THOM SPACES AND A METHOD OF CONSTRUCTING SIMPLY CONNECTED CLOSED SYMPLECTIC NON-FORMAL MANIFOLDS

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1. Introduction

The purpose of this paper is to analyze the formality of symplectic blow-up. Let $\tilde{X}$ be the result of symplectic blow-up of a symplectic compact manifold X along a symplectic submanifold M. We are interested in describing relations between rational homotopy properties of $\tilde{X}$ and those of X and M. In general it seems to be rather difficult to handle this problem, however, the "first step" obstructions to formality, namely, the *triple* Massey products appear to behave "well" enough. The main results of the paper show that the triple Massey products in both $H^*(X)$ and $H^*(M)$ survive under the blow-up construction, and this gives many series of examples of compact symplectic non-formal (and, hence, non-Kähler) manifolds. There are many examples of *non-simply connected* symplectic compact non-formal (non-Kähler) manifolds [16], but until recently there were no examples in the simply-connected case (cf. [11,16]). The first example of this type (in fact, a major breakthrough) was obtained recently by Babenko and Taimanov [1]. However, it is still interesting and important to further investigate formality properties of blow-up. The reason for this is that, on one hand, rational homotopy theory proved to be a powerful tool in symplectic geometry [16], but, on the other hand, in the simply connected case, there is only one (obvious) known homotopic obstruction to the existence of symplectic structures on compact manifolds (the top power of the cohomology class of the given symplectic form must be non-zero). It is very important to explore the boundaries of the homotopic study of symplectic manifolds and to learn if there are other homotopic obstructions. Since, in the simply connected case, there is not many examples of compact symplectic formal (or non-formal) manifolds, the analysis of this and other homotopic properties of blow-ups is crucial. The present paper contains results which essentially give "qualitative" description of the whole problem, as well as a method of constructing simply-connected closed *non-formal* symplectic manifolds. We obtain new examples as well as a general explanation of this "non-formalizing tendency" of simply connected symplectic compact manifolds (as opposed to the formalizing tendency of

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symplectic manifolds noticed by Lupton and Oprea [11]). The manifolds considered by the authors of [11] had "large" symmetry and, conjecturally, this is the main reason why their formality properties differ from that of blow-ups. As a partial evidence for this "dichotomy" we show that compact symplectic G-manifolds endowed with a hamiltonian G-action of cohomogeneity 1 have all triple Massey products zero. The proofs of our results appeared to be a consequence of Gitler's description of the multiplication structure of the cohomology of complex blow-up [5]. Although the author of [5] worked in complex category, rather straightforward modification of his result led to the similar description of the cohomology of symplectic blow-up. This is interesting on its own right. It gives a description of the multiplication in $H^*(\tilde{X})$ in terms of the cohomology rings of $H^*(X)$, $H^*(\tilde{M})$ and maps $\tilde{k}^*$ and $\tilde{t}^*$ together with the Thom isomorphism. Here $\tilde{k}^*$ is a cohomology morphism induced by embedding $\tilde{k}:\tilde{M}\to\tilde{X}$ and $\tilde{t}^*$ is induced by the Thom-Pontryagin map $\tilde{t}^*:T(\tilde{N})\to\tilde{X}$ determined by the Thom space of the normal bundle of $\tilde{k}$. As in the complex case, $\tilde{M}$ denotes the projectivized normal bundle of the embedding $k:M\to X$.

The main results of the paper can be formulated as follows.

Theorem 1. *Let X be any compact symplectic manifold whose minimal model has non-vanishing triple Massey products. Then, its blow-up along any symplectic compact submanifold M is non-formal. In particular, any blow-up of the McDuff manifold $\widetilde{\mathbb{C}P^5}$ along any symplectic submanifold is non-formal.*

Note that by the theorem of Donaldson [4] compact symplectic manifolds always admit symplectic submanifolds of any even codimension, hence we obtain families of examples of simply-connected symplectic non-formal manifolds coming from the corollary below. The corollary is valid due to the fact that the McDuff manifold $\widetilde{\mathbb{C}P^5}$ has non-vanishing triple Massey product by [1].

Corollary 1. *Any blow-up of $\widetilde{CP^5}$ along any Donaldson submanifold is non-formal.*

The next result establishes obstructions to formality coming from a submanifold.

Theorem 2. *Let $\tilde{X}$ be the symplectic blow-up of X along a symplectic submanifold M of codimension $2k$, $k>3$. If the minimal model of M has non-vanishing triple Massey products, $\tilde{X}$ is non-formal.*

Corollary 2. *Let KT denote the Kodaira-Thurston manifold embedded into $\mathbb{C}P^5$ by the Tischler-Gromov theorem. Consider the embedding*

$$KT\hookrightarrow\mathbb{C}P^5\hookrightarrow\mathbb{C}P^6$$

The blow-up $\widetilde{\mathbb{C}P^6}$ of $\mathbb{C}P^6$ along the Kodaira-Thurston manifold is non-formal.

Remark. Taimanov and Babenko proved that the McDuff manifold $\widetilde{\mathbb{C}P^5}$ is non-formal, explicitly calculating the triple Massey products. Our corollary follows from the general theorem, although to use it, we have to raise codimension.

Corollary 3. *Let $I(p,g)\hookrightarrow X$ be any symplectic embedding of codimension >3 of the Iwasawa manifold into compact simply connected symplectic manifold. Then* *the blow-up $\tilde{X}$ of X along $I(p,q)$ for $p>1$, $q>1$ is non-formal. In particular, the Gromov-Tischler embedding*

$$I(p,q) \hookrightarrow \mathbb{C}P^{p+q+2}$$

yields a non-formal simply connected compact symplectic blow-up $\widetilde{\mathbb{C}P}^{p+q+2}_{I(p,q)}$.

Remark. The definition of the Iwasawa manifold is given at the end of the paper (see also [3]).

It is known, that all compact homogeneous symplectic manifolds G/H of compact Lie groups G are Kähler and, hence, formal. The first obstruction to formality also vanishes for G-manifolds of cohomogeneity 1.

Theorem 3. *Let (M,G) be a compact symplectic G-manifold endowed with a hamiltonian action of a compact Lie group G of cohomogeneity 1. Then the minimal model of M has vanishing all triple Massey products.*

Remark. Of course, it would be temptating to conjecture that such manifolds are formal, but the lack of "workable" description of the indeterminacy does not allow to follow the lines of proofs in this paper in case of higher order products.

We complete this introduction by establishing some preliminary facts and notation. However, to make the article reasonably short we prefer to give exact references and keep standard notation rather than write extended introduction.

Basic reference for symplectic geometry is [13], for rational homotopy theory – [6,7,16]. We will work in the category of differential graded algebras over field of characteristic 0 (basically, $\mathbb{R}$), and we will denote this category by *k-DGA*. If we consider morphisms in this category, we call them *DGA-morphisms*. Let us start with the notion of *formality* [6,7,16]. To each CW-complex X there is associated a differential graded algebra $(\mathcal{M}_X, d)$ which is a homotopic invariant and which satisfies the following properties

(1) $(\mathcal{M}_X, d)$ is free in the category of differential graded algebras:

$$(\mathcal{M}_X, d) = (\Lambda(V), d)$$

where V is a space of free generators of $\mathcal{M}_X$

(2) $\Lambda(V)$ is minimal, i.e. there exists a well-ordered basis of V, say $\{x_\alpha, \alpha \in J\}$ such that $|x_\beta| > |x_\alpha| \implies \beta > \alpha$ for any degrees $|x_\beta|, |x_\alpha|$,

(3) there exists a DGA-morphism

$$\rho : (\mathcal{M}_X, d) \to (A_{PL}(X), d_X)$$

with the target algebra the differential graded algebra of the de Rham PL-forms, inducing isomorphism ρ^* on the cohomology level.

By definition, $(\mathcal{M}_X, d)$ is called the *minimal model* of X.

Definition. A space X is called *formal*, if there exists a DGA-morphism

$$\rho : (\mathcal{M}_X, d) \to (H^*(X), 0).$$

Not all CW-complexes are formal. Criteria of formality can be found in [6,7,16]. In our paper we use only one obstruction to formality, namely, *triple Massey products.* Since this obstruction is essential in our considerations, we give details of it. Thus, let (A,d) be any differential graded algebra. Let $[u],[v],[w]\in H^*(A)$ be cohomology classes satisfying the equalities

$$[u][v]=[v][w]=0.$$

Consider the corresponding conditions on the cochain level:

$$dx=uv,\quad dy=vw.$$

One can check that the cochain

$$xw+\bar{u}y,\quad \bar{u}=(-1)^{deg(u)+1}u$$

is in fact a cocycle.

Definition. The set of all cohomology classes which can be constructed in this way is called the *triple Massey product* of $[u],[v],[w]$. This set is denoted by

$$\langle[u],[v],[w]\rangle.$$

Therefore,

$$\langle[u],[v],[w]\rangle=\{[xw+\bar{u}y],\,\text{where}\, u,v,x,y\,\text{vary over the choices}\,\}$$

This means that u,v vary over the choices of cocycles representing the same cohomology classes and x, y over the choices which satisfy $dx=uv, dy=vw$.

Note again that the triple Massey product is a *set* and there is an indeterminacy in the choice of the corresponding cohomology classes. This caution will be important in our considerations. However, this indeterminacy can be completely described. Namely, when defined, $\langle[u],[v],[w]\rangle$ is an element of the cohomology ring

$$H^*(A)/([u],[w])$$

where $([u],[w])\subset H^*(A)$ denotes the ideal in $H^*(A)$ generated by $[u]$ and $[w]$. This can be proved very easily by calculating the difference between cohomology classes representing the triple Massey product and determined by different choices of u,v,w,x,y. Also, one may consult [16] and [12 ,p.289]. The most important observation for us is that *any two cohomology classes representing the same Massey product differ by an element from* $([u],[w])\subset H^*(A)$.

Triple Massey products give an obstruction to formality: *if* $(\mathcal{M}_X,H^*(X))$ *has non-trivial Massey products, X is non-formal.* The proof of this can be found in [16].

We denote by $A_{PL}:\mathbb{T}op^*\to DGA$ the *Sullivan functor* [6] from the category of pointed topological spaces to the category of differential graded Lie algebras.

In the sequel we will use the *Thom spaces* of vector bundles. Basic references are [2,14]. The *blow-up* construction is explained in the next section. Also, in the formulation of our results, we used *the Kodaira-Thurston* and *the McDuff manifold* (these are basic examples in the area of research presented in this article). The Kodaira-Thurston manifold can be described as a nilmanifold of the form

$$KT = (N_3/N_3(\mathbb{Z})) \times S^1$$

where N_3 is a 3-dimensional Heisenberg group of unipotent triangular 3×3-matrices and $N_3(\mathbb{Z})$ denotes the corresponding subgroup of matrices with integer entries. This manifold is symplectic and non-Kähler. The McDuff manifold is obtained by the blow-up procedure applied to the symplectic embedding

$$KT \hookrightarrow \mathbb{C}P^5$$

(which exists by the Gromov-Tischler theorem [8,15]). We denote this manifold by $\widetilde{\mathbb{C}P^5}$. This manifold was the first example of symplectic compact simply connected non-Kähler manifold (McDuff, see [13]). This manifold is also non-formal by [1].

Let us recall, finally, the notions of *hamiltonian G-action* and G-manifolds of cohomogeneity 1. We refer to [10] and [13]. An action of a Lie group on a manifold M is called of *cohomogeneity k*, if the regular (principal) orbits of G have codimension k. Now, in our paper we consider G-manifolds of cohomogeneity 1. Let us recall that if an action of a Lie group on a symplectic manifold (M,ω) preserves the symplectic form ω then there is a Lie algebra homomorphism

$$\mathfrak{g} = Lie(G) \xrightarrow{\mathcal{F}_*} Vect_\omega(M)$$

where $Vect_\omega(M)$ denotes the Lie algebra of symplectic vector fields. The action of G is called *almost hamiltonian* if the image of $\mathcal{F}_*$ lies in the subalgebra $Vect_{\mathrm{Ham}}(M)$ of Hamiltonian vector fields. Finally, if the map $\mathcal{F}_*$ can be lifted to a homomorphism

$$\mathfrak{g} \xrightarrow{\mathcal{F}} C^\infty(M,\mathbb{R})$$

(i.e. $\mathcal{F}_* v = sgrad\mathcal{F}_v$) then the action of G is *Hamiltonian.* The study of symplectic G-manifolds of cohomogeneity 1 was started in [10] in order to get more examples of compact symplectic manifolds via the Lie group theory. It seems that this class of symplectic manifolds is important also in the context of the present paper, as is indicated in Theorem 3. Note, however, that we do not use "geometric" properties of this class directly, it appeares to be enough to use the topological description of it given in [10] (see Theorem 7 below).

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2. A commutative diagram associated to symplectic blow up

Our purpose is to construct an analogue of the commutative diagram of spaces associated to the *complex blow-up* by S. Gitler [5]. Recall that for the complex blow-up $\tilde{X}$ resulting from an embedding of complex manifolds $k : Y \to X$ one can build the commutative diagram of spaces

$$\begin{array}{ccccccc} \tilde{Y} & \xrightarrow{\tilde{k}} & \tilde{N} & \xrightarrow{\tilde{j}} & \tilde{X} & \xrightarrow{\tilde{t}} & T(\tilde{N}) \\ {\scriptstyle g}\downarrow & & \downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \hat{f}}\downarrow \\ Y & \xrightarrow{k} & N & \xrightarrow{j} & X & \xrightarrow{t} & T(N) \end{array} \tag{1}$$

in which f is a blow-up projection, $\tilde{Y} = f^{-1}(Y)$, $\tilde{k}$, k and $j, \tilde{j}$ are inclusions, N and $\tilde{N}$ are the normal bundles of inclusions k and $\tilde{k}$, while inclusions j and $\tilde{j}$ are determined by the tubular neighbourhood theorem. Also, $T(N)$ and $T(\tilde{N})$ denote *the Thom spaces* of the normal bundles of embeddings k and $\tilde{k}$. Further, since $\tilde{Y}$ admits a structure of the projectivized bundle over Y, it is possible to define $\tilde{N} = L$ as a *canonical line bundle*

$$L = \{(y, v, l) | \quad y \in Y, v \in l\}$$

over $\tilde{Y}$. The maps t and $\tilde{t}$ denote the *Thom-Pontryagin (or Browder-Novikov) maps.* These maps are obtained by shrinking the corresponding complements of $\tilde{j}(\tilde{N})$ and $j(N)$ in $\tilde{N}$ and N to points and thus getting maps to one-point compactifications of $\tilde{N}$ and N, i.e., the Thom spaces $T(\tilde{N})$ and $T(N)$ (see [14]).

Now we are going to describe an analogue of this. In case of the symplectic blow-up certain descriptions and identifications have not exactly the same meaning. However, after some modification one can get a diagram which will formally look exactly like (1) and will have *all algebraic properties of the corresponding cohomology diagram.* To do this, we must carefully describe the whole symplectic blow-up construction with the aim of showing which terms in (1) should be replaced or interpreted in a different manner.

Let us begin with a symplectic submanifold $M^{2(n-k)} \subset X^{2n}$ with normal bundle $\nu : E \to M$. We choose ν to be ω-orthogonal to $TM \subset TX$. Therefore, we assume that the fiber of ν is $\mathbb{C}^k$ and the structure group of the associated principal frame bundle is $U(k)$, i.e.

$$U(k) \to P \to M.$$

Consider also the *canonical line bundle*

$$L \to \mathbb{C}P^{k-1}, \quad L = \{(z, l) \in \mathbb{C}^k \times \mathbb{C}P^{k-1} | \quad z \in l\}.$$

Note that $U(k)$ acts on $\mathbb{C}^k$ and $\mathbb{C}P^{k-1}$ by the same rule $z \to Az, [z] \to [Az]$. Hence, we may form the bundles

$$L \to \tilde{E} \to M, \quad \mathbb{C}P^{k-1} \to \tilde{M} \to M$$

with the total spaces

$$\tilde{E} = P \times_{U(k)} L, \quad \tilde{M} = P \times_{U(k)} \mathbb{C}P^{k-1}.$$

Note that $\tilde{E}$ and $\tilde{M}$ will play the role of $\tilde{N}$ and $\tilde{Y}$ in our analogue of (1). Also, $\tilde{E}$ becomes a line bundle over $\tilde{M}$:

$$\tilde{E} = P \times_{U(k)} L \to \tilde{M} = P \times_{U(k)} \mathbb{C}P^{k-1}, \quad [p,(z,l)] \to [p,l]$$

where the square brackets stand for equivalence classes with respect to the $U(k)$-action, $p \in P$, $(z,l) \in L, l \in \mathbb{C}P^{k-1}$. Note also that $\tilde{M}$ is by definition the projectivized normal bundle associated to ν.

Define the map

$$\phi : \tilde{E} \to E, \quad \phi([p,(z,l)]) = [p,z]$$

and consider the sets $\tilde{E}_0 = \tilde{E} - \{\text{the zero section}\}$ and $E_0 = E - \{\text{the zero section}\}$. Also, M and $\tilde{M}$ can be embedded into $\tilde{E}$ and E as zero sections of the vector bundles

$$E \to M, \quad \tilde{E} \to \tilde{M}.$$

These embeddings can be described explicitly:

$$\pi(p) \to [p,0], \quad [p,l] \to [p,(0,l)]$$

for $\pi : E \to M$ (one checks immediately that everything is well-defined). Now, recall the construction of the symplectic blow-up of a symplectic manifold X along a symplectic submanifold M. Take the normal bundle ν and the corresponding disc bundle

$$V \to M.$$

According to the tubular neighbourhood theorem,

$$V \cong W \subset X$$

for the tubular neighbourhood $W \supset M$ of M in X. Set

$$\tilde{V} = \phi^{-1}(V) \subset \tilde{E}$$

and note that ϕ is a diffeomorphism on $\tilde{E}_0$, i.e.

$$\phi|_{\tilde{E}_0} : \tilde{E}_0 \xrightarrow{\cong} E_0.$$

The latter can be seen from the representations

$$\tilde{E}_0 = \{[p,(z,l)], \quad z \neq 0, z \in l\}, \quad E_0 = \{[p,z], \quad z \neq 0\}$$

and the definition of ϕ. In particular, $\tilde{V} \cap \tilde{E}_0 \cong V \cap E_0$, and since $V \cap E_0$ does not contain M realized as a zero section, we get

$$\partial\tilde{V} \cong \partial V.$$

Definition. The *blow-up* $\tilde{X}$ of X along M is the smooth manifold

$$\tilde{X} = \overline{(X - W)} \cup_{\partial\tilde{V}} \tilde{V},$$

where the identification of points is given by

$$\partial\tilde{V} \cong \partial V \cong \partial W.$$

The role of this construction is that it yields new symplectic manifolds: *any symplectic blow-up again carries a symplectic structure* [13].

In the sequel we will need one additional observation that

$$\tilde{V} = \{[p, (z, l)], \quad |z| \leq 1\} = P \times_{U(k)} \tilde{D}$$

where $\tilde{D}$ is the usual complex blow-up of a disc. Now, one can define a projection $\tilde{X} \to X$ separately on $\tilde{V}$ as a restriction of ϕ and on $X - W$ as identity. This projection will be also denoted by ϕ.

One can see that there is a natural map

$$\tilde{k} : \tilde{M} \to \tilde{X} = \overline{(X - W)} \cup_{\partial\tilde{V}} \tilde{V}$$

defined by

$$\tilde{k}([p, l]) = [p, (0, l)] \in \tilde{V} \subset \tilde{X}$$

We claim that the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{k}} & \tilde{X} \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ M & \xrightarrow{k} & X \end{array}$$

is commutative (note that we interpret M as an image of the zero section of E). Indeed, the point $[p, l] \in \tilde{M}$ is embedded into $\tilde{X}$ as $[p, (0, l)]$. Therefore, $\phi([p, (0, l)]) = [p, 0]$. The latter can be identified with a a point in the zero section of E, i.e., with a point in M.

Finally, we have come to

Proposition 1. *There exists a commutative diagram*

$$\begin{array}{ccccccc} \tilde{M} & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{X} & \xrightarrow{\tilde{t}} & T(\tilde{N}) \\ {\scriptstyle\phi}\downarrow & & \downarrow & & {\scriptstyle\phi}\downarrow & & {\scriptstyle\hat{\phi}}\downarrow \\ M & \longrightarrow & N & \longrightarrow & X & \xrightarrow{t} & T(N) \end{array} \tag{1'}$$

Proof. We have just proved the commutativity of the first two squares. The remaining squares are commutative because of the general property of Thom spaces: we consider the normal bundles $\tilde{N}$ and N of embeddings

$$\tilde{M} \hookrightarrow \tilde{X}, \quad M \hookrightarrow X.$$

and embed M and $\tilde{M}$ as zero sections, N and $\tilde{N}$ according to the tubular neighbourhood theorem, then the map between the normal bundles is induced by ϕ and hence $\hat{\phi}$ is the corresponding map of one-point compactifications. Note that we have more, namely, N is identified with E (it is by definition) and $\tilde{N}$ is identified with $\tilde{E}$ considered as a line bundle over $\tilde{M}$. Indeed, one can write an obvious inclusion $\tilde{M} \to \tilde{V}$ and recall the representation of $\tilde{V} = \{[p,(z,l)], \quad |z| \leq 1\} \subset \tilde{X}$ diffeomorphic to the tubular neighbourhood of $\tilde{M}$ in $\tilde{X}$. Since $\tilde{V}$ (open) is obviously diffeomorphic to $\tilde{E}$, the result follows. Note that we did not change notation for the "open" bundle $\{[p,(z,l)], |z| < 1\}$ corresponding to $\tilde{V}$.

The following results are direct corollaries of [5] and Proposition 1.

Theorem 4. *We have short exact sequences*

$$0 \longrightarrow H^q(T(N)) \xrightarrow{\alpha} H^q(X) \oplus H^q(T(\tilde{N})) \xrightarrow{\beta} H^q(\tilde{X}) \longrightarrow 0$$

$$\alpha(y) = (t^* y, \hat{\phi}^* y), \quad \beta(x,z) = \phi^* x - \tilde{t}^* z.$$

The sequence is not multiplicative.

In the sequel we denote the Thom class of the vector bundle $\tilde{N}$ by U. Note that $U \in H^2(T(\tilde{N}))$. It is known that $\tilde{t}^*(U) = \zeta$, where ζ is identified with the Euler class of the vector bundle $\tilde{N}$ (considered now as a real 2-bundle over M). Consider the usual cup-product $H^*(T(\tilde{N})) \otimes H^*(M) \to H^*(T(\tilde{N}))$, and the Thom isomorphism determined by it, i.e. $H^p(M) \to H^{p+2}(T(\tilde{N})), v \to U \cup v$.

Since $\tilde{M}$ is a projectivized vector bundle over M of rank k, the cohomology of $\tilde{M}$ is given as follows

$$H^*(\tilde{M}) \cong H^*(M)[\zeta]/I$$

where I is an ideal generated by

$$\zeta^k - c_1\zeta^{k-1} + \ldots + (-1)^{k-1}c_{k-1}\zeta + (-1)^k c_k.$$

Here c_j denote the Chern classes of the vector bundle $E \to M$ (see [16, p. 127]).

Theorem 5. *The multiplication rule in the cohomology of the symplectic blow-up $\tilde{X}$ of X along M is given as follows. In the direct sum of vector spaces*

$$H^*(\tilde{X}) \cong \phi^* H^*(X) \oplus_{v.s.} \tilde{H}^*(M)[\zeta]/(\zeta^k - c_1\zeta^{k-1} + \ldots + (-1)^{k-1}c_{k-1}\zeta + (-1)^k c_k)$$

one multiplies elements as indicated below:

(i) *in $H^*(X)$, in accordance with the multiplication rule in $H^*(X)$,*
(ii) *in $\tilde{H}^*(M)[\zeta]/(\zeta^k - c_1\zeta^{k-1} + \ldots + (-1)^{k-1}c_{k-1}\zeta) = \tilde{H}^*(M)[\zeta]/I$ by*

$$[u\zeta^p] \cdot [v\zeta^q] = \tilde{t}^*((U \cup \zeta^{p-1}u) \cdot (U \cup \zeta^{q-1}v))$$

(iii) *for $\phi^* u \in H^*(X)$ and $[v\zeta^p] \in \tilde{H}^*(M)[\zeta]/I$, by the formula*

$$\phi^* u \cdot [v\zeta^p] = [k^* uv\zeta^p]$$

where $k^* : H^*(X) \to H^*(M)$ is induced by the embedding k.

Proof of Theorems 4 and 5. Consider, first, the manifolds

$$X = \overline{X - W} \cup_{\partial W} W \quad \text{and} \quad \tilde{X} = \overline{X - W} \cup_{\partial \tilde{V}} \tilde{V}$$

together with the accompanying homology ladder of Mayer-Vietoris sequences (with integral coefficients)

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H_i(\overline{X-W}) \oplus H_i(\tilde{V}) & \longrightarrow & H_i(\tilde{X}) & \longrightarrow & H_{i-1}(\partial\tilde{V}) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\ldots & \longrightarrow & H_i(\overline{X-W}) \oplus H_i(W) & \longrightarrow & H_i(X) & \longrightarrow & H_{i-1}(\partial W) & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
H_{i-1}(\overline{X-W}) \oplus H_{i-1}(\tilde{V}) & \longrightarrow & \ldots \\
\downarrow & & \downarrow \\
H_{i-1}(\overline{X-W}) \oplus H_{i-1}(W) & \longrightarrow & \ldots
\end{array}$$

Recall that, if Z is an oriented m-manifold with boundary ∂Z, then there is a natural isomorphism

$$H_m(Z, \partial Z) \cong H_{m-1}(\partial Z) \cong \mathbb{Z}.$$

This isomorphism implies that the inclusion $\partial Z \to Z$ induces the *zero* map $H_{m-1}(\partial Z) \to H_{m-1}(Z)$. Thus, in the Mayer-Vietoris ladder above we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_{2n}(\tilde{X}) & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \cong\downarrow & & \downarrow \\
0 & \longrightarrow & H_{2n}(X) & \xrightarrow{\cong} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Therefore, we have proved that the blow-up projection ϕ is an isomorphism in $2n$-homology over $\mathbb{Z}$:

$$\phi_* : H_{2n}(\tilde{X}) \xrightarrow{\cong} H_{2n}(X)$$

Of course, this is also true for cohomology and this has the effect that the induced map on cohomology $\phi^* : H^*(X) \to H^*(\tilde{X})$ is *injective*. To see this, suppose that $\phi^*(\alpha) = 0$ for $\alpha \in H^j(X)$. By Poincaré duality, there exists a $\beta \in H^{2n-j}(X)$ with $\alpha \cup \beta$ the (cohomology) fundamental class $\nu \in H^{2n}(X)$. But then, the isomorphism above gives a contradiction

$$\begin{aligned}
0 &\neq \phi^*(\nu) \\
&= \phi^*(\alpha) \cup \phi^*(\beta) \\
&= 0 \cup \phi^*(\beta) \\
&= 0.
\end{aligned}$$

Hence we have proved the following two facts:

(i) ϕ^* is injective in cohomology and preserves orientation;
(ii) $\phi : \tilde{X} \to X$ is a *degree one map* (by definition).

Having all this in mind we can proceed as follows. By (i), ϕ is a degree one map and, therefore, one can define the *Gysin map*

$$f! : H^*(\tilde{X}) \to H^*(X), \quad [X] \cap f!\tilde{x} = f_*([\tilde{X}] \cap \tilde{x})$$

(where $\cap$ denotes the cap-product and $[X]$ and $[\tilde{X}]$ denote the fundamental homology classes). One can easily check that $f!$ is a left invertible of f^*: $f!f^* = \mathrm{id}_{H^*(X)}$ and hence

$$H^*(\tilde{X}) = f^*H^*(X) \oplus (\mathrm{Ker}\, f!).$$

This check-up is made, e.g., in [5]. Indeed, one can notice that

$$[X] \cap f!f^*x = f_*([\tilde{X}] \cap f^*x) = f_*[\tilde{X}] \cap x = [X] \cap x$$

as required. In the same vein, one can prove the following relation.

Lemma 1. *For any $y \in (Ker\, f!)$ and $x \in H^*(X)$*

$$y \cup f^*x \in Ker\, f!.$$

Proof. See [5].

Now, to proceed further, we need the diagram of spaces given by Proposition 1. Consider the decompositions

$$\tilde{X} = (\tilde{X} - \tilde{M}) \cup \tilde{N}, \quad (\tilde{X} - \tilde{M}) \cap \tilde{N} = \tilde{N} - \tilde{M}$$

$$X = (X - M) \cup N, \quad (X - M) \cap N = N - M$$

They satisfy the excision property and therefore one can write the Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H^{q-1}(\tilde{N} - \tilde{M}) & \xrightarrow{\tilde{\alpha}} & H^q(\tilde{X}) & \xrightarrow{\tilde{j}^*} & H^q(\tilde{N}) \oplus H^q(\tilde{X} - \tilde{M}) & \xrightarrow{\tilde{\beta}} & \\
\uparrow & & \cong \uparrow f_2^* & & \phi^* \uparrow & & \bar{f}^* \oplus f_1^* \uparrow & & \\
\dots & \longrightarrow & H^{q-1}(N - M) & \xrightarrow{\alpha} & H^q(X) & \xrightarrow{j^*} & H^q(N) \oplus H^q(X - M) & \xrightarrow{\beta} &
\end{array}$$

$$\begin{array}{ccc}
H^q(\tilde{N} - \tilde{M}) & \longrightarrow & \dots \\
f_2^* \uparrow & & \\
H^q(N - M) & \longrightarrow &
\end{array}$$

where f_1^* and f_2^* are isomorphisms, ϕ^* and $\bar{f}^*$ are monomorphisms. Indeed, f_1^* and f_2^* are induced by diffeomorphisms $\tilde{X} - \tilde{M} \to X - M$ and $\tilde{N} - \tilde{M} \to N - M$. Note that the latter manifolds are diffeomorphic, since $\tilde{E}_0$ and E_0 are diffeomorphic. Then it follows that

$$\tilde{X} - \tilde{M} = \overline{(X - W)} \cup_{\partial \tilde{V}} (\tilde{V} - \tilde{M}) \cong \overline{(X - W)} \cup_{\partial V} (V - M) = X - M$$

and the diffeomorphism follows from the fact that $\tilde{V} - \tilde{M} \cong V - M$. As fas as $\bar{f}^*$ is concerned, one can notice that $\tilde{M}$ is a projectivized normal bundle over M and hence

$$H^*(\tilde{M}) \cong H^*(M)[\zeta]/I, \quad I = (\zeta^k - c_1\zeta^{k-1} + ...(-1)^k c_k)$$

where $c_1, ..., c_k$ denote the Chern classes of N. This is known and we refer to [16]. Also, $\bar{f}^*$ is the same as injection

$$g^* : H^*(M) \to H^*(\tilde{M}) = H^*(M)[\zeta]/I.$$

The excision property can be checked, since it is satisfied, e.g. for any pair of topological spaces K_1, K_2 such that $K_1 \cup K_2 = \mathrm{Int}_{K_1\cup K_2} K_1 \cup \mathrm{Int}_{K_1\cup K_2} K_2$ (this is well known). Also, as a map in the Mayer-Vietoris sequence, $\tilde{j}^* = (\tilde{j}_1^*, \tilde{j}_2^*)$, where $\tilde{j}_1^* : H^*(\tilde{X}) \to H^*(\tilde{N}) \cong H^*(\tilde{M})$ is induced by inclusion $\tilde{N} \hookrightarrow \tilde{X}$ and $\tilde{j}_2^* : H^*(\tilde{X}) \to H^*(\tilde{X} - \tilde{M})$ is induced by $\tilde{X} - \tilde{M} \hookrightarrow \tilde{X}$.

Lemma 2. *The map $\tilde{j}^*$ in the Mayer-Vietoris diagram above induces an isomorphism*

$$Ker\,\phi! \cong H^*(\tilde{M})/g^*H^*(M).$$

Proof. The proof follows from diagram chasing. Indeed, consider the following algebraic situation. Let

$$\begin{tikzcd}
... \arrow[r] & A \arrow[r, "\mu"] & B \arrow[r, "\alpha"] & C \arrow[r, "\beta"] & D \arrow[r] & ... \\
... \arrow[r] \arrow[u] & A \arrow[r, "\mu_1"] \arrow[u, "\cong"', "\theta"] & B_1 \arrow[r, "\alpha_1"] \arrow[u, "i_B"'] & C \arrow[r, "\beta_1"] \arrow[u, "i_C"'] & D \arrow[r] \arrow[u, "\cong"', "\eta"] & ... \arrow[u]
\end{tikzcd}$$

be a commutative diagram in which horizontal rows are exact. Assume that θ and η are isomorphisms and that i_B and i_C are monomorphisms. Then $B/i_B(B_1) \cong C/i_C(C_1)$. The proof is accomplished as follows. By exactness, $\mathrm{Im}\mu = \mathrm{Ker}\,\alpha$ and $\mathrm{Im}\,\mu_1 = \mathrm{Ker}\,\alpha_1$.Obviously, $i_B(\mathrm{Im}\,\mu_1) \subset \mathrm{Im}\,\mu$. Also, from the commutativity of the first square and injectivity of i_B it follows the converse inclusion, which gives

$$\mathrm{Im}\,\mu = i_B(\mathrm{Im}\,\mu_1) \implies \mathrm{Ker}\,\alpha = i_B(\mathrm{Ker}\,\alpha_1).$$

The latter implies also the inclusion

$$\mathrm{Ker}\,\alpha \subset i_B(B_1).$$

Consider the decomposition $B = i_B(B_1) \oplus W$. Note that $\alpha|_W$ is injective. From the commutativity of the middle square we get $i_C(\mathrm{Im}\,\alpha_1) = \alpha(i_B(B_1))$ which implies $\alpha(i_B(B_1)) \subset i_C(C_1)$. The latter yields a well-defined map

$$\bar{\alpha} : W \cong B/i_B(B_1) \to C/i_C(C_1).$$

This map is injective. Indeed, assume that $\alpha(w) \in i_C(C_1)$. Then, $\alpha(w) = i_C(c_1)$. By exactness, $\beta\alpha(w) = 0 \implies \beta i_C(c_1) = 0 \implies \beta_1(c_1) = 0$, since η is an isomorphism. Again, by exactness, $c_1 = \alpha_1(b_1)$. Hence,

$$\alpha(w) = i_C(\alpha_1(b_1)) = \alpha(i_B(b_1)) \implies \alpha(w - i_B(b_1)) = 0.$$

Therefore

$$w - i_B(b_1) \in \operatorname{Ker} \alpha \subset i_B(B_1)$$

which is a contradiction. Hence, $\bar{\alpha}$ is injective.

To prove the surjectivity of $\bar{\alpha}$, one must consider the "next" square in the given diagram:

$$\begin{array}{ccccc} C & \xrightarrow{\beta} & D & \xrightarrow{\mu^+} & B^+ \\ {\scriptstyle i_C}\uparrow & & \cong\uparrow{\scriptstyle \theta} & & {\scriptstyle i_{B^+}}\uparrow \\ C_1 & \xrightarrow{\beta_1} & D & \xrightarrow{\mu_1^+} & B_1^+. \end{array}$$

Let $c \in C$. Then $\beta(c) = d \in D$ and $d = \eta(d_1)$. Consider $i_{B^+}(\mu_1^+(d_1))$. Assume that $\mu_1^+(d_1) \neq 0$. Then $i_{B^+}(\mu_1^+(d_1)) \neq 0$. But $i_{B^+}\mu_1^+(d_1) = \mu^+\eta(d_1) = \mu^+(d) = \mu^+\beta(c) = 0$, because of the exactness. Hence, $\mu_1^+(d_1) = 0 \implies d_1 \in \operatorname{Ker} \mu_1^+ \implies d_1 = \beta_1(c_1)$. Therefore, $d = \eta(d_1) = \eta\beta(c_1) = \beta i_C(c_1)$.Finally,

$$\beta(c - i_C(c_1)) = 0 \implies c - i_C(c_1) \in \operatorname{Im} \alpha$$

which implies the surjectivity of $\bar{\alpha}$. Lemma 2 is proved.

Combining all the results above together, we get an *additive* decomposition of $H^*(\tilde{X})$.

Now, the proof of Theorem 4 goes exactly as the proof of the analogous theorem for complex blow-ups in [5]. One can repeat word by word Gitler's argument, since this argument depends only on algebraic properties of the corresponding commutative diagrams of spaces and cohomologies, which appeared to be the same for symplectic blow-ups (see, however, the "Appendix").

Now, we begin the proof of Theorem 5. Property (i) in the formulation of Theorem 5 is obvious. To prove (ii), note that the isomorphism $\operatorname{Ker} \phi! \cong H^*(\tilde{M})/g^*H^*(M)$ was induced by the map $\tilde{j}^*$ (and, hence, by $\tilde{j}_1^*$, since the form of the commutative diagram and the fact that W in the proof was arbitrary). The latter can be identified with $\tilde{k}^*$ (since $\tilde{N}$ is homotopy equivalent to $\tilde{M}$). It means, that we have a sequence

$$H^*(T(\tilde{N})) \xrightarrow{\tilde{t}^*} H^*(\tilde{X}) \xrightarrow{\tilde{k}^*} H^*(\tilde{M}) \xrightarrow{\pi} H^*(\tilde{M})/g^*H^*(M)$$

(with π being the natural projection) which identifies $\operatorname{Ker} \phi!$ with $H^*(\tilde{M})/g^*H^*(M)$. Note that $\tilde{t}^*$ and $\tilde{k}^*$ are *multiplicative, but π is not.*

Now we are using Thom spaces. Since $\tilde{E} \cong \tilde{N}$ is a line bundle with the Euler class ζ, we can use the following fact proved in [14]:

$$\tilde{k}^*\tilde{t}^*(U) = \zeta$$

where U is the Thom class of the line bundle $\tilde{N}$. From the usual properties of the cup product

$$\tilde{k}^*\tilde{t}^*(U \cup \zeta^p y) = \tilde{k}^*\tilde{t}^*(U)\zeta^p y = \zeta^{p+1}y.$$

Therefore, for $\bar{k}^*\tilde{t}^* : H^*(T(\tilde{N}) \to H^*(\tilde{M})/g^*H^*(M)$ determined by $\tilde{k}^*\tilde{t}^*$ and the canonical projection

$$\bar{k}^*\tilde{t}^*(U \cup \zeta^p y) = [\zeta^{p+1}y]$$

and this proves (ii).

To prove (iii), we note that we must find the result of the multiplication $\phi^*x \cdot z$, where $z \in H^*(\tilde{X})$ corresponds to some $[y\zeta^p] \in H^*(\tilde{M})/g^*H^*(M)$ by the map $\pi\tilde{k}^*$. By Lemma 1, since $z \in \operatorname{Ker}\phi!$, $\phi^*x \cdot z \in \operatorname{Ker}\phi!$. Therefore, the result of multiplication of ϕ^*x and z again can be described as the image of $\pi\tilde{k}^*$. Therefore

$$\pi\tilde{k}^*(\phi^*x \cdot z) = \pi(\tilde{k}^*(\phi^*x \cdot z)) = \pi(g^*k^*x \cdot \tilde{k}^*(z)) = \pi(g^*k^*x \cdot y\zeta^p) = [k^*xy\zeta^p].$$

This yields the necessary identification.

Example. Assume that $M \subset X$ is a symplectic submanifold of a compact manifold X such that $k^* : H^*(X) \to H^*(M)$ is surjective. Then one can obtain from Theorem 5 the following form of $H^*(\tilde{X})$ for the symplectic blow-up along M.

Proposition 2. *If $k^* : H^*(X) \to H^*(M)$ is surjective, there exists a surjective homomorphism of algebras*

$$\mu : H^*(X) \otimes \mathbb{R}[\zeta] \to H^*(\tilde{X})$$

$$\mu(u \otimes \zeta^r) = \phi^*u, \quad \textit{if} \quad r = 0 \textit{ and } \quad \mu(u \otimes \zeta^r) = [k^*u\zeta^r], \quad \textit{if } r > 0$$

The kernel of μ is the ideal generated by $(Ker k^*) \otimes \mathbb{R}^+[\zeta]$ and the class

$$z = \zeta^k - \zeta^{k-1}\bar{c}_1 + ... + (-1)^k\bar{c}_k$$

where $\bar{c}_i \in H^*(X)$ are cohomology classes such that $k^*\bar{c}_i = c_i$. The notation $\mathbb{R}^+[\zeta]$ is used for subalgebra of elements of positive degrees.

Proof. As a linear map, μ is surjective, since $[k^*(u)\zeta^q]$ generate the whole $\tilde{H}^*(M)[\zeta]/I$. Hence, it is enough to check that μ is multiplicative. This is done directly, the only non-trivial case being when one multiplies elements $x_1 \otimes \zeta^r$ with $r > 0$ and $x_2 \in H^*(X)$. Then we have

$$\mu((x_1 \otimes \zeta^r) \cdot x_2) = \mu(x_1 \cdot x_2 \otimes \zeta^r) = k^*(x_1 \cdot x_2)\zeta^r$$

$$= k^*(x_1) \cdot k^*(x_2)\zeta^r = \phi^*(x_1) \cdot k^*(x_2)\zeta^r = \mu(x_1) \cdot \mu(x_2 \otimes \zeta^r)$$

according to the proof of (iii) of Theorem 5.

Remark. To get a non-trivial example of surjective k^*, one can take any totally non-cohomology to zero symplectic fibration (TNCZ-fibration)

$$F \to E \to B$$

and blow-up E along the fiber F. Proposition 2 is also contained in [5] in the complex case. The definition and examples of TNCZ-fibrations can be found in [12].

3. Proofs of Theorems 1 and 3

Recall that we need to prove the following.

Any compact symplectic blow-up $\tilde{X}$ of X along M is non-formal, if X has non-vanishing triple Massey products.

Proof. By Theorem 5,

$$H^*(\tilde{X}) = H^*(X) \oplus \tilde{H}^*(M)[\zeta]/(z^k - c_1\zeta k - 1 + ... + (-1)^{k-1}c_{k-1}\zeta + (-1)^k c_k)$$

with the multiplication rules determined by $\tilde{t}^*$ and k^*. The property of this multiplication, which we need, can be reformulated as follows:

$$H^*(\tilde{X}) = \phi^* H^*(X) \oplus_{v.s.} W$$

where

$$\phi^*(x) \cdot w = w' \in W, \quad \text{for any} \quad x \in H^*(X), w \in W$$

Now, denote $(A, d_A) = A_{PL}(X)$ and $(B, d_B) = A_{PL}(\tilde{X})$ and consider the following algebraic situation. Let (A, d_A) and (B, d_B) be differential graded algebras such that there exists a DGA-morphism $\gamma : (A, d_A) \to (B, d_B)$ inducing *monomorphism* in cohomology

$$\gamma^* : H^*(A) \xrightarrow{\text{mono}} H^*(B)$$

Assume that

$$H^*(B) = \gamma^* H^*(A) \oplus_{v.s.} W$$

and

$$\gamma^*[a] \cdot w = w' \in W, \quad \text{for any} \quad [a] \in H^*(A), w \in W.$$

Then, the minimal model $(\mathcal{M}_B, \delta_B)$ of B inherits non-vanishing triple Massey products of $(\mathcal{M}_A, \delta_A)$. We are going to show that any non-trivial Massey product in $H^*(A)$ determines a non-trivial Massey product in $H^*(B)$. Note that in general this is not the case, since the *indeterminacy* in the definition of the cohomology classes representing Massey products. But the restrictions on $H^*(B)$ force this indeterminacy to be "the right one". So, let $[a], [b], [c]$ be cohomology classes such that $[a][b] = [b][c] = 0$ and choose a representative cocycle in $\mathcal{M}_A$

$$u = ay + xc, \quad \delta_A x = ab, \quad \delta_A y = bc.$$

We don't write the right signs here, since it does not influence the argument. Since we have assumed that there exists a non-trivial Massey product, this means that one can choose u in a way to get

$$[u] \notin ([a], [c]) \subset H^*(A).$$

Now, consider the map $\hat{\gamma}$ induced by $\gamma : A \to B$ with $\gamma^* = \hat{\gamma}^*$, i.e. $\hat{\gamma} : \mathcal{M}_A \to \mathcal{M}_B$. Then, the cohomology class

$$\hat{\gamma}^*[u] = [\hat{\gamma}(a)\hat{\gamma}(y) + \hat{\gamma}(c)\hat{\gamma}(x)] \in \gamma^* H^*(A)$$

represents some Massey product of the pair $(\mathcal{M}_B, H^*(B))$. In general, as we have already noticed, this cohomology class may represent vanishing Massey product in

$H^*(B)$. However, in our case we can proceed as follows . Assume that $\hat{\gamma}^*[u] = \langle \hat{\gamma}^*[a], \hat{\gamma}^*[b], \hat{\gamma}^*[c]\rangle = 0$. Then, by definition,

$$\hat{\gamma}^*[u] \in (\hat{\gamma}^*[a], \hat{\gamma}^*[c]) \subset H^*(B).$$

Hence,

$$\hat{\gamma}^*[u] = \hat{\gamma}^*[a]\theta + \hat{\gamma}^*[c]\eta, \quad \theta, \eta \in H^*(B).$$

Because of the assumptions on the multiplication in $H^*(B)$, one can write

$$\hat{\gamma}^*[u] = \hat{\gamma}^*[a](\hat{\gamma}^*(x_1) + w_1) + \hat{\gamma}^*[c](\hat{\gamma}^*(x_2) + w_2)$$

where

$$\theta = \hat{\gamma}^*(x_1) + w_1, \quad \eta = \hat{\gamma}^*(x_2) + w_2, \quad x_i \in H^*(XA), w_i \in W.$$

Therefore,

$$\hat{\gamma}^*[u] = \hat{\gamma}^*[a]\hat{\gamma}^*(x_1) + \hat{\gamma}^*[c]\hat{\gamma}^*(x_2) + (\hat{\gamma}^*[a]w_1 + \hat{\gamma}^*[c]w_2)$$

with the last bracket *belonging to* W, because of our assumptions. However, since $H^*(B)$ is a direct sum of $\gamma^*H^*(A)$ and W, the last bracket must be zero and we have obtained an equality

$$\hat{\gamma}^*[u] = \hat{\gamma}^*[a]\hat{\gamma}^*(x_1) + \hat{\gamma}^*[c]\hat{\gamma}^*(x_2)$$

which can be transferred to $H^*(A)$, but in $H^*(A)$ such equality is impossible.

To complete the proof of the first part of Theorem 1, it is enough to specialize A and B as was indicated above, as de Rham polynomial algebras of X and $\tilde{X}$.

Thus, we have proved that non-vanishing triple Massey products survive under blow-ups along *any* submanifold. In particular, any blow-up of $\widetilde{\mathbb{C}P^5}$ along any symplectic submanifold is nonformal, since, by [1], the McDuff manifold $\widetilde{\mathbb{C}P^5}$ has non-vanishing triple Massey products. This final remark completes the proof of the second part of Theorem 1 as well as Corollary 1.

Proof of Theorem 3. The proof follows from the following two theorems below.

Theorem 6. (Lupton-Oprea, [11]) *Let*

$$\mathbb{C}P^n \to E \to B$$

be any rational fibration with fiber $\mathbb{C}P^n$. Then, E is formal if and only if B is formal.

Theorem 7. (Le Hong Van [10]) *Let (M, G) be a symplectic compact G-manifold of a compact Lie group G. Assume that G-action on M is of cohomogeneity 1 and hamiltonian. Then*

1. *either M is a total space of a $\mathbb{C}P^n$-bundle over a co-adjoint orbit G/G_ξ,*
2. *or M is a symplectic blow-down of such bundle along two symplectic exceptional orbits.*

Now, if M is a $\mathbb{C}P^n$-bundle over a co-adjoint orbit, the formality of M follows from the formality of G/G_ξ by the Lupton-Oprea theorem. Indeed, since a co-adjoint orbit G/G_ξ is a homogeneous symplectic manifold (it has the Kirillov symplectic structure), G_ξ is a cenralizer of a torus in G, and, therefore, is a subgroup of maximal rank. Hence, G/G_ξ is formal (see, e.g. [16]). Also, one can notice that G/G_ξ is, in fact, Kähler by Borel's theorem (see [16]). If M is a blow-down, we cannot state that M must be formal, however, it must have vanishing triple Massey products, since otherwise they would survive under the blow-up construction by Theorem 1, which is a contradiction.

4. Proof of Theorem 2

Recall that we need to prove the following statement.

Let $\tilde{X}$ be the symplectic blow-up of X along a symplectic submanifold M of codimension $2k$, $k>3$. If $(\mathcal{M}_M, H^*(M))$ has non-vanishing triple Massey products, the same is valid for $(\mathcal{M}_{\tilde{X}}, H^*(\tilde{X}))$.

Proof. The proof of this theorem follows from Lemmas 3 and 4 below.

Lemma 3. *Let $\tilde{M}$ denote the projectivization of the complex vector bundle $E\to M$ of rank k. If $k>3$, the non-zero Massey triple products in $H^*(M)$ determine non-zero Massey triple products in $H^*(\tilde{M})$.*

Proof of Lemma 3. It is not difficult to show using the theory of KS-extensions [9], that

$$\mathcal{M}_{\tilde{M}} = (\mathcal{M}_M \otimes \Lambda(\zeta, y), D)$$

$$D\zeta = 0, \quad Dy = \zeta^k - c_1\zeta^{k-1} + ... + (-1)^{k-1}c_{k-1}\zeta + (-1)^k c_k, \quad |\zeta| = 2, |y| = 2k-1$$

where $c_j \in \mathcal{M}_M$ represent the rational Chern classes $c_j(E)$ via isomorphism $H^*(\mathcal{M}_M) \cong H^*(M)$. (alternatively, one can look at the proof of the Lupton-Oprea theorem in [11], or at the proof given in [1]). Let $[a], [b], [c] \in H^*(M)$ denote cohomology classes determining some Massey triple product, i.e.

$$[a][b] = [b][c] = 0.$$

Assume that $\langle [a], [b], [c] \rangle \neq 0$, i.e. there exists at least one cocycle

$$\bar{a}y + xc, \quad dx = ab, \quad dy = bc$$

such that

$$[\bar{a}y + xc] \notin ([a], [c]) \subset H^*(M).$$

Consider the cohomology classes

$$[a]\zeta, [b]\zeta, [c]\zeta \in H^*(\tilde{M}).$$

Note that $H^*(\tilde{M}) = H^*(M) \otimes \mathbb{R}[\zeta]/I$, where $I = (\zeta^k - c_1\zeta^{k-1} + ... + (-1)^k c_k)$, which shows that the multiplication rule in $H^*(\tilde{M})$ is the "natural" one and that these cohomology classes are non-zero. Also, obviously, $[a]\zeta \cdot [b]\zeta = [b]\zeta \cdot [c]\zeta = 0$ and the triple Massey product

$$\langle [a]\zeta, [b]\zeta], [c]\zeta \rangle$$

is well defined. Assume that it is zero. Then, any cohomology class representing it must belong to the ideal $([a]\zeta,[c]\zeta)\subset H^*(\tilde{M})$. However, consider the cocycle

$$\bar{a\zeta}(y\zeta^2)+(x\zeta^2)c\zeta$$

representing the above Massey product. We get

$$\begin{gathered}[\bar{a\zeta}(y\zeta^2)+(x\zeta^2)c\zeta]=[\bar{a}y+xc]\zeta^3=\\ [a]\zeta(\alpha_1+\alpha_2\zeta+...+\alpha_k\zeta^{k-1})+\\ [c]\zeta(\beta_1+\beta_2\zeta+...+\beta_k\zeta^{k-1})\end{gathered}$$

which implies $[\bar{a}y+cx]\zeta^3=[a](\alpha_3+\alpha_{k-1}c_{k-3})\zeta^3+[c](\beta_3+\beta_{k-1}c_{k-3})\zeta^3$ and the rest of the brackets zero, since $H^*(\tilde{M})$ is a free module over $H^*(M)$ generated by $\zeta,...,\zeta^{k-1}$. Finally, $[\bar{a}y+cx]\in([a],[c])\subset H^*(M)$, a contradiction.

The next lemma is completely algebraic.

Lemma 4. *Let $f:A\to B$ be a DGA-morphism. Assume that $[b_1],[b_2],[b_3]\in H^*(B)$ are cohomology classes determining a triple Massey product $\langle[b_1],[b_2],[b_3]\rangle$. Assume that there exist cohomology classes $[a_1],[a_2],[a_3]\in H^*(A)$ such that*

(1) $[a_1][a_2]=[a_2][a_3]=0$
(2) $f^*[a_i]=[b_i]$ *for all* i.

If $\langle[b_1],[b_2],[b_3]\rangle\neq 0$, then $\langle[a_1],[a_2],[a_3]\rangle\neq 0$.

Proof of Lemma 4. It is necessary to prove that there exists at least one cocycle, say, $v\in Z(A)$ representing $\langle[a_1],[a_2],[a_3]\rangle$ such that $[v]\notin([a_1],[a_3])$. Assume that $[v]\in([a_1],[a_3])$ for any v. Note that $f^*[v]$ represents (obviously), the triple Massey product

$$\langle f^*[a_1],f^*[a_2],f^*[a_3]\rangle=\langle[b_1],[b_2],[b_3]\rangle.$$

This is a general naturality property of all Massey products:

$$f^*\langle[\gamma_1],...,[\gamma_m]\rangle\subset\langle f^*[\gamma_1],...,f^*[\gamma_m]\rangle$$

(see [12, p. 293]). This remark, in fact, completes the proof: if $\langle[a_1],[a_2],[a_3]\rangle$ were zero, its image would represent zero in the set $\langle[b_1],[b_2],[b_2]\rangle$. More formal argument is below. Note, that, by the assumption

$$f^*[v]\in(f^*[a_1],f^*[a_3])=([b_1],[b_3])\subset H^*(B)\qquad(*)$$

However, any two cohomology classes representing the given triple Massey product, differ by an element in $([b_1],[b_3])$ (by definition). Since $\langle[b_1],[b_2],[b_3]\rangle\neq 0$, there exists $[u]\in H^*(B)$ such that $[u]$ represents the above Massey product and does not belong to the ideal generated by $[b_1]$ and $[b_3]$. By the previous remark

$$f^*[v]=[u]+\alpha[b_1]+\beta[b_3]$$

which together with $(*)$ yields a contradiction. Lemma is proved.

Consider now $\tilde{X}$, X and M as in the assumptions of the Theorem. Let $[b_1],[b_2],[b_3]\in H^*(M)$ be cohomology classes representing non-zero triple Massey product. By Lemma 3,

$$\langle[b_1]\zeta,[b_2]\zeta,[b_3]\zeta\rangle\neq 0$$

is a non-zero triple Massey product in $H^*(\tilde{M})$. Consider cohomology classes

$$[a_1] = \tilde{t}^*(U \cup [b_1]), \quad [a_2] = \tilde{t}^*(U \cup [b_2]), \quad [a_3] = \tilde{t}^*(U \cup [b_3]) \in H^*(\tilde{X}).$$

From the standard properties of cup-products

$$[a_1][a_2] = \tilde{t}^*(U \cup [b_1]) \cdot t^*(U \cup [b_2]) = \tilde{t}^*(U^2 \cup ([b_1][b_2])) = 0$$
$$[a_2][a_3] = \tilde{t}^*(U \cup [b_2]) \cdot \tilde{t}^*(U \cup [b_3]) = \tilde{t}^*(U^2 \cup ([b_2][b_3])) = 0,$$

which yields the triple Massey product $\langle [a_1], [a_2], [a_3] \rangle \in H^*(\tilde{X})$. Consider $\tilde{k}^* : H^*(\tilde{X}) \to H^*(\tilde{M})$ and evaluate it on $[a_i]$. It will give

$$\tilde{k}^*[a_i] = \tilde{k}^*\tilde{t}^*(U \cup [b_i]) = [b_i]\zeta.$$

Applying Lemma 4 to $\tilde{k}^*$ and $\langle [a_1], [a_2], [a_3] \rangle$, one completes the whole proof.

Also, we get Corollary 2, since it follows from Theorem 2 together with the fact that the Kodaira-Thurston manifold has non-vanishing triple Massey products. The latter is easy, and can be proved for various types of nilmanifolds (see, e.g. [3]). For the Kodaira-Thurston manifold it can be shown as follows. It is known that one of the form in which this manifold is defined is the form of nilmanifold $KT = (N_3/\Gamma) \times S^1$, where N_3 is a 3-dimensional Heisenberg group of unipotent triangular matrices and $\Gamma = N_3(\mathbb{Z})$ is a lattice of the matrices with integer entries. It is known that the minimal model of KT is given as follows

$$\mathcal{M}_{KT} = (\Lambda(x_1, x_2, x_3, z), d)$$
$$dx_1 = dx_2 = dz = 0, \quad dx_3 = x_1 x_2, \quad |x_i| = |z| = 1$$

(see [16]) for details). Therefore, one can notice that there is a non-zero Massey product

$$\langle x_1, x_1, x_2 \rangle \neq 0.$$

Corollary 2 is proved.

Remark. The identity $U^2 \cup ([b_1][b_2]) = (U \cup [b_1]) \cdot (U \cup [b_2])$ follows from the associativity of the cup-product $H^*(T(\tilde{N})) \otimes H^*(\tilde{M}) \to H^*(T(\tilde{N}))$. The latter can be seen from [2, p. 35]. Indeed, the considered cup-product is obtained by the identification of the corresponding cohomologies with the cohomologies of *pairs* by the collapsing map. The cup-product of pairs satisfies the associativity condition (this is well-known).

Proof of Corollary 3. It is enough to notice that the minimal model of the Iwasawa manifold has non-vanishing triple Massey products. This is proved in [3].

Remark. For the convenience of the reader we reproduce the definition of the Iwasawa manifold $I(p, q)$.

$$I(p, q) = (H(1, p) \times H(1, q))/\Gamma$$

where $H(1, p)$ consists of all matrices of the form

$$\alpha = \begin{pmatrix} I_p & A & C \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with I_p being the identity $(p \times p)$-matrix, A, C – $(p \times 1)$-matrices and b a real number. The subgroup Γ consists of the corresponding matrices with integer entries, so Γ is a lattice in the nilpotent group $H(1, p) \times H(1, q)$.

Appendix. Sketch of proof of Theorem 4

As we have already noticed, the proof of Theorem 4 can be accomplished by repeating the proof of [5]. Here we sketch it for the convenience of the reader as well as to show that none of the facts specific for complex blow-up is used.

Lemma 5. *Suppose that E and F are vector bundles over X. Let there be given the embedding $E \to E \oplus F, e \to (e, 0)$ inducing a mapping of Thom spaces $\hat{\imath} : T(E) \to T(E \oplus F)$. We have*

$$\hat{\imath}^* U_{E \oplus F} = U_E \cup \chi(F)$$

where U_E and $U_{U \oplus F}$ are the Thom classes of E and $E \oplus F$, and $\chi(F)$ is the Euler class of F.

We refer to [5], since the very formulation of Lemma 5 shows that it is valid for all vector bundles.

Lemma 6. *The mapping $\hat{\phi} : T(\tilde{N}) \to T(N)$ (in notation of Proposition 1) is the composite*

$$T(\tilde{N}) \to T(g^* N) \to T(N)$$

where

$$0 \to \tilde{N} \to g^* N \to W.$$

Again, this is true for all vector bundles.

Lemma 7. *In the notation of Lemma 6,*

$$\hat{\phi}^* U_N = U_{\tilde{N}} \cup \chi(W).$$

Lemma 8. *The Euler class $\chi(W)$ is given by*

$$\chi(W) = c_{k-1}(N) - \zeta c_{k-2}(N) + ... + (-1)^{k-1} \zeta^{k-1},$$

where ζ is the Euler class of the complex line bundle $\tilde{N} \cong \tilde{E}$.

Proof. Since $\tilde{N}$ is a complex line bundle, its total Chern class is given by $C(N) = 1 + \zeta$. Since $C(W) \cup C(\tilde{N}) = g^* C(N)$ one easily obtains $C(W) = g^* C(N)(1 - \zeta + \zeta^2 + ... + (-1)^s \zeta^s + ...)$ and since $\chi(W) = c_{k-1}(W)$, the result follows.

Sketch of proof of Theorem 4. Obviously, if $\hat{\phi}^*$ is injective, the same is valid for α. Let us show that $\hat{\phi}^*$ is injective. By Lemma 6

$$\hat{\phi}^* \mu = U \cup \chi$$

for the Thom classes μ and U of $T(N)$ and $T(\tilde{N})$ respectively. Note that χ is given by Lemma 7. From the standard properties of Thom spaces and cup-products $\hat{\phi}^*(\mu \cup y) = U \cup \chi \cup y$ for any $y \in H^*(M)$. By the well-known isomorphism $H^*(\tilde{M}) \cong (H^*(M) \otimes \mathbb{R}[\zeta])/I$ where I is given as in Theorem 5. Hence, obviously,

$$U \cup \chi \cup y = 0 \quad \text{if and only if} \quad U \cup \zeta^{k-1} \cup y = 0.$$

The Thom isomorphism shows that this is impossible and $\hat{\phi}^*$ must be injective. The surjectivity of β easily follows from the fact that the classes $[\zeta^p y]$ generate $H^*(\tilde{M})/g^* H^*(M)$ and that $\tilde{k}^* \tilde{t}^* (U \cup \zeta^p) = \zeta^{p+1} y$.

Analogously, one derives the equality $\beta\alpha = 0$ from the commutativity of (1).

It remains to prove that $\operatorname{Ker}\beta = \operatorname{Im}\alpha$. Assume that $\beta(u,z) = 0$ for $(u,z) \in H^*(X) \oplus H^*(T(\tilde{N}))$. It means that $\phi^* u = \tilde{t}^* z$. From the Thom isomorphism, $z = U(y_0 + y_1\zeta + ... + y_{k-1}\zeta^{k-1})$ and, therefore,

$$\tilde{k}^*\tilde{t}^* z = y_0\zeta + ... + y_{k-1}\zeta^k = \tilde{k}^*\phi^* u.$$

Using the identity determined by the ideal I in the ring $H^*(\tilde{M})$ and comparing expressions which are coefficients of ζ^p, one obtains

$$z = \hat{\phi}^*(\mu \cdot y_{k-1}) \quad \text{and} \quad t^*(\mu \cdot y_{k-1}) = u.$$

The sketch of proof is completed.

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