On the presentation of Hodge algebras

and the existence of Hodge algebra structures

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MPI/90– 76

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Dedicated to Professor H. Matsumura on occasion of his 60th birthday

1. Introduction

The concept of Hodge algebras was introduced by C. DeConcini, D. Eisenbud, and C. Procesi [3] in order to unify and extend a large amount of information about specific algebras such as the coordinate rings of Grassmannians, flag manifolds, Schubert varieties, determinantal and Pfaffian varieties, varities of complexes, and varieties of minimal degree. It generalizes the straightening formula of invariant theory, which allows to determine many features of the algebras by a relatively simple combinatorial study of their generators and relations. See e.g. [1], [2], [7], [8] for recent developments in the theory of Hodge algebras.

This paper will deal with two open questions on the structure of Hodge algebras.

The first question is whether the straightening relations give a presentation of the Hodge algebra. This question has an affirmative answer in the graded case [3]. We shall see that in general, this is not true even for the class of ordinal Hodge algebras (algebras with straightening laws). We construct examples which show that there is no general rule for the presentation of Hodge algebras.

The second question is when an algebra over a field k has a Hodge algebra structure. In [6] T. Hibi has proven that graded algebras generated over k by elements of positive degree always have a Hodge algebra structure. According to [3], a necessary condition for the existence of Hodge algebra structures is that the irreducible components of the spectrum of the algebra contain a common rational point. For zero–dimensional algebras, this condition is also sufficient. In fact, the zero–dimensional Hodge algebras are exactly the zero–dimensional local algebras. In general, the aforementioned condition is far from being sufficient. We shall see that the spectrum of an one–dimensional Hodge algebra must have a rational point which is a set–theoretic complete intersection.

ACKNOWLEDGEMENT. The author is grateful to W. Bruns who read the first draft of this paper and gave many useful suggestions.

2. Presentation of Hodge algebras

We first recall some conventions from the theory of Hodge algebras.

Let H be a finite set. A *monomial* on H is an element of N^H, where N denotes the set of non–negative integers. Let M and N be monomials, then their product MN is defined by $\mathrm{MN(x) = M(x) + N(x)}$ for all $x \in H$. We say that N divides M if $\mathrm{N(x) \leq M(x)}$ for all $x \in H$, and in this case, we define the quotient monomial M/N by $\mathrm{M/N(x) = M(x) - N(x)}$. The *support* of M is the set $\mathrm{supp(M)} = \{x \in H \mid \mathrm{M(x)} \neq 0\}$.

An *ideal of monomials* on H is a set $I \subseteq N^H$ such that if $\mathrm{M} \in I$, then $\mathrm{MN} \in I$ for any monomial N. A monomial M is called *standard* with respect to I if $\mathrm{M} \notin I$. A *generator* of I is an element of I which is not divisible by any other element of I. If A is a commutative ring and if there is an injection $\phi : H \longrightarrow A$, then to each monomial M on H we may associate the element $\phi(\mathrm{M}) = \prod \phi(\mathrm{x})^{\mathrm{M}(\mathrm{x})}$ of A. We will usually identify H with $\phi(H)$ and M with $\phi(\mathrm{M})$.

Let R be a commutative ring with identity and A a commutative R–algebra. Let H be a finite poset with an injection $\phi : H \longrightarrow A$, and I an ideal of monomials on H.

DEFINITION. A is called a *Hodge algebra* generated by H and governed by I if the following conditions are satisfied:

(H1) A is a free R–module admitting the set of standard monomials as a basis.

(H2) If N is a generator of I and $\mathrm{N} = \sum \mathrm{r_i M_i}$ $(\mathrm{r_i} \neq 0)$ is the unique expression of N in A as a linear combination of distinct standard monomials, then for each $\mathrm{x} \in \mathrm{supp(N)}$ and for each $\mathrm{M_i}$ there is an element $\mathrm{y} \in \mathrm{supp(M_i)}$ with $\mathrm{y} < \mathrm{x}$ (Such linear combinations are called the *straightening relations* of A).

In particular, if I is the ideal generated by the products of the pairs of incomparable elements of H, A is called an *ordinal Hodge algebra* (or an algebra with straightening law).

We refer the reader to [3] for various interesting properties and examples of Hodge algebras. In the following, if A is assumed to be a Hodge algebra, then H always denotes the generating poset and I is the ideal of non–standard monomials.

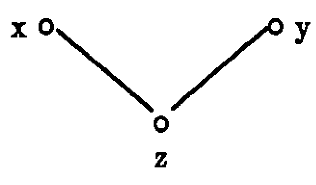
The above definition naturally leads to the question that the straightening relations give a presentation of the Hodge algebra. DeConcini, Eisenbud and Procesi have given an affirmative answer to this question for graded Hodge algebras [3, Proposition 1.1]. The main point of their proof is that one can define a weight for every monomial M on H such that if M is non–standard and if one chooses any generator N dividing M and replaces N by the right side of its straightening relation, then M is expressed as a linear combination of monomials of strictly greater weight. If any of the resulting monomials is non–standard, one repeats this process. In the graded case this process must terminate because there are only finitely many monomials of a given degree.

In general, the replacing process can be infinite and one would obtain an infinite sequence of non–standard monomials with increasing weight. Such a sequence converges to zero in the HA–adic topology, which is separated by [3, Corollary 3.5]. Therefore, a non–standard monomial is likely to be zero if it can not be expressed as a linear combination of standard monomials via the straightening relations. This has led the author to the following counter–example to the above question.

EXAMPLE 2.1. Let A be the algebra

$$k[x,y,z] =: k[X,Y,Z]/(X^2 - XYZ, Y^2 - XYZ, X^3, X^2Y, XY^2, Y^3, Z^2)$$

where k is an arbitrary field. Let H be the poset



Let I be the ideal of monomials on H generated by x^2, y^2, and z^2. Then the standard monomials with respect to I are z, x, y, xz, yz, xyz. If these monomials are not linearly independent in A, there exists a relation

$$az + bx + cy + dxz + eyz + fxy =$$
$$AZ^2 + B(X^2 - XYZ) + C(Y^2 - XYZ) + DX^3 + EX^2Y + FXY^2 + GY^3 \ ,$$

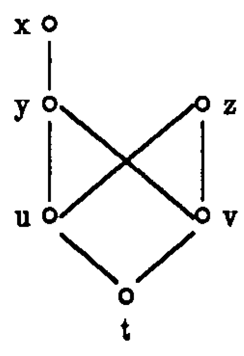
where a, b, c, d, e, f $\in$ k and A, B, C, D, E, F, G $\in$ k[X,Y,Z] . By comparing the coefficients of both sides of the relation, we immediately obtain $a = b = c = d = e = f = 0$, a contradiction. Thus, the standard monomials with respect to I are linearly independent. Since there are the straightening relations $x^2 = xyz$, $y^2 = xyz$, and $z^2 = 0$ for the generators x^2 , y^2 , and z^2 of I and since all the other monomials of I vanish in A , we see that A is a Hodge algebra whose straightening relations do not give a presentation.

It is claimed in [4, Theorem 3.4] that the straightening relations give a presentation for ordinal Hodge algebras. But this is not correct. We will construct below an ordinal Hodge algebra A whose straightening relations do not give a presentation for A . Note that the associated poset is not so simple as in the above example.

EXAMPLE 2.2. Let A be the algebra

$$k[x,y,z,t,u,v] = k[X,Y,Z,T,U,V]/(XZ-UZ, YZ-VZ, XYZ, UV-XYT) \ .$$

Let H be the poset



Let I denote the ideal of monomials on H generated by the products xz, yz, uv of pairs of incomparable elements. Then the standard monomials with respect to I are the monomials $x^m y^n u^r t^s$, $x^m y^n u^r t^s$, $z^m v^n t^r$, $z^m u^n t^r$. If these monomials are linearly dependent, there exists a relation

$$\sum (a_{mnr} X^m Y^n U^r + b_{mnr} X^m Y^n V^r + c_{mn} Z^m U^n + d_{mn} Z^m V^n) =$$

$$D(XZ - UZ) + E(YZ - VZ) + FXYZ + G(UV - XYT)$$

where $a_{mnr}, b_{mnr}, c_{mn}, d_{mn} \in k[T]$ and $D, E, F, G \in k[X,Y,Z,U,V,T]$. Write $G = G' + ZG''$ with $G' \in k[X,Y,U,V,T]$. We can see that

$$\sum (a_{mnr} X^m Y^n U^r + b_{mnr} X^m Y^n V^r) = G'(UV - XYT) \ .$$

Since the left term does not contain any summand divisible by UV , is easy to verify that $G' = 0$. Therefore $a_{mnr} = b_{mnr} = 0$ for all m, n, r and

$$\sum (c_{mn} Z^{m-1} U^n + d_{mn} Z^{m-1} V^n) = D(X-U) + E(Y-V) + FXY + G''(UV-XYT) \ .$$

Since

$$(X-U,Y-V,XY,UV-XYT) = (X-U,Y-V,XY,UV) = (X-U,Y,V) \cap (X,Y-V,U)$$
$$(X-U,Y,V) \cap k[Z,U,V] = Vk[Z,U,V]$$
$$(X,Y-V,U) \cap k[Z,U,V] = Uk[Z,U,V] \ ,$$

we have

$$(X-U,Y-V,XY,UV-XYT)\cap k[Z,U,V] = UVk[Z,U,V] \ .$$

Hence $c_{mn} = d_{mn} = 0$ for all m and n , a contradiction. We have proven that the standard monomials with respect to I are linearly independent. Now we are going to show that every monomial of I can be expressed as a linear combination of standard monomials. By definition, I consists of monomials of the form xzM, yzM, uvM , where M can be any monomials on H. First, using the relations $xyz = 0$, $xzv = zuv = xyzt = 0$, $x^n z = u^n z$, we can easily represent xzM as a monomial whose support is contained in the set {z,u,t} . But such a monomial is standard. Similarly we can also represent yzM as a standard monomial whose support is contained in {z,v,y} and uvM as a standard monomial whose support is contained either in {x,y,u,t} or in {x,y,v,t} . So we can conlcude that A is an ordinal Hodge algebra, whose straightening relations $xz = uz$, $yz = vz$, $uz = xyt$ do not give a representation for A .

Now, one may ask whether the straightening relations plus the vanishing non–standard monomials give a presentation for the Hodge algebra A . If $\dim(A) = 0$, the answer is affirmative.

PROPOSITION 2.3. *Let A be a zero–dimensional Hodge algebra. Then the straightening relations and the vanishing non–standard monomials give a presentation for A .*

Proof. If one applies the aforementioned replacing process only to non–vanishing non–standard monomials, then it must terminate because there are only a finite number of non–vanishing monomials on H .

The following example shows that there is no general rule for the presentation of Hodge algebras, i.e. beside the straightening relations one may need another relations of arbitrary forms.

EXAMPLE 2.4. Let $\mathfrak{a}$ denote the ideal

$$(XZ-UZ+XTF-UTF,\ YZ-VZ+YTF-VTF,\ XYZ+XYTF,\ UV-XYT)$$

of the polynomial ring $k[X,Y,Z,T,U,V]$, where F is an arbitrary polynomial of $k[T]$. Let B be the algebra

$$k[x,y,z,t,u,v] := k[X,Y,Z,T,U,V]/\mathfrak{a} \ .$$

Let H be the poset given in Example 2.2. We claim that B is an ordinal Hodge algebra. Since there are the straightening relations

$$\begin{aligned} xz &= uz-xtf+utf \\ yz &= vz-ytf+vtf \\ uv &= xyt \ , \end{aligned}$$

where f denotes the image of F in B , we need only to show that the standard monomials form a basis for the k–vector space B . First, we notice that there is a natural transformation ε from B to the ordinal Hodge algebra A of Example 2.2 which is given by the map

$$x \longmapsto x,\ y \longmapsto y,\ z \longmapsto z - tf,\ t \longmapsto t,\ u \longmapsto u,\ v \longmapsto v\ .$$

Let M be any standard monomial on H (supp(M) is a chain in H). Write $M = t^r N$, $r \geq 0$ and $t \notin \text{supp}(N)$. Then

$$\varepsilon(t^r N) = t^r N + t^{r+1} g$$

for some element $g \in A$. We notice that $t^r N$ can not be eliminated by any term of the expression of $t^{r+1} g$ as a linear combination of standard monomials. Since the standard monomials form a basis for the k–vector space A, from the above formula for $\varepsilon(M)$ we can deduce that the elements $\varepsilon(M)$ are linearly independent. It is obvious that any standard monomial can be expressed as a linear combination of the elements $\varepsilon(M)$. Hence the set of the elements $\varepsilon(M)$ also form a basis for the k–vector space A. Thus, the standard monomials must form a basis for the k–vector space B.

3. The existence of Hodge algebra structures

The definition of Hodge algebras is so broad that one may well believe that every finitely generated algebra over a field is a Hodge algebra in some way. T. Hibi has confirmed this first for affine semigroup rings [5] and then for graded algebras [6]. In general, this is not true because of the following simple reason.

LEMMA 3.1. *Let* A *be a Hodge algebra over a field* k. *Then* A *satisfies the condition:*
(∗) *The irreducible components of* Spec(A) *contain a common rational point.*

Proof. Let $P = HA$. Then P is a prime ideal because $A/P \cong \mathrm{k}$ by the definition of Hodge algebras. Further, $\cap P^n = 0$ by [3, Corollary 5.3]. By Krull's Intersection Theorem, $\cap P^n$ is the intersection of all primary components of the zero ideal of A which are contained in P. Hence we can conclude that all associated ideals of A are contained in P.

One may ask whether condition (*) is also sufficient for the existence of Hodge algebra structures. If $\dim(A) = 0$, the answer is affirmative. In this case, condition (*) simply means that A is a local ring.

PROPOSITION 3.2. *Let* A *be a zero-dimensional local algebra over a field* k . *Then* A *has a Hodge algebra structure.*

Proof. We go by induction on the length $\ell(A)$. Let P be the maximal ideal of A . Note that the zero ideal of A is a P–primary ideal. If $\ell(A) = 1$, then $A = \mathrm{k}$ and there is nothing to prove. If $\ell(A) > 1$, we can find an element $\mathrm{x} \in P$ such that $0:\mathrm{x} = P$. Since $\mathrm{x}A$ is a P–primary ideal, $A/\mathrm{x}A$ satisfies the assumption of the theorem again. Therefore we may assume that $A/\mathrm{x}A$ is a Hodge algebra generated by a finite poset K and governed by an ideal J of monomials on K. Let H denote the poset $\{\mathrm{x}\} \cup \mathrm{K}$ where the order is induced from the order of K and x is smaller than any element of K. Let I denote the ideal of monomials on H generated by J and all monomials of the form xM . We embed K in A by mapping K to a fixed set of pre–images of elements of K . Since every element of $\mathrm{x}A$ can be uniquely written in the form ax , $\mathrm{a} \in \mathrm{k}$, the set of standard monomials with respect to I forms a basis for A . Further, if N is a generator of I, N is either a generator of J or $\mathrm{N} = \mathrm{xy}$ for some $\mathrm{y} \in H$. In the first case, the straightening relation for N is derived from the straightening relation for N in $A/\mathrm{x}A$ by adding some term ax which depends on the embedding of K in A. In the second case, $\mathrm{xy} = 0$ is the straightening relation for N.

If $\dim(A) > 0$, condition (*) is no longer sufficient for the existence of a Hodge algebra structure. We shall see that if A is an one–dimensional Hodge algebra over a field k, then $P := HA$ is a set–theoretic complete intersection. We start with the following result which generalizes [3, Proposition 1.2 (b)].

LEMMA 3.3. *Let A be a Hodge algebra generated by a poset H and governed by an ideal I of monomials. Let J be an ideal of monomials such that if $\mathrm{M} \in J$ and $\mathrm{M} = \sum \mathrm{r_i M_i}$ is the unique expression of* M *as a linear combination of distinct standard monomials,* $\mathrm{M_i} \in J$ *for all i. Then A/JA is a Hodge algebra generated by H and governed by $I \cup J$, where H is embedded in A/JA by the natural epimorphism from A to A/JA.*

Proof. Let N be an arbitrary generator of $I \cup J$. If $\mathrm{N} \in J$, $\mathrm{N} = 0$ in A/JA. If $\mathrm{N} \in J$, then N is a generator of I and there is a straightening relation $\mathrm{N} = \sum \mathrm{r_i M_i}$ in A. It is easily seen that $\mathrm{N} = \sum \mathrm{r_i M_i}$, $\mathrm{M_i} \in J$, is the straightening relation of N in A/JA. Similarly, every monomial of $I \cup J$ can be expressed as a linear combination of distinct standard monomials in A/JA. Thus, to show that A/JA is a Hodge algebra generated by H and governed by $I \cup J$, we need only to show that the standard monomials with respect to $I \cup J$ are linearly independent in A/JA. Suppose that there exists a non–trivial relation $\sum \mathrm{s_i N_i} \in JA$ between standard monomials $\mathrm{N_i} \notin J$. Since every element of JA is a linear combination of standard monomials in J, there is a linear combination $\sum \mathrm{r_i M_i}$ of standard monomials $\mathrm{M_i} \in J$ such that $\sum \mathrm{s_i N_i} = \sum \mathrm{r_i M_i}$. Hence we obtain a non–trivial relation between standard monomials with respect to I in A, a contradiction.

COROLLARY 3.4. *Let* A *be a Hodge algebra generated by a poset* H *and governed by an ideal* I. *Let* x *be a minimal element of* H. *Then the following statements hold for any positive integer* n:

(*i*) $A/x^n A$ *is a Hodge algebra generated by* H *and governed by the ideal of monomials generated by* I *and* x^n.

(*ii*) $A/(0{:}x^n)$ *is a Hodge algebra generated by* H *and governed by the ideal of monomials generated by* I *and the monomials contained in* $0{:}x^n$.

Proof. If N is any generator of I with $x \in \mathrm{supp}(N)$, then N must be zero because there is no element of H strictly smaller than x. Thus, for any standard monomial M, $x^n M$ is either a standard monomial or zero. Now, let f be an arbitrary element of A and $f = \sum r_i M_i$ is the expression of f as a linear combination of distinct standard monomials. Then $x^n f = \sum r_i x^n M_i$, $x^n M_i \neq 0$, is the expression of $x^n f$ as a linear combination of distinct standard monomials. As a consequence, the assumption of Lemma 3.3 is satisfied for the ideal of monomials generated by x^n. Further, if $x^n f = 0$, we must have $x^n M_i = 0$ for all i. Thus, the ideal $0{:}x^n$ is generated by the monomials contained in it, and the ideal of these monomials satisfies the assumption of Lemma 3.3.

THEOREM 3.5. *Let* A *be an one–dimensional Hodge algebra over a field. Then* $P = HA$ *is the radical of a principal ideal of* A.

Proof. We will go by induction on the number of elements of H. Let x be a minimal element of H. Then A/xA is a Hodge algebra by Corollary 3.4 (i). By the induction hypothesis, P/xA is the radical of a principal ideal. Therefore P is the radical of a principal ideal if x is nilpotent. If x is not nilpotent, let n be a positive integer such that $0{:}x^n = 0{:}x^{n+1}$. Then $x^n A \cap (0{:}x^n) = 0$ and $0{:}x^n$ is the intersection of all primary components of the zero ideal of A whose associated prime ideals do not contain x. Thus, x^n is a non–zerodivisor modulo $0{:}x^n$. By a two–fold application of Corollary 3.4, $A/(0{:}x^n) + x^nA$ is a zero–dimensional Hodge algebra. Hence $(0{:}x^n) + x^nA$ must be a P–primary ideal by Lemma 3.1. On the other hand, since A/xA is a Hodge algebra, using the induction hypothesis we may assume that there is an element $y \in P$ such that (x,y) is a P–primary ideal. Let m be a positive integer such that $y^m \in (0{:}x^n) + x^nA$. Then (x^n,y^m) is a P–primary ideal, too. Write $y^m = z + ax^n$ for some elements $z \in 0{:}x^n$ and $a \in A$. Put $u = x^n + z$. Since $x^nz \in x^nA \cap (0{:}x^n) = 0$, we have $x^{2n} \in uA$. From this it follows that $y^{2m} \in uA$. Hence $uA \supseteq (x^{2n},y^{2m})$ is a P–primary ideal.

COROLLARY 3.6. *Let A be a two–dimensional Hodge algebra over a field. Then $P = HA$ is a set–theoretic complete intersection if A is a domain.*

Proof. Let x be a minimal element of H. Then $\dim(A/xA) = 1$ and A/xA is a Hodge algebra by Corollary 3.4. By Theorem 3.5, P/xA is a set–theoretic complete intersection of A.

REMARK. C. Weibel [9] [10] has described the locus of set–theoretic complete intersection points of affine varieties. He has shown that for an affine curve over an algebraically closed field, this locus is countable or the whole curve itself [9] and that every maximal ideal of a finitely generated algebra over an algebraic extension of a finite field is always a set–theoretic complete intersection [10].

We now construct an algebra which satisfies condition (∗) of Lemma 3.1 but which has no Hodge algebra structure.

EXAMPLE 3.7. Let B be a polynomial ring of more than two variables over an algebraically closed field of characteristic zero. Then one can find a prime ideal q of B with $\dim(B/q) = 1$ and a maximal ideal $p \supset q$ of B such that p/q is not a set–theoretic complete intersection of B/q (see [9]). Let $q_1, q_2 \subset p$ be another prime ideals of B with $\dim(B/q_1) = \dim(B/q_2) = 1$ such that q_2 is not contained in any prime ideal $p' \supset (q, q_1)$, $p' \neq p$. Put $a = q \cap q_1 \cap q_2$. Then $P = p/a$ is not a set–theoretic complete intersection of the algebra $A = B/a$. Since P is the only maximal ideal of A which contains all associated prime ideals of A, there is no Hodge algebra structure on A by Lemma 3.1 and Theorem 3.5.

The ideal $P = HA$ is always a set–theoretic complete intersection if A is a graded Hodge algebra. The same also holds if A is an ordinal Hodge algebra [3, Theorem 6.3]. Therefore it is reasonable to raise the following question.

QUESTION. Is HA a set–theoretic complete intersection for any Hodge algebra A ?

We do not know, even in the case $\dim(A) = 1$, whether the existence of a maximal ideal P such that $A/P = \mathrm{k}$, P contains all associated prime ideals of A and P is a set–theoretic complete intersection is sufficient for the existence of a Hodge algebra structure on A. One may hope to use the inductive method to lift a Hodge algebra structure on $A/\mathrm{x}A$ to A as Hibi [6] has done in the graded case. In particular, following the proof of Proposition 3.2, one can reduce the lifting problem to the case in which x is a non–zero divisor.

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