Geometry of free products, cycloidal groups and polynominal maps

Ravi S. Kulkarni

Department of Mathematics
Indiana University

Bloomington, Indiana 47405
U S A

Max-Planck-Institut für
Mathematik
Gottfried-Claren-Straße 26

D - 5300 Bonn 3

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by

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§ 1 Introduction. (1.1) Let $\Gamma = \ast_{i\in I}\Gamma_i$ be a free product of groups and Φ a subgroup of Γ. By Kurosh's theorem $\Phi = F * \ast_{i\in I}(\ast_{j\in J_i}\Phi_{ij})$ where F is a free subgroup and Φ_{ij}'s are conjugate to subgroups of Γ_i, $i \in I$, $j \in J_i$. The rank of F is a well-defined invariant of Φ as a subgroup of Γ. We shall say that Φ is a C_r-subgroup if the rank of F is r.

(1.2) The notion of a C_o-subgroup generalizes the notion of a cycloidal subgroup of genus 0 introduced by Petersson in the context of the modular group $(\approx \mathbb{Z}_2 * \mathbb{Z}_3)$, cf. [10], [11]. In the more general context of fuchsian groups we shall introduce the notions of a cycloidal group and its cycloidal and t-cycloidal subgroups. First, by abuse of the classical terminology, let us mean by a fuchsian group any properly discontinuous, orientation-preserving group of homeomorphisms of $\mathbb{R}^2$. (These are precisely the discrete groups of isometries of the Euclidean plane or the hyperbolic plane. The ones acting on the hyperbolic plane are classically known as fuchsian groups, cf. [2], [15].) If Γ is a fuchsian group then $\Gamma\backslash\mathbb{R}^2$ is a surface and the orbit map $\mathbb{R}^2 \longrightarrow \Gamma\backslash\mathbb{R}^2$ is a branched covering. We call Γ a cycloidal group if $\Gamma\backslash\mathbb{R}^2 \approx \mathbb{R}^2$. It is not difficult to see that a cycloidal group is a countable product of finite cyclic groups, and conversely any such product can be realized as a cycloidal group uniquely upto topological equivalence.

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This implies in particular that for a subgroup of finite index in a free product of finitely many finite cyclic groups the rank of its free part decomposes into finer intrinsic invariants arising from the genus and the number of ends of the corresponding branched covering surface.

(1.3) A subgroup of a cycloidal group is called a cycloidal subgroup if it is cycloidal in its own right as a fuchsian group. One easily sees, cf. §2, that a subgroup Φ of a cycloidal group Γ is cycloidal $\Leftrightarrow$ Φ is generated by elements of finite order $\Leftrightarrow$ Φ is a C_o-subgroup of Γ in the sense of (1.1).

(1.4) Let Γ be a finitely generated cycloidal group and Φ a subgroup of Γ of finite index. Then $\Phi\backslash\mathbb{R}^2$ is an orientable surface of finite type - say, with t ends and genus g. Then we call Φ a t-cycloidal subgroup* of Γ. Clearly a t-cycloidal subgroup is cycloidal iff $t=1$ and $g=0$.

(1.5) The connection of cycloidal groups with polynomial maps arises as follows. Let $p: \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial map. Then $\delta = \{z \in \mathbb{C} \mid p'(z) = 0\}$ is called its singular set and $\beta = p(\delta)$ is called its branch set. Let $\beta = \{x_1,\dots,x_n\}$. For each $y_{ij} \in p^{-1}(x_i)$ let d_{ij} be the order of the zero of $p(z) - x_i = 0$, at y_{ij}. Let $m_i = \ell.c.m._j\{d_{ij}\}$, and $m_{ij} = m_i/d_{ij}$.

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* In this terminology, a cycloidal subgroup in the sense of Petersson [10] is precisely a 1-cycloidal subgroup of the modular group.

Then by the theory of universal branched coverings one gets an embedding of $\ast_{i,j}\mathbb{Z}_{m_{ij}}$ in $\ast_i\mathbb{Z}_{m_i}$ of index equal to the degree of p.

Conversely let Γ be a finitely generated cycloidal group and Φ a cycloidal subgroup of finite index d in Γ. Then $\Phi\backslash\mathbb{R}^2 \xrightarrow{p} \Gamma\backslash\mathbb{R}^2$ is a d-sheeted branched covering. Identify $\Gamma\backslash\mathbb{R}^2 \approx \mathbb{R}^2$ with $\mathbb{C}$ in some way, and pull back the complex structure on $\Phi\backslash\mathbb{R}^2$ via p. From the fact that p is a proper map it follows from elementary complex analysis that $\Phi\backslash\mathbb{R}^2$ is biholomorphic to $\mathbb{C}$ and p is a polynomial map of degree d.

More generally, proper holomorphic maps among Riemann surfaces are connected with the finite-index inclusions among fuchsian groups in the above manner. For example, a natural generalization of a polynomial map is a meromorphic function on a closed Riemann surface with a single pole. Such functions were especially studied by Weierstrass at the end of the 19th century. These precisely correspond to 1-cycloidal subgroups of finite index in finitely generated cycloidal groups. Also arbitrary meromorphic functions on a closed Riemann surface correspond to t-cycloidal subgroups, $t = 1,2,..$ of finite index in finitely generated cycloidal groups.

It may be remarked that conversely the finite-index inclusions among fuchsian groups lead to a classification of the topological types of proper holomorphic maps among Riemann surfaces.

(1.6) The interconnections noted above suggest that the combinatorial theory of free products would have implications in fuchsian groups and thereon in the topological genesis of the global aspects of the classical theory of a complex variable, and conversely the latter discipline would render some "geometric" insight in the former. For example, Kurosh's theorem provides an elementary viewpoint on fuchsian groups which appears to be absent in the standard literature on this subject, and conversely the group-theoretic investigations on the modular group substantially extend to large classes of free products of groups. And indeed the rank of the free part of a subgroup of a free product has a close analogy with the genus of a branched covering surface. The number of ends of a covering surface, however, reflect additional arithmetic properties which are not apriori evident.

Now we briefly describe the results in this paper.

(1.7) <u>A realization theorem</u>. Let $\Gamma = \overset{*}{\Pi}_{i=1}^{k} \mathbb{Z}_{n_i}$ and $\Phi = F_{2g} * \overset{*}{\Pi}_{u=1}^{L} \mathbb{Z}_{m_u}$. Then Φ can be realized as a 1-cycloidal subgroup of Γ iff it admits a finite-index embedding in Γ.

For the conditions on n_i's, m_u's and g ensuring a finite-index embedding of Φ in Γ see (3.2). If $g = 0$, such an embedding corresponds to a polynomial map and from this viewpoint an equivalent result is due to Thom [13]. However when $g = 0$, <u>any</u> finite-index inclusion of Φ in Γ is automatically 1-cycloidal so this result already follows from theorem 1 of [6]. In case Γ

is the modular group, the above theorem, stated in a different language, is due to Millington [8]. The full theorem would also follow from theorem 2 of [6] and (5.2) of [1]. The proof presented here is different from the Thom's indicated proof in [13] for the special case $g = 0$, or the algebraic methods of [1] or [8]. Actually it is quite elementary. We simply do the "1-dimensional" construction in our proof of the extension of Kurosh's theorem in [6] more carefully so that its "2-dimensional" thickening, cf. [6], (A1.1) is a compact orientable surface with one boundary component. The same method may be used to give geometric proofs of some of the algebraic results in [1].

(1.8) Extensions of some work of Rosenberger and Zieschang

In [12], [14] these authors have studied free products of cyclic groups by the Nielsen method. In [12], (cf. also [14] prop. 4.4) Rosenberger obtains a nice criterion for a subgroup Φ of $\Gamma = *_{i=1}^{n} \mathbb{Z}_{m_i}$ to be of finite index when m_i's are primes and Φ is generated by elements of finite order. He asks if the primality condition may be dropped. When Φ is interpreted as a cycloidal subgroup one can recover Rosenberger's criterion, cf. (4.1), from the thickened diagram for Φ, and this does not use the primality condition. In [14], theorem 4.6 Zieschang proves that there are infinitely many subgroups of finite index in $\Gamma = *_{i=1}^{n} \mathbb{Z}_{m_i}$, $m_i \geq 2$, $n \geq 2$ which are generated by elements of finite order. For the modular group, in a different language, this result is due to Petersson [10]. These results are special cases on C_r-subgroups of essentially arbitrary free products, cf. (4.2).

(1.9) Normal subgroups A normal C_r-subgroup of infinite index can exist only when $r = 0$ or ∞. Normal C_o-subgroups have a simple description, cf. (5.2), which implies that in $\Gamma = \ast_{i=1}^n \Gamma_i$, $|\Gamma_i| < \infty$, there are only finitely many normal C_o-subgroups. For $2 \leq r < \infty$ and Γ_i's finitely generated, $\Gamma = \ast_{i=1}^n \Gamma_i$ admits only finitely many normal C_r-subgroups. Moreover the quotients by normal C_r-subgroups, $2 \leq r < \infty$, in any free product have a faithful integral representation of degree r — this shows that there are only finitely many isomorphism-types of these quotients, cf. (5.8). (There is a clear analogy between these results and the famous results of Schwarz and Hurwitz on the finiteness of the automorphism group of a closed Riemann surface of genus ≥ 2.) As for $r = 1$, an arbitrary free product Γ admits infinitely many normal C_1-subgroups iff there is a surjective morphism, cf. (2.4), of Γ onto $\mathbb{Z}_2 * \mathbb{Z}_2$.

The case of t-cycloidal normal subgroups of finitely generated cycloidal groups has deeper arithmetic aspects. The number of 1-cycloidal normal subgroups of $\ast_{i=1}^n \mathbb{Z}_{m_i}$ is finite ; if m_i's are pairwise coprime this number is $\Pi_{i=1}^n d(m_i)$ where $d(k)$ denotes the number of divisors of a natural number k. The latter statement generalizes Newman's theorem 3 in [9] for the modular group. A much deeper Newman - Greenberg theorem, cf. [9], [3], asserts that the number of t-cycloidal normal subgroups of the modular group is finite for all t. This theorem extends verbatim to $\ast_{i=1}^n \mathbb{Z}_{m_i}$ if m_i's are pairwise coprime, cf. (6.3); and if m_i's are not pairwise coprime and $\Gamma \neq \mathbb{Z}_2 * \mathbb{Z}_2$ there exist infinitely many values of t, for each of which there exist infinitely many normal t-cycloidal subgroups, cf. (6.4).

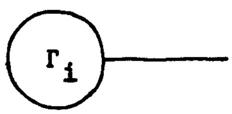
(1.10) Some of the "1-dimensional" considerations in this paper extend to graph-amalgamated products as well. But such extensions are seldom verbatim and they also present some new phenomena. We shall take them up in a later publication.

§ 2 Preliminaries

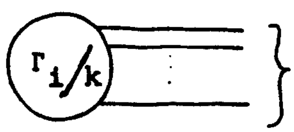
(2.1) Let $\Gamma = *_{i\in I}\Gamma_i$ with $|\Gamma_i| < \infty$ and $\Phi \leq \Gamma$. Then Φ is a C_o-subgroup of Γ iff it is a generated by elements of finite order.

This is clear from the Kurosh's theorem, since the free part of Φ is also its homomorphic image.

(2.2) Diagrams (cf. [6], § 4). Let $\Gamma = *_{i\in I}\Gamma_i$ and (X_i, x_i) a connected CW-complex with $\pi_1(X_i, x_i) \approx \Gamma_i$. Attach a copy I_i of the unit interval [o,1] to X_i by identifying o with x_i, thus obtaining a space $\bar{X}_i$. Then $I_i \subseteq \bar{X}_i$ is called the edge of $\bar{X}_i$. Let X be the space obtained from $\cup_{i\in I}\bar{X}_i$ by identifying the end-points 1's of all the edges to a single point * which serves as the base-point of X. Equip X with the CW-topology w.r.t.$\bar{X}_i$'s i.e. a subset A in X is closed iff $A \cap \bar{X}_i$ is closed for all $i \in I$. Then $\pi_1(X_i\ ) \approx \Gamma$. The diagram corresponding to $\bar{X}_i$ is

(2.2.1) 

It is also denoted by X_{Γ_i}. The diagram for a k-fold connected cover of $\bar{X}_i$ is

(2.2.2) 

k edges.

A diagram for X is obtained from those in (2.2.1) by identifying the end-points 1's which becomes a base-point again denoted by *. A diagram for a d-fold connected cover of X is obtained from the various building blocks shown in (2.2.2) by appropriately identifying the end-points of the edges. We shall denote the diagram for X by X_Γ and the diagram corresponding to a connected cover of X corresponding to a subgroup Φ of Γ by X_Φ. We have a canonical projection $p: X_\Phi \longrightarrow X_\Gamma$. Although this is only a symbolic representation of a covering space, the consideration of lifting paths from X_Γ to X_Φ are exactly the same as in the corresponding actual covering spaces. Let Y_Φ be the space obtained from X_Φ by pinching each of the circles to apoint. Then Y_Φ may be considered as a graph. Its vertices are $p^{-1}(*)$ together with the images of the circles in X_Φ, and its edges are just the images of the edges in X_Φ. In X_Φ as well as Y_Φ we fix a base-point, denoted by $\tilde{*}$, lying over $*$. An application of the Van Kampen theorem to X_Φ yields a proof of Kurosh's theorem, and $F \cong \pi_1(Y_\Phi,\tilde{*})$. So

(2.3) Φ is a $C_\circ$-subgroup iff Y_Φ is a tree.

(2.4) Let $\varphi_i: \Gamma_i \longrightarrow G_i$ be homomorphisms, $i \in I$. Then they induce $\varphi: \Gamma \overset{\text{def}}{=} \#_{i\in I}\Gamma_i \longrightarrow G \overset{\text{def}}{=} \#_{i\in I}G_i$. We call such φ or $\{\varphi_i\}_{i\in I}$ a <u>morphism</u> of free products. For $H \leq G$ we have canonical maps $X_{\varphi^{-1}(H)} \longrightarrow X_H$. If each φ_i is surjective it is clear from the second isomorphism theorem for groups that the induced maps on the diagrams $X_{\varphi^{-1}(H)} \longrightarrow X_H$ are "isomorphisms" of the diagrams in the obvious sense.

(2.5) <u>Thickened diagrams</u> (cf. [6], (A1.1))
Let $\Gamma = \#_{i\in I}\mathbb{Z}_{m_i}$ be a countable free product of finite cyclic groups. If $|I| = \infty$ we may take $I = \mathbb{Z}$. in this case we slightly change the definition of X_Γ as follows: we take X_Γ as the space obtained from $\mathbb{R} \cup \bigcup_{i\in\mathbb{Z}} \bar{X}_{\Gamma_i}$ by identifying the end-point of the edge of $\bar{X}_{\Gamma_i}$ to $i \in \mathbb{R}$.

The purpose of this change is that we now always have a proper 1-1 continuous map of X_Γ into $\mathbb{R}^2$. Such a map is a homeomorphism of X_Γ onto its image. Now fill in each circle in X_Γ by the disk it bounds in $\mathbb{R}^2$, and "thicken" each edge and the copy of $\mathbb{R}$, cf. [6], (A1.1), thus obtaining a thickened diagram $\mathbb{X}_\Gamma$. Then $\mathbb{X}_\Gamma$ is a surface with boundary. It is compact when $|I| < \infty$, and in any case $\text{int}\, \mathbb{X}_\Gamma \approx \mathbb{R}^2$. Int $\mathbb{X}_\Gamma$ may be taken as the base of the universal branched cover $\mathbb{R}^2 \longrightarrow \Gamma\backslash\mathbb{R}^2$, thus realizing Γ as a <u>cycloidal</u> group.

(2.6) Let Γ be as in (2.5) and $\Phi \leq \Gamma$. Then thickening X_Φ to $\mathbb{X}_\Phi$ appropriately we obtain an orientable surface with boundary. Also $p: X_\Phi \longrightarrow X_\Gamma$ extends to a projection, again shown by $p: \mathbb{X}_\Phi \longrightarrow \mathbb{X}_\Gamma$. The free part of Φ given by the Kurosh's theorem is generated by the handles and ends of int $\mathbb{X}_\Phi$. So Φ is generated by elements of finite order iff int $\mathbb{X}_\Phi$ is of genus o and has only one end, i.e. int $\mathbb{X}_\Phi \approx \mathbb{R}^2$, i.e Φ is a cycloidal subgroup. Thus, in view of (2.1), we get:

(2.7) Let Γ be a cycloidal group and $\Phi \leq \Gamma$. Then Φ <u>is a cycloidal subgroup of Γ iff it is a C_o-subgroup of Γ</u>.

§ 3 A proof of the realization theorem (1.7)

(3.1) In case $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$ it is easy to estabtish the theorem. We shall henceforth assume $\Gamma \neq \mathbb{Z}_2 * \mathbb{Z}_2$.

(3.2) We recall for Γ, Φ as in (1.7) a set of necessary and sufficient conditions for the existence of finite-index embedding of Φ in Γ, compare [6] theorem 2.

α) Each m_u divides some n_i.

β) $\{\Sigma_{u=1}^{L} \frac{1}{m_u} - L - 2g+1\}/\{\Sigma_{i=1}^{k} \frac{1}{n_i} - k+1\} = d$

is a positive integer

γ) Let $m_o = 1$. By reindexiag if necessary assume that $m_1,..m_\ell$ is a maximal set of distinct m_u's and each m_q occurs b_q times, $q = 1,2...\ell$. Let

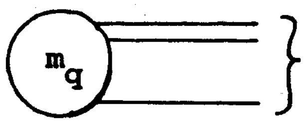
$$\delta_{iq} = n_i/m_q \qquad 1 \le i \le k\ , \quad o \le q \le \ell$$

Then the diophantine system

$$\text{(3.2.1)} \qquad \begin{aligned} \Sigma_{i=1}^{k} x_{iq} &= b_q\ , \qquad q = 1,2...\ell \\ x_{iq} &= 0 \quad \text{if} \quad m_q \nmid n_i \\ \Sigma_{q=o}^{\ell} \delta_{iq} x_{iq} &= d\ , \quad i = 1,2...k \end{aligned}$$

has a solution for x_{iq}'s in non-negative integers.

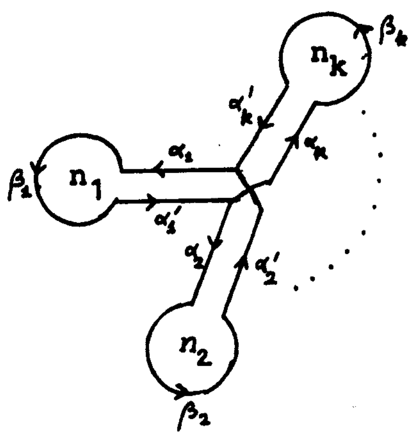
(3.3) Let $x_{iq} = x_{iq}^{o}$ be a solution of (3.2.1) in non-negative integers. Then it is possible to construct a <u>connected</u> diagram from x^o_{iq} copies of

(3.3.1) 

n_i/m_q edges ,

$i = 1,2...k$, $q = 0,1...\ell$ "covering" X_Γ, and any such diagram corresponds to a subgroup of Γ of index d , and $\simeq \Phi$, cf. the proof of theorem 2 of [6]. We shall now do this construction more carefully so that the thickened diagram would be a compact orientable surface with precisely one boundary component.

(3.4) It would be convenient to call the thickening of an edge as an <u>arm</u> - so the thickening of the diagram in (3.3.1) is <u>a disk with arms</u> and we call it <u>a block of level i</u>. Note that $\mathbb{X}_\Gamma \cong$ a closed disk. Fix an orientation on $\mathbb{X}_\Gamma$ and $\partial\mathbb{X}_\Gamma$. This induces orientations in the diagrams in (3.3.1), their thickenings and on the boundaries of these thickenings. The i-th arm of $\mathbb{X}_\Gamma$ has two oriented boundary arcs α_i, α'_i lying in $\partial\mathbb{X}_\Gamma$ which are connected by an oriented circular arc β_i cf. (3.4.1).

(3.4.1) 

So

(3.4.2) $$\partial \mathbb{X}_\Gamma = \alpha_1 * \beta_1 * \alpha_1' * \alpha_2 * \ldots * \alpha_k' .$$

Also the i-th arm meets the i+1 - st arm (i counted mod k) along an arc as shown in (3.4.1).

(3.5) Let $r_i = \Sigma_{q=o}^{\ell} x_{iq}^{o}$. This is the number of blocks of level i.

(3.6) Now we start constructing $\mathbb{X}_\phi$. Take the r_1 blocks of level 1. Start

connecting the arms of the r_2 blocks of level 2 to those of the blocks of level 1 so as to reduce the number of components as quickly as feasible. (While connecting arms we ensure that a lift of α_1' joins a lift of α_2.). We continue this process with the blocks of level(s) 2,3... until we are left with exactly one component. Notice that at every stage in this process upto this point we get a complex ≈ a closed disk. For definiteness suppose s is the integer s.t. the final complex obtained sofar, say D_o, has either exactly used all the blocks of level s-1 ,or else has also used some of the arms of the blocks of level s. Now we have to connect to ∂D_o the remaining arms of the blocks of level s. An elementary observation in the topology of surfaces is that if we have an orientable connected surface M with one boundary component we can attach the arms of any finite number of disks with arms to ∂M so as to obtain an orientable surface with ≤ 2 boundary components. So we attach the remaining arms of the blocks of level s suitably so that we obtain an orientable surface M_s, with ≤ 2 boundary components. There are two cases:

Case 1, $s < k$ and ∂M_s is connected. Then continue the process described above with blocks of level $s+1$.

Case 2, ∂M_s has two components. Then necessarily $s < k$ for otherwise M_s corresponds to a $\mathfrak{X}_\Phi$ and so Φ would have the free part of odd rank which is not the case. Now we make the

Claim: The arcs of ∂M_s along which the arms of the blocks of $(s+1)$-st level are to be connected cannot all lie on a single component.

Assuming the claim let us complete the proof. If each of the blocks of level $s+1$ has only one arm we attach them at appropriate places to ∂M_s thus obtaining M_{s+1}. Now ∂M_{s+1} has two components and $s+1 < k$, and we continue the process. On the other hand suppose that some block of level $s+1$ has ≥ 2 arms. Using the claim we can attach one of these arms to one component of ∂M_s and the second arm to the other component, thus obtaining an orientable surface with a connected boundary. Now we can attach the remaining arms of the blocks of level $s+1$ suitably so that we obtain M_{s+1} with ≤ 2 boundary components. By the argument at the beginning of case 2 in the final stage M_k must have only one boundary component. This would finish the proof modulo the claim.

(3.7) Proof of the claim in (3-6) Notice that each of the arcs $\alpha_i, \beta_i, \alpha_i'$, cf. (3.4.1), has d lifts $\alpha_{i*}, \beta_{i*}, \alpha_{i*}'$ along the boundaries of the blocks of level i. (Here $*$ takes d values but the specific indexing is not relevant). A component of ∂M_s is then of the form

(3.7.1) $\alpha_{1*}\beta_{1*}\alpha'_{1*}\alpha_{2*} \quad \cdots \quad \alpha_{s*} \square \alpha_{1*}\beta_{1*} \cdots \square \cdots \square \cdots \alpha'_{s*} \square$

where $\square$ indicates a place at which an arm of a block of level $s+1$ is to be attached. Now if all these d places lie on a single component, (3.7.1) shows that all the lifts $\alpha_{1*}, \beta_{i*}, \alpha'_{i*}$ also lie in the same component. But that would mean that ∂M_s is connected which is contrary to our hypothesis.

This finishes the proof of the claim and of (1.7).

(3.8) <u>Remark</u>: A purely combinatorial-group-theoretic proof of (1.7) - which, first of all involves the combinatorial-group-theoretic definitions of the genus and ends of a subgroup of a cycloid group - may be fashioned on [8], [1] and [6] .

(3.9) <u>Remark</u>: For the case $g=0$, (1.7) implies the following "high-school-math" fact. Let $\beta = \{x_1, \ldots, x_k\}$ be a set of k distinct complex numbers and suppose $0 \notin \beta$. Let $d = \Sigma_{j=1}^{r_i} d_{ij}$ $i = 1,2 \ldots k$ be k partitions of d. Then there exists a monic polynomial $p(z)$ of degree d, without constant term and branch set β such that

$$p(z) - x_i = \Pi_j (z-y_{ij})^{d_{ij}}$$

for suitable $y_{ij} \in \mathbb{C}$, $i = 1,2,\ldots,k$ iff

$$d(k-1) - \Sigma_{i=1}^{k} r_i + 1 = 0$$

Moreover if this condition is satisfied then there are only finitely many polynomials of this type.

§ 4 Extensions of some results of Rosenberger+ and Zieschang+

(4.1) Let $\Gamma = *_{i=1}^{n}\Gamma_i$, $\Gamma_i = \langle x_i\rangle \cong \mathbb{Z}_{m_i}$, $u = x_1 \ldots x_n$, and Φ a subgroup of Γ generated by elements of finite order. Rosenberger cf. [12], (2.15) has given the following criterion for Φ to be a subgroup of finite index in Γ assuming that m_i's are primes. His proof is by the Nielsen method and he asks if the primality hypothesis may be dropped. Such indeed is the case.

Theorem (Rosenberger's criterion) Let Γ, Φ, u be as above. Then Φ has finite index in Γ iff there exists a positive integer m such that $u^m \in \Phi$. Moreover if m happens to be the least positive integer such that $u^m \in \Phi$ then $(\Gamma:\Phi) = m$.

Proof Construct $\mathbb{X}_\Gamma$ so that taking a base-point in $\partial\mathbb{X}_\Gamma$, u may be represented by $\partial\mathbb{X}_\Gamma$. If there does not exist a positive integer m such that $u^m \in \Phi$ then Φ clearly has infinite index. So suppose that there exists such m and in fact it is the least positive such integer. This means that $(\partial\mathbb{X}_\Gamma)^m$ lifts to a loop in $\mathbb{X}_\Phi$. Of course this lift must lie in $\partial\mathbb{X}_\Gamma$. By minimality of m and since $\partial\mathbb{X}_\Phi$ is a 1-manifold it follows that this lift is a component of $\partial\mathbb{X}_\Phi$. But $\operatorname{int}\mathbb{X}_\Phi \cong \mathbb{R}^2$ so since $\partial\mathbb{X}_\Phi$ is a closed curve, $\mathbb{X}_\Phi \cong$ a closed disk, in particular it is compact. Hence Φ has finite index and

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+ I am thankful to Rosenberger and Zieschang for communicating to me their results.

$$(\Gamma:\Phi) = \deg p|_{\mathrm{int}\ \mathfrak{X}_\Phi} = \deg p|_{\partial\mathfrak{X}_\Phi} = m. \quad \text{q.e.d.}$$

(4.2) Zieschang's theorem 4.6 in [14] which is mentioned in (1.8) is a special case of the following.

Theorem. Let $\Gamma = \Gamma_1 * \Gamma_2 * \Gamma_3 * \ ..$

i) Assume that each of Γ_1 and Γ_2 has a subgroup of finite index ≥ 2. Then Γ has infinitely many C_o-and C_1-subgroups of finite index.

ii) Assume that Γ_1 (resp. Γ_2) has a subgroup of finite index ≥ 2 (resp. ≥ 3) or that each of the Γ_i, $i = 1,2,3$ has a subgroup of index ≥ 2. Then Γ has infinitely many C_r-subgroups of finite index, $r = 0,1,2,\ldots$.

Proof. It is easy to construct appropriate diagrams as in the special case mentioned in (A1.4) of [6]. q.e.d.

§ 5 Normal C_r-subgroups

(5.1) Proposition. Let $\Gamma = \overset{*}{\Pi}_{i\in I}\Gamma_i$ and Φ a normal C_r-subgroup of infinite index, then $r = 0$ or ∞.

Proof. Indeed $G = \Gamma/\Phi$ acts simply transitively on $p^{-1}(*)$ and carries circuits into circuits. Also each edge has one vertex in $p^{-1}(*)$. So if there exists one simple circuit in Y_Φ there exist infinitely many distinct simple circuits. q.e.d.

(5.2) Theorem. Normal C_o-subgroups are precisely the kernels of the morphisms of free products, cf. (2.4).

Proof. Let Φ be a normal sugbroup of $\Gamma = \overset{*}{\Pi}_{i\in I}\Gamma_i$, and $p\colon X_\Phi \longrightarrow X_\Gamma$ be the canonical projection of the associated diagrams. The components of $p^{-1}(X_{\Gamma_i})$ are permuted by Γ/Φ so they correspond to mutually isomorphic regular covering spaces $Z_{ij} \longrightarrow X_i$ which correspond to the same normal subgroup, say θ_i, of Γ_i. Write $G_i = \Gamma_i/\theta_i$ and $\varphi_i\colon \Gamma_i \longrightarrow G_i$ the canonical projections defining a morphism $\varphi\colon\Gamma \longrightarrow G \overset{def}{=} \overset{*}{\Pi}_{i\in I}G_i$. Clearly $\ker\varphi \leq \Phi$. Now suppose that Φ is also a C_o-subgroup. We assert that $\ker\varphi = \Phi$. Indeed let $x \in \Phi$, $x \neq e$. Write x as a word $x_1 \dots x_n$ where each $x_i \neq e$ and the consecutive x_i's belong to different factors of Γ. Suppose if possible that $\varphi(x) \neq e$. Notice that if some $\varphi_i(x_i) = e$ then $x_i \in \Phi$ and since Φ is normal the word x' obtained from x by deleting x_i is still in Φ and $\varphi(x') \neq e$. So by changing notation if necessary we may assume that

$\varphi_i(x_i) \neq e$ for $i = 1,2,..,n$. If $x_i \in \Gamma_j$, this means that a loop in X_{Γ_j} representing x_i does not lift to a loop in any component of $p^{-1}(X_{\Gamma_j})$. Let α_i be a loop based at $*$ representing x_i. Then the loop $\alpha = \alpha_1 * \alpha_2 * .. * \alpha_n$ represents x. Since $x \in \Phi$, α lifts to a loop $\tilde{\alpha}$ in X_Φ based at $\tilde{*}$. Let $\bar{\alpha}$ be the image of $\tilde{\alpha}$ in Y_Φ. Then $\bar{\alpha}$ is a certain circuit in the graph Y_Φ. Now $\tilde{\alpha} = \tilde{\alpha}_1 * \tilde{\alpha}_2 * .. * \tilde{\alpha}_n$ where $\tilde{\alpha}_i$ is a certain lift of α_i. Since $\varphi(x_i) \neq e$, the image $\bar{\alpha}_i$ of $\tilde{\alpha}_i$ in Y_Φ is an arc joining two distinct points of $p^{-1}(*)$. This shows that no edge of $\bar{\alpha}$ is immediately traversed back. But since Φ is assumed to be a C_o-subgroup, Y_Φ is a tree. So this is impossible. This contradiction shows that we must have $\varphi(x) = e$. So $\Phi = \ker \varphi$.

Conversely suppose that $\Phi = \ker \varphi$ where φ is a morphism of the free products $\Gamma = \ast_{i \in I} \Gamma_i \longrightarrow G = \ast_{i \in I} G_i$ defined by the homomorphisms $\varphi_i : \Gamma_i \longrightarrow G_i$. Let H be the identity subgroup of G. By the remark in (2.4) we see that the diagrams X_Φ and X_H are isomorphic. But X_H is just the diagram for the "universal cover" of X_G. So Y_H and hence Y_Φ are trees. So Φ is a normal C_o-subgroup. q.e.d.

(5.3) <u>Corollary</u>. Let $\Gamma = \ast_{i=1}^n \Gamma_i$, $|\Gamma_i| < \infty$. Let $\nu(G)$ denote the number of distinct normal subgroups of G. Then the number of normal C_o-subgroups of Γ is $\Pi_{i=1}^n \nu(\Gamma_i)$. Among them $\Sigma_{i=1}^n \nu(\Gamma_i) - n+1$ are of finite index.

<u>Proof</u>. The first assertion is clear from (5.2). Now note that a normal C_o-subgroup of finite index must correspond to a morphism $\varphi = \{\varphi_i\}$ where at most one $\operatorname{im} \varphi_i \neq e$.

So the number of <u>proper</u> normal C_o-subgroups of finite index is $\Sigma_i\{\nu(\Gamma_i)-1\}$. Together with Γ itself their count is $\Sigma_i \nu(\Gamma_i)-n+1$. q.e.d.

(5.4) <u>Corollary</u> Let $\Gamma = *_{i=1}^{n} \mathbb{Z}_{m_i}$. Let $d(k)$ denote the number of divisors of a positive integer k. Then the number of normal cycloidal subgroups of Γ is $\Pi_{i=1}^{n} d(m_i)$. Among them $\Sigma_{i=1}^{n} d(m_i)-n+1$ are of finite index.

(5.5) Now we pass on to the study of normal C_r-subgroups, $1 \leq r < \infty$. Let $\Gamma = *_{i\in I}\Gamma_i$ and $\Phi \leq \Gamma$ so that $\Phi \approx F_r * *_{i\in I} (*_{j\in J_i} \Phi_{ij})$ where Φ_{ij}'s are subgroups of Γ_i whose conjugates lie in Φ. Write $d_{ij} = (\Gamma_i : \Phi_{ij})$. Then $\{d_{ij}\}_{j\in J_i}$ is a well-defined* collection of positive integers (possibly infinite), associated to Φ as a subgroup of Γ. We shall use this notation below.

(5.6) <u>Lemma</u>. Let $\Gamma = \Gamma_1 * \Gamma_2 * \Gamma_3 * \ldots * \Gamma_n$ where each Γ_i is finitely generated. Fix a non-negative integer r. Then there are only finitely many C_r-subgroups of finite index in Γ satisfying either one of the following conditions

α) $d_{1j} \geq 2$ for $j \in J_1$ and $d_{2j} \geq 3$ for $j \in J_2$

or

β) $d_{ij} \geq 2$ $i = 1,2,3,$ $j \in J_i$.

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* Here we assume that there are no dummy Φ_{ij}'s $\approx e$ in the free product expression.

<u>Proof</u>. Let Φ be a C_r-subgroup of finite index d, and $r_i = |J_i| + \frac{d-\Sigma_j d_{ij}}{|\Gamma_i|}$. Then in Y_Φ there are $d + \Sigma_{i=1}^n r_i$ vertices and dn edges, so

$$r = \text{rank}\ \pi_1(Y_\Phi) = d(n-1) - \Sigma_{i=1}^n r_i + 1$$

We have always $r_i \leq d$. If α) (resp. β)) is satisfied then

$$r_1 \leq \frac{d}{2}\ ,\ r_2 \leq \frac{d}{3} \qquad (\text{resp. } r_i \leq \frac{d}{2}\ ,\ i = 1,2,3)$$

So

$$r \geq \frac{d}{6} + 1 \qquad\qquad (\text{resp. } r \geq \frac{d}{2} + 1).$$

Hence the index of C_r-subgroups satisfying α) or β) is bounded by $6r-1$. Since Γ_i's are assumed to be finitely generated it follows that the number of such C_r-subgroups is finite. q.e.d.

(5.7) <u>Theorem</u> Let $\Gamma = \overset{*}{\Pi}_{i=1}^n \Gamma_i$ where Γ_i's are finitely generated

a) For $2 \leq r < \infty$ there are only finitely many normal C_r-subgroups.

b) For $r = 1$, the number of normal C_1-subgroups is 0 or ∞. It is ∞ iff there exists a surjective morphism of free products $\Gamma \twoheadrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 * e * e * \ldots$.

<u>Proof</u>. By (5.1) the subgroups under consideration are automatically of finite index. So the d_{ij}'s , cf. (5.5), are all finite. Moreover for a normal subgroup d_{ij}'s depend only on i – say, $d_{ij} = d_i$. So (5.6) takes care of the cases where either for two distinct indices i_1, i_2, $d_{i_1} \geq 2$ and $d_{i_2} \geq 3$, or for three distinct values of i, $d_i \geq 2$. The remaining cases, therefore, are precisely

a) all but at most one d_i, say d_1, are 1,

and

b) two of the d_i's, say d_1 and d_2 are 2, and the remaining d_i's are 1.

In case a) we have $r_i = |J_i| = d$ for $i \geq 2$ but then $r = 1 - r_1 \geq 0$ implies that $r = 0$. But we assumed $r \geq 1$—a contradiction.*

In case b) we clearly have a morphism $\Gamma_1 * \Gamma_2 * \ldots \xrightarrow{\lambda} \mathbb{Z}_2 * \mathbb{Z}_2 * e * \ldots$ and Φ has the form $\lambda^{-1}(\bar{\Phi})$ where $\bar{\Phi}$ is a normal subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2 * e * \ldots$.
So by (2.4) X_Φ and $X_{\bar{\Phi}}$ are isomorphic. But it is easy to see that $Y_{\bar{\Phi}}$ is homotopic to a circle, so this case occurs only if $r = 1$.

Conversely if there exists a morphism λ as above one easily constructs infinitely many normal C_1-subgroups. q.e.d.

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* In fact in case a), $d = d_1 = 1$ also.

(5.8) Theorem Let $2 \leq r < \infty$. Then the quotients by normal C_r-subgroups of free products are finite groups having a faithful integral representation of rank r. In particular, there are only finitely many isomorphism-types of such quotients.

Proof. Let $\Gamma = \coprod^{*}_{i\in I}\Gamma_i$ and Φ a normal C_r-subgroup. As remarked in (5.1), then $G \overset{def}{=} \Gamma/\Phi$ is finite. Let $p\colon X_\Phi \longrightarrow X_\Gamma$ be the projection of the diagrams. Then G acts on X_Φ and so also on Y_Φ. Note that G acts simply transitively on $p^{-1}(*)$. Since every edge has an end-point in $p^{-1}(*)$ it follows that G does not fix any edge of Y_Φ. In other words the action is pseudo-free in the sense defined in [4]. Now since the Euler characteristic of Y_Φ is negative it follows, cf. [4], (3.1), that the induced action on $H^*(Y_\Phi,\mathbb{Z}) \cong \mathbb{Z}^r$ is faithful. This proves the first assertion of the theorem. The second is well-known, cf.[4], (9.2.1) q.e.d.

(5.9) Remark. Let Γ be a finitely generated cycloidal group. It is interesting to compare the behavior of the rank of the free part with the genus of a normal subgroup of Γ. The genus of a normal subgroup of infinite index is 0 or ∞. (The proof is similar to (5.1)). From the available information on finite-group-actions on closed surfaces, cf.[5], § 4, one sees that the number of normal subgroups of Γ of finite genus $\neq 1$ is finite. The number of normal subgroups of genus 1 is 0 or ∞; it is ∞ iff Γ has a surjective homomorphism onto $\mathbb{Z}_2 * \mathbb{Z}_3$, $\mathbb{Z}_2 * \mathbb{Z}_4$ or $\mathbb{Z}_3 * \mathbb{Z}_3$.

This behavior of the genus should be contrasted with the behavior of t = the number of ends which has an arithmetic flavor. We study this in the next section.

(5.10) Remark. (5.8) gives a rather crude upper bound for the degree of a faithful integral representation of a finite group. For example, if a group of order d is generated by two elements x,y, say x of order 2 and y of order 3 then it is a quotient of the modular group and has a faithful integral representation of degree $\leq \frac{d}{6} + 1$. A slightly better bound is obtained if one knows also that xy has order $k \geq 7$ then the degree may be chosen to be $\leq 2 + \frac{d}{6} - \frac{d}{k}$. This follows from the fact that such a group acts on a closed orientable surface M of genus $g \geq 2$ so the induced action on $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ is faithful. This oberservation goes back to Hurwitz, cf. [4].

§ 6 Normal t-cycloidal subgroups

(6.1) In this section we extend some results of Newman [9] and Greenberg [3] on the modular group to general finitely generated cycloidal groups. Newman's theorem 3 of [9] classifies normal 1-cycloidal subgroups of the modular group - there are in all 4 in number. A much deeper result of Newman [9] is that there are only finitely many t-cycloidal subgroups of the modular group for $t \leq 11$ —a condition which was later removed by Greenberg [3].

(6.2) Theorem. Let $\Gamma = \ast_{i=1}^{n} \mathbb{Z}_{m_i}$, $m_i \geq 2$. Then it has only finitely many normal 1-cycloidal subgroups. If m_i's are pairwise coprime then their number is $\Pi_{i=1}^{n} d(m_i)$ where $d(k)$ denotes the number of divisors of a natural number k.

Proof. Let Φ be a normal 1-cycloidal subgroup. Then $G = \Gamma/\Phi$ acts on $\mathbb{X}_\Phi$ which is a compact orientable surface with one boundar component. Also G preserves orientation of $\mathbb{X}_\Phi$. Pinching $\partial\mathbb{X}_\Phi$ to a point one may consider G as acting on a closed orientable surface preserving orientation and fixing a point. Such a group is known to be necessarily cyclic. In other words G must be a finite cyclic quotient of Γ. The assertions are now clear. q.e.d.

(6.3) Theorem. Let $\Gamma = \ast_{i=1}^{n} \mathbb{Z}_{m_i}$, m_i's ≥ 2 and pairwise coprime. Then for each $t = 1,2 \ldots$ there are only finitely many normal t-cycloidal subgroups of Γ.

For the proof, first a group-theoretic Lemma.

(6.4) <u>Lemma</u> Let G be a finite group generated by n elements. Let g be an element of G of order N and $t = |G|/N$. The maximal normal subgroup of G contained in $\langle g\rangle$ has index $\leq t^{n+1}$.

<u>Proof</u>. Let H be the maximal normal subgroup contained in $\langle g\rangle$ and $\{x_1,..,x_n\}$ a set of generators of G. Set

$$H_1 = \langle g\rangle \cap \bigcap_{i=1}^{n} x_i\langle g\rangle x_i^{-1} .$$

Clearly $H \leq H_1$ and H_1 has index $\leq t^{n+1}$. We show that $H = H_1$. Notice that $H_1 = \langle g^r\rangle$ for some integer r, and since $g^r \in x_i \langle g\rangle x_i^{-1}$ we can write $g^r = x_i g^s x_i^{-1}$ for some integer s. But g^r, g^s have the same order. So as subgroups of $\langle g\rangle$ we have

$$\langle g^r\rangle = x_i\langle g^s\rangle x_i^{-1} = \langle g^s\rangle.$$

So

$$x_iH_1x_i^{-1} = x_i\langle g^r\rangle x_i^{-1} = x_i\langle g^s\rangle x_i^{-1} = \langle g^r\rangle = H_1 .$$

Hence also $x_i^{-1}H_1x_i = H_1$. Since this holds for all x_i's and every element of G is a word in x_i's it follows that H_1 is normal in G. So $H = H_1$. q.e.d.

(6.5) <u>Proof of</u> (6.3) Let Φ be a normal t-cycloidal subgroup of Γ of index d and let $G \overset{\text{def}}{=} \Gamma/\Phi$. Let $\bar{x}_i$ be a generator of $\mathbb{Z}_{m_i}$ and construct $\mathbb{X}_\Gamma$ so that taking a base-point in $\partial\mathbb{X}_\Gamma$, $\bar{g} = \bar{x}_1 \ldots \bar{x}_n$ is represented by $\partial\mathbb{X}_\Gamma$. Let x_i, g be the images of $\bar{x}_i$, $\bar{g}$ in G. If g has order N, and $d = |G| = (\Gamma:\Phi)$ then clearly $d = N \cdot t$. If H is the maximal normal subgroup contained in $\langle g \rangle$ then by (6.4) H has index $\leq t^{n+1}$. Let α_x denote the automorphism of H induced by the conjugation by $x \in G$.
Then we have $\alpha_{x_i}^{m_i} = 1 = \alpha_{x_1} \circ \alpha_{x_2} \circ \ldots \circ \alpha_{x_n}$.
Since H is cyclic, Aut H is abelian. So since m_i's are assumed to be pairwise coprime we see that $\alpha_{x_i} = 1$ for $i = 1,2 \ldots n$, i.e. H is actually <u>central</u> in G.
Let $H \simeq \mathbb{Z}_k$ and $|G/H| = \ell$. So $H \hookrightarrow G \twoheadrightarrow G/H$ represents an extension which may be regarded as an element of $H^2(G/H;\mathbb{Z}_k)$, cf. [7] ch. 4. Write $k = a \cdot m$ where $a|\ell$ and $(m,\ell) = 1$.
The coefficient map $\mathbb{Z}_k \twoheadrightarrow \mathbb{Z}_m$ induces a map $H^2(G/H;\mathbb{Z}_k) \longrightarrow H^2(G/H;\mathbb{Z}_m)$ which corresponds to a map of the extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H \simeq \mathbb{Z}_k & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 1 \\ & & \downarrow & & \vdots & & \| & & \\ 1 & \longrightarrow & \mathbb{Z}_m & \rightleftarrows & G_1 & \longrightarrow & G/H & \longrightarrow & 1. \end{array}$$

(Since $(m,\ell) = 1$ the bottom line splits, cf. [7] ch.4. § 10.)
So one has surjective maps $\Gamma \twoheadrightarrow G \twoheadrightarrow G_1 \twoheadrightarrow \mathbb{Z}_m$.
Now clearly $m \leq m_1 \ldots m_n \overset{\text{def}}{=} M$, say. So

$$d = |G| = k \cdot |G/H| = a \cdot m \cdot \ell \leq M\ell^2 \leq Mt^{2(n+1)},$$

i.e. the index of Φ is bounded. So there are only finitely many t-cycloidal subgroups for each $t = 1,2,\ldots$. q.e.d.

(6.5) Theorem. Let $\Gamma = \mathbb{Z}_{m_1} * \mathbb{Z}_{m_2} * \ldots * \mathbb{Z}_{m_h}$, $(m_1,m_2) > 1$, and $\Gamma \neq \mathbb{Z}_2 * \mathbb{Z}_2$. Then there are infinitely many values of t for each of which there are infinitely many t-cycloidal subgroups.

Proof. Let p be a prime divisor of (m_1,m_2). First consider the case $p \geq 3$. We have an obvious morphism of free products, cf. (2.4),

$$\mathbb{Z}_{m_1} * \mathbb{Z}_{m_2} * \ldots * \mathbb{Z}_{m_n} \twoheadrightarrow \mathbb{Z}_p * \mathbb{Z}_p * e * \ldots * e.$$

So by the comment in (2.4) applied to their thickenings it suffices to prove the assertion for $\mathbb{Z}_p * \mathbb{Z}_p$. So we just take $\Gamma = \Gamma_1 * \Gamma_2$, $\Gamma_1 = \langle\bar{x}\rangle \cong \mathbb{Z}_p$, $\Gamma_2 = \langle\bar{y}\rangle \cong \mathbb{Z}_p$. Let

$$\Delta = \Delta_{p,p,N} = \langle u,v,w \mid u^p = v^p = w^N = uvw = e\rangle$$

denote a triangle group. So fixing $N \geq 3$ we ensure that this group is infinite. Consider the homomorphisms

$$\lambda: \Gamma \twoheadrightarrow \Delta \quad , \quad \text{s.t. } \lambda(\bar{x}) = u \ , \quad \lambda(\bar{y}) = v \ ,$$

(6.4.1)

$$\mu: \Delta \twoheadrightarrow \mathbb{Z}_p \quad , \quad \text{s.t. } \mu(u) \neq e \neq \mu(v) \ , \ \mu(uv) = e$$

Let Φ be any torsion-free normal subgroup of finite index in Δ so that $\Phi \leq \ker\mu$. As is well-known there exist infinitely many such subgroups. Let

$$\bar{\Phi} = \lambda^{-1}(\Phi) \quad \text{and} \quad G \overset{\text{def}}{=} \Gamma/\bar{\Phi} \cong \Delta/\Phi.$$

If x,y,z denote the images of $\bar{x},\bar{y},\bar{z} \overset{def}{=} \bar{x}\bar{y}$ in G then z has order N (since $\bar{\Phi}$ is torsion-free). If $|G| = N \cdot t$ we see that Φ is a t-cycloidal normal subgroup of Γ. For this t we now produce infinitely many t-cycloidal normal subgroups.

Let q be a prime $\equiv 1(p)$, $q \nmid N$. There exist infinitely many such primes by the well-known Dirichlet's theorem and its extensions. The significance of such primes is that for any power q^{ℓ} there exists an automorphism of order p of $\mathbb{Z}_{q^{\ell}}$. If we fix a generator a of $\mathbb{Z}_{q^{\ell}}$ such an automorphism will have the form $a \longrightarrow a^r$ where $r^p \equiv 1 \ (q^{\ell})$, $r \not\equiv 1 \ (q^{\ell})$. Now by construction there is a surjective homomorphism $G \twoheadrightarrow \mathbb{Z}_p$. Let G act on $\mathbb{Z}_{q^{\ell}}$ via $\mathbb{Z}_p$ in this way and consider the semidirect product

$$H \overset{def}{=} \mathbb{Z}_{q^{\ell}} \rtimes G = \langle a,x,y,z \rangle$$

where the relations include

$$(6.4.1) \quad \begin{cases} a^{q^{\ell}} = x^p = y^p = z^N = xyz = e \ , \ x^{-1}ax = yay^{-1} = a^r, \\ az = za \ . \end{cases}$$

A calculation shows that $x_1 = ax$ has order p. Also $y_1 = y$ has order p, and since z commutes with a, and $q \nmid N$ the element $z_1 = x_1y_1 = axy = az^{-1}$ has order $q^{\ell} \cdot N$. Also $z_1^N = a^N$ is a generator of $\mathbb{Z}_{q^{\ell}}$. So H is generated by two elements x_1, y_1 both of order p such that their product z_1 has order $q^{\ell} \cdot N$. Moreover $|H| = q^{\ell}|G| = q^{\ell} \cdot N \cdot t$. If $\theta : \Gamma \twoheadrightarrow H$ s.t. $\theta(\bar{x}) = x_1$,

$\theta(\bar{y}) = y_1$ we see that $\ker\theta$ is a normal t-cycloidal subgroup of index $q^{\ell}\cdot N\cdot t$. Since there are infinitely many choices for each of q,ℓ and t we have shown that for each of the infinitely many values of t there are infinitely many normal t-cycloidal subgroups.

It remains to consider the case when $p = 2$ but $\Gamma \not\cong \mathbb{Z}_2 * \mathbb{Z}_2$. There are two subcases:

i) $\Gamma = \mathbb{Z}_a * \mathbb{Z}_b * \ldots,$ where $a \equiv b \equiv 0\,(2)$ and $ab > 4$;

ii) $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \ldots$

As before we reduce to

i)' $\Gamma = \mathbb{Z}_a * \mathbb{Z}_b$, where $a \equiv b \equiv 0\,(2)$ and $ab > 4$;

ii)' $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.

<u>Subcase i)'</u> Consider $\mathbb{Z}_a = \langle x\rangle$, $\mathbb{Z}_b = \langle y\rangle$ and

$$\Delta_1 = \Delta_{a,b,N} = \langle u,v,w \mid u^a = v^b = w^N = uvw = e\rangle \; ; \; N \geq 4 .$$

Note that Δ_1 is infinite. Consider the homomorphisms

$$\lambda : \Gamma \longrightarrow \Delta_1 , \quad \text{s.t.} \quad \lambda(x) = u,\ \lambda(y) = v .$$
$$\mu : \Delta_1 \longrightarrow \mathbb{Z}_2 \quad \text{s.t.} \quad \mu(u) \neq e \neq \mu(v),\ \mu(uv) = e .$$

Now start with a normal torsion-free subgroup Φ_1 of Δ_1 of finite index, such that Φ_1 is contained in $\ker\mu$. The argument proceeds exactly as before.

<u>Subcase ii)'</u> Let x_1, x_2, x_3 be a set of generators of Γ ,

each of order 2, and consider a "quadrilateral" group

$$\Psi = \langle u_1, u_2, u_3, u_4 \mid u_1^2 = u_2^2 = u_3^2 = u_4^N = u_1u_2u_3u_4 = e\rangle, \ N \geq 2 \ .$$

Note again that Ψ is infinite. Consider the homomorphisms

$$\lambda : \Gamma \longrightarrow \Psi \quad \text{s.t.} \quad \lambda(x_i) = u_i \ , \ i = 1,2,3 \ ,$$

$$\mu : \Psi \longrightarrow \mathbb{Z}_2 \ \text{s.t.} \ \mu(u_1) \neq e, \quad \mu(u_3) = \mu(u_4) = e \ .$$

Now start with a normal torsion-free subgroup Φ of Ψ of finite index and contained in $\ker \mu$. Hereafter the argument proceeds essentially as before and may be left to the reader.

This completes the proof. q.e.d.

(6.5) Remark: The above argument breaks down for $\Gamma \approx \mathbb{Z}_2 * \mathbb{Z}_2$ precisely at the point that $\Delta_{2,2,N}$ is finite for all N . Indeed it is easy to see that the only normal subgroups Φ of Γ , $e \neq \Phi \neq \Gamma$, are of finite even index. Moreover for any even intege 2k there exists precisely one normal subgroup of index 2k ; it is $\simeq \mathbb{Z}$ and is 2-cycloidal.

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