LINEAR AUTOMORPHISMS THAT ARE SYMPLECTOMORPHISMS

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Abstract. Let $\mathbf{K}$ be a field of real or complex numbers. Let $(X \cong \mathbf{K}^{2n}, \omega)$ be a symplectic space and take $0 < k < n$, $N = \begin{pmatrix} 2n \\ 2k \end{pmatrix}$. Let $L_1, ..., L_N \subset X$ be $2k-$dimensional linear subspaces, which are in a sufficiently general position. We show that if $F : X \to X$ is a linear automorphism, which preserves the form $\wedge^k \omega$ on all subspaces $L_1, ..., L_N$, then F is an ϵ_k-symplectomorphism (i.e. $F^*\omega = c\omega$, where $c^k = 1$). In particular if $\mathbf{K} = \mathbb{R}$ and k is odd then F must be a symplectomorphism. We observe also that the set $\mathcal{A}_{l,2r}$ of all l-dimensional linear subspaces on which the form ω has rank $\leq 2r$ is linear in the Grassmannian $G(l, 2n)$, i.e., there is a linear subspace L such that $\mathcal{A}_{l,2r} = L \cap G(l, 2n)$. In particular the set $\mathcal{A}_{l,2r}$ can be computed effectively.

1. Introduction.

Let $\mathbf{K}$ be a field such that $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$. Let (X, ω) be a symplectic space, i.e., $X \cong \mathbf{K}^{2n}$ is vector space and ω is a bilinear, non-degenerate skew-symmetric form. The symplectic component of a linear subspace $L \subset X$ is defined as the subspace

$$L^\omega = \{x \in X : \omega(x, y) = 0 \forall_{y \in L}\}.$$

This space may not be transversal to L. A subspace $L \subset X$ is called: isotropic if $L \subset L^\omega$, coisotropic if $L^\omega \subset L$, symplectic if $L \cap L^\omega = \{0\}$ and Lagrangian if $L^\omega = L$. L is symplectic if and only if $\omega \mid_L$ is non degenerate form. For any subspace L we have $dim L + dim L^\omega = dim X$ and $(L^\omega)^\omega = L$. For (X, ω) there exists a basis, called the Darboux basis, $u_1, \ldots, u_n, v_1, \ldots, v_n$, such that

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \quad \omega(u_i, v_j) = \delta_{ij}.$$

If $L \subset X$ is a subspace, then there is a basis $u_1, \ldots, u_k, v_1, \ldots, v_k, w_1, \ldots w_l$ of L such that $\omega \mid_L (u_j, v_k) = \delta_{jk}$ and all other pairings $\omega \mid_L (\bullet, \bullet)$ vanish. This basis extends to a Darboux basis for (X, ω) and the integer $2k$ is the rank of $\omega \mid_L$.

We say that a linear automorphism $F : X \to X$ is a symplectomorphism (or symplectic on X) if $F^*\omega = \omega$, i.e. , if $\omega(x, y) = \omega(F(x), F(y))$ for every $x, y \in X$. If $L \subset X$ is a linear subspace, then we say that F is symplectic on L if $\omega(x, y) = \omega(F(x), F(y))$ for every $x, y \in L$. The group of authomorphisms of (X, ω) is called the symplectic group and is denoted by $Sp(X, \omega)$ or in Darboux basis we write $Sp(2n, \mathbb{R})$ as real $2n \times 2n$ matrices A, which satisfy $A^T J_0 A = J_0$, where J_0 is $2n \times 2n$ matrix of ω in Darboux form. If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way $\mathbb{R}^{2n} \ni (x, y) \longrightarrow x + iy$, then multiplication by J_0 in $\mathbb{R}^{2n}$ corresponds to multiplication by i in $\mathbb{C}^n$. And with this identification the complex linear group $GL(n, \mathbb{C})$ is a subgroup of $GL(2n, \mathbb{R})$ and the unitary group $U(n)$ is a maximal, compact subgroup of $Sp(2n, \mathbb{R})$; $U(n) = Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R})$.

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If L is a subspace of (X, ω), then obviously the form $\omega\mid_L$ and its exterior powers (volume form) are symplectic invariants (cf. [1]). The natural questions we may formulate now are following:

1. To how extend such linear subspaces data preserved, determine the group of interesting transformations of symplectic space?
2. Is this possible to collect such partial "information" from the family of subspaces and construct the complete system of invariants for symplectic group?

In this paper, using the methods and ideas introduced in [4], we establish the structural data on subspaces and find the groups of automorphisms defined by them. The criteria for symplectomorphisms are confirmed by the geometrical intuition coming from the standard symplectic geometry. While symplectic structures naturally arise in diverse contexts such as Hamiltonian mechanics, field theory, differential and algebraic geometry we focus on the motivation to extend the considerations including the time reversible (including anti-symplectomorhisms) Hamiltonian systems (cf. [6]) and composite systems with weakened phase space structure.

Let $N = \left(\begin{array}{c} 2n \\ l \end{array}\right)$ and $G = G(l, 2n) \subset \mathbb{P}^{N-1}$ denote a Grassmannian, which parameterizes all $l-$dimensional linear subspaces of X. Let $\{L_1, ..., L_m\} \subset G(l, 2n)$. We say that subspaces $L_1, ..., L_m$ are co-planar, if there is a hyperplane $\Lambda \subset \mathbb{P}^{N-1}$ which contains all of them.

If subspaces $L_1, ..., L_m$ are not co-planar, then we can always choose a sub-family $L_1, ..., L_k$ of not co-planar subspaces with $k = N$ and conversely it is easy to construct a collection $L_1, ..., L_N$ which is not co-planar. Hence we can always assume that $m = N$. We prove:

Theorem. *Let (X, ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < n$ be a natural number. Assume, that F preserves the form $\wedge^k\omega$ on a collection $L_1, ..., L_N$ of $2k-$dimensional subspaces, which are not co-planar. Then F is an ϵ_k-symplectomorphism. (i.e., $F^*\omega = \epsilon\omega$, where $\epsilon^k = 1$). In particular if $\mathbf{K} = \mathbb{R}$ and the number k is odd, then F is a symplectomorphism.*

Corollary. *Let (X, ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < l \leq n$ be natural numbers such that $(k, l) = 1$. Assume, that F preserves forms $\wedge^k\omega$ and $\wedge^l\omega$. Then F is a symplectomorphism.*

We observe also, that the following is true:

Theorem. *Let (X, ω) be a $2n$-dimensional symplectic space and let $0 < l < 2n$ be a natural number. The set $\mathcal{A}_l$ of all l-dimensional linear subspaces on which the form ω degenerates (i.e., has not a maximal rank) is co-planar in the Grassmannian $G(l, 2n) \subset \mathbb{P}^{N-1}$. More precisely, if $\mathcal{A}_{l,2r}$ denote the set of all l-dimensional linear subspaces on which the form ω has rank $\leq 2r$, then the set $\mathcal{A}_{l,2r-2}$ is co-planar in the set $\mathcal{A}_{l,2r}$, i.e., there is a linear projective subspace $L \subset \mathbb{P}^{N-1}$ such that $\mathcal{A}_{l,2r} \cap L = \mathcal{A}_{l,2r-2}$. Moreover, the set $\mathcal{A}_{l,2r}$ is an irreducible algebraic variety and we can find effectively equations of this set.*

At the end we introduce a $2k-$dimensional symplectic volume as $svlol_{2k}(v_1, ..., v_{2k}) = det[\omega(v_i, v_j)]$ and we show the following:

Theorem *Let (X, ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < n$ and $M = 2\begin{pmatrix} 2n \\ 2k \end{pmatrix}$. Assume, that F preserves $2k$ dimensional symplectic volume, on a collection $L_1, ..., L_M$ of $2k-$dimensional subspaces, which are not two-co-planar. Then F is an $\epsilon_{2k}-$symplectomorphism.*

2. The case of $k = 1$.

Let (X, ω) be a symplectic space, i.e., $X \cong \mathbf{K}^{2n}$ is vector space and ω is a bilinear, non-degenerate skew-symmetric form.

Let $L \subset X$ be a $2-$dimensional linear subspace. If vectors v_1, v_2 form a basis of L, then the line $\mathbf{K}(v_1 \wedge v_2)$ is uniquely determined by L and it does not depend on the basis v_1, v_2. This line determines a unique point $\Psi(L)$ in the Grassmannian $G(2, 2n) \subset \mathbb{P}^{N-1}$. Let us recall the notion of co-planar spaces:

Definition 2.1. *Let $\{L_1, ..., L_N\}$ be affine $2-$dimensional subspaces of X. We say that subspaces $\{L_1, ..., L_N\}$ are co-planar, if points $\{\Psi(L_1), ..., \Psi(L_N)\} \subset G(2, 2n)$ are co-planar, i.e., if there is a hyperplane $\Lambda \subset \mathbb{P}^{N-1}$ containing all points $\{\Psi(L_1), ..., \Psi(L_N)\}$.*

Theorem 2.1. *Let X be a symplectic space and let $F : X \to X$ be a linear automorphism. Assume, that F is symplectic on a collection $L_1, ..., L_N$ of $2-$dimensional subspaces, which are not co-planar. Then F is a symplectomorphism.*

Proof. It is well-known (see e.g., [1]), that we can always choose a suitable vector basis $\mathbf{e_1}, ..., \mathbf{e_{2n}}$ in X, (Darboux basis), such that $\omega((\sum v_i \mathbf{e_i}), (\sum w_i \mathbf{e_i})) = \sum_{0<i\le n}(v_i w_{i+n} - v_{i+n} w_i)$. $\omega = \sum_{i=1}^{n} \mathbf{e_i}^* \wedge \mathbf{e_{i+n}}^*$ in the conjugate basis $\mathbf{e_1}^*, ..., \mathbf{e_{2n}}^*$.

Let us denote by $\overline{G(2, 2n)} \subset \bigwedge^2 X := Y$ the set of all vectors $(v \wedge w)$, where $v, w \in X$. It is well-known that the set $\overline{G(2, 2n)}$ is not contained in any linear hyperplane in Y.

Let (v_1^i, v_2^i) be basis of a linear subspace $L_i,\ i = 1, ..., N$. It is easy to see that the linear subspaces $\{L_1, ..., L_N\}$ are co-planar if vectors $\{(v_1^i \wedge v_2^i)\}_{i=1,...,N} \subset \bigwedge^2 X := Y$ are co-planar in Y.

Let (v_1^i, v_2^i) be a basis of a linear space $L_i,\ i = 1, ..., N$ and take $u_i = v_1^i \wedge v_2^i$. Now, let us consider a mapping $R := \bigwedge^2 F : Y \to Y$. In the space Y we have the basis $(\mathbf{e_{ij}} = \mathbf{e_i} \wedge \mathbf{e_j}, 0 < i < j \le 2n)$. For a vector $y = \sum y_{ij}\mathbf{e_{ij}}$ let $\eta(y) = \sum_{i=1}^{n} y_{ii+n}$. Of course η is a linear form on Y.

Observe that $\eta(v \wedge w) = \omega(v, w)$. Consequently, the mapping F is symplectic if and only if for every $v, w \in X$ we have $\eta(v \wedge w) = \eta(R(v \wedge w))$. However the form $\eta(y) - \eta(R(y))$ is linear on Y and by the assumption it does vanish on the set of vectors $u_i,\ i = 1, ..., N$. Since the latter set is not co-planar in Y, we have that the form $\eta(y) - \eta(R(y))$ vanishes identically on Y. This means that $\omega(v \wedge w) = \omega(F(v) \wedge F(w))$ for every $v, w \in X$, i.e., the mapping F is a symplectomorphism. □

From the proof we have the following:

Corollary 2.1. *Let (X, ω) be a $2n$-dimensional symplectic space. Then the set of all 2-dimensional linear subspaces on which the form ω degenerates is co-planar in the Grassmannian $G(2, 2n)$.*

Proof. Indeed, this set is given by the equation $\{y \in G(2, 2n) : \eta(y) = 0\}$. □

Remark 2.1. The Lagrangian Grassmann manifold Λ_2 is isomorphic to a non-singular quadric in $\mathbb{P}^4$. Moreover, if $\mathbf{K} = \mathbb{R}$, then this quadric has the signature $(+ + + - -)$.

Indeed, the Grassmannian $G(2,4)$ is given by the equation $y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23} = 0$ (see e.g., [2], p.211). The manifold Λ_2 is a section of the Grassmannian $G(2,4)$ by the hyperplane $y_{13}+y_{24} = 0$ (cf. the proof of Theorem 2.1). Consequently Λ_2 in homogeneous coordinates $y_{12}, y_{34}, y_{24}, y_{14}, y_{23}$ has the equation $y_{12}y_{34} + y_{24}^2 + y_{14}y_{23} = 0$.

We apply the above results to the case of more general symplectic transformations. Let (X,ω), (Y,μ) be two symplectic spaces, $X \cong \mathbb{K}^{2n}$, $Y \cong \mathbb{K}^{2m}$, $m \geq n$.

Definition 2.2. *Let F be a linear subspace of $X \oplus Y$, $dimF = m+n$. We say that F is an ϵ - symplectic relation if F is a Lagrangian subspace of the product symplectic space $(X \oplus Y, \mu \ominus \epsilon\omega)$, i.e. $\mu \ominus \epsilon\omega \mid_F = 0$, where $\epsilon = \pm 1$, $\mu \ominus \epsilon\omega = \pi_Y^*\mu - \pi_X^*\epsilon\omega$ and $\pi_X(\pi_Y) : X \oplus Y \longrightarrow X(Y)$ are the canonical projections.*

Remark 2.2. If F is an ϵ - symplectic relation of $(X\oplus Y, \mu\ominus\epsilon\omega)$, $m = n$, $\epsilon = +1(-1)$, and $\pi_X \mid_F : F \longrightarrow X$ is onto, then F is graph of a symplectomorphism (anti-symplectomorphism). If $m > n$, $\epsilon = +1(-1)$, and the projection $\pi_X \mid_F$ is onto, then ϵ - symplectic relation F is graph of a linear symplectic reduction (anti-reduction). In this case $W = \pi_Y(F)$ is a coisotropic subspace of (Y,μ), i.e. the μ - complementary space W^μ is a subspace of W, and f is graph of a linear projection of W onto X along W^μ.

Let F be a linear subspace of $X \oplus Y$. Let S be an affine subspace of X. We define the image $F(S)$ of S by F, $F(S) = \{y \in Y : \exists_{x\in S}(x,y) \in F\}$.

Theorem 2.2. *Let (X,ω), (Y,μ) be two symplectic spaces $dimX = 2n$, $dimY = 2m$, $m \geq n$. Let F be a linear subspace of the product symplectic space $(X \oplus Y, \mu \ominus \epsilon\omega)$, $dimF = m+n$. Assume that $\pi_X \mid_F$ is onto and the form $\mu\ominus\epsilon\omega$ is isotropic on a collection of subspaces $(L_1 \oplus F(L_1)), \ldots, (L_N \oplus F(L_N)) \subset X \oplus Y$, where $L_1, \ldots, L_N \subset X$ are not co-planar, 2-dimensional subspaces of X. Then*

1. *F is graph of an ϵ - symplectomorphism if $m = n$,*
2. *F is ϵ - symplectic reduction if $m > n$.*

Proof. First we consider the case $m = n$. By the assumption F is a graph of a linear map $\phi : X \longrightarrow Y$. Let μ and ω be in Darboux form. We see that the corresponding linear forms $\bar{w}$ on $\bigwedge^2 X$; $\bar{\omega}(\sum x_{ij}e_{ij}) = \sum_{i=1}^n x_{ii+n}$, and $\bar{\mu}$ on $\bigwedge^2 Y$, $\bar{\mu}(\sum y_{ij}h_{ij}) = \sum_{i=1}^m y_{ii+m}$, where $e_{ij} = e_i \wedge e_j$, $h_{ij} = h_i \wedge h_j$ and $\{e_i\}$, $\{h_i\}$ are the corresponding Darboux bases on X and Y, define the linear form on $\bigwedge^2 X$,

$$\bar{\mu}(\Phi(x)) - \epsilon\bar{\omega}(x), \tag{2.1}$$

where $\Phi = \bigwedge^2 \phi : \bigwedge^2 X \longrightarrow \bigwedge^2 Y$, $\bar{\omega}(v \wedge w) = \omega(v,w)$, $\bar{\mu}(s \wedge u) = \mu(s,u)$. As in the proof of Theorem 2.1, let (v_1^i, v_2^i) be basis of a linear subspace L_i, $i = 1,\ldots,N$, $u_i = v_1^i \wedge v_2^i$, By assumption the form $\bar{\mu}(\Phi(x)) - \epsilon\bar{\omega}(x)$ vanishes on the set of vectors u_i, $i = 1,\ldots,N$. Since $\{u_i\}_{i=1}^N$ is not co-planar in $\bigwedge^2 X$, we know that the form (2.1) vanishes identically on $\bigwedge^2 X$, i.e. F is Lagrangian so we see that $Ker\phi = \{0\}$, otherwise ω has to be degenerated. Obviously ϕ is an ϵ - symplectomorphism. Indeed $\mu(\phi(v),\phi(w)) - \epsilon\omega(v,w) = 0$ for every $v, w \in X$.

In the case $m > n$ we choose a linear section $\psi : X \longrightarrow Y$ of the canonical projection $\pi_X \mid_F : F \longrightarrow X$ and get the vanishing of the form

$$\mu(\psi(v),\psi(w)) - \epsilon\omega(v,w) = 0$$

for all sections ψ and vectors $v, w \in X$. From this, using the previous arguments, we deduce that F is Lagrangian and so an ϵ - reduction consequently. $\square$

3. The general case

Let $L \subset X$ be a $2k-$dimensional linear subspace. If vectors $v_1, ..., v_{2k}$ form a basis of L, then the line $\mathbf{K}(v_1 \wedge ... \wedge v_{2k})$ is uniquely determined by L and it does not depend on the basis $v_1, ..., v_{2k}$. This line determine a unique point $\Psi(L)$ in the Grassmannian $G(2k, 2n) \subset \mathbb{P}^{N-1}$.

Let us recall the notion of co-planar spaces:

Definition 3.1. *Let $\{L_1, ..., L_N\}$ be linear $2k-$dimensional subspaces of X. We say that subspaces $\{L_1, ..., L_N\}$ are co-planar, if the points $\{\Psi(L_1), ..., \Psi(L_N)\} \subset G(2k, 2n)$ are co-planar, i.e., if there is a hyperplane $\Lambda \subset \mathbb{P}^{N-1}$ containing all points $\{\Psi(L_1), ..., \Psi(L_N)\}$.*

Definition 3.2. *Let (X, ω) be a symplectic space and let $F : X \to X$ be a linear automorphism. We say that F is an ϵ_k-symplectomorphism if $F^*\omega = \epsilon\omega$, where $\epsilon^k = 1$. Additionally we say that F is an anti-symplectomorphism if $F^*\omega = -\omega$.*

Remark 3.1. We can treat an ϵ_k-symplectomorphism as a root from symplectomorphism. Indeed, if F is an ϵ_k-symplectomorphism, then F^k is a symplectomorphism.

Theorem 3.1. *Let (X, ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < n$. Assume, that F preserves the form $\wedge^k\omega$ on a collection $L_1, ..., L_N$ of $2k-$dimensional subspaces, which are not co-planar. Then F is an ϵ_k-symplectomorphism. In particular if $\mathbf{K} = \mathbb{R}$ and the number k is odd, then F is a symplectomorphism.*

Proof. Let us denote by $\overline{G(2k, 2n)} \subset \bigwedge^{2k} X := Y$ the set of all vectors $(v_1 \wedge ... \wedge v_{2k})$, where $v_1, ..., v_{2k} \in X$. It is well-known, that the set $\overline{G(2k, 2n)}$ is not contained in any linear hyperplane in Y.

Let $(v_1^i, ..., v_{2k}^i)$ be a basis of a linear subspace $L_i,\ i = 1, ..., N$. It is easy to see that linear subspaces $\{L_1, ..., L_N\}$ are co-planar if vectors $\{u_i := (v_1^i \wedge ... \wedge v_{2k}^i)\}_{i=1,...,N} \subset \bigwedge^{2k} X := Y$ are co-planar in Y.

In the space Y we have the basis $(\mathbf{e_{i_1 ... i_{2k}}} = \mathbf{e_{i_1}} \wedge ... \wedge \mathbf{e_{i_{2k}}}, 0 < i_1 < i_2 ... < i_{2k} \leq 2n)$. For a vector $y = x_1 \wedge ... \wedge x_{2k} \in \overline{G(2k, 2n)}$ let $y = \sum y_{i_1 ... i_{2k}} \mathbf{e}_{i_1 ... i_{2k}}$. It is easy to see that mappings $y_{i_1 ... i_{2k}}$ treated as functions of vectors $x_1 ..., x_{2k}$ form a basis of the space of all $2k$-linear skew-symmetric forms on X. In particular there are numbers $a_{i_1 ... i_{2k}}$ such that $\wedge^k\omega(y) = \sum a_{i_1 ... i_{2k}} y_{i_1 ... i_{2k}}$. Take $\eta(x_1 \wedge ... \wedge x_{2k}) = \wedge^k\omega(x_1, ..., x_{2k})$. We have $\eta(y) = \sum a_{i_1 ... i_{2k}} y_{i_1 ... i_{2k}}$ and consequently we can treat η as a linear form on the whole of Y.

Now, let us consider a mapping $R := \bigwedge^{2k} F : Y \to Y$. Consequently, the mapping F preserves the form $\wedge^k\omega$ on L_i if and only if $\eta(u_i) = \eta(R(u_i))$. However the form $\eta(y) - \eta(R(y))$ is linear on Y and by the assumption it does vanish on the set of vectors $u_i,\ i = 1, ..., N$. Since the latter set is not co-planar in Y, we have that the form $\eta(y) - \eta(R(y))$ vanishes identically on Y. This means that $\wedge^k\omega(v_1, ..., v_{2k}) = \wedge^k\omega(F(v_1), ..., F(v_{2k}))$ for every $v_1, ..., v_{2k} \in X$, i.e., the mapping F preserves the form $\wedge^k\omega$. Now we prove the following

Lemma 3.1. *Let (X, ω) be a symplectic space. Let $W \subset X$ be a $2k-$dimensional symplectic subspace of X, i.e. $(W, \omega\mid_W)$ is a symplectic space. If*

$$\wedge^k\omega(v, w_1, \ldots, w_{2k-1}) = 0$$

for every $\{w_1, \ldots, w_{2k-1}\}$, $w_i \in W$, then v is complementary to W with respect to ω.

Proof. Because W is symplectic we can choose the basis of X, $\{e_i\}_{i=1}^{2n}$, such that ω is in Darboux form on W, i.e., $\omega\mid_W = \sum_{i=1}^{k} e_i^*\wedge e_{i+k}^*$ and W is generated by vectors $e_i,\ i\le 2k$. Moreover, we can assume that the subspace V, complementary to W is generated by vectors $e_i,\ i>2k$. Now let $v=\sum_{i=1}^{2n} a_ie_i$. Then taking the $(2k-1)-$elements $e_1,\dots,"j",\dots,e_{2k}$ and applying to $\wedge^k\omega$, we get

$$\wedge^k\omega(v,e_1,\dots,"j",\dots,e_{2k}) = \pm a_j\wedge^k\omega(e_1,e_2,\dots,e_{2k}) = \pm a_jk!,$$

so from this and by the assumption we have $a_j=0,\ j=1,...,2k$. This means that $v\in V$. □

Now consider a general vector subspace $V\subset X$ of dimension $2k+2$. Take a general subspace $L_V\subset V$ of dimension $2k$ and let $K_V\subset V$ be a subspace, which is complementary to L_V with respect to ω. Since the form $\wedge^{k+1}\omega$ is proportional to the usual volume form Ω on V and any linear mapping changes the volume form multiplying it by a suitable constant, we have that there is a non-zero constant c such that

$$\Omega(F(v_1),...,F(v_{2k+2})) = c\Omega(v_1,...,v_{2k+2}) \tag{3.1}$$

for every $v_1,...,v_{2k+2}\in V$. Now let us take in (3.1) $v_1,...,v_{2k}\in L_V$ and $v=v_{2k+1}, w=v_{2k+2}\in K_V$.

Computing directly we have (note that v,w are complementary to $v_1,...,v_{2k}$!):

$$\Omega(v_1,...,v_{2k},v,w) = a\omega(v,w)\wedge^k\omega(v_1,...,v_{2k}), \tag{3.2}$$

where a is a suitable non-zero constant. Since F preserves $\wedge^k\omega$, using the Lemma 3.1, we can deduce that the subspace $F(V_L)$ is complementary to the subspace $F(K_L)$ (in the space $F(V)$). In fact $\wedge^k\omega(F(v),F(v_1),\dots,F(v_{2k-1})=0$ for all $v_1,\dots,v_{2k-1}\in L_V$, so the Lemma works.

Hence similarly as before we have

$$\Omega(F(v_1),...,F(v_{2k}),F(v),F(w)) = a\omega(F(v),F(w))\wedge^k\omega(F(v_1),...,F(v_{2k}). \tag{3.3}$$

Now from (3.2) and (3.3) we conclude that

$$\omega(F(v),F(w)) = c\omega(v,w).$$

Since spaces K_V are not all co-planar in $G(2,V)$ we deduce (similarly as in the proof of Theorem 2.1) that $F^*\omega=c\omega$ on V. Finally, since $F^*\wedge^k\omega=\wedge^k\omega=c^k\wedge^k\omega$, we have $c^k=1$. Thus there is only finite number of possibilities on c. Let $\epsilon_1,...,\epsilon_k$ be all roots from 1 of order k (if $\mathbf{K}=\mathbb{R}$, then we consider 1 and -1 only). Now in the Grassmannian $G(2,X)$ consider subsets $S_i:=\{y:\epsilon_i\eta(y)=\eta(\wedge^2F(y))\}$. Since $G(2,X)=\bigcup_{i=1}^k S_i$ we have that there is an index i such that the set S_i is not co-planar in $G(2,X)$. Consequently, it must be $\epsilon_i\eta(y)=\eta(\wedge^2F(y))$ identically, i.e., $F^*\omega=\epsilon_i\omega$ holds on the whole of X. Moreover, if k is odd and $\mathbf{K}=\mathbb{R}$, we have $\epsilon_i=1$. □

Corollary 3.1. *Let (X,ω) be a symplectic space and let $F:X\to X$ be a linear automorphism. Let $0<l<k\le n$ be natural numbers such that $(k,l)=1$. Assume, that F preserves the form $\wedge^k\omega$ and the form $\wedge^l\omega$. Then F is a symplectomorphism.*

Remark 3.2. The simplest example of an ϵ_k-symplectomorphism $F:X\to X$ is the following mapping:

$$J_k(\sum_{i=1}^{n}(v_ie_i+v_{i+n}e_{i+n})) = \sum_{i=1}^{n}(v_ie_i+\epsilon_kv_{i+n}e_{i+n}),$$

in Darboux coordinates, where ϵ_k is a primitive root from 1 of order k. Thus if the linear mapping F preserves the form $\wedge^k\omega$ then it is of the form $F = J_l S$, where S is a symplectomorphism and l is a number, which divides k.

From the proof we have also the following:

Corollary 3.2. *Let (X, ω) be a $2n$-dimensional symplectic space. Then the set of all $2k$-dimensional linear subspaces on which the form ω degenerates (i.e., has not a maximal rank) is co-planar in the Grassmannian $G(2k, 2n)$.*

4. Geometry of the set $\mathcal{A}_{l,2r}$.

Let $\mathcal{A}_{l,2r} \subset G(l, 2n)$ denote the set of all $l-$dimensional linear subspaces of X on which the form ω has rank $\le 2r$. Of course $\mathcal{A}_{l,2r} \subset \mathcal{A}_{l,2r+2}$ if $2r + 2 \le l$. We have the following:

Theorem 4.1. *Let (X, ω) be a symplectic space of dimension $2n$. Then the set $\mathcal{A}_{2k,2k-2} \subset G(2k, 2n) \subset \mathbb{P}^{N-1}$ is an irreducible algebraic subset of $G(2k, 2n)$ and it linearly span a hyperplane in $\mathbb{P}^{N-1}$. More generally, for $r < k$ the set $\mathcal{A}_{2k,2r}$ is also irreducible and linear in the $G(2k, 2n)$, i.e., there is a linear projective subspace $L \subset \mathbb{P}^{N-1}$ such that $\mathcal{A}_{2k,2r} = G(2k, 2n) \cap L$. Moreover the set $\mathcal{A}_{2k,2r}$ can be computed effectively.*

Proof. First assume that $\mathbf{K} = \mathbb{C}$. Let $\mathcal{A} = \mathcal{A}_{2k,2k-2}$ denote the set of all $2k$ dimensional subspaces of rank $< 2k$.

By the Andreotti-Salmon Theorem (see e.g., [5]) the embedded Grassmanian $G(2k, 2n) \subset \mathbb{P}^{N-1}$ is projectively factorial. In particular the Picard group of $G(2k, 2n)$ is generated by the hyperplane section. It can be easily deduced from this that for every hyperplane $H \subset \mathbb{P}^{N-1}$ the set $H \cap G(2k, 2n)$ is an irreducible variety which is not contained in a proper linear subspace of H (cf. [3], Lemma 3.18). Let $H \subset \mathbb{P}^{N-1}$ be a hyperplane, such that $\mathcal{A} = H \cap G(2k, 2n)$ (cf. Corollary 3.2). Thus, by the above, we have that $\mathcal{A}$ linearly span the hyperplane H.

Now let us consider the set $\mathcal{A}_{2k,2r}$. It is irreducible since it contains an orbit of the symplectic group as a dense subset $U = \mathcal{A}_{2k,2r} \setminus \mathcal{A}_{2k,2r-2}$. We show that there is a linear subspace L, such that $\mathcal{A}_{2k,2r} = L \cap G(2k, 2n)$. Let us take a sequence of vectors $(x_1, ..., x_{2k}) \in X^{2k}$. Let $\mathbb{A}$ be the matrix which has (coordinates of) vectors $x_{i_1}, ..., x_{i_{2k-2r}}$ as rows. Let $\delta_{s_1,...,s_{2k-2r}}(x_{i_1}, ..., x_{i_{2k-2r}})$ denote the main minor of the matrix $\mathbb{A}$ determined by columns indexed by $s_1, ..., s_{2k-2r}$. Consider all possible skew-symmetric forms of the type

$$\wedge^{2r}\omega(x_{j_1}, ..., x_{j_{2r}})\delta_{s_1,...,s_{2k-2r}}(x_{i_1}, ..., x_{i_{2k-2r}}),$$

where $\{j_1, ..., j_{2r}\} \cup \{i_1, ..., i_{2k-2r}\} = \{1, ..., 2k\}$. It is not difficult to check that they simultaneously vanish only on vectors $x_1 \wedge ... \wedge x_{2k}$ which belongs to $\mathcal{A}_{2k,2r}$. On the other hand these skew-symmetric functions can be treated as linear forms on $\mathbb{P}^{N-1}$ (cf. the proof of Theorem 3.1). Of course we can find these linear forms effectively (as functions of variables $y_{i_1,...,i_{2k}}$, cf. the proof of Theorem 3.1). Since we know the equations of the Grassmannian $G(2k, 2n)$ (see [2], p. 211), we can computed the set $\mathcal{A}_{2k,2r}$ effectively. This finishes the proof in the case $\mathbf{K} = \mathbb{C}$.

Now we sketch the proof for the case $\mathbf{K} = \mathbb{R}$. As before define the set $\mathcal{A} \subset G(2k, 2n)$ as the set of all linear subspaces of X of dimension $2k$ and of rank $< 2k$. This set has a stratification on smooth subsets $\mathcal{A}_r = \{W \in \mathcal{A} : rank W = 2r\}$, where $r = 0, 1, ..., k-1$ and $\mathcal{A}_i \subset cl(\mathcal{A}_{i+1})$. Moreover, every such subset is homogeneous with respect to the induced action of the group $Spin(2n, \mathbf{K})$. Let us take a real subspace $L \in \mathcal{A}_{k-1}$. Let $H \subset Sp(2n, \mathbb{R})$ be a stabilizer of L in the group $Sp(2n, \mathbb{R})$. Thus dim $\mathcal{A}$ =dim $\mathcal{A}_{k-1}$=

dim $Sp(2n,\mathbb{R})-$dim H. Now let us complexify X. Let $H' \subset Sp(2n,\mathbb{C})$ be a stabilizer of the subspace $L \bigotimes \mathbb{C}$ in the group $Sp(2n,\mathbb{C})$.

Let $\mathcal{A}' \subset G(2k,2n,\mathbb{C})$ be the set of all complex $2k-$dimensional linear subspaces of $X \bigotimes \mathbb{C}$ of dimension $2k$ and of rank $< 2k$. Then $\mathcal{A}'$ has the same (complex) dimension as the orbit of the subspace $L \bigotimes \mathbb{C}$ and this dimension is equal to dim $Sp(2n,\mathbb{C})-$dim H'. But H' contains a complexification of the subgroup H, thus $\dim_{\mathbb{C}} H \geq \dim_{\mathbb{R}} H$ and consequently $\dim_{\mathbb{C}} \mathcal{A}' \leq \dim_{\mathbb{R}} \mathcal{A}$. But in the complex case we have dim $\mathcal{A}'$ =dim $G(2k,2n,\mathbb{C}) - 1$. From this we have immediately that dim $\mathcal{A}$ =dim $G(2k,2n,\mathbb{R}) - 1$. This means that the complexification of the set $\mathcal{A}$ is the set $\mathcal{A}'$. Thus the set $\mathcal{A}$ spans linearly a (real) hyperplane if and only if the set the set $\mathcal{A}'$ spans a (complex) hyperplane. Now we can finish the proof as above. □

From the proof we see that the part of the Theorem 4.1 can be generalized:

Corollary 4.1. *Let (X,ω) be a symplectic space of dimension $2n$. Let l,r be integral numbers such that $l \leq 2n$ and $2r < l$. Then there is a proper linear subspace $L \subset \mathbb{P}^{N-1}$ such that $\mathcal{A}_{l,2r} = L \cap G(l,2n)$. Moreover, we can computed equations of $\mathcal{A}_{l,2r}$ effectively. In particular it is true for the Grassmann Lagrangian manifold $\Lambda_n = \mathcal{A}_{n,0}$.*

Definition 4.1. *Let $\{L_1,...,L_{N-1}\} \subset \mathcal{A}_{2k,2k-2}$ be linear $2k-$dimensional subspaces of X (of rank $< 2k$). We say that subspaces $\{L_1,...,L_{N-1}\}$ are in general position, if they linearly span a hyperplane in $\mathbb{P}^{N-1}$.*

Remark 4.1. By the Theorem above every sufficiently general subset $\{L_1,...,L_{N-1}\} \subset \mathcal{A}_{2k,2k-2}$ is in general position.Moreover, we can find such subspaces L_i with the rank $L_i = 2k-2$.

A slightly more general version of Theorem 3.1 is:

Theorem 4.2. *Let (X,ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < n$. Let $\{L_1,...,L_{N-1}\}$ be linear $2k-$dimensional subspaces of X of rank $< 2k$, which are in general position. Assume, that F transforms subspaces $L_1,...,L_{N-1}$ onto subspaces which also have rank $< 2k$. Then there is a non-zero constant c, such that $F^*\omega = c\omega$.*

Proof. First assume that $\mathbf{K} = \mathbb{C}$. Let L be a $2k$ dimensional subspace of rank $2k$. There is a constant a, such that $F^* \wedge^k \omega = a \wedge^k \omega$ on L. As in the proof of Theorem 3.1, let us denote by $\overline{G(2k,2n)} \subset \bigwedge^{2k} X := Y$ the set of all vectors $(v_1 \wedge ... \wedge v_{2k})$, where $v_1,...,v_{2k} \in X$. Let $\mathcal{A} = \{L_1,...,L_{N-1}\}$.

Let $H \subset \mathbb{P}^N$ be a hyperplane, such that $\mathcal{A}_{2k,2k-2} = H \cap G(2k,2n)$. Thus, by the assumption, we have that $\mathcal{A}$ linearly span the hyperplane H. Since $L \notin H$ we can easily deduced that the set $\mathcal{B} := \{L\} \cup \mathcal{A}$ is not co-planar in Y. By the assumption we have for $W \in \mathcal{B}$:

$$a\eta(u_W) = \eta(R(u_W)),$$

where $u_W \in \overline{G(2k,2n)} \subset \bigwedge^{2k} X := Y$ is a vector determined by W.

This implies that a linear form $a\eta(y)-\eta(R(y))$ does vanish on the set of vectors u_W, $W \in \mathcal{B}$. Since the set $\mathcal{B}$ is not co-planar in Y, we have that the form $a\eta(y) - \eta(R(y))$ vanishes identically on Y. This means that $a \wedge^k \omega(v_1,...,v_{2k}) = \wedge^k \omega(F(v_1),...,F(v_{2k}))$ for every $v_1,...,v_{2k} \in X$, i.e., $F^* \wedge^k \omega = a \wedge^k \omega$.

Now we can repeat word by word the proof of Theorem 3.1 to see that there is a constant c, such that $F^*(\omega) = c\omega$ and $c^k = a$. Since F is a linear automorphism, we have $c \neq 0$. □

Corollary 4.2. *Let (X,ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < n$. Assume, that F transforms $2k$ dimensional subspaces of rank $2k-2$ onto subspaces which also have rank $< 2k$. Then there is a non-zero constant c, such that $F^*\omega = c\omega$.*

Corollary 4.3. *Let (X,ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < l < 2n$ and let $2r+2 \leq l$. Assume, that F transforms the set $\mathcal{A}_{l,2r}$ into the same set. Then there is a non-zero constant c, such that $F^*\omega = c\omega$.*

Proof. Let $\mathcal{B}_{2r+2,2r}$ denote the set of $2r+2$ dimensional subspaces of rank $2r$. Since every subspace from $\mathcal{B}_{2r+2,2r}$ is contained in some subspaces from $\mathcal{A}_{l,2r}$, it is easy to see that F transforms the set $\mathcal{B}_{2r+2,2r}$ into the same set. Hence we are done by Corollary 4.2. □

Corollary 4.4. *Let (X,ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $2 \leq l \leq n$ and assume, that F transforms $l-$dimensional isotropic (e.g. Lagrangian) subspaces onto subspaces of the same type. Then there is a non-zero constant c, such that $F^*\omega = c\omega$.*

5. Symplectic volume

In this section we introduce and study the symplectic volume.

Definition 5.1. *Let (X,ω) be a $2n$-dimensional symplectic space. Let $v_1,...,v_{2k} \in X$ be vectors. By a symplectic $2k$-volume of a a collection $v_1,...,v_{2k}$ we mean the number $svol_{2k}(v_1,...,v_{2k}) := det([\omega(v_i,v_j)]_{1\leq i,j\leq 2k})$.*

The symplectic volume has the following nice property:

Proposition 5.1. *Let (X,ω) be a symplectic space and let $v_1,...,v_{2k} \in X$ be vectors. Then $svol_{2k}(v_1,...,v_{2k}) = 0$ if and only if vectors $v_1,...,v_{2k}$ are either linearly dependent or the space $(W = < v_1,...,v_{2k} >, \omega \mid_W)$ is not symplectic. Moreover, if vectors $v_1,...,v_{2k}$ are linearly independent then* $\operatorname{rank}[\omega(v_i,v_j)]_{1\leq i,j\leq 2k} = \operatorname{rank}\, \omega \mid_W$.

Proof. If vectors $v_1,...,v_{2k}$ are either linearly dependent, then we can easily see (e.g., by Cramer rules) that $det([\omega(v_i,v_j)]_{1\leq i,j\leq 2k}) = 0$.

Assume that vectors $v_1,...,v_{2k}$ are linearly independent. There is a basis $e_1,...,e_{2k}$ of the subspace $(W = < v_1,...,v_{2k} >$ such that the form $\omega' = \omega \mid_W$ has in this basis a canonical form. In particular rank $\omega' =$ rank $[\omega(e_i,e_j)]_{1\leq i,j\leq 2k}$. But there is a non-singular $2k \times 2k$ matrix $\mathbb{A} = [a_{ij}]$, such that $v_i = \sum_{j=1}^{2k} a_{ij}e_j$. Using bilinearity of ω, we have by the direct computation

$$[\omega(v_i,v_j)] = \mathbb{A}[\omega(e_i,e_j)]\mathbb{A}^T.$$

This finishes the proof. □

Theorem 5.1. *Let (X,ω) be a symplectic space and let $v_1,...,v_{2k} \in X$ be vectors. Then $(\wedge^k\omega(v_1,...,v_{2k}))^2 = (k!)^2 svol_{2k}(v_1,...,v_{2k})$.*

Proof. We can assume that $\mathbf{K} = \mathbb{C}$. Let us choose the basis $\{e_1,...,e_{2n}\}$ on X such that $\omega = \sum e_i^* \wedge e_{i+n}^*$. Now for vectors $v_1,...,v_{2k} \in X$ let us consider the determinant $svol_{2k}(v_1,...,v_{2k}) := det([\omega(v_i,v_j)]_{1\leq i,j\leq 2k})$. We can treat this determinant as a polynomial function on X^{2k} of degree $2k$. Since the matrix $[\omega(v_i,v_j)]$ is skew-symmetric we have in fact $svol_{2k}(v_1,...,v_{2k}) = Pf(v_1,...,v_{2k})^2$, where $Pf(v_1,...,v_{2k})$ is also a polynomial (the Pfaffian of $[\omega(v_i,v_j)]$).

Let us note that $Pf(v_1,...,v_{2k})^2 = svol_{2k}(v_1,...,v_{2k}) = 0$ if and only if vectors $v_1,...,v_{2k}$ are either linearly dependent or the space $(W = < v_1,...,v_{2k} >, \omega \mid_W)$ is not symplectic (cf. Proposition 5.1). The same property has the polynomial function $\wedge^k\omega(v_1,...,v_{2k})$. Moreover it is not difficult to check that the latter polynomial is irreducible and of degree k. This means that two polynomials: $Pf(v_1,...,v_{2k})$ and $\wedge^k\omega(v_1,...,v_{2k})$ have the same degree and the same zero-sets. Since the second polynomial is irreducible we have by the Hilbert Nullstellensatz that there is a constant C such that $Pf(v_1,...,v_{2k}) = C \wedge^k \omega(v_1,...,v_{2k})$. In particular we have $C^2(\wedge^k\omega(v_1,...,v_{2k}))^2 = svol_{2k}(v_1,...,v_{2k})$. Moreover, by the direct computations we have $(\wedge^k\omega(e_1,...,e_{2k}))^2 = (k!)^2$ and $svol_{2k}(e_1,...,e_{2k}) = 1$. Thus $C^2 = (k!)^{-2}$ and $(\wedge^k\omega(v_1,...,v_{2k}))^2 = (k!)^2 svol_{2k}(v_1,...,v_{2k})$. □

To state the next result we need additional notion. Let us recall that if $L \subset X$ is a $2k-$dimensional linear subspace. and vectors $v_1,...,v_{2k}$ form a basis of L, then the line $\mathbf{K}(v_1\wedge...\wedge v_{2k})$ is uniquely determined by L and it does not depend on the basis $v_1,...,v_{2k}$. This line determine a unique point $\Psi(L)$ in the Grassmannian $G(2k,2n) \subset \mathbb{P}^{N-1}$, where $N = \begin{pmatrix} 2n \\ 2k \end{pmatrix}$.

Definition 5.2. *Let $\{L_1,...,L_M\}$ be linear $2k-$dimensional subspaces of X. We say that subspaces $\{L_1,...,L_M\}$ are two-co-planar, if the points $\{\Psi(L_1),...,\Psi(L_N)\} \subset G(2k,2n)$ are two-co-planar, i.e., if there are hyperplanes $\Lambda_1,\Lambda_2 \subset \mathbb{P}^{N-1}$ such that the union $\Lambda_1\cup\Lambda_2$ contains all points $\{\Psi(L_1),...,\Psi(L_M)\}$.*

Remark 5.1. Let us take $M = \begin{pmatrix} N \\ 2 \end{pmatrix}$. Let $\{L_1,...,L_M\}$ be linear $2k-$dimensional subspaces of X. It is not difficult to check that if this subspaces are sufficiently general, then they are not two-co-planar.

Theorem 5.2. *Let (X,ω) be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < n$. Assume, that F preserves $2k$ dimensional symplectic volume, on a collection $L_1,...,L_M$ of $2k-$dimensional subspaces, which are not two-co-planar. Then F is an $\epsilon_{2k}-$symplectomorphism.*

Proof. By Theorem above on every L_i we have either $F^*(\wedge^k\omega) = \wedge^k\omega$ or $F^*(\wedge^k\omega) = -\wedge^k\omega$.

Let us denote by $\overline{G(2k,2n)} \subset \bigwedge^{2k} X := Y$ the set of all vectors $(v_1 \wedge ... \wedge v_{2k})$, where $v_1,...,v_{2k} \in X$.

Let $(v_1^i,...,v_{2k}^i)$ be a basis of a linear subspace $L_i,\ i = 1,...,N$. It is easy to see that linear subspaces $\{L_1,...,L_N\}$ are co-planar if vectors $\{u_i := (v_1^i\wedge...\wedge v_{2k}^i)\}_{i=1,...,N} \subset \bigwedge^{2k} X := Y$ are co-planar in Y.

In the space Y we have the basis $(\mathbf{e_{i_1...i_{2k}}} = \mathbf{e_{i_1}} \wedge ... \wedge \mathbf{e_{i_{2k}}}, 0 < i_1 < i_2... < i_{2k} \le 2n)$. For a vector $y = x_1 \wedge ... \wedge x_{2k} \in \overline{G(2k,2n)}$ let $y = \sum y_{i_1...i_{2k}}\mathbf{e}_{i_1...i_{2k}}$. It is easy to see that mappings $y_{i_1...i_{2k}}$ treated as functions of vectors $x_1...,x_{2k}$ form a basis of the space of all $2k$-linear skew-symmetric forms on X. In particular there are numbers $a_{i_1...i_{2k}}$ such that $\wedge^k\omega(y) = \sum a_{i_1...i_{2k}}y_{i_1...i_{2k}}$. Take $\eta(x_1 \wedge ... \wedge x_{2k}) = \wedge^k\omega(x_1,...,x_{2k})$. We have $\eta(y) = \sum a_{i_1...i_{2k}}y_{i_1...i_{2k}}$ and consequently we can treat η as a linear form on the whole of Y.

Now, let us consider a mapping $R := \bigwedge^{2k} F : Y \to Y$. Consequently, $F^* \wedge^k \omega = \pm \wedge^k \omega$ on L_i if and only if $\pm\eta(u_i) = \eta(R(u_i))$. However forms $\eta(y) \pm \eta(R(y))$ are linear on Y and by the assumption vectors $u_i,\ i = 1,...,M$ are contained in the union of kernels of these forms. Since vectors $u_i,\ i = 1,...,M$ are not two-co-planar in Y, we have that either the form $\eta(y) - \eta(R(y))$ or the form $\eta(y) + \eta(R(y))$ vanishes identically on Y. This means that either $\wedge^k\omega(v_1,...,v_{2k}) = \wedge^k\omega(F(v_1),...,F(v_{2k}))$ for every $v_1,...,v_{2k} \in X$ or $-\wedge^k\omega(v_1,...,v_{2k}) = \wedge^k\omega(F(v_1),...,F(v_{2k}))$ for every $v_1,...,v_{2k} \in X$, i.e., $F^*\wedge^k\omega = \pm\wedge^k\omega$.

Now we can repeat word by word the proof of Theorem 3.1 to see that there is a constant c, such that $F^*(\omega) = c\omega$. Moreover, we have $c^k = 1$ or $c^k = -1$. In both cases $c^{2k} = 1$, hence F is an $\epsilon_{2k}-$symplectomorphism. □

Corollary 5.1. *Let X be a symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < n$. Assume, that F preserves $2k$ dimensional symplectic volume. Then F is an $\epsilon_{2k}-$symplectomorphism.*

Corollary 5.2. *Let X be a real symplectic space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < k < n$. Assume, that F preserves $2k$ dimensional symplectic volume. Then F is either a symplectomorphism or it is an anti-symplectomorphism.*

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