The even master system and generalized Kummer surfaces

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Abstract

In this paper we study a generalised Kummer surface associated to the Jacobian of those complex algebraic curves of genus two which admit an automorphism of order three. Such a curve can always been written as $y^2 = x^6 + 2\kappa x^3 + 1$ and $\kappa^2 \neq 1$ is the modular parameter. The automorphism extends linearly to an automorphism on the Jacobian and we show that this extension has a 9_4 invariant configuration, i.e., it has 9 fixed points and 9 invariant theta curves, each of these curves contains 4 fixed points and through each fixed point pass 4 invariant curves. The quotient of the Jacobian by this automorphism has 9 singular points of type A_2 and the 9_4 configuration descends to a 9_4 configuration of points and lines, reminding to the well-known 16_6 configuration on the Kummer surface. Our "generalised Kummer surface" embeds in $\mathbb{P}^4$ and is a complete intersection of a quadric and a cubic hypersurface. Equations for these hypersurfaces are computed and take a very symmetric form in well-chosen coordinates. This computation is done by using an integrable system, the "even master system".

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1. Introduction

Recently several studies were published on the geometric aspects of Hamiltonian systems which are algebraically completely integrable. For a general introduction, see [AvM1]. From the point of view of algebraic geometry, these integrable systems lead to an original approach to study projective embeddings of Abelian varieties and their Kummer varieties, explicit equations for affine parts of these varieties, ... It follows that integrable systems may be used to study and solve some questions in algebraic geometry, especially in curve theory and the theory of Abelian varieties; the present paper is a particular example of such a question.

In order to state this question, let us recall from the classical literature some basic facts about the Kummer surface of the Jacobian of a genus two curve Γ. Such a curve being always hyperelliptic, it carries an involution τ with six fixed points (the Weierstrass points of Γ); this involution extends linearly to the Jacobian of Γ where it has sixteen fixed points and sixteen invariant theta curves (i.e., translates of the Riemann theta divisor), each invariant curve contains six fixed points and each fixed point belongs to six invariant curves, giving a so-called 16_6 configuration of curves and points. The quotient of the Jacobian by this involution is a singular surface, the Kummer surface, and it embeds in $\mathbb{P}^3$ as a quartic surface. An equation for this surface has classically been obtained (in several forms) by purely algebraic methods (see [H]).

Something very analogous happens when the genus two curve Γ has an automorphism σ of order three, in which case the curve has an equation $y^2 = x^6 + 2\kappa x^3 + 1$ (here $\kappa^2 \neq 1$ is the modular parameter as we will show). The symmetry of order three extends to the Jacobian and leads now to a 9_4 configuration as we will prove both directly and by using an analogue of the theta characteristic, which expresses in general the obstruction for a line bundle to descend to a quotient. In the present case this characteristic turns out to be a quadratic form which takes values in $\mathbb{F}_3$ (the field of three elements). Such a configuration, which has essentially only one projective realisation has been considered by Segre and Castelnuovo (see [S] and [C]). The singular surface obtained as the quotient of the Jacobian of Γ by the order three automorphism will be shown to embed now in $\mathbb{P}^4$ as the intersection of a quadric and a cubic hypersurface. The nine singular points are of type A_2 and are part of a 9_4 configuration of lines and points on this surface which, after desingularisation, is a K-3 surface.

The question now is to compute explicit equations for the quadric and cubic hypersurface. To this aim we need to introduce well-adapted coordinates and this is where the integrable system comes in. The system is chosen in such a way that among its invariant surfaces we find the Jacobians corresponding to the genus two curves with an automorphism of order three. Such a system was first constructed by the second author in [V] in analogy with a system introduced by Mumford (see [M2]). It gives on the one hand explicit equations for affine parts of the Jacobians which concern us here, on the other hand it allows us to construct an explicit base for the functions with a pole of order three at one of the invariant theta curves. Among those functions the ones which are invariant by σ are easily determined and the image of the Jacobian in $\mathbb{P}^4$ by these functions is computed from these explicit data. The final result is that in terms of an appropriate base for $\mathbb{P}^4$ — formed by the five fixed points which do not belong to one of the invariant theta curves — the equation for the quadric hypersurface is given by

$$c(y_1 + y_4)(y_2 + y_3 + y_4 - y_0) + \bar{c}(y_2 + y_3)(y_1 + y_3 + y_4 - y_0) = cy_4^2 + \bar{c}y_3^2,$$

while the equation for the cubic hypersurface is given by

$$c^2 y_1 y_4 (y_2 + y_3 - y_0) - \bar{c}^2 y_2 y_3 (y_1 + y_4 - y_0) = 0,$$

where $c = \kappa + 1$ and $\bar{c} = 1 - \kappa$.

The special curves considered here are actually the chiral Potts N-state curves corresponding to $N = 3$ (see [R]). The results and techniques in this paper generalise to all chiral Potts curves. We hope to return to this in the future.

2. An equation for the curve Γ.

We consider a curve Γ of genus two, equiped with an automorphism of order three, denoted by σ. By the Riemann-Hurwitz formula the quotient Γ/σ has genus zero and σ has four fixed points. Since Γ has genus two it is also hyperelliptic; the hyperelliptic involution will be denoted by τ and its fixed points are the six Weierstrass points on Γ. We have the following diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\pi_\sigma", "3:1"'] \arrow[d, "\pi_\tau"', "2:1"] & \mathbb{P}^1 \\ \mathbb{P}^1 & \end{tikzcd}$$

σ necessarily maps Weierstrass points to Weierstrass points, hence the commutator $[\sigma, \tau]$ fixes all these points and we see that $\sigma\tau = \tau\sigma$ since the only automorphisms which fix all Weierstrass points are τ and identity. It follows on the one hand that σ induces on $\mathbb{P}^1$ a fractional linear transformation $\tilde{\sigma}$ of order three, and on the other hand that the four fixed points of σ consist of two τ-orbits. We may therefore suppose that $\tilde{\sigma}$ is given by $\tilde{\sigma}(x) = \epsilon x$, $\epsilon = \exp(\frac{2\pi i}{3})$, by chosing a coordinate x on $\mathbb{P}^1$ such that these two orbits correspond to $x = 0$ and $x = \infty$. The images of the Weierstrass points form two orbits of three points under $\tilde{\sigma}$, which correspond to the roots of the equation $x^3 = \lambda^3$ and $x^3 = \lambda^{-3}$, possibly after a rescaling of x. Obviously $\lambda \neq 0$; since both orbits are different, $\lambda^3 \neq \lambda^{-3}$, i.e., $\lambda^6 \neq 1$. This shows that Γ has an equation

$$\begin{aligned} y^2 &= (x^3 - \lambda^3)(x^3 - \lambda^{-3}), \\ &= x^6 + 2\kappa x^3 + 1, \end{aligned} \tag{1}$$

with $\kappa \neq \pm 1$.

Obviously, every equation of the form (1), with $\kappa \neq \pm 1$ defines a smooth curve of genus two with an automorphism $(x, y) \mapsto (\epsilon x, y)$ of order three; also, if κ in (1) is replaced by $-\kappa$ then an isomorphic curve is obtained. Conversely, let there be given two isomorphic curves Γ and Γ' with respective automorphisms σ and σ' of order three. We may suppose that the isomorphism $\phi: \Gamma \to \Gamma'$ respects the automorphism, i.e., $\phi\sigma = \sigma'\phi$. We claim that if Γ and Γ' are written as above as

$$\begin{aligned} \Gamma&: y^2 = x^6 + 2\kappa x^3 + 1, \\ \Gamma'&: y^2 = x^6 + 2\kappa' x^3 + 1, \end{aligned}$$

then $\kappa^2 = \kappa'^2$. To see this remark that ϕ obviously commutes with τ, hence there is an induced linear transformation $\tilde{\phi}$ which satisfies $\tilde{\phi}(\epsilon x) = \epsilon\tilde{\phi}(x)$, for all $x \in \mathbb{P}^1$. Thus $\tilde{\phi}(x) = \mu x$ and $\phi(x, y) = (\mu x, y)$, giving $\mu^6 = 1$. It follows that $\kappa^2 \neq 1$ can be taken as modular parameter.

The automorphism group of Γ contains a subgroup which is isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z}$, as is seen immediately from (1); it actually coincides with this group, unless $\kappa = 0$ (in which case the group of automorphisms jumps to $D_6 \times \mathbb{Z}/2\mathbb{Z}$). Namely, there is apart from the hyperelliptic involution τ an action of S_3 by means of which the Weierstrass points belonging to one σ-orbit can be at random permuted. For future use we choose an element μ of order two in this symmetry group S_3 corresponding to a transposition in S_3, say $\mu(x,y) = (x^{-1}, yx^{-3})$ and remark that it commutes with τ but not with σ. Its fixed points are the two points in $\pi_\tau^{-1}\{1\}$, hence Γ/μ is an elliptic curve.

We will find it convenient to denote the fixed points of σ, which are mapped by π_τ to 0 (resp. ∞) by o_1 and o_2 (resp. ∞_1 and ∞_2). Then $\tau(\mathrm{o}_1) = \mathrm{o}_2$, $\tau(\infty_1) = \infty_2$ and we may suppose $\mu(\mathrm{o}_1) = \infty_1$ giving also $\mu(\mathrm{o}_2) = \infty_2$. In the same way we denote the Weierstrass points corresponding to the $x^3 = \lambda^3$-orbit by λ_i, $\sigma(\lambda_i) = \lambda_{i+1}$ (indices are taken modulo 3) and the ones corresponding to the $x^3 = \lambda^{-3}$-orbit by $\bar{\lambda}_i$, $\mu(\lambda_i) = \bar{\lambda}_i$. Then the action of $S_3 \times \mathbb{Z}/2\mathbb{Z}$ on these points is contained in the following table.

	order	o_1	o_2	∞_1	∞_2	λ_i	$\bar{\lambda}_i$
σ	3	o_1	o_2	∞_1	∞_2	λ_{i+1}	$\bar{\lambda}_{i-1}$
τ	2	o_2	o_1	∞_2	∞_1	λ_i	$\bar{\lambda}_i$
μ	2	∞_1	∞_2	o_1	o_2	$\bar{\lambda}_i$	λ_i

Table 1

3. The 9_4 configuration on the Jacobian of Γ

Let $J(\Gamma)$ denote the Jacobian of Γ and for a divisor D of degree 0, let $[D]$ denote the correspoding point in $J(\Gamma)$ (i.e., its linear equivalence class). A useful fact about the Jacobian of a curve of genus two is the following: for any fixed Q_1, $Q_2 \in \Gamma$, every element $\omega \in J(\Gamma)$ can be written as $\omega = [P_1 + P_2 - Q_1 - Q_2]$; moreover this representation is unique iff $P_1 \neq \tau(P_2)$, all $P + \tau(P)$ and $Q + \tau(Q)$ $(P, Q \in \Gamma)$ being linearly equivalent, $P + \tau(P) \sim_l Q + \tau(Q)$. In the present case of curves (1) which have an automorphism σ of order three, the cover π_σ associated to σ provides in addition (using the notations of the previous section for the fixed points of σ) the following linear equivalences

$$3o_1 \sim_l 3o_2 \sim_l 3\infty_1 \sim_l 3\infty_2. \tag{2}$$

The automorphism σ extends in a natural way to an automorphism on $J(\Gamma)$, also denoted by σ. It is given and well-defined for $\omega = [P_1 + P_2 - Q_1 - Q_2]$ as follows: $\sigma(\omega) = [\sigma(P_1) + \sigma(P_2) - \sigma(Q_1) - \sigma(Q_2)]$.

Proposition 1 *The automorphism σ has nine fixed points and nine invariant theta curves on $J(\Gamma)$.*

Proof

The principal polarisation on $J(\Gamma)$ is invariant under $\mathrm{Aut}(\Gamma)$, hence the isomorphism $J(\Gamma) \to \hat{J}(\Gamma)$ from $J(\Gamma)$ to its dual $\hat{J}(\Gamma)$ is $\mathrm{Aut}(\Gamma)$-invariant and the second statement follows from the first one.

To count the number of fixed points we use the holomorphic Lefschetz fixed point formula

$$\sum_p (-1)^p \mathrm{trace}\, f^*|_{H^{p,0}(M)} = \sum_{f(p_\alpha)=p_\alpha} \frac{1}{\det(I - B_\alpha)}, \tag{3}$$

for a holomorphic map $f\colon M \to M$, where B_α is the linear part of f at the fixed point p_α. We apply it for $f = \sigma$ and $M = J(\Gamma)$; in this case $H^{p,0}(J(\Gamma))$ may be identified with the p-th anti-symmetric power of the cotangent bundle at any point of $J(\Gamma)$. For the left-hand side in (3), the base of $H^{p,0}(J(\Gamma))$ may thus be taken in a point $[P_1 + P_2 - Q_1 - Q_2]$ as $\{\Omega_1, \Omega_2\} = \{\omega_1(P_1) + \omega_1(P_2), \omega_2(P_1) + \omega_2(P_2)\}$, where $\omega_i = x^{i-1}dx/y$ and $\Omega_1 \wedge \Omega_2$ is a generator for $H^{2,0}(J(\Gamma))$. Since $\sigma^*\Omega_i = \epsilon^i\Omega_i$, $(i = 1, 2)$, the left hand side in (3) gives

$$\sum_{p=0}^{2} (-1)^p \mathrm{trace}\, \sigma^*|_{H^{p,0}(J(\Gamma))} = 1 - \mathrm{trace} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix} + 1 = 3.$$

As for the right hand side, obviously all B_α are equal, in fact

$$B_\alpha = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix} \tag{4}$$

when local coordinates dual to Ω_1 and Ω_2 are picked around the point P_α. Therefore

$$\det(I - B_\alpha) = (1-\epsilon)(1-\epsilon^2) = 3,$$

and the number of fixed points of σ is indeed nine. ∎

Of course these fixed points can also be counted by writing down an explicit list. If we write every point $\omega \in J(\Gamma)$ as $\omega = [P_1 + P_2 - 2\infty_1]$ then $\sigma\omega = \omega$ iff $\sigma(P_1) + \sigma(P_2) \sim_l P_1 + P_2$, i.e., $P_1 = \tau(P_2)$ or P_1 and P_2 are both fixed points for σ. Using (2) we arrive at the following list

$$\{O,\, \mathrm{o}_1 - \mathrm{o}_2,\, \mathrm{o}_2 - \mathrm{o}_1,\, \infty_1 - \infty_2,\, \infty_2 - \infty_1,\, \mathrm{o}_1 - \infty_1,\, \mathrm{o}_2 - \infty_2,\, \mathrm{o}_1 - \infty_2,\, \mathrm{o}_2 - \infty_1\}. \quad (5)$$

The nine invariant curves are then given by the nine translates over these points of the image of Γ in $J(\Gamma)$ by the map $x \mapsto [x - \infty_1]$. Since this curve obviously contains exactly the four fixed points

$$\{O,\, \infty_2 - \infty_1,\, \mathrm{o}_1 - \infty_1,\, \mathrm{o}_2 - \infty_1\},$$

each of the nine invariant curves will contain exactly four fixed points. Dually, every fixed point belongs to four invariant curves since the origin O belongs to the four curves

$$\{x \mapsto [x - \infty_i],\, x \mapsto [x - \mathrm{o}_i],\, i = 1, 2\}$$

Remark that the fixed points form a group F (isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$) which is a subgroup of $J_3(\Gamma)$, the three-torsion subgroup of $J(\Gamma)$. On $J_3(\Gamma)$ there is a non-degenerated alternating form $(\cdot,\cdot)$ induced by the Riemann form corresponding to the principal polarisation. The subgroup $F \subset J_3(\Gamma)$ has the following property.

Proposition 2 *The group F of fixed points of σ on $J(\Gamma)$ is a totally isotropic subgroup of $J_3(\Gamma)$ with respect to the Riemann form $(\cdot,\cdot)$.*

Proof

σ is a symplectic automorphism of $J_3(\Gamma) \cong (\mathbb{Z}/3\mathbb{Z})^4$, which satisfies $1 + \sigma + \sigma^2 = 0$; also $\dim\ker(\sigma - 1) = 2$. It follows that F consists exactly of the elements of the form $\sigma(x) - x$ where $x \in J_3(\Gamma)$. Finally, if $y \in F$, then obviously $(y, \sigma(x) - x) = 0$. ∎

Apart from the Riemann form, which coincides on $J_3(\Gamma)$ with Weil's pairing e_3 (see [LB]) a function can be defined on F with values in the group of cubic roots of unity. It is analogous to Mumford's quadratic form (theta characteristic) on the two-torsion subgroup $J_2(\Gamma)$ of $J(\Gamma)$ and can be defined in complete generality (see [BE]). It mesures the obstruction for a line bundle to descend to the quotient $J(\Gamma)/\sigma$. One can define it as follows. Choose a linearisation of $\mathcal{L}$ with respect to the cyclic group $\mathbb{Z}/3\mathbb{Z}$ generated by σ, i.e., an isomorphism $\phi\colon \mathcal{L} \tilde{\to} \sigma^*(\mathcal{L})$ with $\phi(0) = \mathrm{Id}_{\mathcal{L}(0)}$. When x is a fixed point of σ, then ϕ induces an isomorphism of $\mathcal{L}(x)$ which is multiplication by a root of unity $e(x)$, and $e\colon x \mapsto e(x)$ is the desired function. It depends on the choice of $\mathcal{L}$ itself and not only on the polarisation. If Θ is the (theta) divisor which corresponds to $\mathcal{L}$, i.e., $\mathcal{L} = [\Theta]$, then the corresponding $e = e_\Theta$ may be computed as follows. Let $f = 0$ be a local defining function for Θ in x. Since the divisor Θ is non-singular, the leading part h of f is linear and we have $\sigma^*(h) = e(x)h$. Since the singular points are of type A_2, as is seen from (4), there exist local coordinates $\{u, v\}$ in x such that $\sigma^*(u) = \epsilon u$ and $\sigma^*(v) = \epsilon^2 v$. Therefore we have either $h = u$ and $e(x) = \epsilon$, or $h = v$ and $e(x) = \epsilon^2$. Also if $x \notin \Theta$ then $e(x) = 1$. It follows that e_Θ is explicitly given for all $x \in F$ by

$$e_\Theta(x) = \sigma_{*|T_x\Theta} \text{ or equivalently } e_\Theta(x)v = \sigma_* v \text{ for all } v \in T_x\Theta. \tag{6}$$

The automorphisms μ and τ act on F as well as on the set of invariant theta curves. It is desirable to have a "totally symmetric" theta curve, i.e., invariant by σ, τ and μ. The main observation of this paragraph, from which the 9_4-configuration is a consequence, is the following.

Proposition 3 *There is a unique totally symmetric theta curve among the nine invariant theta curves. The function e_Θ associated to this curve Θ is a quadratic form on F; it is given in a suitable base for F and upon identification of the group of cubic roots of 1 with $\mathbb{F}_3$ by $e_\Theta(r, s) = r^2 - s^2 \pmod 3$.*

Proof

The existence of the curve is clear: since the polarisation is invariant by the group $\mathrm{Aut}(\Gamma)$, we may find an invariant invertible sheaf which gives this polarisation, hence also an invariant divisor. It is unique since if there are two $\mathrm{Aut}(\Gamma)$-invariant curves, then their (two) intersection points must be invariant under $\mathrm{Aut}(\Gamma)$ which is impossible by Table 1. It is easy to identify Θ: it is given by the image of $P \mapsto [P + \infty_1 - 2\infty_2]$. To see this, remark that it can be written as

$$P \mapsto [P + S_1 + \tau(S_1) - 3S_2],$$

independent of the choice of $S_1, S_2 \in \{o_1, o_2, \infty_1, \infty_2\}$. From this representation it is also clear that Θ contains the four points $[\tau(S_1) - S_1]$, $S_1 \in \{o_1, o_2, \infty_1, \infty_2\}$.

Let us determine e_Θ in terms of the base $\{\zeta_1, \zeta_2\}$ where $\zeta_1 = [\infty_2 - \infty_1]$ and $\zeta_2 = [o_2 - o_1]$. Since $\sigma = \tau\sigma\tau$ and $\tau^2 = 1$ it follows using the chain rule that if $\sigma(x) = x$ and $v \in T_{\tau(x)}\Theta$ then

$$e_\Theta(\tau(x))v = \sigma_* v = \tau_*\sigma_*\tau_* v = e_\Theta(x)\tau_*\tau_* v = e_\Theta(x)v,$$

hence $e_\Theta(\tau(x)) = e_\Theta(x)$. In the same way it follows from $\sigma = \mu\sigma^{-1}\mu$ that $e_\Theta(\mu(x)) = e_\Theta(x)^{-1}$. Therefore, if we identify the group of cubic roots of unity with $\mathbb{F}_3$ by $e_\Theta(\zeta_1) = 1$ then e_Θ is given by

$$e_\Theta(r\zeta_1 + s\zeta_2) = r^2 - s^2 \pmod 3).$$

∎

The 9_4 configuration is now described as follows. if ω and ω' are two fixed points, then

$$\omega \in \Theta + \omega' \text{ iff } e(\omega - \omega') \neq 0.$$

It follows that every invariant theta curve passes through four fixed points and that every fixed point belongs to four invariant theta curves. Moreover we have seen that the function e_Θ determines the direction of the tangent to Θ in the fixed points of σ. Therefore, if ω, $\omega' \in F$ then $\Theta + \omega$ and $\Theta + \omega'$ are tangent in a common point $x \in F$ if and only if

$$e_{\Theta+\omega}(x) = e_{\Theta+\omega'}(x).$$

Since $e_{\Theta+\omega}(x) = e_\Theta(x - \omega)$, this condition is rewritten as

$$e_\Theta(x - \omega) = e_\Theta(x - \omega')$$

which is satisfied for $\omega' = 2x - \omega$ (only). We conclude that the four invariant curves running through one fixed point come in two pairs: since any two theta curves always intersect in two points (which may coincide), the curves of one pair are tangent in their unique intersection point and the curves of opposite pairs intersect in two different points (see Figure 1, which also contains the dual picture, equally present in the 9_4-configuration).

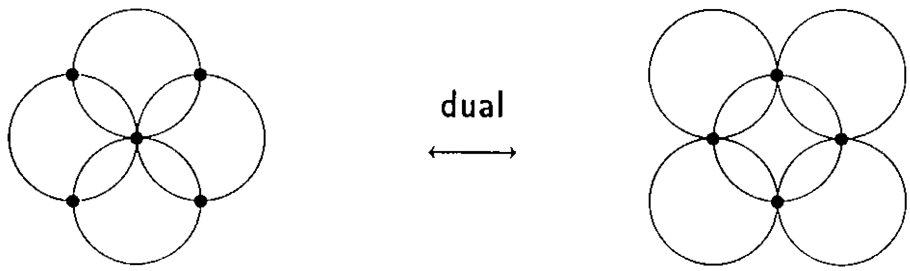

Figure 1: The incidence of points and lines on the 9_4-configuration

4. Equations for $J(\Gamma)/\sigma$ in $\mathbb{P}^4$

In this section we will compute explicit equations for the quotient $S = J(\Gamma)/\sigma$ as an algebraic surface in $\mathbb{P}^4$. Since σ has nine fixed points, it has nine fixed points and we have seen that they are of type A_2. The minimal resolution of these singularities of S leads to a K-3 surface (a generalised Kummer surface), which we will denote by X (see [B]). Let $\pi: J(\Gamma) \to S$ be the quotient and denote by Θ the unique divisor given by Proposition 3.

Proposition 4 *Let M be the divisor on S for which $\pi^*(M) = [3\Theta]$. Then M is very ample and allows to embed S as the complete intersection of a quadric and a cubic threefold in $\mathbb{P}^4$.*

Proof

Since most of the proof is standard, we only give few details. Using the quadratic form e_Θ we see that $\mathcal{L}^{\otimes 3} = [3\Theta]$ descends to an invertible sheaf M on S, i.e., $\pi^*(M) = \mathcal{L}^{\otimes 3}$. Let us denote the line bundle on X which corresponds to M by N. Then using $\mathcal{L} \cdot \mathcal{L} = 2$ we find

$$18 = \mathcal{L}^{\otimes 3} \cdot \mathcal{L}^{\otimes 3} = (\deg \pi) M \cdot M = 3\, M \cdot M,$$

so that $M \cdot M = 6$, which is also the self-intersection of N. Therefore, we find by the Riemann-Roch Theorem (for K-3 surfaces),

$$\chi(N) = \chi(\mathcal{O}_X) + \frac{N \cdot N}{2} = 2 + 3 = 5.$$

It follows moreover from Serre duality and Kodaira vanishing (for K-3 surfaces) that $\dim H^i(X, \mathcal{O}(N)) = 0$ for $i > 0$, so that $\dim H^0(X, \mathcal{O}(N)) = \chi(N) = 5$.

The morphism ϕ_N corresponding to N factorises via the blow-up $p: X \to S$ and is shown to provide an injective morphism $\phi: S \to \mathbb{P}^4$. If we consider now the surjective map

$$\operatorname{Sym} H^0(X, N) \to \oplus_{\ell \geq 0} H^0\left(X, N^{\otimes \ell}\right),$$

whose kernel leads to the defining equations for the image of S in $\mathbb{P}^4$, we see by a dimension count as above that the kernel contains a quadratic as well as an (independent) cubic form. Since the degree of N equals six, we see that the image is the complete intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$. ∎

We will now use the so-called *even master system,* introduced and studied by the second author in [V]. Let us shortly recall what is needed for our purposes. Let us denote by Θ' one of the four invariant theta curves which is tangent to Θ, say $\Theta' = \Theta + [\infty_2 - \infty_1]$. Then every point $\omega \in J(\Gamma) \setminus (\Theta + \Theta')$ is written uniquely as $[P + Q - 2\infty_2]$ and $P, Q \notin \{\infty_1, \infty_2\}$. It follows that we may associate to ω three polynomials $u(x) = x^2 + u_1 x + u_2$, $v(x) = v_1 x + v_2$ and $w(x) = x^4 - u_1 x^3 + w_0 x^2 + w_1 x + w_2$ by†

$$u_1 = -x(P) - x(Q), \qquad u_2 = x(P)x(Q),$$
$$v_1 = \frac{y(P) - y(Q)}{x(P) - x(Q)}, \qquad v_2 = \frac{x(P)y(Q) - x(Q)y(P)}{x(P) - x(Q)}$$

† if $x(P) = x(Q)$ then the definitions of v_1 and v_2 are adjusted in an appropriate way

and $w(x)$ is defined by the fundamental relation $u(x)w(x)+v^2(x) = f(x)$, where $f(x)$ is the right hand side of our equation $y^2 = x^6 + 2\kappa x^3 + 1$ for the curve Γ. The coefficients of this fundamental equation actually lead to affine equations for the affine part $J(\Gamma) \setminus (\Theta + \Theta')$ and are easily written out as

$$\begin{aligned} &w_0 - u_1^2 + u_2 = 0, \\ &w_1 + w_0 u_1 - u_1 u_2 = 2\kappa, \\ &w_2 + w_1 u_1 + u_2 w_0 + v_1^2 = 0, \\ &w_1 u_2 + w_2 u_1 + 2v_1 v_2 = 0, \\ &u_2 w_2 + v_2^2 = 1. \end{aligned} \tag{7}$$

Remark that the action of σ on these coordinates is very simple: the action is diagonal and, if we assign to $(u_1, u_2, v_1, v_2, w_0, w_1, w_2)$ the weights $(1, 2, 2, 3, 2, 3, 4)$, then all equations in (7) are weight homogenous and each variable is multiplied by ϵ as often as given by its weight. Clearly the action leaves the equations (7) invariant.

A second ingredient which we need from [V] is that we may find explicitly in terms of these variables (a base for) the polynomials which have a pole of order three along Θ and are holomorphic elsewhere. This is done by using a vector field on $J(\Gamma)$ and its Laurent solutions which are written down there (we refer to [V] Sect. 6.b for more details). Obviously a weight homogeneous base for these functions can be chosen and functions which are invariant by σ are the ones whose weight is a multiple of three. The list is the following.

$$\begin{aligned} z_0 &= 1, \\ z_1 &= u_1 u_2 - v_2, \\ z_2 &= 2u_1(u_2 + v_1 - u_1^2), \\ z_3 &= 2u_2 v_1^2 + 2v_2^2 + 2u_2 v_1(2u_2 - u_1^2) + 2u_1 v_2(u_1^2 - v_1 - 3u_2) + 2u_2^3, \\ z_4 &= 2v_1^3 - 2(u_1^2 + 4u_2)v_1^2 + 10v_2(u_1 v_1 - v_2) + 2v_1(7u_2 u_1^2 - u_1^4 - 11u_2^2) \\ &\qquad + 2v_2(2\kappa + 15u_1 u_2 - 5u_1^3) + 2(u_1^2 - u_2)^3 - 10u_2^3 - 4\kappa u_1 u_2. \end{aligned} \tag{8}$$

To find the image of $J(\Gamma)$ in $\mathbb{P}^4$ it suffices to eliminate the variables u_i, v_i and w_i from (7) and (8). In fact, from the first three equations of (7) the variables w_i are eliminated linearly and the other equations reduce to

$$\begin{aligned} 2\kappa(u_2 - u_1^2) + 3u_1 u_2^2 - u_1 v_1^2 - 4u_2 u_1^3 + 2v_1 v_2 + u_1^5 = 0, \\ -2\kappa u_1 u_2 + u_2 u_1^4 - u_2 v_1^2 - 3u_1^2 u_2^2 + u_2^3 + v_2^2 = 1, \end{aligned} \tag{9}$$

so it suffices to eliminate u_1, u_2, v_1 and v_2 from (9) and (8) (we have already eliminated the w_i-variables in (8)). In the latter z_1 and z_2 are solved linearly for v_1 and v_2,

$$\begin{aligned} v_1 &= u_1^2 - u_2 + \frac{z_2}{2u_1}, \\ v_2 &= u_1 u_2 - z_1, \end{aligned} \tag{10}$$

and the new equation for z_3, obtained by substituting (10) in (8) is then solved linearly for u_2 as

$$u_2 = \frac{2u_1^2}{z_2^2}\left(z_3 - z_1 z_2 - 2z_1^2\right). \tag{11}$$

After substitution of (10) and (11) in the last equation of (8) and in the equations of (9), we are left with three linear equations in u_1^3, which reflects the fact that $J(\Gamma)$ will be a 3:1 cover of its image in $\mathbb{P}^4$. If we eliminate u_1^3 we arrive at the following two equations:

$$\begin{aligned}
&8z_1^3 - 24\kappa z_1^2 - 4\left(2\kappa z_2 + 6z_3 + z_4\right) z_1 + 4\kappa z_3 - 2\kappa z_2^2 - 3z_2 z_3 - z_2 z_4 = 0,\\
&8z_1^4 - 16\kappa z_1^3 - 4\left(2 + 2\kappa z_2 + 6z_3 + z_4\right) z_1^2 + \left(8\kappa + 4\kappa z_3 - 2\kappa z_2^2 - 4z_2 - 3z_2 z_3 - z_2 z_4\right) z_1\\
&\qquad + 2\kappa z_2 z_3 + 14z_3 + 2z_4 - 2z_2^2 + z_3 z_4 + 5z_3^2 = 0.
\end{aligned}$$

Using the first equation, the second equation can be replaced by

$$\begin{aligned}
8\left(1 - 3\kappa^2\right) z_1^2 + 4\left(-2\kappa + \left(1 - 2\kappa^2\right) z_2 - 6\kappa z_3 - \kappa z_4\right) z_1 + 2\left(1 - \kappa^2\right) z_2^2&\\
- 5z_3^2 - \kappa z_2\left(5z_3 + z_4\right) + 2z_3\left(2\kappa^2 - 7\right) - z_3 z_4 - 2z_4 = 0&,
\end{aligned}$$

or equivalently by ${}^tZAZ = 0$, where

$$A = \begin{pmatrix}
0 & -8\kappa & 0 & 2(2\kappa^2 - 7) & -2\\
-8\kappa & 16(1 - 3\kappa^2) & 4(1 - 2\kappa^2) & -24\kappa & -4\kappa\\
0 & 4(1 - 2\kappa^2) & 4(1 - \kappa^2) & -5\kappa & -\kappa\\
2(2\kappa^2 - 7) & -24\kappa & -5\kappa & -10 & -1\\
-2 & -4\kappa & -\kappa & -1 & 0
\end{pmatrix}.$$

Although at this point these equations for the quadric and cubic hypersurfaces which define S as a subset of $\mathbb{P}^4$ (which we will identify in the sequel with S) may not seem very attractive, we will see that natural coordinates can be picked for $\mathbb{P}^4$ in which these equations take a very symmetric form. Indeed, the choice of base we have picked for $\mathbb{P}^4$ was rather arbitrary: for example, the coordinates of the nine fixed points for σ do not possess special coordinates in terms of the present base. The first interesting observation is here that if the five fixed points for σ which do not lie on Θ are taken as base points for $\mathbb{P}^4$ then the four fixed points on Θ take a simple form and are independent of κ. To see this, let $\alpha = \infty_1 - \infty_2$ and $\beta = o_1 - o_2$ and remark that the points

$$\{O,\ -\alpha - \beta,\ -\alpha + \beta,\ \alpha - \beta,\ \alpha + \beta\}$$

are the five poins which do not lie on Θ. To find their coordinates, use a local parameter t and take $x = t$,

$$y = \pm\left(1 + \kappa t^3\right) + \mathcal{O}\left(t^6\right),$$

picking either sign around o_1 or o_2, and in the same way, $x = t^{-1}$ and

$$y = \pm\left(t^{-3} + \kappa + \frac{1-2\kappa^2}{2}t^3\right) + \mathcal{O}\left(t^4\right)$$

for ∞_1 and ∞_2. Then a careful computation yields the following coordinates:

$$\begin{aligned}
O \quad &: (0\!:0\!:0\!:0\!:1),\\
\alpha \pm \beta &: (0\!:0\!:1\!:\pm 1\!:\mp 3 - 2\kappa),\\
-\alpha \pm \beta &: (1\!:\pm 1\!:\mp 2 - 2\kappa\!:\mp 2\kappa\!:4\kappa^2 \pm 14\kappa + 4).
\end{aligned}$$

We take the points

$$\{O,\ -\alpha-\beta,\ -\alpha+\beta,\ \alpha-\beta,\ \alpha+\beta\}$$

as base points for $\mathbb{P}^4$ (in that order), i.e., $O = (1\!:0\!:0\!:0\!:0)$, etc., with associated coordinates $y_0, \ldots, y_4$. Then the four fixed points on Θ have as coordinates

$$\begin{aligned}
\alpha &= (1\!:1\!:1\!:0\!:0), & -\alpha &= (1\!:0\!:0\!:1\!:1),\\
\beta &= (1\!:1\!:0\!:1\!:0), & -\beta &= (1\!:0\!:1\!:0\!:1),
\end{aligned}$$

and we see that they lie on the (2-dimensional!) plane

$$y_0 = y_2 + y_3 = y_1 + y_4,$$

and it is easy to see that in fact Θ is contained in this plane. The translations τ_α and τ_β correspond to projective transformations of the surface and take in terms of these coordinates the simple form

$$t_\alpha = \begin{pmatrix} -1 & 1 & 1 & 0 & 0\\ -1 & 0 & 1 & 1 & 0\\ -1 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad t_\beta = \begin{pmatrix} -1 & 1 & 0 & 1 & 0\\ -1 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0\\ -1 & 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

from which the equations for the planes to which the other invariant curves belong, are obtained at once. This configuration of nine points in $\mathbb{P}^4$ is characterised by the fact that there exist nine planes with the property that each of these planes contains four of the nine points and every point belongs to four of the planes. Thus we have recovered in a direct way a configuration that has been studied in the work of Segre and Castelnuovo on nets of cubic hypersurfaces in $\mathbb{P}^4$ (see [C] and [S]).

The equations of the quadric and cubic hypersurfaces $\mathcal{Q}$ and $\mathcal{C}$ take in terms of the new coordinates the following symmetric form.

$$\begin{aligned}
&\mathcal{Q}: c(y_1+y_4)(y_2+y_3+y_4-y_0) + \bar{c}(y_2+y_3)(y_1+y_3+y_4-y_0) = cy_4^2 + \bar{c}y_3^2,\\
&\mathcal{C}: \bar{c}^3 y_2^2(y_1+y_3+y_4-y_0) + \bar{c}^2 c y_2\left((y_1+y_4)(y_2-y_0) + y_0y_3 + y_1y_4\right) -\\
&\qquad c^3 y_1^2(y_2+y_3+y_4-y_0) - \bar{c}c^2 y_1\left((y_2+y_3)(y_1-y_0) + y_0y_4 + y_2y_3\right) = 0,
\end{aligned}$$

where $c = \kappa + 1$ and $\bar{c} = 1 - \kappa$. The cubic equation can be simplified in a significant way by adding to it the equation for Q multiplied with $c^2 y_1 - \bar{c}^2 y_2$. The result is

$$c^2 y_1 y_4 (y_2 + y_3 - y_0) - \bar{c}^2 y_2 y_3 (y_1 + y_4 - y_0) = 0.$$

If we define

$$\begin{aligned} x_1 &= -y_1, & x_4 &= y_2, \\ x_2 &= -y_4, & x_5 &= y_1 + y_4 - y_0, \\ x_3 &= y_0 - y_2 - y_3, & x_6 &= y_3, \end{aligned}$$

then S is given as an algebraic variety in $\mathbb{P}^5$ by

$$\begin{aligned} &\mathcal{C} : c^2 x_1 x_2 x_3 + \bar{c}^2 x_4 x_5 x_6 = 0, \\ &\mathcal{Q} : c(x_1 x_2 + x_2 x_3 + x_1 x_3) + \bar{c}(x_4 x_5 + x_5 x_6 + x_4 x_6) = 0, \\ &\mathcal{H} : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0, \end{aligned} \tag{12}$$

and the singular points of S are now the points $\Delta_{ij}\,(i, j = 1, \ldots, 3)$ with a 1 on the i-th place, a -1 on position $3 + j$ and zeroes elsewhere; the nine planes they belong to are given by $\mathcal{H} \cap (x_i = x_{j+3} = 0)$ for $i, j = 1, \ldots, 3$. This presents the 9_4 configuration in the form used by Segre and Castelnuovo. Remark that if one changes the sign of κ in the equations (12) then an isomorphic surface is obtained (interchange $c \leftrightarrow \bar{c}$ and $x_i \leftrightarrow x_{i+3}$ for $i = 1, \ldots, 3$), in agreement with the fact that κ^2 is the modular parameter.

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