BLOCH'S FORMULA FOR SINGULAR SURFACES

MARC LEVINE

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

MPI/SFB 83-27

Let X be a quasi-projective variety with singular locus S. Let $S_1,\dots,S_d$ be the irreducible components of S. We henceforth assume that $\dim(S_i) \leq 1$ for each i, and we order the S_i so that $\dim(S_i) = 1$ for $i = 1,\dots,r$, and $\dim(S_i) = 0$ for $i > r$. Let s_i denote the generic point of S_i, $i = 1,\dots,r$. Let R_S be the semi-local ring of $S_1 \cup \dots \cup S_r$ in X, and let X_S denote $\mathrm{Spec}(R_S)$. If p is a point of X, we let $R(p,S)$ be the localization of R_S at the set $\{f \mid f(p) \neq 0\}$; if $i: X_S \to X$ is the inclusion, then $i^{-1}(\mathrm{Spec}(\mathcal{O}_{X,p})) = \mathrm{Spec}(R(p,S))$. We let $X(p,S)$ denote $\mathrm{Spec}(R(p,S))$.

Let (X,S) be the topological space gotten by removing from X all points x of codimension one which specialize to some s_i, and also removing all points of codimension two which specialize to some point of S. We define a subsheaf F_S of $\coprod_{x \in (X,S)^1} i_{x*}(k(x)^*)$ by

$$F_{S,p} = \left\{ y \in \Big(\coprod_{x \in (X,S)^1} i_{x*}(k(x)^*)\Big)_p \;\middle|\; \begin{array}{l} \text{for } j \geq 2, \text{ and for each } z \text{ in } X^j \\ \text{which specializes to p, and also} \\ \text{specializes to a point of S, there} \\ \text{is an element } t_z \text{ of } i_*(\mathcal{K}_2(X_S))_z \\ \text{with } y = \tilde{T}(t_z) \text{ at } z. \end{array} \right\}$$

Here $\tilde{T}$ is the composition of $i_*(\mathcal{K}_2(X_S)) \to i_*K_2(k(X))$ with the tame symbol map $T: i_*K_2(k(X)) \to \coprod_{x \in X^1} i_{x*}(k(x)^*)$. We note that $\tilde{T}$ actually has image in F_S. The divisor map $\mathrm{div}: \coprod_{x \in X^1} i_{x*}(k(x)^*) \to \coprod_{x \in X^2} i_{x*}(\mathbb{Z})$ restricts to F_S to give a map $\mathrm{div}: F_S \to \coprod_{x \in (X,S)^2} i_{x*}(\mathbb{Z})$. This gives us a complex of sheaves on X

$$G. : \qquad 0 \to i_*(\mathcal{K}_2(X_S)) \xrightarrow{\tilde{T}} F_S \xrightarrow{\mathrm{div}} \coprod_{x \in (X,S)^2} i_{x*}(\mathbb{Z}) \to 0$$

Let $\mathcal{K}_2'$ denote the kernel of $\tilde{T}$.

INTRODUCTION

Let X be a smooth, quasi-projective variety. Bloch's formula

$$H^p(X, \mathcal{K}_p) \cong CH^p(X)$$

relating the K-theory of X with the Chow ring, was proved by Quillen in [Q]. The case p=1 was known to the ancients; Bloch originally proved the case p = in the case of surfaces. The purpose of this note is to show an analogous formula in case X is a singular surface. Collino [C] has constructed groups of cycles mod rational equivalence, $CH^p(X,S)$, in case X is a quasi-projective variety with singular locus S a finite set of points, and has verified that $H^p(X, \mathcal{K}_p) = CH^p(X,S)$ if S is a single point. Our method is a combination of the ideas in [C], together with the original construction of Bloch in [B]. Of especial importance is our lemma 2, which says that a collection of curves and functions $\{(D,f) \mid f \in k(D)^*\}$ on a surface X, with no curve D contained in the singular locus S, trivializes the zero cycle $\sum \mathrm{div}(f)$ in $K_0(X)$ if the col is locally a tame symbol along S (this automatically forces $\sum \mathrm{div}(f)$ to be supported on the smooth locus of X, so its class in K_0 makes sense).

We fix at the outset an algebraically closed field k as ground field for all varieties considered herein.

<u>Proposition 1</u>. The complex G. is a resolution of $\mathcal{K}_2'$. In addition, the map $\mathcal{K}_2 \to i_*(\mathcal{K}_2(X_S))$ factors through $\mathcal{K}_2'$, and $\mathcal{K}_2 \to \mathcal{K}_2'$ is an isomorphism away from the closed points of S.

We first prove the following lemma.

<u>Lemma</u>: Let y be a smooth point of X. Then the Gersten complex for R(y,S):

$$0 \to K_2R(y,S) \to K_2(k(X)) \xrightarrow{T} \coprod_{x \in X(y,S)^1} k(x)^* \xrightarrow{\text{div}} \coprod_{x \in X(y,S)^2} \mathbb{Z} \to 0$$

is exact.

<u>Proof</u>. The proof is essentially the same as in [Q] and [C]; we give a sketch here for the reader's convenience. R(y,S) is a regular ring, and the Gersten complex comes from the spectral sequence arising from the collection of long exact sequences

$$\xrightarrow{\partial} K_d(M^{i+1}) \to K_d(M^i) \to K_d(M^i/M^{i+1}) \xrightarrow{\partial} K_{d-1}(M^{i+1}) \to \quad ,$$

where M^i is the category of coherent R(y,S) modules supported in codimension i. To show the Gersten complex above is exact, we need only show that, for each principal divisor Z on X(y,S), the map

$$K_d(M^i(Z)) \to K_d(M^i)$$

is zero, where $M^i(Z)$ is the full subcategory of M^{i+1} consisting of modules supported on Z.

Let U = Spec(R) be a smooth affine neighborhood of y in X such that Z is represented by a principal divisor Z' on U, defined by an element t of R. Since U is smooth, there is a map $f: U \to \mathbb{A}^{n-1}$ (n = dim(X)) which is smooth on a neighborhood of y, and such that Z' is finite over $\mathbb{A}^{n-1}$.

We have the diagram

$$\begin{array}{ccc} \mathrm{Spec}(R') = Y = U \times_{\mathbb{A}^{n-1}} Z' & \longrightarrow & U \\ s \uparrow\downarrow & & \downarrow \\ \mathrm{Spec}(R/t) = \quad Z' & \longrightarrow & \mathbb{A}^{n-1} \end{array} \quad .$$

After inverting h in R, $h(y) \neq 0$, the ideal I of $s(Z'_h)$ in R'_h is principal, and R'_h is smooth, hence flat, over $\overline{R}_h$ ($\overline{R} = R/t$). Let $f = hf_1f_2$ be in R, with $f_1(y) \neq 0$, and $f_2(s_i) \neq 0$ for each $i = 1,\ldots,r$. We have an exact sequence of functors from $M^i(Z'_f)$ to M^i:

$$0 \to I_f \otimes_{\overline{R}_f} ? \to R'_f \otimes_{\overline{R}_f} ? \to ? \to 0 \quad .$$

As I_f is principal, the map on K groups $K_d(M^i(Z'_f)) \to K_d(M^i)$ is zero. Since $M^i(Z)$ is the direct limit over f as above of the $M^i(Z'_f)$, the map $K_d(M^i(Z)) \to K_d($ is zero. This completes the proof of the lemma.

We now prove proposition 1. Let y be a point of X. We need only check exactness at F_S and at $\coprod i_{x*}(\mathbb{Z})$. We consider three cases.

1) $y = s_i$ some i. Then $F_{S,y} = \coprod_{x \in (X,S)^1} i_{x*}(\mathbb{Z})_y = 0$.

2) y is a closed point of S. Then $\coprod_{x \in (X,S)^2} i_{x*}(\mathbb{Z})_y = 0$, and $\tilde{T}$ is surjective at y by the definition of F_S.

3) y a smooth point of X. By the lemma, we have

$$\begin{aligned} i_*(\mathcal{K}_2(X_S))_y &= H^0(X(y,S), \mathcal{K}_2) \\ &= K_2R(y,S) \quad . \end{aligned}$$

If $t \in F_{S,y}$ has divisor equal to zero in $(\coprod_{x \in (X,S)^2} i_{x*}(\mathbb{Z}))_y$, then t also has divisor equal to zero in $(\coprod_{x \in X^2} i_{x*}\mathbb{Z})_y$, as t is a tame symbol at all x in $X^2 - (X,S)^2$. Thus $t = T_y(x)$ for some x in $K_2(k(X))$, where T_y is the local tame symbol map at y. Also, $t = T_y(x)$ goes to zero in $\coprod_{x \in X(y,S)^1} k(x)^*$, so by the lemma, x comes from $K_2R(y,S)$. This proves exactness at F_S.

Let t be in $(\coprod_{x \in (X,S)^2} i_{x*}(\mathbb{Z}))_y$. Then $t = \mathrm{div}_y(z)$ for some z in $(\coprod_{x \in X^1} i_{x*}(k(x)^*))_y$. As t goes to zero in $\coprod_{x \in X(y,S)^2} \mathbb{Z}$, the image of z in $\coprod_{x \in X(y,S)^1} k(x)^*$ is of the form T(a) for some a in $K_2R(y,S)$. Modifying z by T(a), we may assume that z is in $(\coprod_{x \in (X,S)^1} i_{x*}(k(x)^*))_y$. If u is in $X(y,S)^2$, then $\mathrm{div}_u(z) = 0$, and as u is smooth on X, this implies that $z = T_u(a_u)$, for some a_u in $K_2(k(X))$. Since $T_u(a_u)$ goes to zero in $\coprod_{x \in X(u,S)^1} k(x)^*$, this implies that a_u comes from $K_2R(u,S)$, which is $i_*(\mathcal{K}_2(X_S))_u$ by the lemma, hence z is in $F_{S',y}$, as desired. This proves (a).

For (b), $\mathcal{K}_2$ is mapped to zero by the tame symbol map, hence $\mathcal{K}_2 \to i_*(\mathcal{K}_2(X_S))$ factors through $\mathcal{K}_2'$. Next, note that $\mathcal{K}'_{2,s_i} = i_*(\mathcal{K}_2(X_S))_{s_i} = \mathcal{K}_{2,s_i}$, and ,if y is smooth, the following diagram is commutative:

$$\begin{array}{ccc} i_*(\mathcal{K}_2(X_S))_y = K_2R(y,S) & \hookrightarrow & K_2(k(X)) \\ \downarrow \tilde{T}_y & & \downarrow T_y \\ F_{S',y} & \hookrightarrow & \coprod_{x \in X^1} i_{x*}(k(x)^*)_y \end{array}$$

Thus $\mathcal{K}_2' = \ker(i_*(\mathcal{K}_2(X_S))_y \xrightarrow{\tilde{T}_y} F_{S',y})$ is contained in $\ker(K_2(k(X)) \xrightarrow{T_y} i_{x*}(k(x)^*)_y)$, which is $\mathcal{K}_{2,y}$. As the reverse inclusion is implied by the lemma, (b) is proved.

q.e.d.

If Z is a quasi-projective variety, we let H_Z denote the exact category of torsion coherent $\mathcal{O}_Z$ modules M such that M has a two step resolution

$$0 \to F_1 \to F_0 \to M \to 0$$

by locally free sheaves on Z. I f E is a closed subset of Z, we let $H_Z(E)$ denote subcategory of H_Z consistiong of sheaves supported on E, and we let $H_{(Z,E)}$ denote the subcategory of H_Z consisting of sheaves which are zero at each generic point of E.

Let E be a closed subset of Z, locally defined by a single equation, such that U = Z-E is affine. Grayson [G] has shown there are long exact sequences ($F = k(Z)$; $i \geq 1$)

$$\begin{array}{ccccccccccccc}
\to K_i(H_Z(E)) & \to & K_i(Z) & \to & K_i(U) & \xrightarrow{\partial} & K_{i-1}(H_Z(E)) & \to & K_{i-1}(Z) & \to & K_{i-1}(U) \\
{\scriptstyle i_*}\downarrow & & \| & \nearrow & \downarrow & & {\scriptstyle i_*}\downarrow & & \| & \nearrow & \\
\to K_i(H_Z) & \to & K_i(Z) & \to & K_i(F) & \to & K_{i-1}(H_Z) & \to & K_{i-1}(Z) & \to & K_{i-1}(F) \\
{\scriptstyle j^*}\downarrow & & \downarrow & /\!/\!/ & \| & & {\scriptstyle j^*}\downarrow & & \downarrow & /\!/\!/ & \| \\
\to K_i(H_U) & \xrightarrow[\alpha]{} & K_i(U) & \to & K_i(F) & \to & K_{i-1}(H_U) & \to & K_{i-1}(U) & \to & K_{i-1}(F)
\end{array}$$

The maps i_* and j^* are induced by the inclusion $i: H_Z(E) \to H_Z$ and the restriction $j: H_Z \to H_U$ respectively. It is easily checked that all squares and triangles in the above commute up to sign. We define a map $\delta: K_i(H_U) \to K_{i-1}(H_Z(E))$ by $\delta = \partial \circ \alpha$. A diagram chase shows that δ gives a boundary map forming a long exact sequence $(i \geq 1)$

$$\to K_i(H_Z(E)) \xrightarrow[i_*]{} K_i(H_Z) \xrightarrow[j^*]{} K_i(H_U) \xrightarrow[\delta]{} K_{i-1}(H_Z(E)) \xrightarrow[i_*]{} K_{i-1}(H_Z)$$

By a standard argument, this gives rise to a Mayer-Vietoris sequence $(i \geq 1)$

$$\to K_{i+1}(H_{U \cap V}) \to K_i(H_{U \cup V}) \to K_i(H_U) \oplus K_i(H_V) \to K_i(H_{U \cap V})$$

whenever U and V are affine open subsets of Z with locally principal complements in $U \cup V$.

Let $H_Z^0 = \varinjlim_{\substack{p_1, \ldots, p_n \\ \text{in } Z^2}} H_{Z - \bigcup_i \bar{p}_i}$, and similarly define $H_Z^0(E)$ and $H^0_{(Z,E)}$. If M is a $\mathcal{O}_Z$ module in H_Z, then, after removing a codimension two subset of Z, M breaks up into a direct sum, $M = \bigoplus_{p \in Z^1} M_p$, with M_p supported on $\bar{p}$ minus some proper closed subset. Thus the category H_Z^0 is the direct sum,

$$H_Z^0 = \coprod_{p \in Z^1} H_Z^0(\bar{p})$$

and hence $K_i(H^0_Z) = \coprod_{p \in Z^1} K_i(H^0_Z(\bar{p}))$. In addition, if p is smooth on Z, then $K_i(H^0_Z(\bar{p})) = K_i(k(p))$ by devissage.

For the remainder of the paper, we assume that X is a surface.

<u>Lemma 2</u>. Let X be a quasi-projective surface, and let t be in $H^0(X, F_S)$. Then $[\mathrm{div}(t)] = 0$ in $K_0(X)$.

<u>Proof</u>. Represent t by (D,f), where D is a curve on X, $D \cap S$ is a finite set $p_1 \cup \cdots \cup p_n$, and f is in $k(D)^*$. By adding extra components to D, we may assume that D is locally principal, X-D is affine, and each component S_i of S intersects D. Since $K_1(H^0_X) = \bigoplus_{S_i} K_1(H^0_{\mathcal{X}}(\bar{s}_i)) \oplus \bigoplus_{x \in (X-S)^1} k(x)^*$, t defines an element t^0 in $K_1(H^0_X)$, which has image 0 in each $K_1(H^0(\bar{s}_i))$, and has image f(x) in $k(x)^*$ for x smooth. We first show that there is an affine neighborhood V of $p_1 \cup \cdots \cup p_n$, with locally principal complement, and an element t_V of $K_1(H_V)$ which maps to $\mathrm{res}_V(t^0)$ in $K_1(H^0_V)$. We proceed by induction on n, the case n = 0 being trivial.

Take an affine neighborhood W of $p_1 \cup \cdots \cup p_{n-1}$, with locally principal complement , and an element t_W of $K_1(H_W)$ representing $\mathrm{res}_W(t^0)$. We may assume that $p = p_n$ is not in W. As t is a section of F_S, there is an element x in $i_*(\mathcal{X}_2(X_S))_p$ such that $t = \tilde{T}(x)$ in an affine neighborhood U of p. We may assume that U has locally principal complement in X. The element x determines a element x' of $K_2(k(X))$. We define t_U to be $\partial(x')$, where ∂ is the boundary map from $K_2(k(X))$ to $K_1(H_U)$. If S_i is a one-dimensional component of S, then the following diagram commutes

$$\begin{array}{ccccccc} & & K_2(\mathcal{O}_{X,S_i}) & \xrightarrow{T} & K_1(H_{(U,S_i)}) & \to & K_1(H^0_{(U,S_i)}) = \coprod_{p\in(U-S_i)^1} K_1(H^0(p)) \\ & \nearrow & \downarrow & & \downarrow & & \downarrow \qquad\qquad \uparrow\downarrow \\ i_*(\mathcal{K}_2(X_S))_p & \searrow & K_2(k(X)) & \xrightarrow{T} & K_1(H_U) & \to & K_1(H^0_U) = \coprod_{p\in U^1} K_1(H^0(p)) \end{array}$$

Thus t_U goes to zero in $K_1(H^0_U(S_i))$; similarly, t_U represents $res_U(t^0)$ in $K_1(H^0_U)$.

We now consider the restrictions $res_{U\cap W}(t_W)$ and $res_{U\cap W}(t_U)$. Since both of these represent $res_{U\cap W}(t^0)$, there is a finite set of points $a_1,\dots,a_m$ of $U\cap W$ such that $res_{U\cap W}(t_W) - res_{U\cap W}(t_U)$ goes to zero in $K_1(H_{U\cap W} - \bigcup_i a_i)$. We shrink U by removing a curve C which passes through the a_i's but misses p, and chan notation. We may therefore assume that t_W and t_U restrict to the same element of $K_1(H_{U\cap W})$, hence by Mayer-Vietoris, there is an element $t_{U\cup W}$ of $K_1(H_{U\cup W})$ which restricts to t_U on U and t_W on W. Removing some curves from $U\cup W$, we may assume that $U\cup W$ is affine with locally principal complement, and the induction goes through.

Let then V be an affine neighborhood of $\bigcup_i p_i$, and t_V an element of $K_1(H_V)$ representing $res_V(t^0)$ as above. Let $V' = V-D$. Then $z = res_{V'}(t_V)$ goes to zero in $K_1(H^0_{V'})$, so there is a finite set of points $b_1,\dots,b_m$ such that z goes to zero in $K_1(H_{V'-\bigcup b_i})$ Let C be a locally principal curve on X containing all the b_i's, with X-C affine, but not passing through the finite set S-V. Let U = X- Then $res_{U\cap V}(t_V) = 0$ in $K_1(H_{U\cap V})$, so we can extend t_V to $Y = U\cup V$ to get an element t_Y of $K_1(H_Y)$ which restricts to t_V on V, and to 0 on U. In particular Y is a neighborhood of S in X, and t_Y is an element of $K_1(H_Y)$ which represents $res_Y(t^0)$.

Let $A = X-Y$. Write t as $t = res_Y(t) + t'$. Then t' is supported in the smooth locus of X, hence $div(t') = 0$ in $K_0(X)$. Thus we may assume that $t = res_Y(t)$. As A is contained in the smooth locus of X, we have localization sequences

$$\begin{array}{ccccccc} K_1(X) & \longrightarrow & K_1(Y) & \xrightarrow{\partial} & K_0'(A) & \longrightarrow & K_0(X) \\ \downarrow & & \downarrow{\scriptstyle i^*_{Y-S}} & & \| & & \downarrow \\ K_1(X-S) & \longrightarrow & K_1(Y-S) & \xrightarrow[\partial]{} & K_0'(A) & \longrightarrow & K_0(X-S) \end{array}$$

Let z be the image of t_Y under $K_1(H_Y) \to K_1(Y)$. It is well known that $\partial \circ i^*_{Y-S}(z) = [div(t)]$ as an element of $K_0'(A)$. Thus $\partial(z) = [div(t)]$ goes to zero in $K_0(X)$, as desired.

q.e.d.

We recall that, on a quasi-projective surface X with singular locus S, the group $CH^2(X,S)$ is the free abelian group on the smooth points of X, modulo relations of the form $div(f)$, where f is in $k(D)^*$ for some curve D on X, satisfying the conditions:

1) $D \cap S$ is a finite set
2) D is principal in a neighborhood of each point of $D \cap S$
3) f is a unit in $\mathcal{O}_{D,p}$ for each p in $D \cap S$.

It is easily shown that each zero cycle of the form $div(f)$ for such an f as above goes to zero in $K_0(X)$, hence there is a homomorphism $\gamma : CH^2(X,S) \to K_0(X)$ defined by sending the equivalence class of a smooth point x to the $K_0(X)$ class of the residue field $k(x)$. We have shown in [L] that γ defines an isomorphism of $CH^2(X,S)$ with the subgroup $F_0K_0(X)$ of $K_0(X)$ generated by the classes $[k(x)]$ for x a smooth (closed) point of X.

On the other hand, given such a pair (D,f) satisfying (1)-(3) above, consider a point p of $D \cap S$. Let G be a local defining equation for D, and let F be a function in $\mathcal{O}^*_{X,p}$ which restricts to f on D in a neighborhood of p. We may choose G and F so that both are units in $\mathcal{O}_{X,S_i}$ for each one-dimensional component S_i of S. Then the symbol $\{F,G\}$ defines an element of K_2R_S, hence also an element of $i_*(\mathcal{K}_2(X_S))_p$, and $T_p(\{F,G\}) = (D,f)$ at p. Thus (D,f) defines a global section of F_S . There is therefore a surjection

$$CH^2(X,S) \twoheadrightarrow H^2(\Gamma(G.)) = \coprod_{x \in (X-S)^2} \mathbb{Z} \Big/ \operatorname{div}(H^0(X,F_S))$$

By lemma 2, the map $\gamma: CH^2(X,S) \to K_0(X)$ factors through $H^2(\Gamma(G.))$, hence we have shown

Corollary 3. $CH^2(X,S)$ is isomorphic to $H^2(\Gamma(G.))$.

We now analyze the cohomology of the sheaves in the resolution G. .

Lemma 4. Let A be a one-dimensional reduced semi-local ring, and let $\bar{X} = \operatorname{Spec}(A)$. Then $H^i(\bar{X},\mathcal{K}_2) = 0$ for $i > 0$. (We assume that each residue field of A has at least three elements).

Proof. Let A have closed points $p_1,\ldots,p_n$. Let A_1 be the local ring A_{p_1}, and let A_2 be the semi-local ring $A_{p_2 \cup \cdots \cup p_n}$. Let U_i be the open subset $\operatorname{Spec}(A_i)$ of $\bar{X}$, i=1,2. Since U_1 is local, $H^i(U_1,\mathcal{K}_2) = 0$ for $i > 0$. By induction, we may assume that $H^i(U_2,\mathcal{K}_2) = 0$ for $i > 0$ as well. As $U_1 \cap U_2$ is the ~~single~~ disjoint union of points Spec(F), F the ~~fraction~~ total quotient field of A, $\mathcal{U} = \{U_1, U_2\}$ is a Leray covering of X for $\mathcal{K}_2$. In particular, $H^i(\bar{X},\mathcal{K}_2) = 0$ for $i \geq 2$. To show that $H^1(X,\mathcal{K}_2)$ is zero, we need only show that every element z of $K_2(F)$ can be written as $z = z_1 \cdot z_2$, with z_i in $K_2(A_i)$. We may assume that $z = \{a,b\}$, with a,b in A.

By the Chinese remainder theorem, we may write a and b as

$$a = u_1 \cdot u_2 \; ; \qquad u_i, v_i \in A_i^* \cap A \qquad \text{for } i = 1,2$$
$$b = v_1 \cdot v_2$$

Thus

$$\{a,b\} = \underset{K_2(A_1)}{\{u_1,v_1\}} \cdot \underset{K_2(A_2)}{\{u_2,v_2\}} \cdot \{u_2,v_1\} \cdot \{u_1,v_2\}$$

This reduces us to the case in which a is a unit in A_1 and b is a unit in A_2 (and both a and b are in A). Write b as

$$b = b_0(1 + b_1 a) \; ; \qquad b_0 \in A^*, \;\; b_1 \in A \quad .$$

By the Chinese remainder theorem, there is an element c of A such that $c(1+b_1a) + b_1$ is a unit in A_2, and $t = b_0^{-1} \cdot (1 + ca)$ is a unit in A. Then

1) $1 - tb$ is in $(a)A$
2) $s = (1 - tb)/a$ is a unit in A_2 .

We have

$$\{a,b\} \cdot \{a,t\} = \{a,bt\}$$
$$\{a,bt\} \cdot \{s,bt\} = \{1 - tb, tb\} = 1 \; ,$$

so

$$\{a,b\} = \{t,a\} \cdot \{bt,s\} \in K_2A_1 \cdot K_2A_2 \qquad ,$$

as desired.

Corollary 5. $H^i(X, i_*(\mathcal{K}_2(X_S)) = 0$ for $i > 0$.

Proof. By the above lemma, $H^i(X_S, \mathcal{K}_2(X_S)) = 0$ for $i > 0$, so we need only show that $R^q i_*(\mathcal{K}_2(X_S))_p = 0$ for $q > 0$ and for p in X.

1) p a smooth point of X. Then $R^q i_*(\mathcal{K}_2 X_S)_p = H^q(\mathrm{Spec}(k(X)), \mathcal{K}_2) = 0$ for $q > 0$.

2) p a point of S. Then $R^q i_*(\mathcal{K}_2 X_S)_p = H^q(X_{S/p}, \mathcal{K}_2)$, where $X_{S/p}$ is the open subset of X_S gotten by removing all points s_i which don't specialize to p. By the previous lemma, this cohomology group vanishes for $q > 0$.

This completes the proof of the corollary. q.e.d.

<u>Lemma 6</u>. $H^i(X, F_S) = 0$ for $i > 0$.

<u>Proof</u>. We have the inclusion $F_S \subseteq \coprod_{x \in (X-S)^1} i_{x*}(k(x)^*) \underset{\mathrm{def}}{=} F$; let $\mathcal{C}$ be the cokernel. Then $\mathcal{C}$ is supported at closed points of S, so

$$H^i(X, F_S) = H^i(X, F) = 0 \quad \text{for } i \geq 2$$
$$H^1(X, F_S) = H^0(X, \mathcal{C}) / \mathrm{Im}\, H^0(X, F) \quad .$$

$\mathcal{C}$ is a direct sum of skyscraper sheaves $i_{p*}(\mathcal{C}_p)$, and $\mathcal{C}_p$ is generated by representatives (D,f) in F_p. Take then a curve D passing through p, and a function f in $k(D)^*$. By adding elements of the form $(D', 1)$, we may assume that D is principle in an affine neighborhood U of $D \cap S$, say defined by H in $\Gamma(U, \mathcal{O}_U)$. We may choose U so that U contains each generic point of S. We may also assume that f is a regular function on $D \cap U$. Take a regular function F on U which restricts to f on $D \cap U$. Take N sufficiently large so that, letting m denote the maximal ideal of $\mathcal{O}_{X,p}$, we have

$$F' \in \mathcal{O}_{X,p},\ F' \equiv F \bmod m^N \Rightarrow F'|_D = uf, \text{ with u a unit in } \mathcal{O}_{D,p} \quad .$$

We need only take N so large so that m^{N-1} is contained in $(f)\mathcal{O}_{D,p}$ when restricted to D. ~~where [illegible] is the conductor of [illegible] at p~~. Let L be a line bundle on X, chosen sufficiently ample so that F extends to a global section of L, and $m^N \otimes L$ is generated by global sections. Then there is a section s_0 in $H^0(X, L)$ such that, with respect to some local trivialization of L near p,

i) $s_0|_D = uf$, with u a unit in $\mathcal{O}_{D,p}$

ii) (s_0), the divisor of s_0, does not contain any point of $D \cap S$ other than p, and contains no curve in S

iii) (s_0) contains no generic point of D, nor any point of the finite set $S - U$.

Let s_∞ be a section of L that is non-zero at each point of $D \cap S$, at each generic point of D, at each point of $S-U$, and at each generic point of S. Let G be the rational function s_0/s_∞. Then G satisfies

1) $G|_D = u'f$, with u' a unit in $\mathcal{O}_{D,p}$.

2) H is a unit at each point of $(G) \cap S - \{p\}$.

For (E,t) in $\coprod_{x \in (X-S)^1} k(x)^*$, we denote by $\overline{(E,t)}$ the class of (E,t) as a section of $\mathcal{C}$, and $\overline{(E,t)}_q$ the class of (E,t) in $\mathcal{C}_q$, $q \in X$. As D is Cartier, $\overline{(D,f)}_p = \overline{(D,u'f)}_p$, and so

$$\begin{aligned} \overline{(D,f)}_p &= \overline{(D,u'f)}_p \\ &= \overline{(D,G|_D)}_p \\ &= \overline{(D,G|_D)}_p + \overline{T_p(\{H,G\})}_p \qquad \text{as } \{H,G\} \text{ is in } K_2R_S \\ &= \overline{((s_0),\ h)}_p \qquad h = H|_{(s_0)} \end{aligned}$$

Also, since H is a unit at each other point of $(s_0) \cap S$, $\overline{((s_0),h)}_q = 0$ for each $q \neq p$. Thus $\overline{(D,f)}_p$ is in the image of $H^0(X,F)$, and $H^1(X,F_S) = 0$, as desired.

q.e.d.

We can now prove our main result.

<u>Theorem</u>. Let X be a quasi-projective surface, with singular locus S. Then $CH^2(X,S)$ is isomorphic to $H^2(X,\mathcal{K}_2)$.

<u>Proof</u>. As $\mathcal{K}_2$ and $\mathcal{K}_2'$ are isomorphic off the closed points of S, we have $H^2(X,\mathcal{K}_2) = H^2(X,\mathcal{K}_2')$. By proposition 1, together with lemma 4, corollary 5, and lemma 6, $G.$ is an acyclic resolution of $\mathcal{K}_2'$. Corollary 3 finishes the proof.

q.e.d.

REFERENCES

[B] S. Bloch,"K_2 and algebraic cycles", Ann. of Math. 99(1974) 349-379 .

[C] A.Collino,"Quillen's $\mathcal{K}$-theory and algebraic cycles on almost non-singular varieties", Ill. J. Math.25 #4(1981) 654-666.

[G] D. Grayson,"Higher algebraic K-theory II", Springer LNM No. 551(1976) 217-240

[L] M. Levine,"A geometric theory of the Chow ring on a singular variety", preprint.

[Q] D. Quillen,"Higher algebraic K-theory I" Springer LNM No. 341(1973) 85-147.