x, e)$ from $X \times EC_p$ to $X^p \times EC_p$, which is C_p-equivariant for the trivial action on X and the permutation action on X^p. One easy corollary of Nakaoka's description of the mod-p cohomology of $X^p \times_{C_p} EC_p$ is Quillen's detection lemma [25], which states that the map

$$(\alpha^*, \beta^*) : H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p) \to H^*(X^p \times EC_p; \mathbb{F}_p) \times H^*(X \times BC_p; \mathbb{F}_p)$$

is injective. The result of this section is an integral analogue.

Corollary 6.1. *Let p be a prime, and as usual let X be a finite-type CW-complex, and let α, β be as above. Then the kernel of the map*

$$(\alpha^*, \beta^*) : H^*(X^p \times_{C_p} EC_p) \to H^*(X^p \times EC_p) \times H^*(X \times BC_p)$$

has exponent p, and does not contain any cyclic summand of $H^*(X^p \times_{C_p} EC_p)$. If $H^*(X)$ is expressed as a direct sum of cyclic groups, then the kernel may be described as follows:

For p an odd prime, each cyclic summand of $H^i(X)$ of finite order divisible by p gives rise to one cyclic summand of $\ker(\alpha^*, \beta^*)$ in each degree $pi, pi - 2, \ldots, p(i-1) + 3$, and these summands form the whole of $\ker(\alpha^*, \beta^*)$.

For $p = 2$, each cyclic summand of $H^{2i}(X)$ of finite even order gives rise to one cyclic summand of $\ker(\alpha^*, \beta^*)$ in degree $4i$, and these summands form the whole kernel. If x generates a summand of $H^{2i}(X)$ of even order n, then $n(x \wr 1)$ generates the corresponding summand of the kernel.

Proof. Fix a splitting of $H^*(X)$ as a direct sum of cyclic groups. Note that α^* is the edge map in the spectral sequence whose E_∞-page was described in Theorems 5.1 and 5.1′. It follows that $\ker(\alpha^*)$ consists of a direct sum of cyclic subgroups of order p as described in the statement, which are not direct summands of $H^*(X^p \times_{C_p} EC_p)$, and various cyclic direct summands of $H^*(X^p \times_{C_p} EC_p)$ of order p. Consider the map of Cartan-Leray spectral sequences induced by the following map of p-fold covering spaces:

$$\begin{array}{ccc} X \times EC_p & \longrightarrow & X \times BC_p \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \beta} \\ X^p \times EC_p & \longrightarrow & X^p \times_{C_p} EC_p. \end{array}$$

The spectral sequence for $X \times BC_p$ collapses at the E_2-page, and for $i > 0$, $E_2^{i,j}$ consists of cyclic direct summands of $H^*(X \times BC_p)$ of order p. The map on E_∞-pages is a filtration of β^*, so it follows that the image $\beta^*(\ker(\alpha^*))$ is a sum of cyclic summands of $H^*(X \times BC_p)$ of order p. Thus any element of $\ker(\alpha)$ not generating a cyclic summand of $H^*(X^p \times_{C_p} EC_p)$ is also in $\ker(\beta^*)$, and $\ker(\alpha^*, \beta^*)$ is at least as large as claimed. To see that no cyclic summand of $H^*(X^p \times_{C_p} EC_p)$ of order p may be contained in $\ker(\alpha^*, \beta^*)$, consider the following diagram.

$$\begin{array}{ccc} H^*(X^p \times_{C_p} EC_p) & \xrightarrow{(\alpha^*,\beta^*)} & H^*(X^p) \times H^*(X \times BC_p) \\ \downarrow & & \downarrow \\ H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p) & \xrightarrow{(\alpha^*,\beta^*)} & H^*(X^p) \times H^*(X \times BC_p; \mathbb{F}_p) \end{array}$$

The lower horizontal map is known to be injective, and the kernel of the left-hand map is $p.H^*(X^p \times_{C_p} EC_p)$. Thus the lower composite is injective on any summand of $H^*(X^p \times_{C_p} EC_p)$ of exponent p, and hence so is the top horizontal map. ■

Remark. One could make a more general statement for cochain complexes C^* equipped with a map from C^* to $C^* \otimes C^*$ having properties similar to a diagonal approximation for the cellular cochain complex of a CW-complex. This map is used in the definition of the map β^* above.

7. p-local cohomology for the symmetric group Σ_p.

The methods used in Sections 4, 5, and 6 may be used to compute the cohomology with coefficients $\mathbb{Z}_{(p)}$, the integers localized at p, of $X^p \times_S ES$ for any subgroup S of the symmetric group on a set of size p. If S does not act transitively (or equivalently has order coprime to p), then the Cartan-Leray type spectral sequence for $X^p \times_S ES$ has E_2-page concentrated in the line $E_2^{0,*}$. If S does act transitively, then either the results of sections 4, 5, and 6, together with some transfer argument could be used, or the methods used above could be applied directly. The case when S is the full symmetric group works particularly easily. As examples we give the following, which are analogues of 4.1 and 6.1.

Throughout this section, we let W_* be the chain complex with $\mathbb{Z}_{(p)}$ coefficients for a contractible, free Σ_p-CW-complex. Let $D'(n,i)$ be the complex

$$D'(n,i)^* = D(n,i)^* \otimes \mathbb{Z}_{(p)} = C(n,i)^{*\otimes p} \otimes \mathbb{Z}_{(p)}$$

of $\mathbb{Z}_{(p)}\Sigma_p$-modules, where $D(n,i)$ is as defined in Sections 3 and 4, and without loss of generality, n may be taken to be a power of p. The complexes $D(n,2i)^{*-2pi}$ and $D(n,0)^*$ are isomorphic, as are the complexes $D(n,2i+1)^{*-2pi}$ and $D(n,1)^*$.

Theorem 7.1. *Let p be an odd prime, and let n be a (strictly positive) power of p. Define double cochain complexes $E_0^{*,*}$ and $E_0'^{*,*}$ by*

$$E_0^{i,j} = \mathrm{Hom}_{\Sigma_p}(W_i, D'(n,0)^j), \qquad E_0'^{i,j} = \mathrm{Hom}_{\Sigma_p}(W_i, D'(n,1)^j),$$

where W_* and $D'(n,i)^*$ are as above. The corresponding type II spectral sequences collapse at E_2, and give rise to no extension problems. The E_2-pages are as follows.

$$E_2^{i,j} = \begin{cases} \mathbb{Z}_{(p)}/(pn) & \text{for } i = j = 0, \\ \mathbb{Z}_{(p)}/(p) & \text{for } i = 0,\ j = (2j'+2)(p-1), \\ & \quad \text{or } i = -p,\ j = (2j'+1)(p-1), \\ 0 & \text{otherwise.} \end{cases}$$

$$E_2'^{i,j} = \begin{cases} \mathbb{Z}_{(p)}/(n) & \text{for } i = 0,\ j = 1, \\ \mathbb{Z}_{(p)}/(p) & \text{for } i = 0,\ j = (2j'+2)(p-1), \\ & \quad \text{or } i = p,\ j = (2j'+1)(p-1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Similar to the proof of Theorem 4.1. To calculate the E_1-pages of the spectral sequences, note that as a $\mathbb{Z}_{(p)}\Sigma_p$-module,

$$D(n,0)^{-j} \cong D(n,1)^j \cong \mathrm{Ind}_{\Sigma_j \times \Sigma_{p-j}}^{\Sigma_p}(\epsilon_j),$$

where ϵ_j is the sign representation for Σ_j tensored with the trivial representation of Σ_{p-j}. Using the Eckmann-Shapiro lemma it follows that

$$E_1^{0,0} \cong E_1^{0,-1} \cong E_1'^{0,0} \cong E_1'^{0,1} \cong \mathbb{Z}_{(p)},$$

and that with these exceptions $E_1^{i,j} = E_2^{i,j}$ and $E_1'^{i,j} = E_2'^{i,j}$ as described in the statement. As in Theorem 4.1, the collapse at E_2 follows by comparing with the case $n = 1$ and applying Lemma 1.1. ■

Corollary 7.2. *Let X be a CW-complex of finite type, and let α, β be the maps*

$$X^p \times E\Sigma_p \xrightarrow{\alpha} X^p \times_{\Sigma_p} E\Sigma_p,$$
$$X \times B\Sigma_p \xrightarrow{\beta} X^p \times_{\Sigma_p} E\Sigma_p,$$

analogous to the maps occurring in the statement of Corollary 6.1. If α^*, β^* are the induced maps on cohomology with $\mathbb{Z}_{(p)}$ coefficients, then $\ker(\alpha^*, \beta^*)$ is concentrated in degrees divisible by $2p$, and may be described as follows. Every cyclic summand of $H^{2i}(X;\mathbb{Z}_{(p)})$ of order p^r gives rise to a summand of $\ker(\alpha^*,\beta^*)$ of order p in degree $2pi$. If x generates a summand of $H^{2i}(X;\mathbb{Z}_{(p)})$ of order p^r, then $p^r(x \wr 1)$ generates the corresponding summand of the kernel.

Proof. Similar to that of Corollary 6.1. ■

Remark. Another generalization of the material in Section 4 is to consider other possibilities for W_*. We have made calculations similar to those in Sections 4 and 5 for the case when S is cyclic of order p, and W_* is the chain complex of a sphere with a free S-action. In this case the non-zero entries on the E_2-page of the type II spectral sequence are concentrated at the edges of a rectangle (of height p and width the dimension of the sphere), and the left-hand edge of the rectangle is identical to the start of the 'infinite rectangle' considered in Section 4. This gives enough information to reconstruct the whole E_2-page and the higher differentials may be calculated using Lemma 1.1.

8. The exponent of group cohomology.

In this section we apply the results of previous sections to the question of determining the exponent of the integral cohomology of finite groups. It is simpler to concentrate on p-groups, and we shall do this, although we shall also make some remarks about the case of arbitrary finite groups.

Before starting, we recall the Evens-Venkov theorem ([14,5]), which states that if G is a finite group and H is a subgroup of G, then $H^*(G)$ is a finitely generated ring, and $H^*(H)$ is a finitely generated module for $H^*(G)$.

Definition A. *For G a finite group, say that G enjoys property A when for all i, if there exists j such that $H^j(G)$ contains an element of order p^i, then there exist infinitely many such j.*

Property A was introduced by A. Adem in [1], and by H.-W. Henn in [17], but see also [22, q. no. 754]. No p-group is known which does not have property A, and Adem made conjecture A; the conjecture that all p-groups have property A. Henn also asked whether all p-groups have property A. If G is a finite group with Sylow subgroup G_p, then $H^j(G)$ contains elements of order p^i for infinitely many j if and only if $H^j(G_p)$ does. Thus G enjoys propery A if its Sylow-p subgroup does.

One may reformulate property A in terms of the cohomological exponent $\mathrm{e}(G)$ and eventual cohomological exponent $\mathrm{ee}(G)$ of a finite group G, which we define as follows, where $\exp(-)$ stands for the exponent of an abelian group:

$$\mathrm{e}(G) = \exp\Big(\bigoplus_{i>0} H^i(G)\Big), \qquad \mathrm{ee}(G) = \lim_{j\to\infty} \exp\Big(\bigoplus_{i>j} H^i(G)\Big).$$

Note that for any G, $\mathrm{ee}(G)$ divides $\mathrm{e}(G)$ and $\mathrm{e}(G)$ divides $|G|$. The group G enjoys property A if and only if $\mathrm{e}(G) = \mathrm{ee}(G)$. Adem did not make the above definition, but he pointed out that the Evens-Venkov theorem implies that if $H \leq G$, then $\mathrm{ee}(H)$ divides $\mathrm{ee}(G)$ [1]. It seems to be unknown whether a similar property holds for $\mathrm{e}(G)$, so we make

Conjecture A$^-$. *For G any p-group and H any subgroup of G, $\mathrm{e}(H)$ divides $\mathrm{e}(G)$.*

We call this conjecture A$^-$ because it is weaker than conjecture A.

Proposiion 8.1. *Let G be a p-group. Then*

$$\mathrm{ee}(G \wr C_p) = p \cdot \mathrm{ee}(G) \quad \textit{and} \quad \mathrm{e}(G \wr C_p) = p \cdot \mathrm{e}(G),$$

except that possibly $\mathrm{e}(G \wr C_2) = \mathrm{e}(G)$ if $p = 2$ and $H^*(G)$ contains only finitely many cyclic summands of order $\mathrm{e}(G)$, all of which occur in odd degree.

Proof. First consider the case when p is odd. Express $H^*(G)$ as a direct sum of cyclic subgroups. All of these have order dividing $|G|$, except that $H^0(G) = \mathbb{Z}$. By Theorem 4.2, each p-tuple of cyclic summands of $H^*(G)$, not all equal, of orders $n_1, \ldots, n_p$ gives rise to finitely many cyclic summands of $H^*(G \wr C_p)$ of order h.c.f.$\{n_1, \ldots, n_p\}$. Each cyclic summand of $H^*(G)$ of order p^i also contributes $(p-1)/2$ cyclic summands of $H^*(G \wr C_p)$ of order p^{i+1}, infinitely many summands of order p, and finitely many other summands of order dividing p^i. The only other contribution is from $H^0(G)$, which gives rise to $H^0(G \wr C_p)$, and other summands of order p. The claim follows.

The case when $p = 2$ is similar, relying on Theorem 4.2′. The extra difficulty arises because summands of $H^*(G)$ of order 2^i in odd degree do not contribute any summands of $H^*(G \wr C_2)$ of order 2^{i+1}. It remains to rule out the possibility that $H^*(G)$ contains infinitely many cyclic summands of order $2^i = \mathrm{ee}(G)$, all but finitely many of which occur in odd degrees. However, $H^*(G)$ is a finitely generated ring, and since the square of any generator has even degree, $H^*(G)$ is a finitely generated module for the subring of elements of even degree. Let the supremum of the degrees of a (finite) set of module generators for $H^*(G)$ over the even degree subring be m. Now suppose that all elements of $H^{2j}(G)$ have exponent strictly less than 2^i whenever $j > j_0$. In this case, $H^j(G)$ may contain elements of exponent 2^i only for $j \leq 2j_0 + m$, and so $H^*(G)$ can have only finitely many cyclic summands of order 2^i. ■

Corollary 8.2. *Let G be a p-group. If p is odd, then G enjoys property A if and only if $G \wr C_p$ does. If $p = 2$ and G enjoys property A, then so does $G \wr C_2$.* ■

The following slightly stronger property than property A fits well with wreath product arguments, as can be seen from Theorem 8.3.

Definition A′. *For p an odd prime, say that a p-group G enjoys property A′ if for each $i > 0$, whenever there exists j such that $H^j(G)$ contains a cyclic summand of order p^i, there exist infinitely many such j.*

Say that a 2-group G enjoys property A′ if for each $i > 0$ and $\epsilon = 0, 1$, whenever there exists a j such that $H^{2j+\epsilon}(G)$ contains a cyclic summand of order p^i, there exist infinitely many such j.

I know of no p-groups not enjoying property A′, and it seems reasonable to make conjecture A′, i.e., to conjecture that all p-groups enjoy property A′.

Theorem 8.3. *Let G and S' be p-groups, and let S' act on the finite set Ω, with image $S \leq \Sigma(\Omega)$. If both G and S' enjoy property A′, then so does $G \wr S'$.*

Proof. As at the end of Section 1, we fix a splitting of $H^*(G)$ as a direct sum of cyclic groups indexed by a set A, and let

$$C'^* = \bigoplus_{a \in A} C(a)^*$$

be a direct sum of complexes of the form $C(n,i)^*$ having cohomology isomorphic to $H^*(G)$.

Let $\mathbf{a}$ stand for the S'-orbit (or equivalently S-orbit) in A^Ω containing $(a_1, \ldots, a_l)$, and let $D(\mathbf{a})^*$ be the $\mathbb{Z}S$-subcomplex of $C'^{*\otimes\Omega}$ generated by $C(a_1) \otimes \cdots \otimes C(a_l)$. Then as complexes of $\mathbb{Z}S$-modules,

$$C'^{*\otimes\Omega} = \bigoplus_{\mathbf{a} \in A^\Omega/S} D(\mathbf{a})^*, \tag{3}$$

where the sum is over the S-orbits in A^Ω. For p odd, the isomorphism type of $D(\mathbf{a})^*$ (modulo a shift in degree) depends only on the orders of the summands $H(a_1), \ldots, H(a_l)$

of $H^*(G)$, because no subgroup of S' has a non-trivial sign representation. For $p = 2$, the isomorphism type of $D(\mathbf{a})^*$ (modulo an even shift in degree) depends on the parity of the degrees of the $H(a_i)$ as well as on their orders. In either case, property A$'$ for G implies that each isomorphism type of $D(\mathbf{a})^*$ that occurs in (3) occurs infinitely often, except for $D(\mathbf{a}_0)$, $\mathbf{a}_0 = (a_0, \dots, a_0)$, where $H(a_0) = H^0(G) \cong \mathbb{Z}$. The complex $D(\mathbf{a}_0)$ is isomorphic to the trivial S-module $\mathbb{Z}$ concentrated in degree zero.

Let K be the kernel of the homomorphism from S' onto S, and let W_* be the cellular chain complex for ES'/K, viewed as a complex of free $\mathbb{Z}S$-modules. By Lemma 1.4, $H^*(G \wr S')$ is isomorphic to the cohomology of the total complex of

$$\begin{aligned} E_0^{*,*} &= \mathrm{Hom}_S(W_*, \bigoplus_{\mathbf{a} \in A^\Omega/S} D(\mathbf{a})^*) \\ &= \bigoplus_{\mathbf{a} \in A^\Omega/S} \mathrm{Hom}_S(W_*, D(\mathbf{a})^*) \\ &= \bigoplus_{\mathbf{a} \in A^\Omega/S} E(\mathbf{a})_0^{*,*}. \end{aligned}$$

From the analysis of the $D(\mathbf{a})^*$'s given above, it follows that each isomorphism type of $E(\mathbf{a})_0^{*,*}$ that occurs, occurs infinitely often, except for $E(\mathbf{a}_0)_0^{*,*}$. But

$$E(\mathbf{a}_0)_0^{*,*} \cong \mathrm{Hom}_S(W_*, \mathbb{Z}),$$

and so $H^*\mathrm{Tot}E(\mathbf{a}_0)_0^{*,*}$ is isomorphic to $H^*(S')$. Thus if $\mathbf{a} \neq \mathbf{a}_0$, then any cyclic summand of $H^*\mathrm{Tot}E(\mathbf{a})_0^{*,*}$ is also a summand of $H^*\mathrm{Tot}E(\mathbf{a}')_0^{*,*}$ for infinitely many $\mathbf{a}'$ by property A$'$ for G, while any cyclic summand of $H^*\mathrm{Tot}E(\mathbf{a}_0)_0^{*,*}$ occurs infinitely often by property A$'$ for S'. ∎

For G a p-group, define $\Phi_n(G)$ to be the intersection of the subgroups of G of index at most p^n. Thus $\Phi_1(G)$ is the Frattini subgroup of G, and each $\Phi_n(G)$ is a characteristic subgroup of G. In [21] we pointed out that these subgroups give a group-theoretic description of a good upper bound for $\mathrm{ee}(G)$, as follows.

Proposition 8.4. *Let G be a p-group such that $\Phi_n(G) = \{1\}$. Then $\mathrm{ee}(G)$ divides p^n.*

Proof. If $H_1, \dots, H_m$ are a family of subgroups of G with trivial intersection, then the natural map

$$G \longrightarrow \Sigma(G/H_1) \times \cdots \times \Sigma(G/H_m)$$

sending an element g to the permutation $g'H_i \mapsto gg'H_i$ is injective. If G is a p-group such that $\Phi_n(G)$ is trivial, then G is therefore isomorphic to a subgroup of a product of copies of the Sylow p-subgroup P_n of Σ_{p^n}. Using the results of Section 4, it may be shown that $\mathrm{ee}(P_n) = \mathrm{e}(P_n) = p^n$, and the claim follows. ∎

Remarks. It may be shown that $\mathrm{ee}(P_n) = \mathrm{e}(P_n) = p^n$ without using the results of Section 4—the upper bound by a transfer argument, and the lower bound by exhibiting a suitable subgroup of P_n whose cohomology is known, for example the cyclic group of order p^n.

If G is a finite group with Sylow p-subgroup G_p, then the p-part of $\mathrm{ee}(G)$ is equal to $\mathrm{ee}(G_p)$, so the bound given by Proposition 8.4 may be applied to arbitrary finite groups.

The bound on $\mathrm{ee}(G)$ given by Proposition 8.4 is sharp in many cases including: the extraspecial groups, and the groups presented during the proof of Theorem 3.3; groups of p-rank one; abelian groups; the metacyclic groups whose cohomology has been calculated by Huebschmann in [18]. Moreover, if G has subgroups $H_1, \ldots, H_m$ of index p^i with trivial intersection then the subgroups of the form $G \times \cdots \times H_i \times \cdots \times G$ of G^p intersect trivially and have index p^{i+1} in $G \wr C_p$. It follows from Proposition 8.1 that whenever the bound on $\mathrm{ee}(G)$ is sharp, so is the bound on $\mathrm{ee}(G \wr C_p)$. In the next section we shall exhibit a group for which the bound is not sharp however. This is a 2-group G such that $\mathrm{ee}(G) = 4$ but $\Phi_2(G) \neq \{1\}$.

9. A group whose cohomology has small exponent.

As promised at the end of Section 8, we exhibit a counter-example to the converse of Proposition 8.4. More precisely, we exhibit a 2-group G whose index four subgroups do not intersect trivially, but such that $\mathrm{ee}(G) = 4$. It is known that if H is a p-group such that $\mathrm{ee}(H) = p$, then H is elementary abelian, and so $\Phi_1(H)$ is trivial [1,17,21]. Thus our example is minimal in some sense. The smallest such G that we have been able to find has order 2^7. The necessary calculations are simpler for another example of order 2^9 however, so we shall concentrate on this example and explain the smaller example in some final remarks. We also discuss the case of odd p at the end of the section.

To explain the origin of the example, it is helpful to consider a more general problem. If Γ is a discrete group of finite virtual cohomological dimension (vcd), then $\mathrm{ee}(\Gamma)$ may be defined just as for finite groups. Just as in Proposition 8.4, it may be shown that if the subgroups of Γ of index dividing p^i have torsion-free intersection, then $\mathrm{ee}(\Gamma)$ divides p^i. In this case, groups Γ for which this bound is not tight are already known. Let $\Gamma(n)$ (depending on the prime p as well as the integer n) be the group with presentation:

$$\Gamma(n) = \langle A_1, \ldots, A_n, B_1, \ldots, B_n, C \mid C^p, [A_i, C], [B_i, C], \\ [A_i, A_j], [B_i, B_j], [A_i, B_j]C^{-\delta(i,j)} \rangle.$$

Thus $\Gamma(n)$ is expressible as a central extension with kernel cyclic of order p generated by C and quotient free abelian of rank $2n$. The spectral sequence with $\mathbb{Z}$-coefficients for this central extension collapses. By examining this spectral sequence Adem and Carlson were able to determine the ring $H^*(\Gamma(n))$ [2]. From their calculation it follows that $\mathrm{ee}(\Gamma(p)) = p^p$.

On the other hand, it is easy to show that $\Phi_p(\Gamma)$, the intersection of the subgroups of Γ of index dividing p^p, contains the element C, so is not torsion-free. One way to see this is to note that the centre $Z(\Gamma)$ is generated by C, A_i^p, and B_i^p, while the commutator subgroup Γ' of Γ is generated by C. The quotient $\Gamma/Z(\Gamma)$ is an elementary abelian p-group of rank $2p$. The map

$$\Gamma/Z(\Gamma) \times \Gamma/Z(\Gamma) \to \Gamma'$$

given by $(g, h) \mapsto [g, h]$ may be viewed as an alternating bilinear form on the $\mathbb{F}_p$-vector space $\Gamma/Z(\Gamma)$, whose maximal isotropic subspaces have dimension p. Now any subgroup of Γ of index at most p^p either contains $Z(\Gamma)$, and in particular contains C, or has image in $\Gamma/Z(\Gamma)$ a subspace of dimension greater than p, which cannot be an isotropic subspace. Hence C is a commutator in any subgroup of index p^p that does not contain $Z(\Gamma)$.

The quotient of Γ by the subgroup generated by the A_i^p's and B_i^p's is extraspecial of order p^{2p+1} (in the case $p > 2$, this group has exponent p, and in the case $p = 2$ it is a central product of copies of the dihedral group of order eight). For this group it is known that the bound on the eventual cohomological exponent given by Proposition 8.4 is best possible (see for example Lemma 3.5). Our example is a slightly larger quotient of $\Gamma(2)$ in the case when $p = 2$. Let G be the group with presentation

$$\begin{aligned} G = \langle A_1, A_2, B_1, B_2, C \mid C^2, A_i^4,\ B_i^4,\ [A_i, C], \\ [B_i, C],\ [A_1, A_2],\ [B_1, B_2], [A_i, B_j]C^{\delta(i,j)} \rangle. \end{aligned} \tag{4}$$

Thus G is the quotient of $\Gamma(2)$ by the subgroup generated by A_i^4 and B_i^4. The subgroup Z of G generated by the A_i^2 and B_i^2 is central, and elementary abelian of rank four. The quotient $Q = G/Z$ is the extraspecial group of order 2^5 consisting of a central product of two copies of the dihedral group of order eight.

Our partial calculation of $H^*(G)$ relies on some knowledge of the ring structure of $H^*(Q)$, at least up to degree four. The additive structure of $H^*(Q)$, and of the integral cohomology of all extraspecial 2-groups, was determined by Harada and Kono [16,7]. We find that the description of the ring structure of $H^*(Q)$ given in [7] is incorrect, even in low degrees. All that we shall require concerning $H^*(Q)$ is contained in the following statement.

Lemma 9.1. *Let Q be the quotient G/Z as above, where G has the presentation (4). There is an element χ of $H^4(Q)$ of order eight, and 4χ is expressible as a sum of products of elements of $H^2(Q)$.*

Proof. First we show that there is an element of $H^4(Q)$ of order eight (we could also quote this fact from [16]). Recall that Q has seventeen irreducible real representations, sixteen 1-dimensional ones and one 4-dimensional. Let S be the unit sphere in the 4-dimensional faithful real representation. Then Q acts trivially on $H^*(S)$, so there is an Euler class $e(S) \in H^4(Q)$ defined for S. (Topologically, $e(S)$ is the Euler class of the orientable S-bundle $S \times_Q EQ$ over $BQ = EQ/Q$. Algebraically, $E(S)$ is the extension class in $\mathrm{Ext}^4_{\mathbb{Z}Q}(\mathbb{Z}, \mathbb{Z})$ represented by the chain complex of the Q-CW-complex S.) The centre $Z(Q)$ of Q is cyclic of order two generated by the image of C, and the 4-dimensional faithful real representation restricts to this subgroup as four copies of its non-trivial real representation. It follows that the image of $e(S)$ generates $H^4(Z(Q))$. Now Q is a group of the type discussed in Lemma 3.5, and this lemma implies that $e(S)$ has order eight.

For the rest of the proof, we consider the spectral sequences for the central extension

$$Z(Q) \to Q \to Q/Z(Q) \tag{5}$$

with integer and mod-2 coefficients. The spectral sequence with mod-2 coefficients was solved completely by Quillen [27,7]. Recall that the mod-2 cohomology ring of an elementary abelian 2-group of rank r is a polynomial ring on r generators of degree one. Let $E'^{*,*}_*$

be the spectral sequence for (5) with mod-2 coefficients. Thus $E_2'^{*,*} = \mathbb{F}_2[y_1, \ldots, y_4, z]$, where $y_i \in E_2'^{1,0}$ and $z \in E_2'^{0,1}$. The extension class for Q in $H^2(Q/Z(Q); \mathbb{F}_2)$ is (without loss of generality) $y_1y_2 + y_3y_4$. Thus $d_2(z) = y_1y_2 + y_3y_4$. We shall not need to consider any higher differentials in this spectral sequence.

Now let $E_*^{*,*}$ be the spectral sequence for (5) with $\mathbb{Z}$-coefficients. Then $E_2^{i,j} = H^i(Q/Z(Q); H^j(Z(Q)))$ is more complicated, because the integral cohomology of $Q/Z(Q)$ is more complicated. Except in degree zero, $H^*(Q/Z(Q))$ has exponent two, so the Bockstein map δ and projection map π

$$\delta : H^*(Q/Z(Q); \mathbb{F}_2) \to H^{*+1}(Q/Z(Q)) \qquad \pi : H^*(Q/Z(Q)) \to H^*(Q/Z(Q); \mathbb{F}_2)$$

are (respectively) surjective in positive degrees, and injective in positive degrees. As a ring, $H^*(Q/Z(Q))$ is generated by four elements x_i of degree two, six elements w_{ij} of degree three, four elements v_{ijk} of degree four, and one element $u = u_{1234}$ of degree five. Here the indices satisfy $1 \leq i < j < k \leq 4$, and in terms of δ the elements are

$$x_i = \delta(y_i), \quad w_{ij} = \delta(y_iy_j), \quad v_{ijk} = \delta(y_iy_jy_k), \quad u = \delta(y_1y_2y_3y_4).$$

Note that $E_2^{1,3}$ and $E_2^{3,1}$ are trivial, and that except for $E_2^{0,0}$, each group on the E_2-page has exponent two. It follows that if $\chi \in H^4(Q)$ has order eight, then 4χ yields an element of $E_\infty^{4,0}$ in the spectral sequence. The proof will be complete once we have shown that $E_\infty^{4,0}$ is generated by products of elements of $E_\infty^{2,0}$. Let t be a generator for $E_2^{0,2}$. It is logical to denote the generators of $E_2^{1,2}$ by $[ty_1], \ldots, [ty_4]$. Although there is no element y_i in $E_2^{1,0}$, such relations as $[ty_1]^2 = t^2x_1$ do hold in $E_2^{*,*}$. The group $E_2^{4,0}$ is elementary abelian of rank fourteen, with generators the ten monomials in the x_i's, and the four elements z_{ijk}. The only differential that can hit this group is $d_3 : E_3^{1,2} \to E_3^{4,0}$. To compute this differential, we recall that there is a Bockstein map of spectral sequences:

$$\delta : E_r'^{i,j} \to \begin{cases} E_{r+1}^{i,j+1} & \text{for } j > 0, \\ E_{r+1}^{i+1,j} & \text{for } j = 0, \end{cases}$$

such that the induced map on E_∞-pages is a filtration of the Bockstein $\delta : H^*(Q; \mathbb{F}_2) \to H^{*+1}(Q)$. The map on $E_2^{i,j}$ is given by the map $\delta : H^j(Z(Q); \mathbb{F}_2) \to H^{j+1}(Z(Q))$ for $j > 0$ and by $\delta : H^i(Q/Z(Q); \mathbb{F}_2) \to H^{i+1}(Q/Z(Q))$ for $j = 0$.

Now the differential $d_2 : E_2'^{1,1} \to E_2'^{3,0}$ satisfies $d_2(zy_i) = (y_1y_2 + y_3y_4)y_i$, and so in $E_*^{*,*}$,

$$\begin{aligned} d_3([ty_i]) = d_3\delta(zy_i) &= \delta((y_1y_2 + y_3y_4)y_i) \\ &= \begin{cases} x_1x_2 + z_{134} & \text{for } i = 1, \\ x_1x_2 + z_{234} & \text{for } i = 2, \\ x_3x_4 + z_{123} & \text{for } i = 3, \\ x_3x_4 + z_{124} & \text{for } i = 4. \end{cases} \end{aligned}$$

It follows that $E_4^{4,0}$ is generated by monomials in the x_i's, as required. ■

Corollary 9.2. *The group G with presentation (4) above has $\Phi_2(G) \neq \{1\}$ and* $\mathrm{ee}(G) = 4$.

Proof. The proof that $\Phi_2(G)$ contains the element C (and is in fact equal to the subgroup generated by C) is identical to the proof given above for the group $\Gamma(n)$, so shall be omitted.

The commutator subgroup G' of G is cyclic of order two generated by C, and G/G' is isomorphic to a product of four cyclic groups of order four. Thus G/G' has four 1-dimensional complex representations whose kernels intersect in the trivial group, and hence a free action with trivial action on homology on the torus $U(1)^4$. Letting S be the 3-sphere as in the proof of Lemma 9.1, with G acting via the action of its quotient $Q = G/Z$, it follows that G acts freely with trivial action on homology on $S \times U(1)^4$. By Venkov's proof of the Evens-Venkov theorem [5], it follows that $H^*(G)$ is finite over the subring generated by the Euler classes for these five G-spheres. Thus it suffices to show that the orders of these Euler classes are (at most) four.

$H^2(G) \cong H^2(G/G')$ is isomorphic to four copies of $\mathbb{Z}/4$, and the Euler classes of the four $U(1)$'s generate $H^2(G)$. Thus it remains to check that $e(S)$ has order four in $H^*(G)$. From Lemma 9.1 we know that $e(S) \in H^*(G/Z)$ has order eight, but $4e(S)$ is a sum of products of elements of $H^2(G/Z) = (\mathbb{Z}/2)^4$. The image of $H^2(Q)$ in $H^2(G)$ consists of the elements $2x$ for $x \in H^2(G) = (\mathbb{Z}/4)^4$, and a product of any two such elements $(2x)(2x') = 4xx'$ is zero in $H^4(G)$. ∎

Remarks. With a little more work it can be shown that, with notation as in the proof of Lemma 9.1, $4\chi = x_1x_2 + x_3x_4$. Let $G_2 = G/\langle B_1^2, B_2^2\rangle$, so that G_2 is a quotient of G of order 2^7, and Q is a quotient of G_2. The argument of Corollary 9.2 can be used to show that the image of 4χ in $H^4(G_2)$ is zero, and so $\mathrm{ee}(G_2) = 4$.

The evidence of Adem-Carlson's infinite groups suggests that there should be similar examples for all primes of groups with $\mathrm{ee}(G) = p^p$ and $\Phi_p(G) \neq \{1\}$. The smallest candidates (of the same type as our example for $p = 2$) have order p^{3p+1}, and I have been unable to compute $\mathrm{ee}(G)$ for any of these groups for p odd.

10. Varieties for higher torsion in cohomology.

The description of $H^*(X^p \times_{C_p} EC_p)$ given in Sections 4–6 is complicated. In this section we shall extract some information of a more conceptual nature. We assume that $H^*(X)$ is a finitely generated ring, and describe the ring $H^*(X^p \times_{C_p} EC_p)$ from the point of view of algebraic geometry. We start by defining the varieties that we shall study, and stating some of their elementary properties, before stating our main result as Theorem 10.6. It is convenient to use cohomology with $\mathbb{Z}_{(p)}$-coefficients, so throughout this section the coefficients for cohomology are $\mathbb{Z}_{(p)}$ when omitted.

Fix an algebraically closed field k of characteristic $p > 0$. Let R be a finitely generated commutative $\mathbb{Z}_{(p)}$-algebra, and define $V(R)$ to be the variety of all ring homomorphisms from R to k, with the Zariski topology. For each $j \geq 0$, let $I_j = I_j(R)$ be the annihilator in R of the element p^j, or equivalently the ideal of R generated by the elements of order dividing p^j:

$$I_j = \sum_{p^j r = 0} Rr = \mathrm{Ann}_R(p^j).$$

Let $V_j(R)$ be the corresponding subvariety of $V(R)$. Since R is Noetherian, there exists j_0 such that the I_j are all equal for $j \geq j_0$, and we define $V_\infty(R) = V_{j_0}(R)$. Thus the $V_j(R)$'s are included in each other as follows:

$$V(R) = V_0(R) \supseteq V_1(R) \supseteq \cdots \supseteq V_{j_0}(R) = V_\infty(R).$$

A ring homomorphism $f : R \to S$ induces maps $f^* : V_j(S) \to V_j(R)$ for all j.

For any finite type CW-complex X, let $H^\bullet(X)$ stand for the subring of $H^*(X)$ consisting of elements of even degree. If $H^*(X)$ is a finitely generated $\mathbb{Z}_{(p)}$-algebra, then $H^\bullet(X)$ is an algebra of the type considered above. The main result of the section, Theorem 10.6, gives a description of $V_i(H^\bullet(X^p \times_{C_p} EC_p))$ for such X in terms of the $V_j(H^\bullet(X))$'s. Note that the case $i = 0$ is not a special case of Quillen's work on equivariant cohomology rings [26], because we do not assume that X is finite. The case $i = 0$ does however follow from Nakaoka's description of $H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p)$ together with some of the commutative algebra from the appendix to [26]. The idea of studying the V_i for $i > 0$ came from a paper of Carlson, who considered a special case which we shall discuss in the next section [11]. Proposition 10.1 is Proposition B.8 of [26].

Proposition 10.1. *([26]) Let $f : R \to S$ be a homomorphism of finitely generated commutative $\mathbb{F}_p$-algebras, and assume that f has the following properties:*

i) For $r \in \ker(f)$ there exists n such that $r^n = 0$;

ii) For $s \in S$, there exists n such that $s^{p^n} \in \mathrm{Im}(f)$.

Then f induces a homeomorphism from $V(S)$ to $V(R)$. ∎

Remark. A homomorphism f having properties i) and ii) is known as an F-isomorphism.

Proposition 10.2. *Let X be a finite-type CW-complex. Then the following are equivalent.*

i) $H^*(X)$ is a finitely generated $\mathbb{Z}_{(p)}$-algebra;

ii) $H^*(X; \mathbb{F}_p)$ is a finitely generated $\mathbb{F}_p$-algebra, and the torsion in $H^*(X)$ has bounded exponent.

Proof. Let I_j be the ideal of $H^*(X)$ of elements annihilated by p^j. Under hypothesis i), $H^*(X)$ is Noetherian, so there exists j_0 such that $I_j = I_{j_0}$ for all $j \geq j_0$. Now the map from $H^*(X)$ to $H^*(X; \mathbb{F}_p)$ has image $H^*(X)/(p)$, a finitely generated $\mathbb{F}_p$-algebra, and cokernel isomorphic to the ideal I_1, which is a finitely generated module for $H^*(X)$, and hence also for $H^*(X)/(p)$. Thus i) $\Rightarrow$ ii).

Conversely, assume that ii) holds. Let p^j annihilate the torsion in $H^*(X)$, and let R_j be the image of the map $x \mapsto x^{p^{j+1}}$ from $H^*(X; \mathbb{F}_p)$ to itself. Then R_j is a finitely generated ring, and $H^*(X; \mathbb{F}_p)$ is a finitely generated R_j-module. Since p^j annihilates torsion in $H^*(X)$, the Bockstein spectral sequence with E_1-page $E_1^i = H^i(X; \mathbb{F}_p)$ collapses at the E_{j+1}-page, so that any element of $H^*(X; \mathbb{F}_p)$ which is a cycle for the Bockstein β_1 and for each higher Bockstein $\beta_2, \ldots, \beta_j$ is in the image of $H^*(X)$. But if $x \in H^*(X; \mathbb{F}_p)$ is a cycle for β_i, then

$$\beta_{i+1}(x^p) = p\beta_{i+1}(x)x^{p-1} = 0,$$

so by induction, the subring R_j consists of universal cycles in the Bockstein spectral sequence.

Moreover, each of the non-zero differentials in the Bockstein spectral sequence is an R_j-linear map. Hence the universal cycles in the Bockstein spectral sequence, or equivalently, the image of $H^*(X)$ in $H^*(X;\mathbb{F}_p)$, form an R_j-submodule of $H^*(X;\mathbb{F}_p)$. It follows that this image, which is isomorphic to $H^*(X)/(p)$, is a finitely generated ring. Now take a finite set of elements of $H^*(X)$ mapping to a set of generators for $H^*(X)/(p)$. The $\mathbb{Z}_{(p)}$-subalgebra of $H^*(X)$ generated by these elements is the whole of $H^*(X)$, since a proper $\mathbb{Z}_{(p)}$-submodule of the finitely generated module $H^i(X)$ cannot have image the whole of $H^i(X)/(p)$. ■

Remark. The case when X is a Moore space with $H_n(X;\mathbb{Z})$ isomorphic to the rationals shows that the assumption that X has finite type is necessary in Proposition 10.2. I do not know of any finite type X such that $H^*(X;\mathbb{F}_p)$ is finitely generated, but $H^*(X)$ is not finitely generated.

Proposition 10.3. *Let X and X' be CW-complexes of finite type whose $\mathbb{Z}_{(p)}$-cohomology is a finitely generated algebra, and let $i \geq 0$ be a positive integer. Then the obvious maps induce homeomorphisms as shown below:*

i) $V_0(H^\bullet(X;\mathbb{F}_p)) \cong V_0(H^\bullet(X))$;

ii) if $p = 2$, *then* $V_0(H^*(X;\mathbb{F}_2)) \cong V_0(H^\bullet(X;\mathbb{F}_2))$;

iii) $V_i(H^\bullet(X \times X')) \cong V_i(H^\bullet(X) \otimes H^\bullet(X'))$.

Proof. Firstly, note that for any R, the natural map $R \to R/(p)$ induces a homeomorphism $V_0(R/(p)) \to V_0(R)$. Thus by Proposition 10.1, for i) it suffices to show that the map from $H^\bullet(X)/(p)$ to $H^\bullet(X;\mathbb{F}_p)$ is an F-isomorphism. The kernel of this map is trivial, and it was shown during the proof of Proposition 10.2 that if p^j annihilates the torsion in $H^*(X)$, then for any $x \in H^\bullet(X;\mathbb{F}_p)$, $x^{p^{j+1}}$ is in the image of $H^\bullet(X)/(p)$. Part ii) also follows easily from Proposition 10.1, because the kernel of the inclusion of $H^\bullet(X;\mathbb{F}_2)$ in $H^*(X;\mathbb{F}_2)$ is of course trivial, and the square of any element is in the image.

For iii), we first consider the case $i = 0$. Let f stand for the map from $H^*(X)\otimes H^*(X')$ to $H^*(X \times X')$. Since X and X' are finite type CW-complexes, there is a Künneth exact sequence

$$0 \longrightarrow H^*(X) \otimes H^*(X') \xrightarrow{f} H^*(X \times X') \longrightarrow \mathrm{Tor}^{*-1}(H^*(X), H^*(X')) \longrightarrow 0 \qquad (6)$$

which splits, and the Tor-term consists of torsion elements of bounded exponent.

Consider the commutative diagram given below. Each of the maps is injective (for $f/(p)$ this follows from the fact that the Künneth sequence (6) is split). The maps π and $\pi \otimes \pi$ are F-isomorphisms by the proof of i), and the map labelled 'inc.' is an F-isomorphism because it is injective and the pth power of any element is in its image. Now $f/(p)$ is injective, and every other map in the diagram is an F-isomorphism. It follows that $f/(p)$ is also an F-isomorphism.

Since $V_i(R)$ is a subvariety of $V_0(R)$, it follows that for general i the map

$$f : V_i(H^\bullet(X \times X')) \to V_i(H^\bullet(X) \otimes H^\bullet(X'))$$

is a homeomorphism onto its image. To show that this map is surjective, it suffices to show that if $y \in H^\bullet(X \times X')$ satisfies $p^i y = 0$, then there exists $x \in H^\bullet(X) \otimes H^\bullet(X')$

$$
\begin{array}{ccc}
(H^\bullet(X) \otimes H^\bullet(X'))/(p) & \xrightarrow{\cong} & H^\bullet(X)/(p) \otimes H^\bullet(X')/(p) \\
\Big\downarrow{\scriptstyle f/(p)} & & \Big\downarrow{\scriptstyle \pi\otimes\pi} \\
H^\bullet(X \times X')/(p) & & H^\bullet(X;\mathbb{F}_p) \otimes H^\bullet(X';\mathbb{F}_p) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \text{inc.}} \\
H^\bullet(X \times X';\mathbb{F}_p) & \xrightarrow{\cong} & (H^*(X;\mathbb{F}_p) \otimes H^*(X';\mathbb{F}_p))^{\text{even}}
\end{array}
$$

with $p^i x = 0$ and such that $f(x) = y^N$ for some N. (This is because $\phi \in V_0(R) \setminus V_i(R)$ if and only if there exists r with $p^i r = 0$ and $\phi(r) \neq 0$.) From the fact that $f/(p)$ is an F-isomorphism, we obtain $x' \in H^\bullet(X) \otimes H^\bullet(X')$ and $y' \in H^\bullet(X \times X')$ such that $f(x') = y^{p^n} + py'$. Now pick r sufficiently large that p^r annihilates the torsion in $H^*(X)$ and $H^*(X')$. Then p^r annihilates the Tor-term in (6), so $f(x'^{p^r}) - y^{p^{n+r}}$ is in the image of f. Thus there exists $x'' \in H^*(X) \otimes H^*(X')$ such that $f(x'') = y^{p^{n+r}}$. Express $x'' = x_{\mathrm{e}} + x_{\mathrm{o}}$, where $x_{\mathrm{e}} \in H^\bullet \otimes H^\bullet$ and $x_{\mathrm{o}} \in H^{\mathrm{odd}} \otimes H^{\mathrm{odd}}$. Now $p^i x_{\mathrm{e}} = p^i x_{\mathrm{o}} = 0$, and either p is odd in which case $x_{\mathrm{o}}^2 = 0$, and

$$x''^{p^i} = x_{\mathrm{e}}^{p^i} + p^i x_{\mathrm{e}}^{p^i - 1} + x_{\mathrm{o}}^2(\cdots) = x_{\mathrm{e}}^{p^i},$$

or $p = 2$, in which case $x_{\mathrm{o}}^2 \in H^\bullet(X) \otimes H^\bullet(X')$, $2x_{\mathrm{o}}^2 = 0$, and

$$x''^{2^i} = x_{\mathrm{e}}^{2^i} + 2^i x_{\mathrm{e}}^{2^i - 1} x_{\mathrm{o}} + 2x_{\mathrm{o}}^2(\cdots) + x_{\mathrm{o}}^{2^i} = x_{\mathrm{e}}^{2^i} + x_{\mathrm{o}}^{2^i}.$$

In either case, $x = x^{p^i}$ is an element of $H^\bullet(X) \otimes H^\bullet(X')$ such that $f(x) = y^{p^N}$ for some N, as required. ∎

Proposition 10.4. *Let R be a finitely generated commutative $\mathbb{Z}_{(p)}$-algebra, with an action of a finite group G, and write R^G for the G-fixed points in R. Then for each i, the natural map gives rise to a homeomorphism $V_i(R)/G \to V_i(R^G)$.*

Proof. For $i = 0$, this is a standard result, see for example chapter 5 of [4]. It is also a special case of Lemma 8.11 of [26] (view the group G as a category with one object and the G-action on R as a functor from G-to-$\mathbb{Z}_{(p)}$-algebras). Given the case $i = 0$, the general case follows provided that $V_i(R)$ maps surjectively to $V_i(R^G)$. Let ϕ be any element of $V(R)$ whose restriction to R^G lies in $V_i(R^G)$, and let r be an element of R such that $p^i r = 0$. Then $Q_r(r) = 0$, where $Q_r(X)$ is the monic polynomial

$$Q_r(X) = \prod_{g \in G} (X - g.r).$$

The coefficients in this polynomial lie in R^G, and (apart from 1, the leading coefficient) are annihilated by p^i. Thus

$$\phi(r^{|G|}) = \phi(f_r(r)) = \phi(0) = 0,$$

and so $\phi \in V_i(R)$. ∎

Proposition 10.5. *Let R be a finitely generated commutative $\mathbb{Z}_{(p)}$-algebra, let C_p act on $S = R^{\otimes p}$ by permuting the factors, and let $\mu : S \to R$ be the multiplication homomorphism. Define $U \subseteq V(S)$ by*

$$U = \{\phi \in V(S) \mid \forall x \in S, \phi\Big(\sum_{g \in C_p} g.x\Big) = 0\}.$$

Then $U = \mu^*(V(R))$, and U maps injectively to $V(S)/C_p = V(S^{C_p})$.

Proof. First note that $\mu : S \to R$ is C_p-equivariant for the given action on S and the trivial action on R. Now if $\phi \in \mu^*(V(R))$, or equivalently, $\phi = \psi \circ \mu$ for some $\psi : R \to k$, then for any $x \in S$,

$$\phi\Big(\sum_{g \in C_p} g.x\Big) = \psi\Big(\sum_{g \in C_p} g.\mu(x)\Big) = \psi(p.\mu(x)) = 0.$$

Thus $\mu^*(V(R)) \subseteq U$.

For the converse, given $r \in R$, let $x_1 = r \otimes 1 \otimes \cdots \otimes 1$, and let $x_2, \ldots, x_p$ be the images of x_1 under the C_p-action. It suffices to show that if $\phi \in U$, then for any r, $\phi(x_1) = \phi(x_2) = \cdots = \phi(x_p)$. Let $\sigma_i = \sigma_i(x_1, \ldots, x_p)$ be the ith elementary symmetric function in $x_1, \ldots, x_p$. For $1 \leq i \leq p-1$, the stabilizer in Σ_p of a subset of size i has order coprime to p, and hence there exists y_i such that

$$\sigma_i = \sum_{g \in C_p} g.y_i.$$

Now if $\phi \in U$, then $\phi(\sigma_i) = 0$ for $1 \leq i \leq p-1$, and hence each $\phi(x_j)$ is a root of the equation $X^p - \phi(\sigma_p) = 0$, which has all of its roots equal.

The last claim follows since $\mu^*(V(R))$ consists of points of $V(S)$ fixed by the C_p-action. ■

Theorem 10.6. *If X is a finite type CW-complex such that $H^*(X)$ is finitely generated as a $\mathbb{Z}_{(p)}$-algebra, then so is $X^p \times_{C_p} EC_p$, and $V_i(H^\bullet(X^p \times_{C_p} EC_p))$ may be described as follows:*

Let γ^*, δ^*, be the ring homomorphisms

$$H^*(X \times BC_p) \xrightarrow{\gamma^*} H^*(X) \qquad (H^*(X)^{\otimes p})^{C_p} \xrightarrow{\delta^*} H^*(X)$$

induced by the projection $X \times EC_p \to X \times BC_p$ and the diagonal map $X \times EC_p \to X^p \times EC_p$. Then

$$V_0(H^\bullet(X^p \times_{C_p} EC_p)) \cong \varinjlim(V_0(H^\bullet(X \times BC_p)) \xleftarrow{\gamma} V_0(H^\bullet(X)) \xrightarrow{\delta} V_0(H^\bullet(X)^{\otimes p})/C_p),$$

and for $i > 0$,

$$V_i(H^\bullet(X^p \times_{C_p} EC_p)) \cong \varinjlim(V_{i-1}(H^\bullet(X)) \xleftarrow{\mathrm{Id}} V_i(H^\bullet(X)) \xrightarrow{\delta} V_i(H^\bullet(X)^{\otimes p})/C_p).$$

Remark. Roughly one could write

$$\begin{aligned} V_0(H^\bullet(X^p \times_{C_p} EC_p)) &= V_0(H^\bullet(X \times BC_p)) \cup V_0(H^\bullet(X)^{\otimes p})/C_p, \\ V_i(H^\bullet(X^p \times_{C_p} EC_p)) &= V_{i-1}(H^\bullet(X)) \cup V_i(H^\bullet(X)^{\otimes p})/C_p. \end{aligned}$$

Proof. Let α^* and β^* be the homomorphisms defined in Section 6, so that there is a commutative diagram:

$$\begin{array}{ccc} H^*(X^p \times_{C_p} EC_p) & \xrightarrow{\alpha^*} & H^*(X^p)^{C_p} \\ \big\downarrow{\scriptstyle \beta^*} & & \big\downarrow{\scriptstyle \delta^*} \\ H^*(X \times BC_p) & \xrightarrow{\gamma^*} & H^*(X). \end{array}$$

Use the same notation for the corresponding homomorphisms of mod-p cohomology.

Obviously, X is of finite type if and only if $X^p \times_{C_p} EC_p$ is of finite type. In this case, Nakaoka's theorem given above as Theorem 2.1 (together with the remarks that follow it), implies that $H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p)$ is isomorphic to the graded tensor product

$$H^*(X^p; \mathbb{F}_p)^{C_p} \otimes H^*(BC_p; \mathbb{F}_p)$$

modulo the ideal generated by elements of the form

$$\sum_{g \in C_p} g^*(x) \otimes y, \qquad x \in H^*(X^p; \mathbb{F}_p), \quad y \in H^i(BC_p; \mathbb{F}_p), \quad i > 0.$$

Let η^* stand for the projection map

$$\eta^* : H^*(X^p; \mathbb{F}_p)^{C_p} \otimes H^*(BC_p; \mathbb{F}_p) \longrightarrow H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p).$$

Note that there is no analogue of η^* for $\mathbb{Z}_{(p)}$-cohomology. Note also that there is a commutative diagram:

$$\begin{tikzcd} H^*(X^p; \mathbb{F}_p)^{C_p} \otimes H^*(BC_p; \mathbb{F}_p) \arrow[r, "\eta^*"] \arrow[dr, "\delta^* \otimes 1"'] & H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p) \arrow[d, "\beta^*"] \\ & H^*(X; \mathbb{F}_p) \otimes H^*(BC_p; \mathbb{F}_p). \end{tikzcd}$$

It follows that $H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p)$ is finitely generated if $H^*(X; \mathbb{F}_p)$ is. If p^i annihilates torsion in $H^*(X)$, then either by the results of Section 4 or a transfer argument, p^{i+1} annihilates the torsion in $H^*(X^p \times_{C_p} EC_p)$. Hence by Proposition 10.2, if $H^*(X)$ is finitely generated, then so is $H^*(X^p \times_{C_p} EC_p)$.

For the case $i = 0$, it is convenient to work with mod-p cohomology, and deduce the result for $\mathbb{Z}_{(p)}$-cohomology from Proposition 10.3. Let ϕ be an element of $V_0(H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p))$, and let y be the image in $H^2(X^p \times_{C_p} EC_p; \mathbb{F}_p)$ of a generator for $H^2(BC_p; \mathbb{F}_p)$. If y' is any element of $H^*(X^p \times_{C_p} EC_p; \mathbb{F}_p)$ such that $\alpha^*(y') = 0$, then y'^2 is in the ideal generated by y. (To see this, recall that α^* is the edge homomorphism in the Cartan-Leray spectral sequence for $X^p \times_{C_p} EC_p$, and that the ideal generated by y maps on to $E_2^{i,j}$ for $i \geq 2$.) Thus if $\phi(y) = 0$, then ϕ factors through $H^\bullet(X^p; \mathbb{F}_p)^{C_p}$, or in other words $\phi \in \alpha(V(H^\bullet(X^p; \mathbb{F}_p)^{C_p}))$, and this factorisation is of course unique.

On the other hand, if $\phi(y) \neq 0$, then for all $x \in H^\bullet(X^p; \mathbb{F}_p)$,

$$\phi(\sum_{g \in C_p} g^*(x)) = 0.$$

In particular, Proposition 10.5 in the case $R = H^\bullet(X; \mathbb{F}_p)$ shows that there exists $\phi' \in V(H^\bullet(X; \mathbb{F}_p) \otimes H^\bullet(BC_p; \mathbb{F}_p))$ such that $\phi \circ \eta^* = \phi' \circ (\delta^* \otimes 1)$. Since η^* is surjective, it follows that $\phi = \phi' \circ \beta^*$.

Thus if $\phi(y) \neq 0$ then ϕ factors through the image of β^*. If R is the image of $H^\bullet(X; \mathbb{F}_p)$ under the map $x \mapsto x^p$, then the image of β^* contains $R \otimes H^\bullet(BC_p; \mathbb{F}_p)$, as this is the subring generated by the image of $H^\bullet(BC_p; \mathbb{F}_p)$ and elements of the form $x \wr 1$. However, any homomorphism from $R \otimes H^\bullet(BC_p; \mathbb{F}_p)$ to k extends uniquely to $H^\bullet(X \times BC_p; \mathbb{F}_p)$ by Proposition 10.1, and so ϕ factors uniquely through $H^\bullet(X \times BC_p; \mathbb{F}_p)$.

Finally, if ϕ factors through both $H^\bullet(X^p; \mathbb{F}_p)^{C_p}$ and $H^\bullet(X \times BC_p; \mathbb{F}_p)$, then $\phi(y) = 0$, and so ϕ factors uniquely through $H^\bullet(X; \mathbb{F}_p)$. This completes the claim in the case $i = 0$.

For the case when $i > 0$, we consider the information given by the Cartan-Leray spectral sequence with $\mathbb{Z}_{(p)}$-coefficients. The results of Section 5 imply that any element x of $H^*(X^p \times_{C_p} EC_p)$ yielding an element of $E_\infty^{m,n}$ for $m > 0$, has order p. Hence if ϕ is an element of $V_1(H^\bullet(X^p \times_{C_p} EC_p))$, ϕ factors through $H^\bullet(X^p)^{C_p}$. Thus for each $i > 0$,

$$\alpha(V(H^\bullet(X^p)^{C_p})) \supseteq V_i(H^\bullet(X^p \times_{C_p} EC_p)) \supseteq \alpha(V_i(H^\bullet(X^p)^{C_p})).$$

For simplicity we shall restrict attention to H^{2p*}, the subring of $H^\bullet$ consisting of elements in degrees divisible by $2p$. (The inclusion of H^{2p*} in $H^\bullet$ induces a homeomorphism from $V_i(H^\bullet)$ to $V_i(H^{2p*})$ for each i.)

As in previous sections we fix a decomposition of $H^*(X)$ as a direct sum of cyclic groups, and use this to split the E_∞-page of the Cartan-Leray spectral sequence as a direct sum of pieces as described in Section 5. Refine the resulting decomposition of $H^*(X^p \times_{C_p} EC_p)$ to a splitting into cyclic summands. If x generates a cyclic summand of $H^{2p*}(X^p \times_{C_p} EC_p)$ of order p^i, there are only two possibilities: Either x is contained in a summand of the spectral sequence concentrated in $E_*^{0,*}$, and its image in $H^{2p*}(X^p)^{C_p}$ generates a cyclic summand of order p^i, or x is contained in a summand of the spectral sequence of the type discussed in Theorems 5.1 and 5.1′, and its image in $H^{2p*}(X^p)^{C_p}$ generates a cyclic summand of order p^{i-1}. In the first case, there exists $x' \in H^{2p*}(X^p)$ such that

$$x = \sum_{g \in C_p} g.x',$$

and so (as in the proof of Proposition 10.5) $\delta^*(x) = 0$. It follows that

$$V_i(H^{2p*}(X^p \times_{C_p} EC_p)) \supseteq \alpha(V_i(H^{2p*}(X^p)^{C_p})) \cup \alpha \circ \delta(V_{i-1}(H^{2p*}(X)),$$

and it remains only to prove the opposite inequality.

If $\phi \in V(H^{2p*}(X)) \setminus V_{i-1}$, then there exists $x \in H^{2p*}(X)$ such that $p^{i-1}x = 0$ and $\phi(x) \neq 0$. In this case, $y = x \wr 1$ is an element of $H^{2p*}(X^p \times_{C_p} EC_p)$ such that $p^i y = 0$ and $\phi \circ \delta^* \circ \alpha^*(y) \neq 0$. Thus

$$\alpha \circ \delta(\phi) = \phi \circ \delta^* \circ \alpha^* \notin V_i(H^{2p*}(X^p \times_{C_p} EC_p)).$$

If

$$\phi \in V(H^{2p*}(X^p)) \setminus (\delta(V(H^{2p*}(X))) \cup V_i(H^{2p*}(X^p))),$$

then there exists $x, x' \in H^{2p*}(X)^{\otimes p}$ such that $p^i.x = 0$ but $\phi(x) \neq 0$, and (by Proposition 10.5) $\phi(\sum_{g \in C_p} g.x') \neq 0$. It suffices to find $y \in H^{2p*}(X)^{\otimes p}$ such that $p^i.y = 0$ and $\phi(\sum_{g \in C_p} g.y) \neq 0$. By choosing y_j to be a combination of various products of elements of the form $g.x$, we may ensure that for $1 \leq j \leq p-1$,

$$\sum_{g \in C_p} g.y_j = \sigma_j(g_1.x, \ldots, g_p.x),$$

where $g_1.x, \ldots, g_p.x$ are the images of x under the C_p-action, and σ_j stands for the jth elementary symmetric function. Each y_j satisfies $p^i.y_j = 0$, so if there exists j such that

$$\phi\Big(\sum_{g \in C_p} g.y_j\Big) = \phi\Big(\sigma_j(g_1.x, \ldots, g_p.x)\Big) \neq 0,$$

then $y = y_j$ will do. Otherwise, it follows (as in the proof of Proposition 10.5) that for each $g \in C_p$, $\phi(g.x) = \phi(x)$. In this case, $y = xx'$ has the required properties. ∎

11. Carlson's $W_i(G)$.

As in the last section we take $\mathbb{Z}_{(p)}$-coefficients for cohomology unless otherwise stated, and fix an algebraically closed field k of characteristic p, upon which the definition of $V(-)$ depends. For G a finite group, note that EG may be taken to be of finite type. For G a finite group and Y a finite G-CW-complex, Quillen showed that the equivariant cohomology ring $H^*(Y \times_G EG)$ is finitely generated [26]. For such G and Y, define varieties $W_i(G, Y)$, $W_i(G)$ by

$$W_i(G, Y) = V_i(H^*(Y \times_G EG)), \qquad W_i(G) = W_i(G, \{*\}),$$

where $V_i(-)$ is as defined in Section 10. Note that $W_i(G, Y)$ is a covariant functor of the pair (G, Y), and that if H is a subgroup of G, then the finiteness of $H^*(H)$ as an $H^*(G)$-module implies that the fibres of the map $W_i(H) \to W_i(G)$ are finite. In [26], Quillen gave a complete description of $W(G, Y) = W_0(G, Y)$. The subvarieties $W_i(G)$ for $i > 0$ were introduced by Carlson in [11]. Very little is known about $W_i(G)$ and $W_i(G, Y)$ in general. The results of Section 10 of course have Corollaries concerning $W_i(G, Y)$. For example, if Y' is a finite H-CW-complex, then by Proposition 10.3 iii),

$$W_i(G \times H, Y \times Y') \cong W_i(G, Y) \times W_i(H, Y').$$

For most of this section we use examples constructed as wreath products to shed some light on questions raised by Carlson concerning $W_i(G)$. The lower bound we give for the size of $W_1(G)$ in Theorem 11.9 is independent of the rest of the paper however. The following three statements are corollaries of Theorem 10.6.

Corollary 11.1. *Let G be a finite group and Y a finite G-CW-complex, and let $G \wr C_p$ act on Y^p, where G^p acts component-wise and C_p by permuting the factors. Let γ be the inclusion of the G-space Y as the diagonal in Y^p, which is equivariant for the diagonal map from G to G^p, let C_p act trivially on Y, and let δ be the identity map on Y viewed as an equivariant map from the G-space Y to the $G \times C_p$-space Y. Then*

$$W_0(G \wr C_p, Y^p) = \varinjlim(W_0(G^p, Y^p)/C_p \xleftarrow{\gamma} W_0(G, Y) \xrightarrow{\delta} W_0(G \times C_p, Y)),$$

and for $i > 0$,

$$W_i(G \wr C_p, Y^p) = \varinjlim(W_i(G^p, Y^p)/C_p \xleftarrow{\gamma} W_i(G, Y) \xrightarrow{\mathrm{Id}} W_{i-1}(G, Y)).$$

Proof. This is just Theorem 10.6 in the case $X = Y \times_G EG$. ∎

Corollary 11.2. *With notation as in Corollary 11.1, and $i > 0$,*

$$\dim W_0(G \wr C_p, Y^p) = \max\{p. \dim W_0(G, Y),\ 1\},$$
$$\dim W_i(G \wr C_p, Y^p) = \max\{p. \dim W_i(G, Y),\ \dim W_{i-1}(G, Y)\}.$$

∎

Proposition 11.3. *Let $m = \sum_j m_j p^j$, where $0 \le m_j < p$. Then*

$$\dim W_i(\Sigma_m) = \sum_{j \ge i+1} m_j p^{j-i-1}.$$

In particular,

$$\dim W_i(\Sigma_{p^n}) = \begin{cases} 0 & \text{if } i \ge n, \\ p^{n-i-1} & \text{otherwise.} \end{cases}$$

Proof. If G is a finite group with Sylow p-subgroup P, then the map $W_i(P) \to W_i(G)$ has finite fibres (by the Evens-Venkov theorem) and is surjective since the kernel of the transfer is an ideal forming a complement to the image of $H^*(G)$ in $H^*(P)$. The Sylow p-subgroup of Σ_m is isomorphic to

$$(P_1)^{m_1} \times (P_2)^{m_2} \times \cdots,$$

where P_n is the Sylow p-subgroup of Σ_{p^n}. Thus it suffices to prove the assertion for P_n. This follows from Corollary 11.2 by induction, since $P_n \cong P_{n-1} \wr C_p$. ∎

Carlson also introduced the cohomological exponential invariant, *che*(G) of a finite group G, defined by

$$che(G) = \min\{O(x_1) \cdots O(x_r) \mid H^*(G; \mathbb{Z}) \text{ is a finitely generated module for the subring generated by the } x_i\}.$$

The theorem that we quote as Theorem 3.4 is equivalent to the statement that $|G|$ divides *che*(G). Let $\nu_p(che(G))$ stand for the number of times that p divides *che*(G). Carlson showed that

$$\nu_p(che(G)) = \sum_{i \ge 0} \dim W_i(G) \qquad (7),$$

and gave an example of a group G of order p^5 with $che(G) = p^6$. Using Corollary 11.2 and the description of (the p-part of) *che*(G) given by (7), it is easy to find groups for which $|G| < che(G)$. One such example is contained in Proposition 11.4. We give another different example in Proposition 11.10.

Proposition 11.4. *Let $G = (C_p^n) \wr C_p$, so that G is a split extension with kernel $(C_p)^{pn}$, quotient C_p, and the quotient acts freely on an $\mathbb{F}_p$-basis for the kernel. Then*

$$che(G) = p^{pn+n}, \quad \textit{whereas} \quad |G| = p^{pn+1}.$$

■

The generalized Frattini subgroups $\Phi_n(G)$, defined in Section 8 (just before Proposition 8.4) may be used to give an upper bound for $W_n(G)$. Before stating our result in Theorem 11.6, we recall a theorem of Carlson from [11, section 4].

Theorem 11.5. *Let G be a finite group and H a normal subgroup of G. The inverse image of 0 under the map $W(G) \to W(G/H)$ is equal to the image of $W(H)$.* ■

Remark. The element $0 \in W(G)$ is the homomorphism $H^*(G) \to k$ which sends all elements in positive degree to zero.

Theorem 11.6. *Let G be a p-group. Then $W_n(G)$ is contained in the image of $W(\Phi_n(G))$.*

Proof. The group $H^i(G/\Phi_n(G))$ has exponent dividing p^n for all but finitely many i by Proposition 8.4. Thus if $\phi \in W_n(G)$, then ϕ maps to 0 in $W(G/\Phi_n(G))$. The claim follows from Theorem 11.5. ■

Carlson asked if, for G a p-group, $W_n(G)$ always contains the image of $W(\Phi^n(G))$, where $\Phi^n(G) = \Phi(\Phi^{n-1}(G))$ is the nth iterated Frattini subgroup of G. Proposition 11.7 shows that this is consistent with the upper bound of Theorem 11.6. However, in Proposition 11.8 we construct wreath products G such that $W_n(G)$ does not contain the image of $W(\Phi^n(G))$, answering Carlson's question negatively.

Proposition 11.7. *Let G be a p-group. Then $\Phi_n(G) \geq \Phi^n(G)$.*

Proof. Given H, a subgroup of G of index $p^r \leq p^n$, let

$$H = H_r < H_{r-1} < \cdots < H_1 < H_0 = G$$

be a chain of subgroups of G such that $|H_i : H_{i+1}| = p$. Then for each i, $H_{i+1} \geq \Phi(H_i)$, and so by induction $H_r \geq \Phi^r(G) \geq \Phi^n(G)$. ■

Proposition 11.8. *Let S be an elementary abelian p-group of rank r, acting freely transitively on Ω, and let G be the wreath product $C_{p^n} \wr S$ for some $n > 0$. Then*

$$\dim W(\Phi^n(G)) = p^r - 1, \quad \textit{whereas} \quad \dim W_n(G) \leq p^{r-1}.$$

In particular, $W_n(G)$ cannot contain the image of $W(\Phi^n(G))$ unless $p = 2$ and $r = 1$.

Proof. Recall that for H a subgroup of G, the map $W(H)$ to $W(G)$ is finite, and so preserves dimensions. Recall also that for G a p-group, $\Phi(G)$ can be defined to be the minimal normal subgroup H of G such that G/H is elementary abelian. Now G as in the statement can be generated by $r+1$ elements, r of which map to a generating set for S, and one element of the form

$$(g, 1, \ldots, 1) \in (C_{p^n})^\Omega \leq C_{p^n} \wr S = G.$$

A homomorphism from G onto $(C_p)^{r+1}$ may be constructed, which shows that G cannot be generated by fewer than $r+1$ elements. It follows that $\Phi(G)$ has index p^{r+1}, and is an index p subgroup of $(C_{p^n})^\Omega$. Thus

$$\Phi(G) \cong (C_{p^n})^{p^r-1} \times C_{p^{n-1}}, \qquad \Phi^n(G) \cong (C_p)^{p^r-1}.$$

This proves the first claim. For the second claim, note that G is a subgroup of the wreath product $C_{p^n} \wr \Sigma_{p^r} \leq \Sigma_{p^{n+r}}$, and so $\dim W_n(G)$ is bounded by $\dim W_n(\Sigma_{p^{n+r}})$, which equals p^{r-1} by Proposition 11.3. ■

The only general lower bound that I have been able to find is the following bound for $W_1(G)$, which is related to Proposition 1 of [21].

Theorem 11.9. *Let G be a p-group, and let $H = Z(G) \cap \Phi(G)$, the intersection of the centre and the Frattini subgroup of G. Then $W_1(G)$ contains the image of $W(H)$.*

Proof. H is central, so is abelian. Let K be the elements of H of order dividing p. Then K is a subgroup of H and $W(K)$ maps homeomorphically onto $W(H)$. It will therefore suffice to show that any element of $H^*(G)$ of order p has trivial image in $H^*(K)$.

For this, we claim that the image of $H^*(G;\mathbb{F}_p)$ in $H^*(K;\mathbb{F}_p)$ is contained in the image of $H^*(K)$. Assuming this for now, let $x \in H^*(G)$ have order p, and recall the Bockstein long exact sequence

$$\cdots H^*(G) \xrightarrow{\times p} H^*(G) \longrightarrow H^*(G;\mathbb{F}_p) \xrightarrow{\delta} H^{*+1}(G) \xrightarrow{\times p} H^{*+1}(G) \cdots.$$

Since $px = 0$, there exists y such that $\delta(y) = x$. Now $\delta(\mathrm{Res}^G_K(y)) = \mathrm{Res}^G_K(x)$. By the claim, the image of $H^*(G;\mathbb{F}_p)$ in $H^*(K;\mathbb{F}_p)$ is contained in the kernel of δ, and so $\mathrm{Res}^G_K(x) = 0$ as required.

It remains to prove the claim made in the last paragraph. For this we consider the spectral sequence with $\mathbb{F}_p$-coefficients for the central extension

$$K \longrightarrow G \longrightarrow G/K.$$

The E_2-page is isomorphic to $H^*(K;\mathbb{F}_p) \otimes H^*(G/K;\mathbb{F}_p)$. For clarity, suppose that p is odd (the case $p = 2$ is similar but not identical). In this case,

$$H^*(K;\mathbb{F}_p) \cong \Lambda[K^\#] \otimes \mathbb{F}_p[K],$$

the tensor product of the exterior algebra on $K^\# = \mathrm{Hom}(K, \mathbb{F}_p)$, generated in degree 1, with the algebra of polynomial functions on K (where the monomials have degree 2). The Bockstein $H^*(K;\mathbb{F}_p) \to H^{*+1}(K;\mathbb{F}_p)$ maps H^1 isomorphically to the degree 2 piece of $\mathbb{F}_p[K]$.

Since K is contained in $\Phi(G)$, every element of $H^1(G;\mathbb{F}_p) \cong \mathrm{Hom}(G, \mathbb{F}_p)$ restricts trivially to K. It follows that

$$d_2 : E_2^{0,1} \cong K^\# \longrightarrow E_2^{2,0} \cong H^2(G/K;\mathbb{F}_p)$$

must be injective. Note also that the generators for $\mathbb{F}_p[K]$ are cycles for d_2 by the Serre transgression theorem. It follows that $E_3^{0,*}$, the cycles for d_2 in $E_2^{0,*}$, is isomorphic to $\mathbb{F}_p[K]$. This subring of $H^*(K;\mathbb{F}_p)$ is contained in the image of $H^*(K)$. The image of $H^*(G;\mathbb{F}_p)$ in $H^*(K;\mathbb{F}_p)$ is isomorphic to $E_\infty^{0,*}$, so is a subring of the image of $H^*(K)$ as required. ■

Remark. Note that Theorems 11.6 and 11.9 together describe $W_1(G)$ completely for any p-group G such that $\Phi(G)$ is central. This includes the two examples discussed by Carlson in [11]. As a further example we make the following statement, whose proof we leave as an exercise.

Proposition 11.10. *Let G be the group with presentation*

$$G = \langle a_1, \dots, a_n \mid a_i^p = 1, [a_i, a_j] \text{ is central} \rangle.$$

Then G is a finite p-group, with

$$\dim W_0(G) = \binom{n}{2} + 1, \qquad \dim W_1(G) = \binom{n}{2},$$

$$\nu_p(|G|) = \binom{n+1}{2}, \qquad \nu_p(che(G)) = n^2 - n + 1.$$

∎

Remark. The author knows of no example in which $W_i(G)$ is not equal to the image of $W(H)$ for some subgroup H of G. One might conjecture that this is always the case, or more weakly one might conjecture that for any G, $W_i(G)$ is equal to a union of images of $W(E)$ for some collection of elementary abelian subgroups E of G. By section 12 of [26], this weaker conjecture is equivalent to the following: Let J_i be the ideal of $H^*(G; \mathbb{F}_p)$ generated by the image of I_i, the ideal in $H^*(G)$ used to define $W_i(G)$. Then the radical of J_i is closed under the action of Steenrod's reduced powers $\mathcal{P}^i$.

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