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ON THE TOPOLOGY OF SOME SIMPLE IND-SCHEMES

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Abstract. Arising from a function on an affine space $\mathbb{A}^d$ is a function, the 0^{th} Fourier mode, on its ind-scheme of loops $\mathcal{L}\mathbb{A}^d$. We compute the suitably renormalised cohomology of the nearby fibre to this function in the case where the function is homogeneous and has an isolated singularity at the origin. We find reduced Milnor fibre cohomology. The result can equivalently be stated as cohomological contractability of an associated ind-scheme.

1. Introduction

We will deal in this note with the topological properties of some infinite dimensional spaces arising from formal loops, $\mathcal{L}X$, into a scheme X, as defined for example by Kapranov and Vasserot in [5]. These are algebro-geometric analogues of the unbased loop spaces, $LM = Map(S^1, M)$ of maps from a circle to a manifold, as studied for example by topologists and string theorists. Intuitively speaking, they are best understood as the mapping space $Map(D^*, X)$ from a formal punctured disc to X. It bears remarking that this is not literally the case. To the best of our knowledge, the topology of the complex points of formal loop spaces is very poorly understood in all but a small number of cases (notably reductive groups G as studied beautifully in [4]). Indeed it seems that even their sets of connected components remain mysterious. (cf. forthcoming work of the author [2] for more on this question in a special case.) We do not address the question of the topology of loop spaces $\mathcal{L}X$ directly, but rather deal with certain ind- schemes built from loops. It turns out that these are easier to deal with.

More precisely, given a variety X and a function on it, F, we can produce a natural function on its loop space, called the 0th *Fourier mode*, $\hat{F}$. We study the topology of this function in the case of the simplest possible singularity, i.e. the cone on a smooth projective hypersurface. In fact, we'll see that there is essentially no more cohomology than that contained in the constant loops. The proof is elementary and makes use of an obvious inductive structure on the ind-scheme in question. We will prove below (after introducing the relevant notions) that the *renormalised cohomology* of the nearby fibre of

$\hat{F}$ on $\mathcal{L}\mathbb{A}^d$ is Milnor fibre cohomology (cf. [6]) concentrated in degree $d-1$, i.e that we have an isomorphism

$$H^*_{ren}(\{\hat{F}=\epsilon\}(\mathbb{C}),\mathbb{Q}) \cong \mathbb{Q}^{\mu(F)}\big[d-1\big].$$

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2. Basics

We will recall here the basic notions and objects involved in the statement of our main theorem. We will work throughout over the field $\mathbb{C}$ of complex numbers. By a *space* we will mean a pre-sheaf on the category of affine $\mathbb{C}$-schemes. We will deal in particular with *ind-schemes*, which are filtered colimits (taken inside the pre-sheaf category) of schemes along closed finite codimensional embeddings. Given a set S, throughout we will write $\mathbb{A}\{S\}$ for the affine space with coordinates indexed by S.

2.1. Loop Spaces.

We now define some of our basic objects of study. X will denote an arbitrary space below.

Definition 2.1.

- *The* arc space *of X, denoted $\mathcal{L}^+X$, is defined as the space with A-points $\mathcal{L}^+X(A) = X(A[[t]])$*
- *The* loop space*, denoted $\mathcal{L}X$, is defined as the space with A-points $\mathcal{L}X(A) = X(A((t)))$.*
- *The space of* negative loops*, denoted $\mathcal{L}^-X$ is defined as the complement of $\mathcal{L}^+X$ inside $\mathcal{L}X$.*
- *There is an evident $\mathbb{G}_m$ action on $\mathcal{L}X$. Functions on $\mathcal{L}X$ of weight w for this action will be said to be of* conformal weight *w.*

We record some simple and well-known facts below. For a more detailed introduction to these objects we recommend [5] and its companion papers.

Lemma 2.1. (1) *If X is a scheme then the functor $\mathcal{L}^+X$ is representable by a scheme.*

(2) *If X is moreover affine, then the functor $\mathcal{L}X$ is representable by an ind-scheme, and in fact by an ind- affine scheme, that is to say a filtered colimit of affine schemes along closed embeddings.*

(3) *If X is smooth then both the loop and the arc space are formally smooth.*

Proof. In the case where X is affine, (1) and (2) reduce to the case of $\mathbb{A}^1$ by compatibility of the relevant functors with formation of arbitrary limits. For (1) we note that $\mathcal{L}^+\mathbb{A}^1$ is represented by the infinite dimensional affine space $\mathbb{A}\{\mathbb{Z}_{\geq 0}\}$. For (2) we note that $\mathcal{L}\mathbb{A}^1$ is represented by the ind- affine scheme $\varinjlim_n \mathbb{A}\{\mathbb{Z}_{\geq -n}\}$. To conclude the proof of (1) we observe Zariski gluability of $\mathcal{L}^+$. (3) is a trivial consequence of the infinitesimal lifting criterion for formal smoothness. □

Remark. Giving an ind- affine scheme is equivalent to giving a complete and *linearly topologized* $\mathbb{C}$-algebra, i.e. a complete topological algebra with a neighbourhood basis at 0 consisting of ideals.

Example. The case of $\mathcal{L}\mathbb{A}^1$ corresponds to the linearly topologized algebra

$$\varprojlim_n \mathbb{C}\big[z_i \mid i \in \mathbb{Z}_{\geq -n}\big],$$

The function z_i is of conformal weight i.

We'd like now to introduce the functions on the loop space that we'll consider. We'll assume X to be an affine scheme in what follows. If $X = spec(A)$, then we will abusively write $X\hat{\times}D^* = spec(A((t)))$, where $D^* = spec(\mathbb{C}((t)))$ is the formal punctured disc. We extend the definition of $(-)\hat{\times}D^*$ to ind- affine schemes by imposing compatibility with filtered colimits. Note that if an ind- affine scheme $\mathcal{X}$ corresponds to a linearly topologized algebra A, then the ind- affine scheme $\mathcal{X}\hat{\times}D^*$ corresponds to the algebra, $A\{t\}$, of *topological Laurent series* over A, i.e. the algebra of formal series in t and t^{-1} with negative tails tending to 0; $\{\sum_i a_i t^i \mid a_i \to 0, i \to -\infty\}$.

Definition 2.2. (1) *The definition of $\mathcal{L}X$ produces a map $\mathcal{L}X\hat{\times}D^* \to X$. We call this the* evaluation map *and denote it ev.*

(2) *If F is a function on X, then expanding $ev^*(F)$ as a topological Laurent series over functions on $\mathcal{L}X$, we define the coefficient of t^i to be the ith* Fourier mode *of F. In particular we will concentrate on the 0th Fourier mode and denote it $\hat{F}$.*

Example. In the case of $\mathbb{A}^1$, ev can be understood very easily. It simply corresponds to the function $\sum_i z_i t^i$ on $\mathcal{L}\mathbb{A}^1$. As such we see for example that $\widehat{z^2} = \sum_i z_i z_{-i}$. In general, working with coordinates $\{z^j\}_j$, one substitutes the sums $z^j(t) = \sum_i z_i^j t^i$ into F and formally expands, then takes the constant term.

2.2. **Cohomology of Ind-Schemes.** Here we give a brief account of how to assign renormalised cohomology groups to suitably nice ind-schemes. We require a notion of smoothness and the data of a *dimension theory*, to be recalled below.

Definition 2.3. *If X is a scheme then we call it* smooth *if it can be represented as the filtered limit of smooth finite type schemes.*

Remark. This implies formal smoothness of X but is not equivalent to it.

We extend this to ind-schemes in an evident manner:

Definition 2.4. *If $\mathcal{X}$ is an ind-scheme, we call it* smooth *if it can be represented as the filtered colimit of smooth schemes in the sense of the definition above.*

Remark. This is a strong condition, and indeed is not fulfilled by the loop space to an arbitrary smooth scheme. For example, it was observed by Contou-Carrère (cf. [3]) that $\mathcal{L}\mathbb{G}_m$ is non-reduced, and thus a fortiori not smooth in this sense.

Henceforth, for a scheme X, when we write $H^*(X)$ it is to be understood as $H^*(X(\mathbb{C}), \mathbb{Q})$. As explained in [1], a codimension d embedding, $X \to Y$, of smooth and possibly infinite dimensional schemes induces a *Gysin map* on cohomologies, $H^*(X) \to H^{*+2d}(Y)$. We require one more definition:

Definition 2.5. *Let $\mathcal{Y} = \varinjlim_j \mathcal{Y}^j$ be an ind-scheme. Then a* dimension theory, *Δ, for $\mathcal{Y}$ is a rule assigning an integer $\Delta(j)$ to each scheme $\mathcal{Y}^j$ in such a way that for $i < j$, the codimension of $\mathcal{Y}^i$ inside $\mathcal{Y}^j$ is equal to $\Delta(j) - \Delta(i)$.*

Remark.

- It is clear that a dimension theory always exists for $\mathcal{Y}$ and the set of such is a $\mathbb{Z}$-torsor.
- Given two presentations of $\mathcal{Y}$ as a colimit of schemes along finite codimensional closed embeddings, a dimension theory with respect to one presentation canonically produces a dimension theory with respect to the other.

We are finally in the position to define the cohomology of a smooth ind-scheme, $\mathcal{Y}$, equipped with a dimension theory Δ.

Definition 2.6. *We define* $H^*_\Delta(\mathcal{Y}) = \varinjlim_j H^{*+2\Delta(j)}(\mathcal{Y}^j)$, *where* $\{\mathcal{Y}^j\}_j$ *is a given smooth presentation of* $\mathcal{Y}$ *and the limit is taken with respect to the Gysin maps.*

Remark. This doesn't depend on the presentation. The dependence on the choice of dimension theory is only up to a shift in degree. We will often drop the Δ subscript in what follows, and instead write H^*_{ren}, which we'll refer to as *renormalised cohomology.*

We are now in the position to state the main theorem of this note:

Theorem 2.2. *Let* F *be a homogenous function on* $\mathbb{A}^d$ *with an isolated singularity at* 0. *Let* $\mathcal{X}$ *denote the nearby fibre of the* 0*th Fourier mode on* $\mathcal{L}\mathbb{A}^d$. *Fixing the dimension theory assigning* 0 *to* $\mathcal{L}^+\mathbb{A}^d \cap \mathcal{X}$, *the renormalised cohomology of* $\mathcal{X}$ *is computed as*

$$H^*_{ren}(\mathcal{X}) = \mathbb{Q}^{\mu(F)}\big[d-1\big],$$

where $\mu(F)$ *denotes the Milnor number of the singularity.*

Remark.

- Letting $\mathcal{X}^- = \mathcal{X} \cap \mathcal{L}^-\mathbb{A}^d$, the above is equivalent to the vanishing of the renormalised cohomology of $\mathcal{X}^-$.
- Of course, to make sense of $H^*_{ren}(\mathcal{X})$ we should prove that $\mathcal{X}$ is smooth; this is very easy as will be seen in the next section.

3. The Computation

3.1. **The Set-Up.** For ease of reference we recall the set-up. Let $F \in \mathcal{O}(\mathbb{A}^d)$ be a homogenous polynomial with isolated singularity at 0. The degree of homogeneity of F will be denoted δ. The coordinates on the affine space $\mathbb{A}^d$ will be denoted with upper indices, $\{z^1, ..., z^d\}$. The loop space $\mathcal{L}\mathbb{A}^d$ is then $\varinjlim_n \mathbb{A}\{z^i_j \mid i \in \big[1,d\big], j \in \mathbb{Z}_{\geq -n}\}$ and it carries the function $\hat{F} = CT(F(z^1(t), ..., z^d(t)))$, where CT denotes the constant term and $z(t) = \sum_i z_i t^i$. The closed subscheme $\{\hat{F} = 1\}$ is denoted $\mathcal{X}$.

Lets say something about the smooth structure: we write $\mathcal{X}^n_m = \mathcal{X} \cap \mathcal{L}^n_m\mathbb{A}^d$, where $\mathcal{L}^n_m\mathbb{A}^d$ is defined to be $\mathbb{A}\{z^i_j \mid i \in \big[1,d\big], j \in \big[-n,m\big]\}$. Evidently $\mathcal{X} = \varinjlim_n \varprojlim_m \mathcal{X}^n_m$.

We now record a trivial lemma, guaranteeing that we may define the renormalised cohomology of $\mathcal{X}$.

Lemma 3.1. *The family* $\{\mathcal{X}^n_m\}_{n,m}$ *gives a smooth presentation for* $\mathcal{X}$*, i.e. every space* $\mathcal{X}^n_m$ *is smooth.*

Proof. Recall that F is homogeneous of weight $\delta > 0$ and hence the $\mathbb{G}_m$ action on $\mathbb{A}^d$ induces one on $\mathcal{L}\mathbb{A}^d$ for which $\hat{F}$ has weight δ. Each $\mathcal{X}^n_m$ is then the nearby fibre of a homogeneous function on an affine space. However Euler's formula guarantees that the singularities of such a function are in the central fibre, so we are done. □

We will write $\mathcal{X}^n = \varprojlim_m \mathcal{X}^n_m$ in what follows. We've seen above that $\mathcal{X}^n$ is a smooth (infinite type) scheme and of course the colimit of the $\mathcal{X}^n$ is $\mathcal{X}$.

3.2. **Proof of Theorem 2.2.**

Proof. We'll break the proof up into a few steps.

(*Step 1*) Observe that if we bound the conformal degree of our variables below by $-n$, then homogeneity of F implies that no variables of conformal degree strictly greater than $n(\delta-1)$ can show up in $\hat{F}$ (more properly $\hat{F}$ restricted to $\mathcal{X}^n$). This is simply because the conformal weight would become strictly positive otherwise. This immediately implies that we have an isomorphism:

$$\mathcal{X}^n \xrightarrow{\sim} \mathcal{X}^n_{n(\delta-1)} \times \mathbb{A}\{z^i_j \mid i \in [1,d], j > n(\delta-1)\}.$$

(*Step 2*). We wish to calculate the Gysin maps in cohomology for the inclusions $\mathcal{X}^n \hookrightarrow \mathcal{X}^{n+1}$. Now, assume given two smooth finite type spaces X and Y and a closed inclusion (relative to $\mathbb{A}\{\mathbb{Z}_{\geq 0}\}$), $X \times \mathbb{A}\{\mathbb{Z}_{\geq 0}\} \hookrightarrow Y \times \mathbb{A}\{\mathbb{Z}_{\geq 0}\}$. Then to compute the Gysin map in cohomology for this inclusion we need only do it for $X \hookrightarrow Y$.

Using (*step 1*) we see that we need to compute the Gysin map for the closed inclusion $\mathcal{X}^n_{(n+1)(\delta-1)} \hookrightarrow \mathcal{X}^{n+1}_{(n+1)(\delta-1)}$. We denote by $\mathcal{U}^n$ the open complement to this inclusion. So as to avoid having to write $(n+1)(\delta-1)$ repeatedly we hereby denote this number N. Recall the Gysin exact sequence:

$$\cdots \to H^*(\mathcal{X}^n_N) \to H^{*+2d}(\mathcal{X}^{n+1}_N) \to H^{*+2d}(\mathcal{U}^n) \to H^{*+1}(\mathcal{X}^n_N) \to \cdots ;$$

we'd like to use this to compute the Gysin maps by computing the cohomology of $\mathcal{U}^n$.

(*Step 3*) Here we compute the cohomology of $\mathcal{U}^n$ explicitly. This is the main technical step and relies crucially on the hypothesis that F be a homogenous isolated singularity.

Claim: $\mathcal{U}^n$ has the same cohomology as $\mathbb{A}^d \backslash \{0\}$, i.e. the cohomology of S^{2d-1}.

In order to prove this we firstly define $\mathcal{U}^n(j)$ to be the locus of non-vanishing of $\partial_{z^j_N} \hat{F}$. Monomials showing up in $\hat{F}$ can contain at most one variable of conformal degree N for obvious degree reasons. This implies that $\hat{F}$ looks like

$$\sum_j z^j_N (\partial_{z^j_N} \hat{F}) + \mathcal{O}(< N),$$

where the big $\mathcal{O}$ notation denotes a sum of monomials with no conformal degree N terms. Further, let us note that the chain rule *together with the homogeneity assumption* imply that we have

$$\partial_{z^j_N} \hat{F} = \partial_j F(z^1_{-(n+1)}, ..., z^d_{-(n+1)}).$$

Putting the above together we see that we have an isomorphism

$$\mathcal{U}^n(j) \cong \mathbb{A}\{z^i_l \mid i \in [1,d]\backslash\{j\}, l \in [-n, N]\} \times \{\partial_j F \neq 0\},$$

where the second factor on the left hand side is understood to be contained in the affine space $\mathbb{A}^d = \mathbb{A}\{z^1_{-(n+1)}, ..., z^d_{-(n+1)}\}$.

We claim further that $\mathcal{U}^n = \bigcup_j \mathcal{U}^n(j)$. To see this, recall that $\mathcal{U}^n(j)$ is the locus of non-vanishing of $\partial_j F(z^1_{-(n+1)}, ..., z^d_{-(n+1)})$ and the simultaneous vanishing locus of these partials is precisely the locus of vanishing of all the functions $z^i_{-(n+1)}$, since F was assumed to have an isolated singularity at the origin. That is to say the locus of vanishing is precisely $\mathcal{X}^n_N$, which is the complement to $\mathcal{U}^n$, and so the claim holds.

We'll now note the following technical (and easy) observation.

Suppose we're given topological spaces, X, Y and Z with $X \subset Y \times Z$. Suppose further that we are given open decompositions $X = \bigcup_i X_i$ and $Z = \bigcup_i Z_i$, for $i \in [1, d]$ with $X_i \subset Y \times Z_i$, and finally that projection onto the second factor, $X \to Z$ restricts to a cohomology isomorphism on each multiple interection of the X_i. Then we have an isomorphism $H^*(X) \cong H^*(\bigcup_j Z_j)$. To see this note that the projection induces a morphism of convergent Mayer-Vietoris spectral sequences which is an isomorphism at the E^1-page.

Take $X = \mathcal{U}^n$, $Y = \mathbb{A}\{z^i_j \mid i \in [1,d], j \in [-n,N]\}$ and $Z = \mathbb{A}\{z^1_{-(n+1)},...,z^d_{-(n+1)}\}$. The decomposition of $\mathcal{U}^n$ is of course given by the $\mathcal{U}^n(j)$ and that of $\mathbb{A}\{z^1_{-(n+1)},...,z^d_{-(n+1)}\}$ by the non-vanishing of the various partials. Applying the above we deduce a cohomology isomorphism $H^*(\mathcal{U}^n) \cong H^*(\bigcup_j\{\partial_j F \neq 0\})$. Finally, the singularity at 0 is isolated and hence we have $\bigcup_j\{\partial_j F \neq 0\} = \mathbb{A}^d \backslash \{0\}$ and the claim is proven.

Step 4 We can now conclude the proof quite easily. Examining the Gysin sequence

$$\cdots \to H^*(\mathcal{X}^n_N) \to H^{*+2d}(\mathcal{X}^{n+1}_N) \to H^{*+2d}(\mathcal{U}^n) \to H^{*+1}(\mathcal{X}^n_N) \to \cdots$$

we see stabilisation in the limit defining H^*_{ren} at $n = 0$ for any $* > 0$, because $H^{>2d-1}(\mathcal{U}^n) = 0$ by the result above. By Milnor's results (cf. [6]) this will produce Milnor fibre cohomology concentrated in $* = d - 1$.

The same argument implies stabilisation at $n = 1$ for $* = 0$. It then suffices to prove that this is 0, i.e. that $H^{2d}(\mathcal{X}^1) = 0$, and as above this amounts to showing $H^{2d}(\mathcal{X}^1_{\delta-1}) = 0$. To do this we can simply observe that $\mathcal{X}^1_{\delta-1}$ is the nearby fibre of an isolated singularity on an affine space of dimension $d\delta$ and so all cohomology is in middle dimension according again to a theorem of Milnor (cf [6]). If H^{2d} were not zero then we'd deduce $d\delta - 1 = 2d$ and this case is easily handled by itself as it implies $d = 1$.

This concludes the proof of Theorem 2.2. □

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