Multiplicities of Principal Curvatures of Isoparametric Hypersurfaces

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Dedicated to Professor S.S.Chern on his eighty-fifth birthday

Abstract – Using K-theory we prove in this paper that, if $m_- \leq m_+$ are two natural numbers satisfying $m_- + m_+ + 1 \neq 2^i$ for any i and $m_- = 5, 6$ or $7(mod 8)$, then $m_-, m_+; m_-, m_+$ are the multiplicities of principal curvatures of an isoparametric hypersurface with $g = 4$ in the unit sphere S^{n+1} if and only if $m_+ + m_- + 1$ is divisible by $\delta(m_-)$. Moreover, if $m_-, m_+; m_-, m_+$ are the multiplicities of an isoparametric hypersurface with $g = 4$ and $m_- = 3(mod 8)$, then $m_+ + m_- + 1$ is either divisible by $\frac{\delta(m_-)}{2}$ or equal to 2^i for some $i \in \mathbb{N}$, where $n = 2(m_+ + m_-)$ and $\delta(m)$ is an integral function satisfying that $\delta(m + 8k) = 2^{4k}\delta(m)$ and

m	1	2	3	4	5	6	7	8
$\delta(m)$	1	2	4	4	8	8	8	8

§1. Introduction

A hypersurface in the unit sphere S^{n+1} is called isoparametric if it has constant principal curvatures, that is, the eigenvalues of the shape operator are constant. E.Cartan [Ca] classified first isoparametric hypersurfaces with 1, 2 or 3 distinct principal curvatures. A remarkable theorem of Münzner [Mu] says that the number g of distinct principal curvatures of an isoparametric hypersurface in S^{n+1} must be 1, 2, 3, 4 or 6. If g is even, by [Mu] the multiplicities of the g principal curvatures satisfy that $m_1 = m_3 = \cdots = m_{g-1}$ and $m_2 = m_4 = \cdots = m_g$. We denote by m_+ for m_1 and m_- for m_2. The dimension and its multiplicities are related by the formula $2n = g(m_- + m_+)$ and the homology of the focal manifolds and the hypersurface depend only on the multiplicities m_+ and m_-.

In [F-2], the author proved that any isoparametric hypersurface with $g = 6$ is either diffeomorphic to $S^3 \times S^3/\mathbb{Q}_8$ or homeomorphic to the normal sphere bundle of an embedding of the quadric $X_5(2)$ in S^{13}(even true for Dupin hypersurface), where $\mathbb{Q}_8 \subset S^3$ is the quaternionic subgroup. Moreover, by a correspondence with Mark Mahowald, in the latter case the hypersurface is actually homotopy equivalent to the homogeneous space G_2/T^2, where T^2 is a maximal torus in G_2.

The case of $g = 4$ is completely different and delicated. By using the orthogonal representations of Clifford algebra, Ferus-Karcher-Münzner [FKM] constructed infinite many nonhomogeneous isoparametric hypersurfaces with $g = 4$. To remind the reader, let us recall the simple construction. As usual we use $Cl^{0,m+1}$ to denote the Clifford algebra spanned by $1, e_0, \cdots, e_m$. For any nontrivial $(n+2)$-

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dimensional orthogonal representation of $Cl^{0,m+1}$, $e_0, \cdots, e_m$ give rise matrices $P_0, \cdots, P_m$ satisfying that $P_i^2 = I$ and $P_iP_j = -P_jP_i$ for $i \neq j$. Define

$$f(x) = \langle x, x\rangle^2 - 2\sum_{i=0}^{m}\langle P_i(x), x\rangle^2,\ x \in \mathbb{R}^{n+2}.$$

The function f maps the unit sphere to $[-1, 1]$ and satisfies the Cartan-Münzner equations, i.e, $\|df\|^2$ and the Laplacian Δf are both functions of f. By [Mu][FKM], for a regular value $c \in [-1, 1]$, the hypersurface $M = f^{-1}(c)$ defines an isoparametric hypersurface with four distinct principal curvatures and multiplicities m, $\frac{n}{2} - m$; m, $\frac{n}{2} - m$. Its scalar curvature is constant and equal to $n^2 - 4n$. Notice that all irreducible representations of $Cl^{0,m+1}$ have the same dimension $2\delta(m)$ where $\delta(m)$, $m \geq 1$ is an integral function satisfying that $\delta(m + 8k) = 2^{4k}\delta(m)$ and

m	1	2	3	4	5	6	7	8
$\delta(m)$	1	2	4	4	8	8	8	8

Therefore $(n+2)$ must be a multiple of $2\delta(m)$. We refer to [FKM] for more details. Some hypersurfaces constructed above are even nonhomogeneous up to homotopy however its topology are quite nicely understood [W]. The author proved generally that isoparametric hypersurfaces are iterated sphere bundles in many cases [F1]. As posed by S.T. Yau in [Ya], it is a wide open problem in classical differential geometry to classify all nonhomogenous isoparametric hypersurfaces in the unit sphere with $g = 4$. To achieve this, the first important step is to study which pairs of natural numbers can be realized as the multiplicities of principal curvatures of isoparametric hypersurfaces.

The first progress to this question was achieved by Abresch in [Ab]. He obtains the following interesting result:

Theorem(Abresch [Ab]) *Given an isoparametric hypersurface in S^{n+1} with $g = 4$, then the pair (m_-, m_+)-w.r.g. we may assume that $m_- \leq m_+$- satisfies one of the following conditions below*

4A *$m_+ + m_- + 1$ is divisible by 2^k: $=min\{2^\sigma | 2^\sigma > m_-, \sigma \in N\}$.*

4B1 *m_- is a power of 2 , and $2m_-$ divides $m_+ + 1$.*

4B2 *m_- is a power of 2, and $3m_- = 2(m_+ + 1)$.*

Each condition corresponds to a topological different kind of examples.

Throughout the rest of this paper we assume that $m_- \leq m_+$. A direct corollary from the calculations in [Ab] is that in the situations **4B1** and **4B2,** m_- must be among $\{1, 2, 4, 8\}$[Ta]. Compare with the constructions in [FKM] mentioned above it follows that every pair m_+, m_- where $m_- = 1, 2, 4, 8$ satisfying **4B1** is indeed realizable as the multiplicities of an isoparametric hypersurface. For the family **4B2**, by [F-1], only the pairs $(2, 2)$ and $(4, 5)$ can be realized as the multiplicities of isoparametric hypersurfaces. Notice that (m_-, m_+) are the multiplicities of an isoparamertric hypersurface if $m_- + m_+ + 1$ is divisible by $\delta(m_-)$.

In this paper I am going to show the following theorem, we should like to remind the reader that $n = m_- + m_+$ there.

Theorem A: *Let m_-, m_+ be the multiplicities of an isoparametric hypersurface with $g = 4$ in S^{2n+1}. Suppose that $m_- + m_+ + 1 \neq 2^i$ for any i. Then $m_- + m_+ + 1$ is divisible by $\delta(m_-)$ if $m_- = 5, 6, 7(mod 8)$ and $m_- + m_+ + 1$ is divisible by $\frac{\delta(m_-)}{2}$ if $m_- = 3(mod 8)$.*

Compare this with [FKM] we have the following immediately corollary:

Corollary B: *If $m_- = 5, 6, 7(mod 8)$ and $m_- + m_+ + 1 \neq 2^i$ for any i. Then m_-, m_+ are the multiplicities of an isoparametric hypersurface with $g = 4$ in S^{2n+1} if and only if $m_- + m_+ + 1$ is divisible by $\delta(m_-)$.*

Applying this result to equifocal hypersurface in rank one symmetric spaces in the sense of [TT] we have the following theorem. The proof follows immediately from Theorem A and 7.1.

Corollay C: *If $m_- = 5, 6(mod 8)$ and $m_- + m_+ + 1 \neq 2^i$ for any i. Then m_-, m_+ are the multiplicities of an equifocal hypersurface with $g = 2$ in $\mathbb{C}P^n$ ($\mathbb{H}P^n$) if and only if $m_- + m_+ + 1$ is divisible by $\delta(m_-)$.*

The idea to show Theorem A is the following:

Let $\widetilde{M}$ be an isoparametric hypersurface in S^{2n+1} with $g = 4$ and $\widetilde{F_\pm}$ being the focal manifolds corresponding to the maximum and minimal principal curvatures respectively. Following [Ab], the hypersurface $\widetilde{M}$ as well as $\widetilde{F_\pm}$ are all invariant under the antipodal involution. We obtain the quotient manifolds M and $F_\pm$ in $\mathbb{R}P^{2n+1}$.

Let $\gamma_\pm$ denote the normal bundles of the focal manifolds $F_\pm$. $\dim \gamma_\pm = m_\pm + 1$. If $m_- = 3, 5, 6, 7(mod 8)$, we can show first that the restriction of γ_+ on a n-dimensional skeleton $F_+^{(n)}$ is stable equivalent to $2k\xi$ for an even $2k$ where ξ is the restriction on F_+ of the Hopf line bundle over $\mathbb{R}P^{2n+1}$. As the total Stiefel-Whitney class $w(\gamma_+) = (1+w)^{n+1}$ where $w \in H^1(F_+, \mathbb{Z}_2)$ is the restriction of the generator of $H^1(\mathbb{R}P^{2n+1}, \mathbb{Z}_2)$, we can conclude that $2k \geq n + 1$. Notice that $2k\xi$ admits a complex structure and so we may regard it as $k\eta$ where η is the complexification of ξ. For dimension reasoning, we can write $k\eta = \{k\eta - (k - [\frac{n}{2}] - 1)\} + (k - [\frac{n}{2}] - 1)$ where $\{k\eta - (k - [\frac{n}{2}] - 1)\}$ is the complex vector bundle over $F_+^{(n)}$ of dimension $[\frac{n}{2}] + 1$ and stable equivalent to $k\eta$. By $n = m_+ + m_-$ and $\dim \gamma_+ = m_+ + 1$ we obtain that the pullback of $\{k\eta - (k - [\frac{n}{2}] - 1)\}$ to M has at least $m_- + 1$ linearly independent sections. Consider now the tensor bundle $k\eta \otimes \eta \to M^{(n)} \times \mathbb{R}P^{m_-}$, by Becker [Be] this complex bundle admits a nowhere zero section. Then we use the Adams operations in $\mathbb{Z}_2$-equivariant K-theory to show that k is divisible by $2^{[\frac{m_-+1}{2}]-1}$.

Let $\mathbf{E}_i (1 \leq i \leq 4)$ denote the four focal distributions and recall that $p_i^* \gamma_i =$

$\mathbb{E}_i \oplus \varepsilon^1$. Notice the following very crucial relations for the focal distributions:

$$TM = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \oplus \mathbb{E}_4.$$

Moreover, there exist involutions $T_i : M \to M$ such that $T_1^*\mathbb{E}_1 = \mathbb{E}_3$ and $T_2^*\mathbb{E}_2 = \mathbb{E}_4$. By using K-theory once again we prove that $2(n+1) = 4k(mod 2^{\frac{n-1}{2}})$. Combine the above we can conclude Theorem A.

We remark that the current methods does not work for Dupin hypersurface in sphere form, basically because it is unknown if the hypersurface and focal manifolds are invariant under the antipodal involution. In concluding this section, we should like to pose an open question, namely, *if $m_+, m_-; m_+, m_-$ are the multiplicities of a Dupin hypersurface in space form, what condition is satisfied by m_+ and m_-?*

The organization of this paper is as the follows: In §2 we give a very brief review to some necessary facts on isoparametric hypersurfaces. In §3 and §4, we develop the K-theory and KO-theory for isoparametric hypersurfaces in $\mathbb{R}P^{2n+1}$ with $g = 4$, this constitutes the main tool used in the proof. In §5 we use Adams operation to show the divisiblity property of k mentioned above. In §6 we give the proof of Theorem A. In §7 we apply our Theorem A to equifocal hypersurfaces in rank one symmetric spaces.

The present work was motivated by a program with G.Thorbergsson. I would like to express my sincere thanks to him for many enlightening discussions. I thank Brian Smyth for pointing out some English mistakes and Max-Planck Institute for Mathematics at Bonn for its hospitality during the preparation of this paper.

§2. A Brief Rieview on Isoparametric Hypersurfaces

In this section we present some properties about isoparametric hypersurfaces in spheres needed in the subsequent arguments. We refer to [Ab] and [Mu] for more details.

Definition 2.1: *Let $M \subset S^{n+1}(1)$ be a codimension 1 Riemannian submanifold in the unit sphere and ∇ stand for the convariant derivative of a connection on the sphere. Choose a unit normal vector field v of M, we define the shape operator $A_v : TM \to TM$ on the tangent bundle of M by*

$$A_v(e) = \nabla_e(v), \forall e \in T_xM,$$

The eigenvalues of A_v is called the principal curvatures of M. We say that M is isoparametric if and only if all its principal curvatures are constant.

Notations: $g := \#\{$ distinct principal curvatures $cot\phi_i$ of $M\}$.
$\mathbb{E}_i \subset TM$: distribution of the curvature $cot\phi_i$.
$m_i := dim\mathbb{E}_i$ the multiplicity of the curvature $cot\phi_i$.

By Codazzi-Gauss equations we have

$$\phi_i = \phi(M) + \frac{i\pi}{g}(mod\mathbb{Z}\pi)$$

and $m_i = m_{i+2}$, where $\phi(M)$ is a constant depending only on M.

Let $pr_\omega : M \to M_\omega$, $p \to exp_p \omega v$ denote the projection maps along the normal geodesics.
(i): If $\omega \notin \{\phi_i : 1 \leq i \leq g\} + \mathbb{Z}\pi$, then pr_ω is a diffeomorphism and the parallel surface M_ω is also isoparametric with curvature $\cot(\phi_i - \omega)$ on $d\{pr_\omega(\mathbb{E}_i)\}$. These give a family of isoparametric hypersurfaces diffeomorphic to each other.
(ii): If $\omega = \phi_i(mod\mathbb{Z}\pi)$, then $\mathbb{E}_i = \ker d\{\ pr_\omega\}$ and M_ω is a focal manifold of dimension $n - m_i$. The fibres of pr_ω are the integral surfaces of the distribution $\mathbb{E}_i$. This gives a sphere bundle $S^{m_i} \to M \to M_{\phi_i}$.

For an isoparametric hypersurface M, we can choose an ϕ_k and define a function $f : S^{n+1} \to \mathbb{R}$ by setting $f(p) = cos\{g(\omega - \phi_k)\}$ for $p \in M_\omega$. We can extend this function to $\mathbb{R}^{n+2}$ by $f(x) = |x|^g \cdot f(\frac{x}{|x|})$. This gives a function satisfying the Cartan-Münzner's equations:

$$\begin{aligned} &|\text{grad} f|^2 = g^2|x|^{2g-2} && (1) \\ &\Delta f = \tfrac{1}{2}g^2(m_1 - m_2)|x|^{g-2} && (2) \end{aligned}$$

where m_1, m_2 are the multiplicities of M which satisfies that $n = \frac{g}{2}(m_1 + m_2)$. Moreover, by [Mu], the converse also holds. We call a function satisfying these equations an isoparametric function.

Proposition 2.2(Münzner): *$M \subset S^{n+1}$ is isoparametric if and only if there exists a smooth function $f : \mathbb{R}^{n+2} \to \mathbb{R}$ satisfying the Cartan-Münzner equations such that $M = f|_{S^{n+1}}^{-1}(0)$.*

f maps S^{n+1} to $[-1, 1]$ and the regular values are $(-1, 1)$. There are exactly two focal manifolds $F_\pm = f^{-1}(\pm 1)$. Let $B_\pm = f^{-1}(\pm[0, 1])$. Thus we have immediately that $S^{n+1} = B_+ \cup B_-$ and $B_\pm$ is a disk bundle of dimension $m_\pm + 1$ over the focal manifold $F_\pm$ with boundary M. Obviously $B_\pm$ is exactly the normal bundle of $F_\pm$ in S^{n+1}.

The homology of isoparametric hypersurface and its focal manifolds can be presented in terms of the multiplicities m_+ and m_- [Mu]. In particular, if $g = 4$ and the focal manifolds are all orientable, the result shows that $F_\pm$ has the same homology as the product of two spheres. As the author proved in [F1], in many cases one of the focal manifold is homeomorphic to a sphere bundle. When g is even, the focal manifolds and the hypersurface M are all invariant under the antipodal map [Ab]. The reason is simply that the isoparametric function f determining M is an even function. This gives corresponding data in $\mathbb{R}P^{n+1}$ and a bundle decomposition. For brevity we use still the same notation to stand for the hypersurface

and its focal manifolds in $\mathbb{R}P^{n+1}$. If $g = 4$ and let $\mathbb{E}_i (1 \leq i \leq 4)$ denote the four principal distributions in $\mathbb{R}P^{n+1}$ and let $p_i : M \to F_i (F_i = F_{i+2})$ be the projections along normal geodesics which are sphere bundles. Notice that $p_i^* B_+ = \mathbb{E}_i \oplus \varepsilon^1$ if i is odd, and $p_i^* B_- = \mathbb{E}_i \oplus \varepsilon^1$ if i is even. Obviously $TM = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \oplus \mathbb{E}_4$.
By [Ab], there exist periodic two diffeomorphisms $T_i : M \to M$ such that $T_1^* \mathbb{E}_1 = \mathbb{E}_3$ and $T_2^* \mathbb{E}_2 = \mathbb{E}_4$. Moreover, $p_1 \circ T_1 = p_3$ and $p_2 \circ T_2 = p_4$. By [Ab] one has actually the following

Proposition 2.3 *$T_i^* : H^*(M, \mathbb{Z}_2) \to H^*(M, \mathbb{Z}_2)$ is the identity for $i = 1$ and 2.*

If T_i inducing the identity on the integral cohomology ring, we point out that the Pontryagin classes of all focal distributions must be zero. In fact, if $\mathbb{E}_i$ $(1 \leq i \leq 4)$ are the distributions of a hypersurface in sphere. Then $T_1^* \mathbb{E}_1 \oplus \mathbb{E}_1 \oplus T_2^* \mathbb{E}_2 \oplus \mathbb{E}_2$ is stably trivial. One can check easily that both of $T_1^* \mathbb{E}_1 \oplus \mathbb{E}_1$ and $T_2^* \mathbb{E}_2 \oplus \mathbb{E}_2$ must have trivial Pontryagin classes. This implies that the Pontryagin classes $P_i(\mathbb{E}_1) = P_i(\mathbb{E}_2) = 0$ for $i \geq 1$ provided $T_1^* = T_2^* = id$.
It is an interesting question to investigate the charateristic classes(e.g: Pontryagin class) of the focal distributions. Compare with the examples in [FKM] we point out that:

Proposition 2.4. *There is an isoparametric hypersurface M with the smaller multiplicity m such that $i^* E_+ \in \widetilde{KO}(S^m) (\cong \mathbb{Z}$ and $\mathbb{Z}_2$ by $m = 0 (mod 4)$ and $1, 2 (mod 8))$ is a generator, where E_+ is a focal distribution and $i : S^m \to M$ represents a generator of $\pi_m(M) \cong \mathbb{Z}$.*

Consequently, $p_k(E_+) = (2k-1)! gcd(k+1, 2) x$ if $m = 4k$, where $x \in H^{4k}(M, \mathbb{Z}) \cong \mathbb{Z}$ is a generator.

§3. K-theory of Isoparametric Hypersurfaces in $\mathbb{R}P^{2n+1}$

Let $M \subset \mathbb{R}P^{2n+1}$ denote an isoparametric hypersurface with four distinct principal curvatures and $F_k (k \in \{\pm 1\} = \{\pm\})$ being the two focal manifolds. Let m_k denote the multiplicities of the principal curvatures. Recall that $n = (m_1 + m_{-1})$. Let γ_k denote the normal bundle of F_k in $\mathbb{R}P^{2n+1}$. By [Ab] [F1], the necessary and sufficient conditions for m_+ and m_- being the multiplicities of principal curvatures of an isoparametric hypersurface have been obtained if $min(m_k, m_{-k}) \leq 8$. Thus throughout the following sections we may assume that $min(m_k, m_{-k}) \geq 9$. Applying [Ab] we have $2^4 | 2(n+1)$. Moreover, in this situation both focal manifolds and its normal bundles in $\mathbb{R}P^{2n+1}$ are spin. The cohomology ring of F_k and M are as the follows[Ab]:
Theorem 3.0.(Abresch)

$$H^*(M, \mathbb{Z}_2) \cong \mathbb{Z}_2[w, x_+, x_-] / \{w^{n+1} = 0, x_+^2 = 0, x_-^2 = \begin{pmatrix} n+1 \\ m_+ \end{pmatrix} x_- w^{m_+}\},$$

where $degw = 1$, $degx_+ = m_-$ and $degx_- = m_+$.
$H^*(F_k) =$ the subalgebra generated by w and x_k. Moreover,

$Sqw = w(1+w)$;
$Sqx_+ = x_+ + w^{m_-}\beta_+$;
$Sqx_- = x_-(1+w)^{n+1} + w^{m_+}\beta_-$;
where $\beta_\pm$ *are two appropriate elements in* $\mathbb{Z}_2[w]/(w^{n+1})$.

Let ρ_2 denote the mod 2 reduction and $a \in H^2(F_{\pm k},\mathbb{Z}) \cong \mathbb{Z}_2$ denote a generator. By using universal coefficients theorem it is standard to check that

$$H^{m_k}(F_{-k},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } m_k \text{ is odd.} \\ \mathbb{Z}\oplus\mathbb{Z}_2 & \text{if } m_k \text{ is even} \end{cases}$$

Therefore one can choose x_k above is in the image of ρ_2 of an integral class in $H^{m_{-k}}(F_k,\mathbb{Z})$. Obvious $\rho_2(a) = w^2$. Thus $\rho_2(a^{\frac{n-1}{2}}) = w^{n-1} \neq 0$ and $\rho_2(a^{\frac{n+1}{2}}) = 0$. In other words, in the graded ring $H^*(F_{\pm k},\mathbb{Z})$ we have $a^{\frac{n-1}{2}} \neq 0$. By Poincarè duality there is a primitive element $z \in H^n(M,\mathbb{Z})$ so that $\rho_2(z) = w^n$, $\{zx_+x_-\}[M] = 1$ and $z^2 = 0$. In the graded ring $H^*(M,\mathbb{Z})$, $x_k^2 = R_k(a,x_k)$ where $R_k(a,b)$ is a a certain polynomial so that degree in b is less than 2. Let

$$\mathbb{G}R = \mathbb{Z}[a,x_+,x_-]/\{2a=0, x_\pm^2 = R_\pm(a,x_\pm)\}$$

denote a graded ring where deg $x_k = m_{-k}$ and deg $a = 2$. It is not hard to check that

Proposition 3.1 *The ring* $H^*(M,\mathbb{Z})$ *is generated by* a, x_+, x_- *and* z. $H^*(F_k,\mathbb{Z})$ *is a direct summand of* $H^*(M,\mathbb{Z})$ *generated by* a, x_{-k} *and* z. *Moreover,*
(i). $H^i(M,\mathbb{Z}) \cong \mathbb{G}R^i$ *if* $i \neq n, n+m_+, n+m_-$ *and* $2n$.
(ii). $H^i(M,\mathbb{Z}) \cong \mathbb{G}R^i \oplus \mathbb{Z}$ *if* $i = n, n+m_+, n+m_-$ *and* $2n$. *The final factor is generated by* z, zx_+, zx_- *and* zx_+x_- *respectively.*

Let η be the complexification of the Hopf line bundle over F_k and M(i.e, the restriction of the standard Hopf bundle over $\mathbb{R}P^{2n+1}$). Let $x = \eta - 1 \in \widetilde{K}(F_k)$, $\widetilde{K}(M)$ and $\widetilde{K}(\mathbb{R}P^{2n+1})$. If X is a closed manifold, we use $P: X \to S^{\dim X}$ to denote the pinch map and $\ell \in \widetilde{K}(S^{2j}) \cong \mathbb{Z}$ denote a generator.

Let $E \to X$ denote a principal Spin$(n+1)$-bundle. Spin(n) acts freely on E. Let $\mathbb{P}(E)$ denote the orbits space. Notice that $\mathbb{P}(E)$ can be identified with the sphere bundle of E associated with the standard representation. Let $\hat{E}$ denote the principal Spin(n) bundle $E \to \mathbb{P}(E)$. When $n = 0(mod2)$, let $\Delta_+ \in R(Spin(n))$ denote one of the irreducible spin representation. This representation gives rise an associated vector bundle $\hat{E}\times_{\Delta_+}\mathbb{C}^{2^h} \to \mathbb{P}(E)$, where $2^h = a_n^c$ is the Radon-Hurewicz number. Let $y \in \widetilde{K}(\mathbb{P}(E))$ denote the stable class of the vector bundle above. By Bott[Bo], the restriction of y at the fibre S^n is a generator of $\widetilde{K}(S^n) \cong \mathbb{Z}$. Applying Leray-Hirsch type theorem(c.f: Dold[Do]) it follows that $K(\mathbb{P}(E))$ is a free $K(X)$-module with generator 1 and y.

If m_{-k} is even, applying the construction above to the sphere bundle $p_{-k} : M \to F_{-k}$ we obtain immediately that $K(M)$ is a free $K(F_{-k})$-module generated by 1 and y. In general we have the following

Theorem 3.2 *Let m_{-k} be an even. Then*
(i) $\widetilde{K}(F_{-k}) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\frac{n-1}{2}}}$ generated as amodule by $z = P^!(\ell)$ and x.
(ii) $\widetilde{K}(M) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_{2^{\frac{n-1}{2}}} \oplus \mathbb{Z}_{2^{\frac{n-1}{2}}}$ generated as amodule by y, z, yz, x and xy.
(iii) $\widetilde{K}(F_k) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\frac{n-1}{2}}} \oplus \mathbb{Z}_{2^{\frac{n-1}{2}}}$ as a direct summand of $\widetilde{K}(M)$ by the map p_k^*.

Proof: First of all, recall $\widetilde{K}(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}_{2^n}$ generated by x which has filtration 2. The first Chern class $c_1(x) = a \neq 0$. Thus $x \in \widetilde{K}(F_{\pm k})$ is nonzero and in the Atiyah-Hirzebruch spectral sequence(abbreviated as AHSS) for $\widetilde{K}(F_{\pm k})$, the term $\mathbb{E}_2^{2,-2} = H^2(F_{\pm k}, \mathbb{Z})$ survives to E_∞. This implies that $d_r(a) = 0$ for each r. The derivative property of differential applies to conclude that $d_r(a^i) = 0$, $\forall i \geq 1$. Recall m_{-k} is even. By 3.1 $H^{2j-1}(F_k, \mathbb{Z}) = 0$ for $2j \leq n$ and therefore if $i \leq \frac{n-1}{2}$, a^i survives to E_∞ in the AHSS for $K(F_k)$.

By [AH], $rank K(F_k)$ is equal to the sum of even dimensional Betti numbers. For an odd j, $H^j(F_k, \mathbb{Z})$ is nonzero only if $j = n$ and $j = \dim F_k$. At these dimensions, the groups are both $\mathbb{Z}$. Thus the term $x_k \in H^{m_{-k}}(F_k, \mathbb{Z}) = \mathbb{E}_2^{m_{-k}, -m_{-k}}$ survives to E_∞. This proves $E_2^{p,-p} = E_\infty^{p,-p}$, $\forall p \leq n-1$. We claim more generally the AHSS for $K(F_k)$ collapses. It is enough to prove the differential $d_r : E_r^{n,-n-1} \to E_r^{n+r,-(n+r)}$ vanishes for each r. Let $z \in \mathbb{E}_2^{n,-n-1} = H^n(F_k, \mathbb{Z}) \cong \mathbb{Z}$ be a generator. Notice $z(mod 2) = w^n$. Let r be the smallest natural number so that $d_r(z) \neq 0$. Then there is an $j \in \mathbb{N}$ so that $d_r(z) = x_k a^j$. Apply themod 2 reduction to both sides we get $d_r(w^n) = x_k w^{2j}$, here d_r is a differential in the AHSS for $K(F_k, \mathbb{Z}_2)$. Recall that for a CW-complex X, there is a spectral sequence with $\mathbb{E}_2$-terms as $H^*(X, \mathbb{Z}_2)$ and strong converge to $K(X, \mathbb{Z}_2) := K(X \wedge \mathbb{R}P^2)$(c.f: [H]). Now $d_r(w^n) = 0$ by derivative property of differential and so we get a contradiction. Thus the AHSS collapses and the graded rings $\oplus G_q K(F_k)$ and $H^{even}(F_k, \mathbb{Z})$ are isomorphic. Notice in the former ring, $x_k a^{\frac{n-1}{2}} \neq 0$, $a^{\frac{n-1}{2}} \neq 0$.

Let $\alpha \in \widetilde{K}(F_k)$ be a class so that

$$ch(\alpha) = x_k + \text{ higher terms.}$$

Then $x \in G_2 K(F_k) \cong \mathbb{Z}_2$ is a generator, $\alpha \in G_{m_{-k}} K(F_k) \cong H^{m_{-k}}(F_k, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ generates the free part. Notice $x^2 = 2x$. Consequently x and αx span a subgroup of $K(F_k)$ of order at least 2^{n-1}. Compare the order of the torsion summand we obtain that $tor\widetilde{K}(F_k) \cong \mathbb{Z}_{2^{\frac{n-1}{2}}} \oplus \mathbb{Z}_{2^{\frac{n-1}{2}}}$ is generated by x and αx. Therefore $\widetilde{K}(F_k) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\frac{n-1}{2}}} \oplus \mathbb{Z}_{2^{\frac{n-1}{2}}}$.

By the Mayer-Vietoris exact sequence

$$\widetilde{K}(\mathbb{R}P^{2n+1}) \xrightarrow{(i_k^!, -i_{-k}^!)} \widetilde{K}(F_k) \oplus \widetilde{K}(F_{-k}) \xrightarrow{p_k^! + p_{-k}^!} \widetilde{K}(M) \to \widetilde{K}^1(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}$$

it follows that the exponent of $tor\widetilde{K}(M)$ is $2^{\frac{n-1}{2}}$. As we pointed out prior to the theorem, $K(M) \cong K(F_{-k})[1,y]$ and so $tor\widetilde{K}(M) \cong \mathbb{G}\oplus\mathbb{G}$, where $\mathbb{G} = torK(F_{-k})$. Thus $\mathbb{G}$ is of order at least $2^{\frac{n-1}{2}}$. On the other hand, by Atiyah-Hirzebruch spectral sequence for $\widetilde{K}(F_{-k})$ we obtain readily that $tor\widetilde{K}(F_{-k})$ is generated by x and its order is at most $2^{\frac{n-1}{2}}$. Thus $\mathbb{G} \cong \mathbb{Z}_{2^{\frac{n-1}{2}}}$ and $\widetilde{K}(F_{-k}) \cong \mathbb{Z}\oplus\mathbb{G} \cong \mathbb{Z}\oplus\mathbb{Z}_{2^{\frac{n-1}{2}}}$. It is obvious to check z is a generator of the free part. This proves (i) and (ii).

Apply the Mayer-Victoris sequence again (iii) follows. ♣

Fix a CW-complex structure on F_k and M. Let $F_k^{(n)}$ and $M^{(n)}$ denote the n-skeleton of F_k and M respectively. By the proof above it is readily to see that the restriction of x in $\widetilde{K}(F_k^{(n)})$ and $\widetilde{K}(M^{(n)})$ generates a cyclic group of order $2^{\frac{n-1}{2}}$.

Corollary 3.3: *x generates a cyclic groups of order $2^{\frac{n-1}{2}}$ in $\widetilde{K}(F_{\pm k}^{(n)})$ and $\widetilde{K}(M^{(n)})$ respectively.*

§4.Real K-theory of Isoparametric Hypersurfaces in $\mathbb{R}P^{2n+1}$

This section is going to deal with the real K-theory of the focal manifolds and the hypersurface. We adopt the same notation in §3. Recall that for any CW complex X, there is a Atiyah-Hirzebruch spectral sequence with $\mathbb{E}_2$ -terms and E_∞-terms are:

$$\mathbb{E}_2^{p,q} = \widetilde{H}^p(X;KO^q(*))$$
$$E_\infty^{p,q} = G_p\widetilde{KO}^{p+q}(X) = \widetilde{KO}_p^{p+q}(X)/\widetilde{KO}_{p+1}^{p+q}(X)$$

where $\widetilde{KO}_p^n(X) = Ker\left\{\widetilde{KO}^n(X) \to \widetilde{KO}^n(X^{p-1})\right\}$. The periodic property implies that the differentials $d_r^{p,q+8} = d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. The differentials d_2 and d_3 may be presented in terms of primary Steenrod operations as the following:

$$\begin{aligned} d_2^{p,-8t} &= Sq^2 : H^p(X;\mathbb{Z}) \to H^{p+2}(X,\mathbb{Z}_2)\\ d_2^{p,-8t-1} &= Sq^2 : H^p(X;\mathbb{Z}_2) \to H^{p+2}(X,\mathbb{Z}_2)\\ d_3^{p,-8t-2} &= \delta_2 Sq^2 : H^p(X;\mathbb{Z}_2) \to H^{p+3}(X,\mathbb{Z})\\ d_3^{p,q} &= 0 \text{ if } q \neq 0,-1,-2(mod 8)\end{aligned}$$

where δ_2 is the Bockstein coboundary operator. We use $\varepsilon : \widetilde{KO}^*(X) \to \widetilde{K}^*(X)$ and $\rho : \widetilde{K}^*(X) \to \widetilde{KO}^*(X)$ to denote the complexification and realization homomorphism. Recall $\rho\varepsilon = 2$ and $\varepsilon\rho(x) = x + x^*$, where x^* is the conjugate of x.

Let $T(\gamma_k)$ denote the Thom complex of γ_k which is homeomorphic to $\mathbb{R}P^{2n+1}/F_{-k}$. We have a Thom isomorphism $\phi : KO^*(F_k) \cong \widetilde{KO}^*(\mathbb{R}P^{2n+1}/F_{-k})$. For clarity we consider the following four cases by $m_k \equiv 0,1,2,3(mod 8)$. Let $x = \xi - 1 \in \widetilde{KO}(F_{\pm k}), \widetilde{KO}(M)$, where ξ is the Hopf real line bundle. Notice $\phi(n) = \frac{n-1}{2}$ if $n = -1(mod 8)$. By Theorem 3.2 and the following commutative diagram

$$\begin{array}{ccc}\widetilde{KO}(\mathbb{R}P^{2n+1}) & \xrightarrow{i_k^!} & \widetilde{KO}(F_k) \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ \widetilde{K}(\mathbb{R}P^{2n+1}) & \xrightarrow{i_k^!} & \widetilde{K}(F_k)\end{array}$$

we obtain readily

Lemma 4.0 *$Imi_k^!$ is cyclic and of order at least $2^{\frac{n-1}{2}}$ for $k = \pm 1$.*

Let $E \to X$ denote a principal Spin$(n+1)$-bundle as in §3 and $\mathbb{P}(E)$ denote the orbits space of the obvious free $Spin(n)$ action on E. Recall $\mathbb{P}(E) = S(V)$, where $S(V) = E \times_\rho S^n$ is the associated sphere bundle of E with respect to the standard representation. Let $\hat{E}$ denote the principal Spin(n) bundle $E \to \mathbb{P}(E)$. If $n = 0(mod 4)$. The spin representations $\Delta_\pm \in RO(Spin(n))$ gives an associated vector bundle $\hat{E} \times_{\Delta_+} \mathbb{R}^{2^h} \to \mathbb{P}(E)$, where $2^h = a_n$ is the Radon-Hurewicz number. In particular, if $n = 8k+8$ and $8k+4$, $a_n = 2^{4k+3}$ and $a_n = 2^{4k+2}$ respectively. Let $y \in \widetilde{KO}(\mathbb{P}(E))$ denote the stable class of the vector bundle above.

By Quillen[Qu], the nonzero Stiefel-Whitney classes of y are those of degree 2^h and $2^h - 2^i$ for $r \le i \le h$, where $r = 0, 2$ by $n = 0(mod 8)$ and $n = 4(mod 8)$. In particular, $w_j(y) = 0$ if $1 \le j < 2^{h-1}$.

If $min\{m_k, m_{-k}\} \ge 9$, the bundle γ_k is spin of dimension $m_k + 1$. $n = m_k + m_{-k} = -1(mod 8)$. When $m_k \equiv 0(mod 4)$, by the prior construction we have an $y \in \widetilde{KO}(M)$.

Theorem 4.1. *Let $m_k \equiv 0(mod 8)$. Then*

(i) $\widetilde{KO}(F_k) \cong \mathbb{Z}_{2^{\phi(n)}}$ generated by x.

(ii) $\widetilde{KO}(M) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}}$ generated by y, x and xy. Moreover, the homomorphism $p_{-k}^! : \widetilde{KO}(F_{-k}) \to \widetilde{KO}(M)$ is an isomorphism.

Proof: Recall first that $m_{-k} \equiv 7(mod 8)$. By Proposition 3.1 we obtain

(1) For $p \equiv 1, 2(mod 8)$, $\mathbb{E}_2^{p,-p} \cong H^p(F_k, \mathbb{Z}_2)$ is generated by w^p and $w^{p-m_{-k}}x_k$ whenever $p \ge m_{-k}$.

(2) For $p = 0(mod 4)$, $\mathbb{E}_2^{p,-p} \cong H^p(F_k, \mathbb{Z})$ is generated by $a^{\frac{p}{2}}$.

(3) For other p, the $\mathbf{E}_2$-terms are identically zero.

We claim that $w^{p-m_{-k}}x_k \in \mathbb{E}_2^{p,-p}$ does not survive to $\mathbb{E}_4$ if $p \equiv 1, 2(mod 8)$.

If $p \equiv 1(mod 8)$, notice $Sq^2(w^{p-m_{-k}}x_k) = w^{p-m_{-k}+2}x_k (\text{mod } w^{p+2})$ as $\binom{p - m_{-k}}{2} \equiv 1(mod\ 2)$. Therefore $w^{p-m_{-k}}x_k \notin \mathbb{E}_3^{p,-p} = \ker d_2$.

If $p = 2(mod 8)$, $d_3(w^{p-m_{-k}}x_k) = a^{\frac{p-m_{-k}+1}{2}+1}x_k \ne 0$ and so $w^{p-m_{-k}}x_k \notin \mathbb{E}_4^{p,-p}$. This shows that $\mathbf{E}_4^{p,-p}$ is at most of order 2 when $p = 0, 1, 2, 4(\text{mod } 8)$ so that $p \le n$, and zero otherwise. Therefore $|\widetilde{KO}(F_k)| \le 2^{\phi(n)}$, where $\phi(n) := \#\{p = 0, 1, 2, 4(\text{mod } 8); p \le n\}$.

Notice $\phi(n) = \frac{n-1}{2}$ and so by 4.0 $\widetilde{KO}(F_k) \cong \mathbb{Z}_{2^{\phi(n)}}$ and generated by x. (i) follows.

By Bott[Bo], $KO(M)$ is a free $KO(F_k)$-module with basis 1 and y. Thus

$\widetilde{KO}(M) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}}$. By AHSS as above it is easy to check that $tor K(F_{-k})$ is of order at most $2^{2\phi(n)}$. Applying 4.0 and the Mayer-Vietoris sequence

$$\widetilde{KO}(\mathbb{R}P^{2n+1}) \to \widetilde{KO}(F_k) \oplus \widetilde{KO}(F_{-k}) \to \widetilde{KO}(M) \to \widetilde{KO}^1(\mathbb{R}P^{2n+1}) = 0$$

it follows that $\widetilde{KO}(F_{-k}) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}}$ and the homomorphism $p^!_{-k}$: $\widetilde{KO}(F_{-k}) \to \widetilde{KO}(M)$ is an isomorphism. This proves (ii). ♣

Theorem 4.2 *Let $m_k = 3(mod 8)$. Then*

(i) $\widetilde{KO}(M) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by y, x, yx, y_1 and y_2, , where y_1 and y_2 are two order 2 classes with filtrations $2m_k + m_{-k}$ and $2m_k + m_{-k} - 1$ respectively. Moreover, the homomorphism $p_k^* : \widetilde{KO}(F_k) \to \widetilde{KO}(M)$ is injective onto the direct summand $\mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}}$.

(ii) $\widetilde{KO}(F_{-k}) \cong \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and imp^*_{-k} is a direct summand of $\widetilde{KO}(M)$.

Proof: By AHSS for $\widetilde{KO}(F_{-k})$ as above one can check readily that $\mathbb{E}_4^{p,-p} \cong \mathbb{Z}_2$ if $p = 0, 1, 2, 4(mod 8)$ and $p \leq n$ presented by w^p and $a^{\frac{p}{2}}$ by $p = 1, 2(mod 8)$ and $p = 0,\ 4(mod 8)$. All other $\mathbb{E}_4$ terms vanish except $\mathbb{E}_4^{p,-p}$ where $p = \dim F_{-k}$ and $\dim F_{-k} - 1$. Now we show the last two terms survive in E_∞. Consider the Thom isomorphism

$$KO(F_{-k}) \cong \widetilde{KO}(T(\gamma_{-k} \oplus \varepsilon^3)) \cong \widetilde{KO}^{-3}(\mathbb{R}P^{2n+1}/F_k).$$

Notice that the terms $\mathbb{E}_4^{p,-p}$ correspond to $\mathbb{E}_4^{2n+1,-2n-4}$ and $\mathbf{E}_4^{2,-2n-3}$ respectively in the AHSS of $\widetilde{KO}^{-3}(\mathbb{R}P^{2n+1}/F_k)$. By [Fu], $\widetilde{KO}^{-3}(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ contributed by w^{2n+1} and a^n of the E_∞ terms. It is easy to see that the natural homomorphism

$$\widetilde{KO}^{-3}(\mathbb{R}P^{2n+1}/F_k) \to \widetilde{KO}^{-3}(\mathbb{R}P^{2n+1})$$

is surjective. Combine these we obtain readily that $x_{-k}w^n$ and $x_{-k}a^{\frac{n-1}{2}}$ survive to E_∞. It is easy to check the others $\mathbb{E}_4$-terms survive to E_∞. This proves that $(\mathbb{E}_4, d_4)$ collapses and so the order of $\widetilde{KO}(F_{-k})$ is equal to $2^{\phi(n)+2}$.

Claim $\widetilde{KO}(F_{-k}) \cong imi^*_{-k} \oplus \widetilde{KO}^{-3}(F_k)$ *and the exponent of* $\widetilde{KO}^{-3}(F_k)$ *is* 2.

Consider the exact sequence

$$\widetilde{KO}(\mathbb{R}P^{2n+1}) \xrightarrow{i^!_{-k}} \widetilde{KO}(F_{-k}) \to \widetilde{KO}^1(\mathbb{R}P^{2n+1}/F_{-k}) \to \widetilde{KO}^1(\mathbb{R}P^{2n+1}) = 0.$$

Note the Thom isomorphism $\widetilde{KO}^1(\mathbb{R}P^{2n+1}/F_{-k}) = \widetilde{KO}^1(T(\gamma_k)) \cong \widetilde{KO}^{-3}(F_k)$. Consequently there is an extension

$$0 \to imi^*_{-k} \to \widetilde{KO}(F_{-k}) \to \widetilde{KO}^{-3}(F_k)$$

which splits. Otherwise, there is an $z \in \widetilde{KO}(F_{-k})$ so that $x = 2z$. Thus the Stiefel-Whitney class $w_1(x) = 0$ and we reach a contradiction.

Obviously $K^{-3}(F_k)$ is torsion free and so the composition $\rho \circ \varepsilon : \widetilde{KO}^{-3}(F_k) \to \widetilde{KO}^{-3}(F_k)$ is zero. Therefore $2\widetilde{KO}^{-3}(F_k) = 0$ and the claim follows.

Similarly, the AHSS argument shows easily $|tor\widetilde{KO}(F_k)| \leq 2^{2\phi(n)}$. Consider the exact sequence

$$\widetilde{KO}(\mathbb{R}P^{2n+1}) \xrightarrow{i_k^!} \widetilde{KO}(F_k) \to \widetilde{KO}^1(\mathbb{R}P^{2n+1}/F_k) \to \widetilde{KO}^1(\mathbb{R}P^{2n+1}) = 0$$

and the Thom isomorphism $\widetilde{KO}^1(\mathbb{R}P^{2n+1}/F_k) = \widetilde{KO}^1(T(\gamma_{-k})) \cong KO^{-4}(F_{-k})$. We claim that $\widetilde{KO}^{-4}(F_{-k}) \cong \mathbb{Z}_{2^{\phi(n)}}$. Combine 4.0 this implies $imi_k^! \cong \mathbb{Z}_{2^{\phi(n)}}$ and there is an extension

$$\mathbb{Z}_{2^{\phi(n)}} \to \widetilde{KO}(F_k) \to \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}}.$$

It is readily to verify that this extension splits and so $\widetilde{KO}(F_k) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}}$. Notice that $KO^{-4}(F_{-k})$ fits in the exact sequence

$$\widetilde{KO}^{-4}(\mathbb{R}P^{2n+1}) \xrightarrow{i_{-k}^!} \widetilde{KO}^{-4}(F_{-k}) \to \widetilde{KO}^{-3}(\mathbb{R}P^{2n+1}/F_{-k}) \xrightarrow{j} \widetilde{KO}^{-3}(\mathbb{R}P^{2n+1}) \cong Z_2 \oplus \mathbb{Z}_2.$$

It is easy to show that j is surjective and $\widetilde{KO}^1(F_k) \cong \widetilde{KO}^{-3}(\mathbb{R}P^{2n+1}/F_{-k})$ is of order at most 4 by AHSS. Thus $\widetilde{KO}^{-4}(F_{-k}) \cong imi^*_{-k}$ is cyclic. By the AHSS for $\widetilde{KO}^{-4}(F_{-k})$ it follows that the nontrivial $\mathbb{E}_4$-terms are $\mathbb{E}_4^{p,-p-4} \cong \mathbb{Z}_2$ presented by w^p and $a^{\frac{p}{2}}$ if $p = 5, 6(mod 8)$ and $p = 0(mod 4)$ respectively for $p \leq n$.

Compare with the AHSS of $K(\mathbb{R}P^{2n+1})$ via the inclusion map $i_{-k} : F_{-k} \to \mathbb{R}P^{2n+1}$ it is easy to check that all of these $\mathbb{E}_4$-terms survive to E_∞, in other words, the spectral sequence $(\mathbb{E}_4, d_4)$ collapses. Thus $\widetilde{KO}^{-4}(F_{-k}) \cong imi^*_{-k} \cong \mathbb{Z}_{2^{\phi(n)}}$. This proves that $\widetilde{KO}(F_k) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}}$.

Apply 4.0 again we obtain easily that imi^*_{-k} is either $\mathbb{Z}_{2^{\phi(n)}}$ or $\mathbb{Z}_{2^{\phi(n)+1}}$. On the other hand, the Mayer-Vietoris sequence argument implies easily that $\widetilde{KO}(M) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{G}$ where $\mathbb{G} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2$ by $imi^*_{-k} = \mathbb{Z}_{2^{\phi(n)}}$ or $\mathbb{Z}_{2^{\phi(n)+1}}$ respectively. If the latter case happens, then one of $x_{-k}w^n$ and $x_{-k}a^{\frac{n-1}{2}}$ must be killed in the AHSS for $\widetilde{KO}(M)$. It is not difficult to see this is impossible by inspecting the AHSS. Thus $imi^*_{-k} \cong \mathbb{Z}_{2^{\phi(n)}}$ and so $\widetilde{KO}(M) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This completes the proof. ♣

Theorem 4.3 *Let $m_k = 1(mod 8)$. Then*

(i) $\widetilde{KO}(F_k) \cong \mathbb{Z}_{2^{\phi(n)}}$ generated by x.

(ii) $\widetilde{KO}(F_{-k}) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, the two $\mathbb{Z}_2$-summands are generated by y_1 and y_2 corresponding to x_{-k} and $x_{-k}w$ in the graded ring respectively. The free part is generated by $P^*(\ell)$, where $\ell \in \widetilde{KO}(S^{2m_k+m_{-k}}) \cong \mathbb{Z}$ is a generator.

(iii) $p^!_{-k} : \widetilde{KO}(F_{-k}) \to \widetilde{KO}(M)$ *is an isomorphism.*

Proof: Consider the AHSS for $\widetilde{KO}(F_k)$; The only nontrivial $\mathbb{E}_2$ terms are as the follows:

If $p = 0(mod4)$, $\mathbb{E}_2^{p,-p} \cong H^p(F_k, \mathbb{Z})$ is generated by $a^{\frac{p}{2}}$ (and $x_k a^{\frac{p-m_{-k}}{2}}$ if $p \geq m_{-k}$). If $p = 1, 2(mod8)$, $\mathbb{E}_2^{p,-p} \cong H^p(F_k, \mathbb{Z}_2)$ is generated by w^p(and $x_k w^{p-m_{-k}}$ if $p \geq m_{-k}$). Notice that

$$\begin{array}{ll} Sq^2(x_k w^{p-m_{-k}}) = x_k w^{p-m_{-k}+2}(mod w^{p+2}) & \text{if } p = 1(mod8), \\ Sq^2(x_k w^{p-m_{-k}-2}) = x_k w^{p-m_{-k}}(mod w^{p+2}) & \text{if } p = 2(mod8), \\ Sq^2(x_k a^{\frac{p-m_{-k}}{2}}) = x_k w^{p-m_{-k}+2}(mod w^{p+2}) & \text{if } p = 0(mod8), \\ \beta Sq^2(x_k a^{\frac{p-m_{-k}}{2}}) \neq 0 & \text{if } p = 4(mod8) \end{array}$$

We assert therefore that $x_k a^{\frac{p-m_{-k}}{2}}$ and $x_k w^{p-m_{-k}}$ do not survive to E_∞ for every p. Thus the order of $\widetilde{KO}(F_k)$ is at most $2^{\phi(n)}$. By 4.0 (i) follows.

ASSERTION: In the AHSS for $\widetilde{KO}(F_{-k})$, $x_{-k} a^{\frac{p-m_k}{2}}$ and $x_{-k} w^{p-m_k}$ do not survive to E_∞ for $p \geq m_k + 2$, where $p \equiv 0(mod4)$ and $1, 2(mod8)$ respectively. x_{-k} and $x_{-k}w$ contribute to E_∞.

The proof of the first half can be verified just same as above which shows that the torsion of $\widetilde{KO}(F_{-k})$ is of order at most $2^{\phi(n)+2}$.

To show x_{-k} and $x_{-k}w$ survive, notice first the differential $d_r : E_r^{p-r,-p+r-1} \to E_r^{p,-p}$ vanishes for each r if $p = m_k$ and $m_k + 1$. In fact, $\mathbb{E}_2^{p-r,-p+r-1}$ is generated by w^{p-r} or $a^{\frac{p-r}{2}}$ for these p, this can be verified via comparing with the AHSS of $\mathbb{R}P^{2n+1}$.

Next we have to prove the differentials $d_r : E_r^{p,-p} \to E_r^{p+r,-p-r+1}$ for $p = m_k$ or $m_k + 1$ vanish. For this, note first $d_r = 0$ for $r = 2$ and 3. To complete the proof, it is enough to show $E_r^{p+r,-p-r+1} = E_\infty^{p+r,-p-r+1}$ if $r \geq 4$. Similar as in the proof of 4.3, we consider the Thom isomorphism

$$\widetilde{KO}(\mathbb{R}P^{2n+1}/F_k) = \widetilde{KO}(T(\gamma_{-k})) \cong \widetilde{KO}^1(T(\gamma_{-k} \oplus \varepsilon)) \cong KO^1(F_{-k}).$$

Note that $E_r^{i,-i+1}$ corresponds to $\mathbb{E}_2^{i+v,-i+1+v}$ in the AHSS of $\widetilde{KO}(\mathbb{R}P^{2n+1}/F_k)$, here $v = m_{-k} + 2$. Comparing with the AHSS of $\widetilde{KO}(\mathbb{R}P^{2n+1})$ via the projection map $\mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}/F_k$ the proof of the assertion follows.

Consequently, $\widetilde{KO}(F_{-k} - pt)$ is of order $2^{\phi(n)+2}$. We claim the following exact sequence splits

$$0 \to \widetilde{KO}(S^{2m_k+m_{-k}}) \xrightarrow{P^!} \widetilde{KO}(F_{-k}) \xrightarrow{i^!} \widetilde{KO}(F_{-k} - pt) \to 0$$

Consider the commutative diagram:

$$\begin{array}{ccc} \widetilde{KO}(F_{-k}) & \xrightarrow{P_!} & \widetilde{KO}(S^{2m_k+m_{-k}}) \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ \widetilde{K}(F_{-k}) & \xrightarrow{P_!} & \widetilde{K}(S^{2m_k+m_{-k}}) \end{array}$$

Note ε above is surjective. Let $z \in \widetilde{KO}(S^{2m_k+m_{-k}})$ be a generator. Applying the Hirzebruch-Riemann-Roch theorem we conclude that $P_!(P^!\varepsilon(z)) = \varepsilon(z)$ and thus $P_! : \widetilde{K}(F_{-k}) \longrightarrow \widetilde{K}(S^{2m_k+m_{-k}})$ is surjective since $\hat{A}(F_{-k}) = 1(\ sigF_{-k} = 0)$. In particular, the exact sequence splits. Thus $\widetilde{KO}(F_{-k}) \cong \widetilde{KO}(F_{-k} - pt) \oplus \mathbb{Z}$.

Let X denote the restriction of the sphere bundle $M \to F_k$ at its 2-skeleton, where we may assume the 2-skeleton of F_k is an embedded $\mathbb{R}P^2$. Thus X has the homotopy type of $\mathbb{R}P^2 \times S^{m_k}$ as this sphere bundle is spin. The AHSS argument above shows that the restriction

$$tor\widetilde{KO}(M) \to \widetilde{KO}(X) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

is surjective. Thus $tor\widetilde{KO}(M)$ contains at least three nontrivial direct summand. Notice the exponent of $torKO(M)$ is at least $2^{\phi(n)}$ by 4.0. On the other hand, by the Mayer-Vietoris sequence we obtain that $p^!_{-k} : \widetilde{KO}(F_{-k}) \to \widetilde{KO}(M)$ is surjective. Thus $|tor\widetilde{KO}(M)| \leq 2^{\phi(n)+2}$. Therefore $\widetilde{KO}(M) \cong \mathbb{Z}\oplus\mathbb{Z}_{2^{\phi(n)}}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$. By the Mayer-Vietoris exact sequence once again we have $\widetilde{KO}(M) \cong \widetilde{KO}(F_{-k}) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This proves (ii) and (iii). ♣

The proof of the above theorems implies also the following

Corollary 4.4 *Let $T : M \to M$ is a diffeomorphism inducing identity on the $\mathbb{Z}_2$-cohomology ring. If $m_k = 1, 3(mod8)$, then $T^*(x) = x$, $T^*(y_1) = y_1$ and $T^*(y_2) = y_2$.*

Finally let us consider the case of $m_k = 2(mod8)$.

Theorem 4.5 *Let $m_k = 2(mod8)$. Then*

(i) $\widetilde{KO}(F_k) \cong \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}$ generated by x and $P^*(\ell)$.

(ii) $\widetilde{KO}(F_{-k}) \cong \mathbb{Z}_{2^{\phi(n)}} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with generators x, y_1, y_2, y_3 and y_4.

(iii) $\widetilde{KO}(M) \cong \mathbb{Z}\oplus\mathbb{Z}_{2^{\phi(n)}}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$. $imp^*_{\pm k}$ are both direct summands.

Proof: Note first that $\dim F_k = 4(mod8)$. As in 4.1, by considering the differentials d_2 and d_3 it follows that $\widetilde{KO}(F_k)$ contains the torsion of order at most $\mathbb{Z}_{2^{\phi(n)}}$. Compare with (4.0) it follows that $tor\widetilde{KO}(F_k) \cong \mathbb{Z}_{2^{\phi(n)}}$. Thereby $\widetilde{KO}(F_k) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^{\phi(n)}}$ where the free part corresponds to the top dimensional cell. This proves (i).

To prove (ii), note that $x_{-k}w^{p-m_k}$ and $x_{-k}a^{\frac{p-m_k}{2}}$ do not survive to $\mathbb{E}_4$ if $p \neq m_k, m_k + 2, m_k + n - 1$ or $m_k + n$. For the rest several terms, we assert that

ASSERTION *$x_{-k}, x_{-k}a, x_{-k}w^n$ and $x_{-k}a^{\frac{n-1}{2}}$ survive to E_∞ in the AHSS for $\widetilde{KO}(F_{-k})$. Moreover, the latter two terms have filtrations m_k+n and m_k+n-1 respectively.*

Obviously the four terms do survive to $\mathbb{E}_4$. The differentials $d_r : E_r^{p-r,-p+r-1} \to E_r^{p,-p}$ where $p = m_k$ or m_k+2 and $r \geq 4$ are both identically zero since $\mathbb{E}_2^{p-r,-p+r-1}$ is either 0 or generated by w^{p-r}(or $a^{\frac{p-r}{2}}$) and the differential d_r in the spectral sequence for $\widetilde{KO}^{-1}(\mathbb{R}P^{2n+1}) = \mathbb{Z} \oplus \mathbb{Z}_2$ is zero for $r \geq 4$.

Next we consider the differential $d_r : E_r^{p,-p} \to E_r^{p+r,-p-r+1}$. Notice that the terms $\mathbb{E}_4^{i,-i+1} = 0$ except $i = m_k$, m_k+1 and m_k+n. Moreover, $E_\infty^{m_k+n,-m_k-n+1} \cong \mathbb{Z}$. Thus $d_r : E_r^{p,-p} \to E_r^{p+r,-p-r+1}$ is zero for $p = m_k$ or m_k+2 and so x_{-k} and $x_{-k}a$ survive to E_∞.

To show the last two terms survive, we need only to prove that $d_r : E_r^{p,-p-1} \to E_r^{p+r,-p-r}$ does not hit $x_{-k}w^n$ and $x_{-k}a^{\frac{n-1}{2}}$ for each r where $p = m_k+n-r-1$ or m_k+n-r. Under the Thom isomorphism

$$KO(F_{-k}) \cong \widetilde{KO}(T(\gamma_{-k} \oplus \varepsilon^2)) \cong \widetilde{KO}^{-2}(\mathbb{R}P^{2n+1}/F_k),$$

$x_{-k}w^n$ and $x_{-k}a^{\frac{n-1}{2}}$ correspond to w^{2n+1} and a^n in the AHSS for $\widetilde{KO}^{-2}(\mathbb{R}P^{2n+1}/F_k)$. By [Fu] w^{2n+1} and a^n survive in the AHSS for $\widetilde{KO}^{-2}(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This proves the assertion. Consequently the order of $\widetilde{KO}(F_{-k})$ is $2^{\phi(n)+4}$.

Consider the exact sequence:

$$\widetilde{KO}(\mathbb{R}P^{2n+1}) \xrightarrow{i^!_{-k}} \widetilde{KO}(F_{-k}) \to \widetilde{KO}^1(\mathbb{R}P^{2n+1}/F_{-k}) \to \widetilde{KO}^1(\mathbb{R}P^{2n+1}) = 0.$$

By 4.0 and comparing the AHSS of $\widetilde{KO}(M)$ with $\widetilde{KO}(F_{-k})$ it is not hard to show that $im i^*_{-k} \cong \mathbb{Z}_{2^{\phi(n)}}$. Thus $\widetilde{KO}^1(\mathbb{R}P^{2n+1}/F_{-k}) \cong \widetilde{KO}^1(T(\gamma_k)) \cong KO^{-2}(F_k) = KO^{-2}(pt) \oplus \widetilde{KO}^{-2}(F_k)$ is of order 16. As in the proof of 4.2 one can check easily the extension above splits and so $\widetilde{KO}(F_{-k}) = im i^!_{-k} \oplus \widetilde{KO}^1(\mathbb{R}P^{2n+1}/F_{-k})$.

Now we are going to show that $KO^{-2}(F_k) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Recall for each CW complex, there is a long exact sequence(c.f: [Bo])

$$\cdots \widetilde{KO}^i(X) \to \widetilde{KO}^{i-1}(X) \xrightarrow{\varepsilon} \widetilde{K}^{i-1}(X) \xrightarrow{p_1} \widetilde{KO}^{i+1}(X)$$

Consider the following exact commutative diagram:

$$\begin{array}{ccccc}
\widetilde{KO^{-2}}(\mathbb{R}P^{2n+1}) & \xrightarrow{\varepsilon} & \widetilde{K^{-2}}(\mathbb{R}P^{2n+1}) & \xrightarrow{p_1} & \widetilde{KO}(\mathbb{R}P^{2n+1}) \\
\downarrow i^!_k & & \downarrow i^!_k & & \downarrow i^!_k \\
\widetilde{KO^{-2}}(F_k) & \xrightarrow{\varepsilon} & \widetilde{K^{-2}}(F_k) & \xrightarrow{p_1} & \widetilde{KO}(F_k)
\end{array}$$

Notice the homomorphism $i^!_k$ at the right square are both surjective to the torsion part, $\mathbb{Z}_{2^{\phi(n)}}$. By [Fu], $\widetilde{K^{-2}}(\mathbb{R}P^{2n+1}) \xrightarrow{p_1} \widetilde{KO}(\mathbb{R}P^{2n+1})$ is a multiple two homomorphism. So the restriction of $p_! : \widetilde{K^{-2}}(F_k) \to \widetilde{KO}(F_k)$ on the torsion summand is also a multiple two map. Therefore at the bottom line, $im\varepsilon = 2^{\phi(n)-1}x$. Compose with the realization homomorphism $\rho : \widetilde{K^{-2}}(F_k) \to \widetilde{KO}^{-2}(F_k)$ we have $\rho(2^{\phi(n)-1}x) = 0$ if $2^{\phi(n)-1} \geq | \widetilde{KO^{-2}}(F_k) |$. Thus $\rho\varepsilon = 0$ and so $2\widetilde{KO^{-2}}(F_k) = 0$ if $n \geq 8$. This proves that $\widetilde{KO^{-2}}(F_k) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $KO^{-2}(F_k) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This proves (ii). By the Mayer-Vietoris exact sequence one can conclude (iii) readily. This completes the proof. ♣

similarly we have

Corollary 4.6 *Let $T : M \to M$ is a diffeomorphism inducing identity on the $\mathbb{Z}_2$-cohomology ring. If $m_k = 2(mod8)$, then $T^*(x) = x$, $T^*(y_i) = y_i$ for $1 \leq i \leq 4$.*

5. Geometric Dimensions

Recall that the geometric dimension of a stable bundle V over a connect complex, denoted $g.\dim V$, is the minimum fibre dimension of all vector bundles stable equivalent to V. In this section, let X denote a connected n-dimensional CW complex with $\pi_1(X) \cong \mathbb{Z}_2$. Let ξ be the unique nontrivial real line bundle over X and $x = \varepsilon(\xi) - 1$. We are addressed to discuss the $g.\dim kx$ for $k \in \mathbb{N}$ motivated by studying the multiplicities of isoparametric hypersurfaces in sphere. The question itself has also some interests even in the case of $X = \mathbb{R}P^n$, this is exactly the generalized vector field problem arised by Atiyah-Bott-Shapiro[ABS]. It has other application in differential topology, e.g, the immersion problem of $\mathbb{R}P^n$ in Euclidean space of minimum dimension. Some important tools in algebraic topology were applied to attack it. Among them, perhaps the most successful method update is to use the BP-homology theory[Da]. We will use Adams operations in equivariant K-theory to prove the following result which plays a key role in the proof of theorem A.

Theorem 5.1 *Let X be a n-dimensional CW-complex with $\pi_1(X) \cong \mathbb{Z}_2$ and $E = k\varepsilon(\xi)$. Assume $n = -1(mod8)$. Suppose that*
(i). $x = \varepsilon(\xi) - 1 \in \widetilde{K}(X)$ generates a cyclic subgroup of order $2^{\phi(n)}$.
(ii). The total Stiefel-Whitney class $w(E) = (1+w)^{n+1}$ where $w \in H^1(X, \mathbb{Z}_2) \cong \mathbb{Z}_2$ is a generator and $w^n \neq 0$.
(iii). E has $2m + (2k - n - 1)$ linearly independent sections.
Then $\nu_2(k) \geq m - 1$ if $m = 1(mod2)$ and $\nu_2(k) \geq m - 2$ if $m = 0(mod2)$.

Notice that (iii) above is equivalent to saying $g.\dim_{\mathbb{R}} E \leq n + 1 - 2m$. Let us recall a result of Becker[Be] which says if $E \to X$ is a real vector bundle and has l linearly independent sections, then $E \otimes \xi \to X \times \mathbb{R}P^{l-1}$ admits a nowhere zero section, where ξ is the Hopf line bundle $\xi \to \mathbb{R}P^{l-1}$. For our E in 5.1, by (ii) we obtain immediately that $2k \geq n + 1$. Let $r = [\frac{n}{2}]$ and $E_0 = E - (k - r - 1)$. For the dimension reasoning E_0 is also a virtual complex vector bundle. By (iii)

E_0 possesses $2m$ linearly independent sections. Thus $\rho(E_0)\otimes\xi$ and so $E_0\otimes\varepsilon(\xi)$ admits a nonzero section. For the sake of simple, in the rest let us consider E_0 instead of E. Notice that the Gysin exact sequence

$$0 \to K^{-1}(X\times \mathbb{R}P^{2m-1}) \xrightarrow{p^!} K^{-1}(S(E\otimes\varepsilon(\xi))) \xrightarrow{\delta} K^0(E\otimes\varepsilon(\xi)) \to 0$$

splits exact.

Let $t = X\times \mathbf{C} \to X$ denote the trivial complex line bundle with a nontrivial involution $(x,z)\to(x,-z)$. We regard X as a trivial $\mathbb{Z}_2$ space. Notice t gives an element in the equivariant K -group $K_{\mathbf{Z}_2}(X)\cong K(X)\otimes R(\mathbb{Z}_2)$, where $R(\mathbb{Z}_2)$ is the complex representation ring of $\mathbb{Z}_2$. We write Et for $E\otimes t$. Observe that
$X\times \mathbb{R}P^{2m-1}\approx S(mt)/\mathbf{Z}_2$,
$S(E\otimes\varepsilon(\xi))\approx S(mt)\times_X S(Et)/\mathbb{Z}_2$,
$E\otimes\varepsilon(\xi)\approx S(mt)\times_X Et/\mathbf{Z}_2$,
where $S(mt)\times_X S(Et)$ is the following pullback bundle:

$$\begin{array}{ccc} S(mt)\times_X S(Et) & \xrightarrow{\pi} & S(Et) \\ \downarrow p & & \downarrow p \\ S(mt) & \xrightarrow{\pi} & X \end{array}$$

Under this identification, the exact sequence above may be identified with the exact sequence

$$0\to K^{-1}_{\mathbf{Z}_2}(S(mt)) \xrightarrow{p^!} K^{-1}_{\mathbf{Z}_2}(S(mt)\times_X S(Et)) \xrightarrow{\delta} K^0_{\mathbf{Z}_2}(S(mt)\times_X Et)\to 0$$

which fits into the following commutative diagram(**):

$$\begin{array}{ccccccc}
 & & K^{-1}_{\mathbf{Z}_2}(X) & \longrightarrow & K^{-1}_{\mathbf{Z}_2}(S(Et)) & \longrightarrow & K^{0}_{\mathbf{Z}_2}(Et) \\
 & & \downarrow \pi^! & & \downarrow & & \downarrow \pi^! \\
 & & K^{-1}_{\mathbf{Z}_2}(S(mt)) & \xrightarrow{p^!} & K^{-1}_{\mathbf{Z}_2}(S(mt)\times_X S(Et)) & \xrightarrow{\delta} & K^{0}_{\mathbf{Z}_2}(S(mt)\times_X Et) \\
 & & \downarrow \delta_1 & & \downarrow \delta_1 & & \downarrow \delta_1 \\
K^{0}_{\mathbf{Z}_2}(mt\times_X Et) & \xrightarrow{p^!} & K^{0}_{\mathbf{Z}_2}(mt) & \xrightarrow{p^!} & K^{0}_{\mathbf{Z}_2}(mt\times_X S(Et)) & \xrightarrow{\delta} & K^{1}_{\mathbf{Z}_2}(mt\times_X Et) \\
\downarrow \pi^! & & \downarrow \pi^! & & \downarrow \pi^! & & \\
K^{0}_{\mathbf{Z}_2}(Et) & \xrightarrow{p^!} & K^{0}_{\mathbf{Z}_2}(X) & \xrightarrow{p^!} & K^{0}_{\mathbf{Z}_2}(S(Et)) & &
\end{array}$$

Obviously all homomorphisms in the above diagram commute with the Adams operation $\psi^p (p\in\mathbb{N})$ since they are induced by maps. For a bundle F based on X, let $U_F\in K(F)$ denote its Thom class. Recall that the Bott *canibalistic charateristic class* $\rho_p(F)\in K(X)$ is defined by the identity:

$$\psi^p(U_F)=\rho_p(F)U_F.$$

For a line bundle L, $\rho_p(L) = 1 + L^* + \cdots + (L^*)^{p-1}$.

Conventions: *For every G-complex vector bundle E based on a connected complex X, $S(E)$ always stands for the sphere bundle and U_E for the Thom class, where G is a compact Lie group.*
(i) $\forall b \in K_G^*(X)$ and $\forall u = aU_E \in K_G^*(E) \cong K_G^*(X)$, $a \in K_{\mathbb{Z}_2}^*(X)$, define $bu := (ba)U_E \in K_G^*(E)$.
(ii) $\forall b \in K_G^*(X)$ and $\forall u \in K_G^*(S(E))$, define $ub := u\pi^! b \in K_G^*(S(E))$ where $\pi : S(E) \to X$ is the projection.

Let $\varphi_p = \psi^p - \rho_p(Et)$. With those conventions in mind, it is easy to see that φ_p is natural with respect to maps in the diagram(**). Obviously we have

Lemma 5.2 *Let $(U, u) \in K_{\mathbb{Z}_2}^0(mt) \times K_{\mathbb{Z}_2}^{-1}(S(mt) \times_X S(Et))$ satisfying $\delta_1 u = p^! U$. If there is an $b \in K_{\mathbb{Z}_2}^0(X)$ such that $\psi^p(b) = b$, $p \in \mathbb{N}$, and $\delta u = \pi^!(aU_{Et})$. Then there is an $\alpha \in K_{\mathbb{Z}_2}^{-1}(S(mt))$ such that $\varphi_p(u) = p^!(\alpha)$.*

Now let us turn back to the circumstance in 5.1. Let $s : X \times \mathbb{R}P^{2m-1} \to S(E \otimes \varepsilon(\xi))$ is a section. s gives rise an equivariant section to the sphere bundle $S(mt) \times_X S(Et) \to S(mt)$. For an $\alpha \in K_{\mathbb{Z}_2}^{-1}(S(mt))$ as in 5.2, notice $\alpha = s^!$ $p^!(\alpha) = s^! \varphi_p(u) = \varphi_p(s^! u)$. By 5.2

$$\varphi_p(u) = p^!(\alpha) = p^* \varphi_p(s^! u) = \varphi_p(p^! \beta),$$

where $\beta = s^* u \in K_{\mathbb{Z}_2}^{-1}(S(mt))$. Applying δ_1 to this equality we get $\delta_1 \varphi_p(u) = p^! \delta_1 \varphi_p(\beta) = \varphi_p(\delta_1 u) = p^! \varphi_p(U)$ and so

$$p^!(\varphi_p(U) - \delta_1 \varphi_p(\beta)) = 0.$$

Let A and $B \in K_{\mathbb{Z}_2}^0(X)$ be the unique classes so that $A \cdot U_{mt} = U \in K_{\mathbb{Z}_2}^0(mt)$ and $\delta_1 \beta = B \cdot U_{mt}$. Let $\varphi_p(1) := \rho_p(mt) - \rho_p(Et)$. Notice that $\varphi_p(\theta) = \varphi_p(1)\theta$, $\forall \theta \in K_{\mathbb{Z}_2}^0(X)$. Thus we have

$$\varphi_p(1)(A - B) \cdot U_{mt} \in im\{p^* : K_{\mathbb{Z}_2}^0(mt \times_X Et) \to K_{\mathbb{Z}_2}^0(mt)\}.$$

Note that $B \cdot U_{mt} \in ker\{\pi^* : K_{\mathbb{Z}_2}^0(mt) \to K_{\mathbb{Z}_2}^0(X)\}$.
Inserting the Thom isomorphism, the left below corner of the diagram (**) gives a commutative diagram

$$\begin{array}{ccc} K^0(X) \otimes R(\mathbb{Z}_2) & \xrightarrow{\lambda_{-1}Et} & K^0(X) \otimes R(\mathbb{Z}_2) \\ \downarrow \lambda_{-1}mt & & \downarrow \lambda_{-1}mt \\ K^0(X) \otimes R(\mathbb{Z}_2) & \xrightarrow{\lambda_{-1}Et} & K^0(X) \otimes R(\mathbb{Z}_2) \end{array}$$

where $\lambda_{-1}mt$ and $\lambda_{-1}Et$ are the K-theoretical Euler classes of mt and Et respectively.

This proves

Lemma 5.3 *$\varphi_p(1)(A-B)=0(mod\lambda_{-1}Et)$, $\lambda_{-1}mt(B)=0$. Moreover, B does not depend on the integer $p\in\mathbb{N}$.*

Proof of Theorem 5.1: Let $r=[\frac{n}{2}]$. Notice that the Euler classes $\lambda_{-1}(mt)=(1-t)^m$ and $\lambda_{-1}(Et)=\frac{(1-\varepsilon(\xi)t)^k}{(1-t)^{k-r-1}}$. Set $X=1-\varepsilon(\xi)$ and $T=1-t$. Then $X^2=2X$ and $X^{r+1}=0$. we have therefore

$\lambda_{-1}(Et)=\frac{(T+Xt)^k}{T^{k-r-1}}=\sum_{i=1}^r\begin{pmatrix}k\\i\end{pmatrix}t^iX^iT^{r+1-i}$.

Inserting the relation $t^2=1$, i.e, $T^2=2T$ we obtain that

$\lambda_{-1}(Et)=T^{r+1}+XT^r\sum_{i=1}^r\begin{pmatrix}k\\i\end{pmatrix}$.

Let $\epsilon_{k,r}=\sum_{i=1}^r\begin{pmatrix}k\\i\end{pmatrix}$. Notice that $XT^r\in K(X)\otimes R(\mathbb{Z}_2)$ is of order two since $2XT^r=X^{r+1}T=0$. So we may regard $\epsilon_{k,r}$ as amod 2 number. We claim $\epsilon_{k,r}=0(mod2)$. In fact, by $w(E)=(1+w)^{2(r+1)}=(1+w)^{2k}(modw^{2(r+1)})$ we get that

$$\begin{pmatrix}2k\\2i\end{pmatrix}\equiv\begin{pmatrix}2(r+1)\\2i\end{pmatrix}\text{ if }2i\le 2(r+1)-1.$$

Thus

$$\epsilon_{k,r}\equiv\sum_{i=0}^{r+1}\begin{pmatrix}r+1\\i\end{pmatrix}\equiv 2^{r+1}(mod2)\equiv 0$$

and so $\lambda_{-1}(Et)=T^{r+1}$. Set $A=T^{r-m+1}\in K^0_{\mathbf{Z}_2}(X)$ which corresponds to an $U\in K^0_{\mathbf{Z}_2}(mt)$ under the Thom isomorphism. We claim that there is an $u\in K^{-1}_{\mathbf{Z}_2}(S(mt)\times_X S(Et))$ satisfying 5.2.

To show this, notice that Et is the pullback of a bundle over $\mathbb{R}P^n$ by the classifying map $f:X\to\mathbb{R}P^n$ for the universal cover of X and $\psi^p(e)=e$ if p is odd, $\forall e\in K^0_{\mathbf{Z}_2}(\mathbb{R}P^n)$. We may regard t as a line bundle over $\mathbb{R}P^n$ as well and let $U^0\in K^0_{\mathbf{Z}_2}(mt)$ denote the class corresponds to $T^{r-m+1}\in K^0_{\mathbf{Z}_2}(\mathbb{R}P^n)$. Obviously $U=f^*U^0$. By the commutative diagram (**) where $X=\mathbb{R}P^n$ and $\lambda_{-1}(mt)\cdot A=T^{r+1}\in im(\lambda_{-1}(Et))$ it follows that there exists an $u^0\in K^{-1}_{\mathbf{Z}_2}(S(mt)\times_{\mathbf{R}P^n}S(Et))$ satisfying 5.2. By naturality $u=f^*u^0$ is the desired element.

Therefore, by 5.3, for p odd there exists an $B\in\ker\{T^m:K^0_{\mathbf{Z}_2}(X)\to K^0_{\mathbf{Z}_2}(X)\}$ independent of $p\in\mathbb{N}$ such that

$$\varphi_p(1)(A-B)=0(mod\lambda_{-1}(Et))\qquad(5.4)$$

Notice that $\rho_3(mt)=(1+t+t^2)^m$ and $\rho_3(Et)=\frac{(1+\varepsilon(\xi)t+t^2)^k}{(1+t+t^2)^{k-r-1}}$. In the localized ring $K^0_{\mathbf{Z}_2}(X)\otimes\mathbb{Z}_{(3)}$, one can check readily that

$\rho_3(Et)=\sum\begin{pmatrix}k\\i\end{pmatrix}(-t)^iX^i(3-T)^{r+1-i}$

$=3^{r+1}+\frac{1-3^{r+1}}{2}T+\frac{1-3^k}{2}\{1-\frac{1+3^{k-r-1}}{2}T\}3^{-(k-r-1)}X.$

$\rho_3(mt) = 3^m + \frac{1-3^m}{2}T$
$\varphi_3(1) = (3^m - 3^{r+1}) + \frac{3^{r+1}-3^m}{2}T - \frac{1-3^k}{2}\{1 - \frac{1+3^{k-r-1}}{2}T\}3^{-k+r+1}X.$
This implies that $\varphi_3(1)T = \frac{1-3^k}{2}TX$.
Similarly we have
$\rho_{-1}(mt) = (-1)^m t^m$ and
$\rho_{-1}(Et) = (-1)^{r+1}t^{r+1}\varepsilon(\xi)^k$
Therefore
$\varphi_{-1}(1) = (-1)^m t^m\{1 + (-1)^{m+r}t^{r+m+1}\varepsilon(\xi)^k\}$.
Notice that $Tt = -T$. We obtain $\varphi_{-1}(1)T = T(1-\varepsilon(\xi)^k) = 0$ as $k = 0(mod2)$.

Note that r is odd. $\varphi_{-1}(1) = 0$ and $\varphi_{-1}(1) = -(2-T)$ by $m = 0(mod2)$ and $m = 1(mod2)$ respectively. Therefore
$\varphi_3(1)A = 2^{r-m}\varphi_3(T) = 2^{r-m}\frac{1-3^k}{2}TX$
$\varphi_{-1}(A) = 0.$
By 5.4, there is an
$B \in im\delta K^{-1}_{\mathbb{Z}_2}(S(mt)) \otimes \mathbb{Z}_{(3)} \cong \ker\{T^m : K^0_{\mathbb{Z}_2}(X) \otimes \mathbb{Z}_{(3)} \to K^0_{\mathbb{Z}_2}(X) \otimes \mathbb{Z}_{(3)}\}$
such that
$\varphi_3(1)A \equiv \varphi_3(1)B(modT^{r+1})$
$\varphi_{-1}(1)A \equiv \varphi_{-1}(1)B \equiv 0(modT^{r+1})$.
Let $B = a + bX + (\alpha + \beta X)T$, then

$$\varphi_{-1}(B) = \pm(2-T)(a+bX) \equiv 0(modT^{r+1})$$

if $m = -1(mod2)$ and so $a = 0$, $(2-T)bX \equiv 0(modT^{r+1})$. Thus $bX = 0$. As $B \in \ker T^m$ and so $\alpha T^{m+1} + \beta XT^{m+1} \equiv 0$. This shows that $\alpha = 0$ and $\beta = 2^{r-m}c$ for an $c \in \mathbb{Z}$. Recall

$$\varphi_3(B) = \varphi_3(2^{r-m}cXT) = 2^{r-m+1}c\frac{1-3^k}{2}XT.$$

Thus we have

$$\varphi_3(A) - \varphi_3(B) = 2^{r-m}(1-2c)\frac{1-3^k}{2}XT \equiv 0(modT^{r+1}).$$

This implies then $\nu_2(\frac{1-3^k}{2}) \geq m$.
In other words, if m is odd, then $m \leq \nu_2(k) + 1$ and so $\nu_2(k) \geq m - 1$.
For m even, we consider $m-1$ instead of m and can conclude that $\nu_2(k) \geq m-2$. This proves the theorem.

§ 6. **Proof of Theorem A**

Now we are ready to prove theorem A advertised in §1. Recall that $F_k(k = \pm)$ are the focal manifolds of an isoparametric hypersurface M in $\mathbb{R}P^{2n+1}$ and $\mathbb{E}_i$ for $1 \leq i \leq 4$ are the focal distributions as §2. Let γ_k is the normal bundle of F_k in $\mathbb{R}P^{2n+1}$ whose codimension is $m_k + 1$. Again $p_k : M \to F_k$ stands for the projection. Let $F_k^{(n)}$ be the n-skeleton of a triangulation of F_k. By 4.1, 4.2, 4.3 and 4.5 the image of $[\gamma_+] \in \widetilde{KO}(F_+)$ at $\widetilde{KO}(F_+^{(n)})$ must be kx for some $k \in \mathbb{N}$ since the terms other than $\mathbb{Z}_{2^{\phi(n)}}$ have very high filtrations(exist only if $m_- = 3, 5(mod8)$).

By [Ab], the total Stiefel-whitney class of γ_+ is $(1+w)^{n+1}$. Thus $k = 0(mod 2)$ and $k \geq n+1$ as $n = -1(mod 8)$ as we assumed at the very beginning. Note that $\mathbb{E}_1 \oplus 1 = p_+^*(\gamma_+)$. Let $X := M^{(n)}$ be the n-skeleton. Thus $k\xi \to X$ has at least $(k-n-1)+(m_-+1)$ linearly independent sections by dim
$E_1 = m_+$. Applying 5.1 we obtain $\nu_2(k) \geq [\frac{m_-+1}{2}]$ if $[\frac{m_-+1}{2}] = 1(\text{mod } 2)$ and $\nu_2(k) \geq [\frac{m_-+1}{2}] - 1$ if $[\frac{m_-+1}{2}] = 0(\text{mod } 2)$.

Now let us consider γ_-. If $m_- = 7(mod 8)$, by 4.1 there are integers $l, a, b \in \mathbb{N}$ so that $\gamma_- = lx + ay + byx \in \widetilde{KO}(F_-)$. Recall that $p_-^*(\gamma_-) = \mathbb{E}_2 \oplus 1$ and $T_2^*\mathbb{E}_2 = \mathbb{E}_4$, $T_1^*\mathbb{E}_1 = \mathbb{E}_3$, where T_i is as §2. Now $[\mathbb{E}_1] = kx$ and so $T_1^*([\mathbb{E}_1]) = kx$. Thus $E_1 \oplus E_2 \oplus T_1^*E_1 \oplus T_2^!E_2 \cong TM$ gives an equation

$$2kx + [\mathbb{E}_2] + T_2^*[\mathbb{E}_2] = 2(n+1)x \in \widetilde{KO}(M).$$

Assume that $T_2^*(y) = \delta y + ix + jxy$. Obviously the coefficient δ must be ± 1. Inserting into the equation above we have

$$\{2(k+l)+(a+2b)i\}x + a(1+\delta)y + \{b(1+\delta)+(a+2b)j\}xy = 2(n+1)x \in \widetilde{KO}(M) \quad (6.1)$$

In particular, $a(1+\delta) = 0$ and $b(1+\delta)+(a+2b)j$ is divisible by $2^{\phi(n)}$. By Quillen[Qu] the Stiefel-Whitney classes $w_s(y) = 0$ if $1 \leq s < 2^{[\frac{m_+}{2}]-2}$ and so $w_s(T_2^*(y)) = 0$. Recall that $m_+ \geq m_- \geq 9$ and $m_+ = 0(mod 8)$. Thus $\dim F_- = 2m_+ + m_- < 2^{[\frac{m_+}{2}]-2}$ and so $w(y) = 1$. One can check then readily $w(xy) = 1$. Thus $w(ix) = 1$. This implies that i is divisible by $2^{i_0} \geq n+1$, where i_0 is the minimal number satisfying this inequality. Similarly, by $w(\gamma_-) = 1$ we obtain that l is divisible by $2^{i_0} \geq n+1$ too. The equation (6.1) shows also that

$$2(k+l) + (a+2b)i = 2(n+1)(mod 2^{\phi(n)}).$$

Thus $2(n+1)$ is divisible by $min\{2^{[\frac{m_-+1}{2}]}, 2^{i_0}\}$. This implies that either $n+1 = 2^{i_0}$ or $n+1$ is divisible by $2^{[\frac{m_--1}{2}]}$. Notice that $\delta(m_-) = 2^{[\frac{m_--1}{2}]}$. This proves the case of $m_- = 7(\text{mod } 8)$.

The proof of other cases are similar, except one should note that, in case of $m_- = 3, 5(mod 8)$, $T_1^*[\mathbb{E}_1] + [\mathbb{E}_1]$ is also equal to $2kx$ by corollary 4.4 and 4.6. This completes the proof.♣

§7. Equifocal Hypersurfaces in Rank one Symmetric Spaces

In recent years, various generalized concept of isoparametric hypersurfaces in ambient spaces other than space forms have been greatly studied. Among them perhaps the equifocal hypersurfaces in symmetric spaces are the most natural one. An important property for the hypersurface is that every geodesic in the ambient space normal to the hypersurface is closed of constant length l and containing $2g$ focal points. The number g is a natural generalization of the number of distinct principal curvatures. By using Hopf fibrations $S^{2n+1} \to \mathbb{C}P^n$ and $S^{4n+3} \to \mathbb{H}P^n$ we can lift an equifocal hypersurface to an isoparametric hypersurface in the sphere and the length of normal geodesic is $2l$. Using Münzner's remarkable theorem it follows that the number g must be among $\{1, 2, 3\}$ if the ambient space is $\mathbb{C}P^n$ and $\mathbb{H}P^n$. The multiplicities of the hypersurface are the same as those of the isoparametric hypersurface in the sphere and therefore our theorem A can be applied to settle the problem on the multiplicities of isoparametric hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{H}P^n$. For equifocal hypersurface in the Cayley plane, we need some extra argument to conclude the multiplicities. Through some work with G.Thorbergsson we actually know that g must be either 1 or 2 in this case.

First let us give some examples of equifocal hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{H}P^n$ along the line of [FKM].

Proposition 7.1 *(i): If $m_- = 2, 3, 4, 5(mod8)$ or $6(mod8)$, $m_- \geq 1$. Then for any $k \geq 1$ so that $m_+ = k\delta(m_-) - m_- - 1 \geq 1$, there exists an equifocal hypersurface in $\mathbb{C}P^{m_-+m_+}$ with $g = 2$ and multiplicities $m_-, m_+; m_-, m_+$.*
(ii): If $m_- = 0, \pm1(mod8)$ and $m_- \geq 1$. Then for an even $2k \geq 1$ and $m_+ = 2k\delta(m_-) - m_- - 1 \geq 1$, there exists an equifocal hypersurface in $\mathbb{C}P^{m_-+m_+}$ with $g = 2$ and multiplicities $m_-, m_+; m_-, m_+$.
(iii): If $m_- = 3, 4, 5(mod8)$, $m_- \geq 1$. Then for any $k \geq 1$ so that $m_+ = k\delta(m_-) - m_- - 1 \geq 1$, there exists an equifocal hypersurface in $\mathbb{H}P^{\frac{m_-+m_+-1}{2}}$ with $g = 2$ and multiplicities $m_-, m_+; m_-, m_+$.
(iv): If $m_- = \pm2(mod8)$, $m_- \geq 1$. Then for any $k \geq 1$ so that $m_+ = 2k\delta(m_-) - m_- - 1 \geq 1$, there exists an equifocal hypersurface in $\mathbb{H}P^{\frac{m_-+m_+-1}{2}}$ with $g = 2$ and multiplicities $m_-, m_+; m_-, m_+$.
(v): If $m_- = 0, \pm1(mod8)$, $m_- \geq 1$. Then for any $k \geq 1$ so that $m_+ = 4k\delta(m_-) - m_- - 1 \geq 1$, there exists an equifocal hypersurface in $\mathbb{H}P^{\frac{m_-+m_+-1}{2}}$ with $g = 2$ and multiplicities $m_-, m_+; m_-, m_+$.

Proof: Notice the relations among the Clifford algebras $Cl^{0,m+8k} = Cl^{0,m} \otimes \mathbb{R}(2^{4k})$ for $k, m \geq 1$ and

m	1	2	3	4	5	6	7	8
$Cl^{0,m}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$

Thus, if $m = 8i+2$, $8i+3$, $8i+4$, $8i+5$, $8i+6$, $Cl^{0,m+1}$ have irreducible complex representations of $\mathbb{C}$-dimensions 2^{4i+1}, 2^{4i+2}, 2^{4i+2}, 2^{4i+3} and 2^{4i+3} respectively. Let $e_0, \cdots, e_m$ denote an orthogonal basis of $(\mathbb{R}^{m+1}, -x_0^2 - \cdots - x_m^2)$ and $ke_0, \cdots, ke_m$ gives $m+1$ complex matrixes $P_0, \cdots, P_m \in U(n)$. We define an isoparametric function

$$F(x) = \langle x, x\rangle^2 - 2\sum_{i=0}^m \langle P_i(x), x\rangle^2,\ x \in \mathbb{C}^n.$$

The restriction $F|_{S^{2n-1}} : S^{2n-1} \to [-1, 1]$ satisfies the Cartan-Münzner equations and so for each regular value $c \in (-1,1)$, the hypersurface $F^{-1}(c)$ is isoparametric[FKM]. Notice that $F(e^{i\theta}x) = F(x)$ and so $F^{-1}(c)$ is invariant under the standard circle action on the sphere. This gives an equifocal hypersurface in $\mathbb{C}P^{n-1}$. The proof of (i) follows.

To show (ii), we need only to complexify the standard irreducible representaions. This produces the extra factor 2 in (ii). The others are similar. This completes the proof. ♣

Recall an equifocal hypersurface with multiplicities m_1, m_2 also must be a S^{m_1}-bundle over a focal manifold M_1 as well as a S^{m_2}-bundle over another focal manifold M_2. The following result settle the multiplicities problem for euifocal hypersurface in Cayley plane separately since we can not reduce this case to the similar problem in sphere via the Hopf fibration.

Theorem 7.2. *Let $M \subset \mathbb{Q}P(2)$ be an equifocal hypersurface with multiplicities $m_1 \le m_2$. Suppose that all of the focal manifolds are orientable. Then either $(m_1, m_2) = (7, 15)$, M is diffeomorphic to S^{15}, and the number of focal points on each normal geodesic is equal to 2 or $(m_1, m_2) = (4, 7)$, M is a S^4-bundle over homotopy 11-sphere as well as a S^7-bundle over $\mathbb{H}P^2$.*

Proof: Let S, S_1 and S_2 denote the sequences $\{m_1, m_2, u = m_1 + m_2, u + m_1, u + m_2, 2u, \cdots\}$, $\{m_1, u, u + m_1, 2u, \cdots\}$ and $\{m_2, u, u + m_2, 2u, \cdots\}$. M_1 and M_2 will denote the two corresponding focal manifolds. The coefficients of every homology or cohomology groups will be the rational. The proof is divided into the following steps.

Step I. $7, 15 \in S$.
Let $P = P(\mathbb{Q}P(2), M \times p)$ denote the path space consists of all paths from p to M in $\mathbb{Q}P(2)$, p is not a focal point. Consider the Leray-Serre cohomology spectral sequence (LSSS) of the fibration $\Omega\mathbb{Q}P(2) \to P \to M$. Notice that the rational cohomology groups of $\Omega\mathbb{Q}P(2)$ occur only in the dimensions $\{0, 7, 22, \cdots\}$. By [T-T] the reduced cohomology groups of the path space P occur nontrivially only in the dimensions in the sequence S. The argument is to compare this fact with the spectral sequence.

If the differential $d_8 : E_8^{0,7} \to E_8^{8,0}$ vanishes, then the term $E_8^{0,7}$ survives in the ∞-term and so $7 \in S$. Also the term $E_8^{15,0} \cong H^{15}(M)$ survives and therefore $15 \in S$. If d_8 is nontrivial, then this means that $E_8^{8,0}$ is nonzero. By Poincaré duality, $H^7(M)$ is nontrivial and it survives in the ∞-term. Thus $7 \in S$ and $E_8^{8,7} \cong H^8(M)$ is of rank at least 1. The differential $d_8 : E_8^{8,7} \to E_8^{16,0} = 0$ and so $E_\infty^{8,7} \cong E_8^{8,7}$. Thus $15 \in S$. This completes the step I.

By step I above it follows that $m_1 \ne m_2$.

Step II. $2 \le m_1 \le m_2 \le 6$ can not happen.
Consider the fibration $\Omega\mathbb{Q}P(2) \to P_1 = P(\mathbb{Q}P(2), M_1 \times p) \to M_1$ and its Leray

Serre spectral sequence. Note $\dim M_1 = 15 - m_1 \leq 13$. The LSSS argument shows easily that $H^q(M) \cong \mathbb{Q}$ if $q = 0$ or $q \leq 6$ and $q \in S_2$.

In particular, $E_8^{m_2,7} \to E_8^{m_2+8,0} \cong H^{7-u}(M_1) = 0$ if $7 - u \neq 0$ since $7 - u \leq 2 < m_2$. Either $E_8^{m_2,7}$ survives and so $m_2 + 7 \in S_2$ or $u = 7$. In either cases, note that $H^8(M_1) \cong H^{7-m_1}(M_1) \cong \mathbb{Q}$. Thus $E_8^{8,7} \cong \mathbb{Q}$ which survives in the ∞-term. This shows that $15 \in S_2$.

If $m_2 + 7 \in S_2$ and $u \neq 7$, then $u + m_1 = 7$. Comparing with the relations $15 \in S_2$(i.e, $15 = ku$ or $ku + m_2$,) we conclude that either $u|15$ or $22 = (k+2)u$. Therefore $u = 5, (m_1, m_2) = (2, 3)$ or $u = 11, (m_1, m_2) = (5, 6)$.

The cases $u = 7$ and $15 = ku$ or $ku + m_2$ are impossible.

For the two possibilities above, the dimensions of the focal manifolds are $12, 13$ and $9, 10$. In the first case, the Euler numbers $\chi(M) = 0$, $\chi(M_1) = 0$. $H^6(M_2) = 0$ by an analogous spectral sequence for the path space $P_2 = P(\mathbb{Q}P(2), M_2 \times p)$. Thus $\chi(M_2)$ is even since $\dim M_2 = 12$. In the second case, $\chi(M) = 0$, $\chi(M_2) = 0$ and $\chi(M_1)$ is even since the intersection form of M_1 is skew symmetric by $dim M_1 = 10$. Note $\chi(\mathbb{Q}P(2)) = 3$. This contradicts with the following identity of the Euler numbers

$$\chi(\mathbb{Q}P(2)) - \chi(M_1) - \chi(M_2) + \chi(M) = 0.$$

This completes the step II.

Step III. $2 \leq m_1 \leq 6$ and $m_2 \geq 6$ implies $(m_1, m_2) = (4, 7)$.
Notice $\dim M_1 = 15 - m_1$ and $H^*(M_1) = 0$ if $* \neq 0$ and ≤ 6 by the spectral sequence as before. By the duality it follows easily that M_1 has the rational homotopy type of S^{15-m_1}(In fact it is a homotopy sphere). M_2 is a manifold of dimension $15 - m_2 \leq 8$. For $q \leq 6$, $H^q(M_2) = \mathbb{Q}$ only if $q = 0$ or $q = m_1$. By duality it follows that M_2 is either homological equivalent to $\mathbb{C}P(2)$ or $\mathbb{H}P(2)$. These implies $m_1 = 2$ or 4 and $m_2 = 11$ or 7 respectively. Notice that the hypersurface M are sphere bundles over the two focal manifolds. In the former case, M is a S^{11}-bundle over $\mathbb{C}P(2)$ as well as a S^2-bundle over S^{13}. Obvious it is impossible since they have different homology groups. In the latter case, M is a S^7-bundle over a homological $\mathbb{H}P(2)$ as well as a S^4-bundle over a homotopy 11-sphere. It is easy to check the focal manifold with the homology of $\mathbb{H}P^2$ is actually diffeomorphic to $\mathbb{H}P^2$. The proof for step III is completed.

Combining the steps I, II and III, if $m_1 > 1$, then either $(m_1, m_2) = (4, 7)$ or $m_2 > m_1 \geq 7$ and thus $m_1 = 7$ by step I. Furthermore, the latter case implies that either $m_2 = 8$ or 15. If $m_2 = 8$, the Euler number $\chi(M_1) = 2$ and $\chi(M) = \chi(M_2) = 0$. This contradicts with the Euler number formula as above. Thus $(m_1, m_2) = (7, 15)$ if we can exclude the cases of $m_1 = 1$.

Step IV. $m_1 \neq 1$.
Suppose not, $\dim M_1 = 14$ and $\dim M_2 = 15 - m_2 \leq 14$. The formula $3 = \chi(\mathbb{Q}P(2)) = \chi(M_1) + \chi(M_2)$ shows that m_2 must be odd since $\chi(M_1)$ must be

even for the intersection form of M_1 is even type. There are only four cases, $(1,1)$, $(1,3)$, $(1,5)$ or $(1,7)$ by step I. $(1,1)$ and $(1,5)$ are impossible since otherwise, the dimension of M_2 is either 14 or 10, the Euler number of M_2 is therefore even for the same reasoning. To exclude the cases $(1,3)$ and $(1,7)$, notice that the middle dimensional homology group of M_2 is zero in either cases. Thus the Euler number is even again. This contradicts with the above identity again. This completes Step IV.

Combining these steps it follows that $(m_1, m_2) = (7,15)$ or $(4,7)$ as we mentioned above and consequently $M = S^{15}$ or a sphere bundle over $\mathbb{H}P^2$. Notice that, in the former case, there is a free action of the dihedral group $\mathbb{D}_{2g}$ on M. By Milnor[Mi] it is possible only if $g = 1$ and then $\mathbb{D}_{2g} = \mathbb{Z}_2$, where $2g$ is the number of focal points on normal geodesic.

This completes the proof. ♣

Remark One can generalize the concept of equifocal hypersurface in symmetric spaces to hypersurfaces in Blaschke manifolds. The same result presented above holds for hypersurfaces in Blaschke manifolds modeled on $\mathbb{Q}P^2$.

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