Koszul duality in deformation quantization and Tamarkin's approach to Kontsevich formality

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Abstract

Let α be a quadratic Poisson bivector on a vector space V. Then one can also consider α as a quadratic Poisson bivector on the vector space $V^*[1]$. Fixed a universal deformation quantization (prediction some weights to all Kontsevich graphs [K97]), we have deformation quantization of the both algebras $S(V^*)$ and $\Lambda(V)$. These are graded quadratic algebras, and therefore Koszul algebras. We prove that for some universal deformation quantization, independent on α, these two algebras are Koszul dual. We characterize some deformation quantizations for which this theorem is true in the framework of the Tamarkin's theory [T1].

Introduction

This paper is devoted to the theorem that there exists a universal deformation quantization compatible with the Koszul duality, as it is explained in the Abstract. Let us firstly formulate it here in a more detail, and then outline the main ideas of the proof.

0.1

Let V be a finite-dimensional vector space over $\mathbb{C}$. We denote by $T_{poly}(V)$ the graded Lie algebra of polynomial polyvector fields on V, with the Schouten-Nijenhuis bracket. For a $\mathbb{Z}$-graded vector space W denote by $W[1]$ the graded space for which $(W[1])^i = W^{1+i}$, that is, the space "shifted to the left". The following simple statement is very fundamental for this work:

Lemma. *There is a canonical isomorphism of graded Lie algebras $\mathcal{D}\colon T_{poly}(V) \to T_{poly}(V^*[1])$.*

The map $\mathcal{D}$ maps a bi-homogeneous polyvector field γ on V, $\gamma = x_{i_1}\dots x_{i_k}\frac{\partial}{\partial x_{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{j_\ell}}$ to the polyvector field $\mathcal{D}(\gamma) = \xi_{j_1}\dots\xi_{j_\ell}\frac{\partial}{\partial \xi_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial \xi_{i_k}}$ on the space $V^*[1]$. Here $\{x_i\}$ is a basis in V^*, and $\{\xi_i\}$ is the dual basis in $V[-1]$. $\square$

It is a good place to recall the Hochschild-Kostant-Rosenberg theorem by which the cohomological Hochschild complex of the algebra $A = S(V^*)$ endowed with the

Gerstenhaber bracket has cohomology isomorphic to $T_{poly}(V)$ as graded Lie algebra. That is, the Gerstenhaber bracket becomes the Schouten-Nijenhuis bracket on the level of cohomology.

This theorem is related to the lemma above (which is certainly clear just straightforwardly, without this more sophisticated argument), as follows. Consider the algebras $A = S(V^*)$ and $B = \Lambda(V) = Fun(V^*[1])$. The algebras A and B are Koszul dual (see, e.g. [PP]). Bernhard Keller proved in [Kel1] (see also the discussion below in Sections 1.4-1.6) that the cohomological Hochschild complexes $\mathrm{Hoch}^\bullet(A)$ and $\mathrm{Hoch}^\bullet(B)$ are quasi-isomorphic with all structures when A and B are quadratic Koszul and Koszul dual algebras. (For the Hochschild cohomology it was known before, see the references in loc.cit). In our case $HH^\bullet(A) = T_{poly}(V)$ and $HH^\bullet(B) = T_{poly}(V^*[1])$.

0.2

The isomorphism $\mathcal{D}$ from the lemma above does not change the grading of the polyvector field, but it maps i-polyvector fields with k-linear coefficients to k-polyvector fields with i-linear coefficients. In particular, it maps quadratic bivector fields on V to quadratic bivector fields on $V^*[1]$. Moreover, $\mathcal{D}$ maps a Poisson quadratic bivector on V to a Poisson quadratic bivector on $V^*[1]$, because it is a map of Lie algebras.

In [K97], Maxim Kontsevich gave a formula for deformation quantization of algebra $S(V^*)$ by a Poisson bivector α on V (the vector spaced V should be finite-dimensional). His formula is organized as a sum over admissible graphs, and each graph is taken with the Kontsevich weight W_Γ. In particular, this W_Γ depends only on the graph Γ and does not depend on dimension of the space V.

Consider now all these complex numbers W_Γ as undefined variables. Then the associativity gives an infinite number of quadratic equations on W_Γ. Kontsevich's paper [K97] then shows that these equations have at least one complex solution. Actually there is a lot of essentially different solutions, as is clear from [T] (see the discussion in Section 3 of this paper). Any such deformation quantization is called *universal* because the complex numbers W_Γ do not depend on the vector space V.

The case of a quadratic Poisson bivector α is distinguished, by the following lemma:

Lemma. *Let $S(V^*)_\alpha$ be a universal deformation quantization of $S(V^*)$ by a quadratic Poisson bivector α. Then the algebra $S(V^*)_\alpha$ is graded. This means that for $f \in S^i(V^*)[[\hbar]]$ and $g \in S^j(V^*)[[\hbar]]$, the product $f \star g \in S^{i+j}(V^*)[[\hbar]]$.*

Proof. By the Kontsevich formula,

$$f \star g = f \cdot g + \sum_{k \geq 1} \hbar^k \sum_{\Gamma \in G_{k,2}} W_\Gamma B_\Gamma(f,g) \tag{1}$$

where $G_{k,2}$ is the set of admissible graphs with two vertices on the real line and k vertices in the upper half-plane (see [K97], Section 1 for details). Now each graph Γ from $G_{k,2}$

has k vertices at the half-plane, and $2k$ edges. One can compute the grading degree of $B_\Gamma(f, g)$ as follows. It is the sum of degrees of quantities associated with all vertices (which is $\deg f + \deg g + k \deg \alpha = \deg f + \deg g + 2k$) minus the number of edges (equal to $2k$ by definition of an admissible graph) because each edge differentiate once, and then decreases the degree by 1). This difference is equal to $\deg f + \deg g$. $\square$

In particular, for quadratic deformation quantization the map $x_i \cdot x_j \mapsto x_i \star x_j$ gives a $\mathbb{C}[[\hbar]]$-linear endomorphism of the space $S^2(V)[[\hbar]]$ which is clearly non-degenerate. We can find an inverse to it, then we can present the star-algebra as the quotient of the tensor algebra $T(V^*)$ by the set of *quadratic* relations $R_{ij} \in V^* \otimes V^*$, one relation for each pair of indices $1 \le i < j \le \dim V$. We conclude, that the Kontsevich deformation quantization of $S(V^*)$ by a quadratic Poisson bivector is a graded quadratic algebra.

0.3

We actually get two quadratic associative algebras for any universal deformation quantization, one is the deformation quantization of $S(V^*)$ by the quadratic Poisson bivector α, and another one is the deformation quantization of $\Lambda(V) = Fun(V^*[1])$ by the quadratic Poisson bivector $\mathcal{D}(\alpha)$. Denote these two algebras by $S(V^*) \otimes \mathbb{C}[[\hbar]]_\alpha$ and $\Lambda(V) \otimes \mathbb{C}[[\hbar]]_{\mathcal{D}(\alpha)}$.

In the present paper we prove the following result:

Theorem. *There exists a universal deformation quantization such that the two algebras $S(V^*) \otimes \mathbb{C}[[\hbar]]_\alpha$ and $\Lambda(V) \otimes \mathbb{C}[[\hbar]]_{\mathcal{D}(\alpha)}$ are Koszul dual as algebras over $\mathbb{C}[[\hbar]]$. In particular,*

$$\mathrm{Ext}^\bullet_{S(V^*)\otimes\mathbb{C}[[\hbar]]_\alpha - Mod}(\mathbb{C}[[\hbar]], \mathbb{C}[[\hbar]]) = \Lambda(V) \otimes \mathbb{C}[[\hbar]]_{\mathcal{D}(\alpha)} \tag{2}$$

and

$$\mathrm{Ext}^\bullet_{\Lambda(V)\otimes\mathbb{C}[[\hbar]]_{\mathcal{D}(\alpha)} - Mod}(\mathbb{C}[[\hbar]], \mathbb{C}[[\hbar]]) = S(V^*) \otimes \mathbb{C}[[\hbar]]_\alpha \tag{3}$$

The Tamarkin's deformation quantization defined from any Drinfeld's associator (which is clearly universal) satisfies the condition of Theorem.

See Section 1 of this paper for an overview of Koszul duality, and of Koszul duality over a discrete valuation ring.

0.4

Now let us outline our strategy how to prove this theorem.

We firstly "lift the Theorem" on the level of complexes. We do it as follows.

Let A and B be two associative algebras, and let K be a dg $B - A$-module (this means that it is a left B-module and left A-module, and the left action commutes with the right action). Define then a differential graded category with 2 objects, a and b, as follows. We set $\mathrm{Mor}(a, a) = A$, $\mathrm{Mor}(b, b) = B$, $\mathrm{Mor}(b, a) = K$, $\mathrm{Mor}(a, b) = 0$. To make this a dg category the only what we need is that A and B are algebras, and K is a $B-A$-module. Denote this category by $\mathsf{cat}(A,B,K)$, see Section 5 for more detail.

Consider the Hochschild cohomological complex $\mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K))$ of this dg category. There are natural projections $p_A\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(A)$ and $p_B\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(B)$. The B.Keller's theorem [Kel1] gives sufficient conditions for p_A and p_B being quasi-isomorphisms. These conditions are that the natural maps

$$B \to \mathrm{RHom}_{Mod-A}(K,K) \tag{4}$$

and

$$A^{opp} \to \mathrm{RHom}_{B-Mod}(K,K) \tag{5}$$

are quasi-isomorphisms.

An easy computation shows that in the case when A is Koszul algebra, $B=A^{!opp}$ is the opposite to the Koszul dual algebra, and K is the Koszul complex of A, the Keller's condition is satisfied (see Section 5).

Consider the case when $A=S(V^*)\otimes\mathbb{C}[[\hbar]]$ and $B=\Lambda(V)\otimes\mathbb{C}[[\hbar]]$. Denote in this case the category $\mathsf{cat}(A,B,K)$ where K is the Koszul complex of A, just by cat. Consider the following solid arrow diagram diagram:

$$\begin{tikzcd}
 & \mathrm{Hoch}^\bullet(A) & \\
T_{poly}(V) \arrow[ur, "\mathcal{U}^S"] \arrow[dr, "\mathcal{U}^\Lambda"'] \arrow[rr, dotted, "\mathcal{F}"] & & \mathrm{Hoch}^\bullet(\mathsf{cat}) \arrow[ul, "p_A"'] \arrow[dl, "p_B"] \\
 & \mathrm{Hoch}^\bullet(B) &
\end{tikzcd} \tag{6}$$

The right "horn" was just defined. The maps $\mathcal{U}^S$ and $\mathcal{U}^\Lambda$ in the left "horn" are the following. We consider some *universal* L_∞ map $\mathcal{U}\colon T_{poly}(V)\to\mathrm{Hoch}^\bullet(S(V^*))$. This means that we attribute some complex numbers to each Kontsevich graph in his formality morphism in [K97], but which are not necessarily the Kontsevich integrals (but the first Taylor components is fixed, it is the Hochschild-Kostant-Rosenberg map). The word "universal" again means that these numbers are the same for all spaces V. Then we apply this L_∞ morphism to our space V, it is $\mathcal{U}^S$, and the composition of $\mathcal{D}\colon T_{poly}(V)\to T_{poly}(V^*[1])$ with the L_∞ morphism $\mathcal{U}\colon T_{poly}(V^*[1])\to\mathrm{Hoch}^\bullet(\Lambda(V))$, constructed from the same universal L_∞ map.

The all solid arrows (6) are quasiisomorphisms. Therefore, they are homotopically invertible (see Section 4), and we can speak about the homotopical commutativity of this diagram.

Theorem. *There exists a universal L_∞ morphism $\mathcal{U}\colon T_{poly}(V)\to\mathrm{Hoch}^\bullet(S(V^*))$ such that the solid arrow diagram (6) is homotopically commutative. The L_∞ morphism cor-*

responded by Tamarkin's theory (see Sections 2 and 3) to any choice of the Drinfeld associator satisfies this property.

We firstly explain why our theorem about Koszul duality follows from this theorem, and, secondly, how to prove this new theorem.

0.5

If we know the homotopical commutativity of the solid arrow diagram (of quasi-isomorphismes) (6), we can construct the dotted arrow $\mathcal{F}$ which is a G_∞ quasi-isomorphism $\mathcal{F}\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(\mathsf{cat})$, which divides the diagram into two homotopically commutative triangles. Then, if α is a quadratic Poisson bivector field on V, the L_∞ part of $\mathcal{F}$ defines a solution of the Maurer-Cartan equation $\mathcal{F}_*(\alpha)$ in $\mathrm{Hoch}^\bullet(\mathsf{cat})$. A solution of the Maurer-Cartan equation in $\mathrm{Hoch}^\bullet(\mathsf{cat})$ deforms the following four things simultaneously: the algebra structures on A and B, the differential in K, and the bimodule structure. Using very simple arguments we then can prove that this deformed complex K is a free resolution of the deformed A, and the deformed bimodule isomorphisms (4)-(5) give the Koszul duality theorem. See Section 7 for detail.

0.6

Here we outline the main ideas of Theorem 0.4. First of all, the two maps p_A and p_B in the right "horn" of the diagram (6) are maps of B_∞ algebras (see [Kel1]). Here B_∞ is the braces dg operad, which acts on the Hochschild cohomological complex of any algebra or dg category. Formally it is defined as follows: a B_∞ module structure on X is a dg bialgebra structure on the cofree coalgebra cogenerated by $X[1]$ such that the coalgebra structure coincides with the given one. The action of B_∞ on the Hochschild complex $\mathrm{Hoch}^\bullet(A)$ of any dg algebra (or dg category) A is constructed by Getzler-Jones [GJ] via the braces operations.

Now define analogously the dg operad B_{Lie}. A vector space Y is an algebra over B_{Lie} iff there is a dg Lie bialgebra structure on the free Lie coalgebra cogenerated by $Y[1]$ such that the Lie coalgebra structure coincides with the given one. The operads B_{Lie} and B_∞ are quasi-isomorphic by the Etingof-Kazhdan (de)quantization. Any quasi-isomorphism of operads $B_{\mathrm{Lie}} \to B_\infty$ depends on the choice of Drinfeld associator [D].

The operad B_{Lie} is quasi-isomorphic to the Gerstenhaber homotopical operad G_∞, as is explained in [H], Section 6 (see also discussion in Section 2 of this paper). Finally, the Gerstenhaber operad is Koszul, and G_∞ is its Koszul resolution. Thus, any B_∞ algebra can be considered as a G_∞ algebra. As G_∞ is a resolution of the Gerstenhaber operad G, all three dg operads G_∞, B_∞, and B_{Lie}, are quasi-isomorphic to their cohomology G.

(There is no canonical morphism from B_{Lie} to B_∞. Any such quasi-isomorphism gives a G_∞ structure on the Hochschild cohomological complex of any dg category. Any Drinfeld associator [D] gives, via the Etingof-Kazhdan (de)quantization, such a morphism of operads.)

Now consider the entire diagram (6) as a diagram of G_∞ algebras and G_∞ maps, where the G_∞ action on the Hochschild complexes is as above, it depends on the choice of a map $B_{\mathrm{Lie}} \to B_\infty$. Then, if our diagram is homotopically not commutative, it defines some G_∞ automorphism of $T_{poly}(V)$.

This G_∞ automorphism is clearly $\mathrm{Aff}(V)$-equivariant. First of all, we prove that on the level of cohomology the diagram (6) is commutative. This is in a sense the only new computation which we make in this paper (see Section 5).

Thus, we can take the logarithm of this automorphism, which is G_∞-derivation. By the Tamarkin's G_∞-ridigity of $T_{poly}(V)$, any Aff-equivariant derivation is homotopically inner. But any inner derivation acts non-trivially on cohomology! On the other hand, a G_∞-morphism homotopically equivalent to identity, acts by the identity on cohomology. This proves that our diagram is homotopically commutative. The only property of this diagram which we have used is that it is defined over G_∞ and is commutative on the level of cohomology.

0.7

When the author started to attack this problem, he started to prove the homotopical commutativity of the diagram (6) by "physical" methods. Namely, the Kontsevich's formality in the original proof given in [K97] is a particular case of the AKSZ model on open disc [AKSZ], also called by Cattaneo and Felder the "Poisson sigma-model". As usual in open theories, we should impose some boundary conditions for the restrictions of the fields to the circle $S^1 = \partial D^2$. Maxim Kontsevich considers the boundary condition "$p = 0$" on all arcs. This, certainly together with other mathematical insights, led him in [K97] to the formality theorem.

Our idea was to divide S^1 by two parts, fixing two points $\{0\}$ and $\{\infty\}$ (in the Kontsevich's case only $\{\infty\}$ is fixed). Then, we impose the boundary condition "$x = 0$" on all left arcs, and "$p = 0$" on all right arcs. This seems to be very reasonable, and the author hoped to construct in this way an L_∞ quasi-isomorphism $\mathcal{F}$ (the dotted arrow in (6)), making the two triangles homotopically commutative.

Somehow, he did not succeed in that. From the point of view of this paper, it seems that the reason for that is the following.

The author worked with the Kontsevich's propagator in [K97], namely, with

$$\varphi(z_1, z_2) = \frac{1}{2\pi i} d\mathrm{Log} \frac{(z_1 - z_2)(z_1 - \overline{z}_2)}{(\overline{z}_1 - z_2)(\overline{z}_1 - \overline{z}_2)} \tag{7}$$

(here z_1 and z_2 are distinct points of the complex upper half-plane).

In this paper we deal with the Tamarkin's quantization. Conjecturally (see [K99]) when this formality morphism is constructed from the Knizhnik-Zamolodchikov Drinfeld's associator, it coincides (as a universal L_∞ morphism, see above) with the L_∞ morphism, constructed from the "another Kontsevich's propagator". This is "the half" of (7):

$$\varphi_1(z_1, z_2) = \frac{1}{2\pi i} d\mathrm{Log}\frac{z_1 - z_2}{\overline{z}_1 - z_2} \tag{8}$$

Kontsevich proved (unpublished) that this propagator also leads to an L_∞ morphism from $T_{poly}(V)$ to $\mathrm{Hoch}^\bullet(S(V^*))$. If this conjecture (that the Tamarkin's theory in the Knizhnik-Zamolodchikov case gives this propagator) is true, we should try to elaborate the physical idea described above (with the two boundary conditions) for this propagator. The reason is that it is not a priori clear that the Kontsevich's first propagator $\varphi(z_1, z_2)$ comes from any Drinfeld's associator, and therefore from the Tamarkin's theory.

We are going to come back to these questions in the sequel.

0.8

We tried to make the exposition as self-contained as possible. In particular, we prove in Section 2.4 the main Lemma in the Tamarkin's proof [T1] of the Kontsevich formality, because we use it here not only for the first cohomology as in [T1] and [H], and also for 0-th cohomology. We give a simple proof of it for all cohomology for completeness. As well, we reproduce in Section 4.2 the proof of Keller's theorem from [Kel1], because in [Kel1] some details are omitted. Nevertheless, in one point we did not overcome some vagueness. This is the using of the homotopical relation for maps of dg operads or algebras over dg operads. Some implications like "homotopical maps of dg operads induce homotopical morphisms of algebras" in Section 3, are stated without proofs. Finally in Section 5 we give a construction of the homotopical category of dg Lie algebras through the "right cylinder" in the sense of [Q], which is suitable for the proof of the Main Theorem in Section 7.

0.9

The paper is organized as follows:

In Section 1 we develop the Koszul duality for algebras over a discrete valuation rings. Our main example is the algebras over $\mathbb{C}[[\hbar]]$, and we should justify that the main theorems of Koszul duality for associative algebras hold in this context;

In Section 2 we give a brief exposition of the Tamarkin's theory [T1]. The Hinich's paper [H] is a very good survey, but we achieve some more clarity in the computation of deformation cohomology of $T_{poly}(V)$ over the operad G_∞ of homotopy Gerstenhaber algebras. As well, in the Appendix after Section 2.5 we give a deduction of the existence of Kontsevich formality over $\mathbb{Q}$ from its existence over $\mathbb{C}$, which differs from the Drinfeld's approach [D]. This deduction seems to be new;

In Section 3 we touch some unsolved problems in the Tamarkin's theory and leave them unsolved, wee only need to know here that any map of operads $G_\infty \to B_\infty$ defined up to homotopy, defines a *universal* G_∞ map $T_{poly}(V) \to \mathrm{Hoch}^\bullet(S(V^*))$ where the G_∞

structure on $\mathrm{Hoch}^\bullet(S(V^*))$ is defined via the map of operads. The deformation quantizations for which our Main Theorem is true belong to the image of the map $\mathfrak{X}$ defined there;

In Section 4 for introduce differential graded categories, give a construction of the Keller's dg category from [Kel1] associated with a Keller's triple, and reformulate our Main Theorem in this new setting. We get a more general statement, which is, however, more easy to prove;

A very short Section 5 is just a place to relax before the long computation in Section 6, here we recall the explicit construction [Sh3] of the Quillen's homotopical category via the right cylinder. The advantage of this construction is that it is immediately clear from it that two homotopical L_∞ maps map a solution of the Maurer-Cartan equation to gauge equivalent solutions (Lemma 5.2);

In Section 6 we construct the Hochschild-Kostant-Rosenberg map for the Keller's dg category. This computation is done in terms of graphs, closed to the ones from [K97]. Originally the author got this computation truing to construct the L_∞ morphism $\mathcal{F}\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(\mathsf{cat}(A, B, K))$ dividing the diagram (6) into two commutative triangles, by "physical" methods. The computation here is the only what the author succeed to do in this direction;

The final Section 7 is the culmination of our story. Here we deduce the Main Theorem on Koszul duality in deformation quantization from Theorem 4.4. The idea is that from the diagram (6) associates with a quadratic Poisson bivector α on V a solution of the Maurer-Cartan equation in the Hochschild complex of the Keller's dg category. This Maurer-Cartan elements defines an A_∞ deformation of the Keller's category, and, in particular, deforms the Koszul complex. This is enough to conclude that the two deformed algebras are Koszul dual.

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1 Koszul duality for algebras over a discrete valuation ring

Here we give a brief overview of the Koszul duality. Our main reference is Section 2 of [BGS]. In loc.cit., the zero degree component A_0 is supposed to be a (non-commutative) semisimple algebra over the base field k. For our applications in deformation quantization, we should consider algebras over $\mathbb{C}[[\hbar]]$. For this reason, we show that the theory of Koszul duality may be defined over an arbitrary commutative discrete valuation ring. This result seems to be new, although L.Positselski announced in [P] that the zero degree component A_0 may be an arbitrary algebra over the base field.

1.1

The main classical example of Koszul dual algebras are the algebras $A = S(V^*)$ and $B = \Lambda(V)$, where V is a finite-dimensional vector space over the base field k. In general, suppose A_0 is a fixed k-algebra. Koszulness is a property of a graded algebra

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \dots \tag{9}$$

that is,

$$A_i \cdot A_j \subset A_{i+j} \tag{10}$$

In our example with $S(V^*)$ and $\Lambda(V)$ the algebra $A_0 = k$, it is the simplest possible case. In general, all A_i are A_0-bimodules.

There is a natural projection $p\colon A \to A_0$ which endows A_0 with a (left) A-module structure. Denote by $A - \mathrm{Mod}$ the category of all left A-modules, and by $A - \mathrm{mod}$ the category of graded left A-modules.

The A-module A_0 always has a free resolution in $A - \mathrm{mod}$

$$\dots \to M_2 \to M_1 \to M_0 \to 0 \tag{11}$$

such that M_i is a graded A-module generated by elements of degrees $\geq i$. Indeed, the bar-resolution

$$\dots \to A \otimes_k A_+^{\otimes 2} \otimes_k A_0 \to A \otimes_k A_+ \otimes_k A_0 \to A \otimes_k A_0 \to 0 \tag{12}$$

obeys this property. (Here $A_+ = A_1 \oplus A_2 \oplus \dots$). This motivates the following definition:

Definition. A graded algebra (9) is called *Koszul* if the A-module A_0 admits a projective resolution (11) in $A - \mathrm{mod}$ such that each M_i is finitely generated by elements of degree i.

For our example with the symmetric and the exterior algebra, such a resolution is the following *Koszul complex*:

$$\cdots \to S(V^*)\otimes \Lambda^3(V)^* \to S(V^*)\otimes \Lambda^2(V)^* \to S(V^*)\otimes V^* \to S(V^*) \to 0 \tag{13}$$

with the differential

$$d=\sum_{i=1}^{\dim V} x_i \otimes \frac{\partial}{\partial \xi_i} \tag{14}$$

Here $\{x_i\}$ is a basis in the vector space V^* and $\{\xi_i\}$ is the corresponding basis in $V^*[1]$. The differential is a $\mathfrak{gl}(V)$-invariant element, it does not depend on the choice of basis $\{x_i\}$ of the vector space V^*.

1.2

Here we explain some consequences of the definition of Koszul algebra, leading to the concept of Koszul duality for quadratic algebras. In Sections 1.2.1-1.2.3 A_0 may be arbitrary finite-dimensional algebra over the ground field k, and in Sections 1.2.4-1.2.6 we suppose that A_0 is a semisimple finite-dimensional algebra over k (see [BGS]).

1.2.1

Let A be a graded algebra. Then the space $\mathrm{Ext}^\bullet_{A-\mathrm{Mod}}(A_0,A_0)$ is naturally *bigraded*. We write $\mathrm{Ext}^n_{A-\mathrm{Mod}}(A_0,A_0)=\oplus_{a+b=n}\mathrm{Ext}^{a,b}(A_0,A_0)$. From the bar-resolution (12) we see that for a general algebra A the only non-zero $\mathrm{Ext}^{a,b}(A_0,A_0)$ appear for $a\le -b$ (here a is the cohomological grading and b is the inner grading). In the Koszul case the only nonzero summands are $\mathrm{Ext}^{a,-a}$. Let us analyze this condition for $a=1$ and $a=2$.

1.2.2

Lemma. *Suppose A is a graded algebra.*

1. *If $\mathrm{Ext}^1_{A-\mathrm{Mod}}(A_0,A_0)=\mathrm{Ext}^{1,-1}(A_0,A_0)$ (that is, all $\mathrm{Ext}^{1,-b}=0$ for $b>1$), the algebra A is 1-generated. The latter means that the algebra A in the form of (9) is generated over A_0 by A_1;*
2. *if, furthermore, $\mathrm{Ext}^2_{A-\mathrm{Mod}}(A_0,A_0)=\mathrm{Ext}^{2,-2}(A_0,A_0)$ (that is, $\mathrm{Ext}^{2,-\ell}(A_0,A_0)=0$ for $\ell\ge 3$), the algebra A is quadratic. This means that $A=T_{A_0}(A_1)/I$ where I is a graded ideal generated as a two-sided ideal by $I_2=I\cap A_2$.*

See [BGS], Section 2.3.

This Lemma implies that any Koszul algebra is quadratic. So, in fact the Koszulness is a property of quadratic algebras.

1.2.3

From now on, we use the notation $I = I_2$ for the intersection of the graded ideal I in $T_{A_0}(A_1)$ with A_2. Any quadratic algebra is uniquely defined by the triple $(A_0, A_1, I \subset A_1 \otimes_{A_0} A_1)$.

Using the bar-complex (12), it is very easy to compute the "diagonal part" $\oplus_\ell \mathrm{Ext}^{\ell,-\ell}(A_0, A_0) \subset \mathrm{Ext}^\bullet_{A-\mathrm{Mod}}(A_0, A_0)$ for any algebra A. Let us formulate the answer.

Define from a triple (A_0, A_1, I) another triple $(A_0^\vee, A_1^\vee, I^\vee)$, as follows. Suppose A_1 and I are flat A_0-bimodules. We set $A_0^\vee = A_0$, $A_1^\vee = \mathrm{Hom}_{A_0}(A_1, A_0)[-1]$. Define now $I^\vee$. Denote firstly $A_1^* = \mathrm{Hom}_{A_0}(A_1, A_0)$. There is a pairing $(A_1 \otimes_{A_0} A_1) \otimes (A_1^* \otimes A_0 A_1^*) \to A_0$ which is non-degenerate. Denote by I^* the subspace in $A_1^* \otimes_{A_0} A_1^*$ dual to I. Denote by $I^\vee = I^*[-2]$, it is a subspace in $A_1^\vee \otimes_{A_0} A_1^\vee$. The triple $(A_0, A_1^\vee, I^\vee)$ generates some quadratic algebra, denote it by $A^\vee$.

Let now A be any 1-generated not necessarily quadratic algebra. Then the *quadratic part* A^q is well-defined. Let A be a quotient of $T_{A_0}(A_1)$ by graded not necessarily quadratic ideal. We define A^q as the quadratic algebra associated with the triple $(A_0, A_1, I \cap A_2)$. There is a canonical surjection $A^q \to A$ which is an isomorphism in degrees 0, 1, and 2.

Lemma. *Let A be a 1-generated algebra over A_0. Then the diagonal cohomology $\oplus_\ell \mathrm{Ext}^{\ell,-\ell}(A_0, A_0)$ as algebra is canonically isomorphic to the algebra opposed to $(A^q)^\vee$. Here by the opposed algebra to an algebra B we understand the product $b_1 \star^{\mathrm{opp}} b_2 = b_2 \star b_1$.*

It is a direct consequence from the bar-resolution (12).

In particular, let now a graded 1-generated algebra A be Koszul. Then $\mathrm{Ext}^\bullet_{A-\mathrm{Mod}}(A_0, A_0) = (A^\vee)^{\mathrm{opp}}$. This follows from the identity $A = A^q$ for a quadratic algebra A (in particular, for Koszul A), and from the equality of the all Exts to its diagonal part for any Koszul algebra.

1.2.4

The inverse is also true, under an assumption on A_0.

Lemma. *Suppose A_0 is a simple finite-dimensional algebra over k and A is a quadratic algebra over A_0. Then if $\mathrm{Ext}^\bullet_{A-\mathrm{Mod}}(A_0, A_0)$ is equal to its diagonal part, then A is Koszul. In particular, if $\mathrm{Ext}^\bullet_{A-\mathrm{Mod}}(A_0, A_0) = (A^{q\vee})^{\mathrm{opp}}$, then A is Koszul.*

See [BGS], Proposition 2.1.3.

Let us comment why we need here a condition on A_0. The projective resolution which we need to prove that A is constructed inductively. We construct a resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to 0 \tag{15}$$

satisfying the property of Definition 1.2 and such that the differential is injective on P_i^i. We set $P_0 = A$. To perform the step of induction, set $K = \ker(P_i \to P_{i-1})$. We have:

$\mathrm{Ext}^{i+1}_{A-\mathrm{Mod}}(A_0, A_0) = \mathrm{Hom}_{A-\mathrm{Mod}}(K, A_0)$. From the condition of lemma we conclude that K is generated by the part K^{i+1} of inner degree $i+1$ (here for simplicity we suppose that A has trivial cohomological grading). Then we put $P_{i+1} = A \otimes_{A_0} K^{i+1}$. But then we need to check that the image of the map $P_{i+1} \to P_i$ is K. For this we necessarily need to know that K^{i+1} is a flat left A_0-module. For this it is sufficiently to know that K is. So we need a theorem like the following: the kernel of any map of good (flat, etc.) A_0-modules is again a flat A_0-module. It does not follow from any general things, it is a property of A_0. It is the case if *any* module is flat, as in the case of a finite-dimensional simple algebra. For another possible condition, see Section 1.3.

1.2.5

Proposition. *Suppose A is a quadratic algebra defined from a triple (A_0, A_1, I) where A_1 and I are flat A_0-bimodules. Suppose A is Koszul. Then $(A^\vee)^{\mathrm{opp}}$ is also Koszul.*

Remark. It is clear that A is Koszul iff A^{opp} is Koszul.

We give a sketch of proof, which is essentially given by the construction of the Koszul complex. For $A = S(V^*)$ the Koszul complex is constructed in (13).

Let $A = (A_0, A_1, I)$ be a quadratic algebra, and let A_1 and $I \subset A_1 \otimes_{A_0} A_1$ be flat A_0-bimodules. We define the Koszul complex

$$\dots \to K_3 \to K_2 \to K_1 \to K_0 \to 0 \tag{16}$$

We set

$$K_i = A \otimes_{A_0} K_i^i \tag{17}$$

where

$$K_i^i = \bigcap_\ell A_1^{\otimes \ell} \otimes_{A_0} I \otimes_{A_0} A_1^{\otimes i-\ell-2} \tag{18}$$

In particular, $K_0^0 = A_0$, $K_1^1 = A_1$, $K_2^2 = I$. The differential $d\colon K_i \to K_{i-1}$ is defined as the restriction of the map $\hat{d}\colon A \otimes_{A_0} A_1^{\otimes i} \to A \otimes_{A_0} A_1^{\otimes (i-1)}$ given as

$$a \otimes v_1 \otimes \dots \otimes v_i \mapsto (av_1) \otimes v_2 \otimes \dots \otimes v_{i-1} \tag{19}$$

Clearly $d^2 = 0$. The complex (16) is called the *Koszul complex* of the quadratic algebra A.

Lemma. *Let $A = (A_0, A_1, I)$ be a quadratic algebra, A_1 and I flat A_0-bimodules. Suppose, additionally, that A_0 is a finite-dimensional semisimple algebra over k. Then its Koszul complex is acyclic except degree 0 iff A is Koszul.*

See [BGS], Theorem 2.6.1 for a proof. *In the proof it is essential that the modules K_i^i a flat left A_0-bimodules.* In general the tensor product of two flat modules is flat, but there is no theorem which guarantees the same about the intersection of two flat submodules.

In the case which is considered in [BGS], any module over a finite-dimensional semisimple algebra is flat.

Let us note that the part "only if" also follows from Lemma 1.2.4.

The Proposition follows from this Lemma easily.

Indeed, it is clear that $K_i = A \otimes_{A_0} [(A^!)^*]^i[-i]$ and that the Koszul complex K of a Koszul algebra satisfies the Definition 1.1. Then the dual complex K^* also satisfies the Definition 1.1 and can be written as $K^* = A^* \otimes_{A_0} A^!$. We immediately check that it coincides with the Koszul complex of the quadratic algebra $A^!$ because $(A^!)^! = A$ for any quadratic algebra A. Then from its acyclicity follows that $A^!$ is Koszul.

1.2.6

We summarize the discussion above in the following theorem.

Theorem. *Let $A = (A_0, A_1, I)$ be a quadratic algebra, A_1 and I be flat A_0-bimodules, and A_0 be semisimple finite-dimensional algebra over k. Then A is Koszul if and only if the quadratic dual $A^!$ is also Koszul, and in this case*

$$\mathrm{Ext}^i_{A-\mathrm{Mod}}(A_0, A_0) = [(A^!)^{\mathrm{opp}}]_i[-i] \tag{20}$$

and

$$\mathrm{Ext}^i_{A^!-\mathrm{Mod}}(A_0, A_0) = [A^{\mathrm{opp}}]_i[-i] \tag{21}$$

for any integer $i \geq 0$. $\square$

1.3

In the context of deformation quantization, all our algebras are algebras over the formal power series $\mathbb{C}[[\hbar]]$, therefore, $A_0 = \mathbb{C}[[\hbar]]$. The theory of Koszul algebras as it is developed in [BGS] does not cover this case. In this Subsection we explain that in the theory of Koszul duality A_0 may be any commutative discrete valuation ring (see [AM], [M]). L.Positselski announced in [P] that A_0 may be any algebra over k.

Recall the definition of a discrete valuation ring.

Definition. A commutative ring is called a discrete valuation ring if it is an integrally closed domain with only one nonzero prime ideal. In particular, a discrete valuation ring is a local ring.

The two main examples are the following:

(1) Let C be an affine algebraic curve, and let $p \in C$ be a non-singular point (not necessarily closed). Then the local ring $\mathcal{O}_p(C)$ is a discrete valuation ring (recall that in dimension 1 integrally closed=nonsingular);

(2) let C and p be as above; we can consider the completion of the local ring $\mathcal{O}_p(C)$ by the powers of the maximal ideal. Denote this ring by $\widehat{\mathcal{O}}_p(C)$, this is a discrete valuation ring. In particular, $k[[\hbar]]$ is a discrete valuation ring.

It is known that any discrete valuation ring is Noetherian and is a principal ideal domain (see [M], Theorem 11.1).

To extend the theory of Section 1.2 to the case when A_0 is a discrete valuation ring we need to know that the intersection of flat submodules over a discrete valuation ring (Section 1.2.5), and the kernel of a map of flat modules over a discrete valuation ring (Section 1.2.4) are flat. This is guaranteed by the following, more general, result:

Lemma. *Let R be a discrete valuation ring, and let M be a flat R-module. Then any submodule of M is again flat.*

Proof. Let R be a ring and M is an R-module. Then M is flat if and only if for any finitely generated ideal $I \subset R$ the natural map $I \otimes_R M \to R \otimes_R M$ is injective (see [M], Theorem 7.7). Any ideal in a discrete valuation ring is principal ([M], Theorem 11.1); therefore flatness of a module over a discrete valuation ring is the same that torsion-free (a module M is called torsion-free if $x \neq 0$, $m \neq 0$ implies $xm \neq 0$). So now our Lemma follows from the fact that a submodule over a torsion-free module is torsion-free. $\square$

Remark. If R is any local ring and M is a finite R-module, then flatness of M implies that M is free ([M], Theorem 7.10). Nevertheless, in dimension ≥ 2 a submodule of a free module may be not free. For example, one can take the (localization of the) coordinate ring of a curve in $\mathbb{A}^2$.

Combining the Lemma above with the discussion of Section 1.2, we get the following Theorem:

Theorem. *Let $A = (A_0, A_1, I)$ be a quadratic algebra, with A_0 a commutative discrete valuation ring, and A_1, I flat A_0-modules. Then A is Koszul iff $A^!$ is, and in this case*

$$\mathrm{Ext}^i_{A-\mathrm{Mod}}(A_0, A_0) = [(A^!)^{\mathrm{opp}}]_i[-i] \tag{22}$$

and

$$\mathrm{Ext}^i_{A^!-\mathrm{Mod}}(A_0, A_0) = [A^{\mathrm{opp}}]_i[-i] \tag{23}$$

for any integer $i \geq 0$. $\square$

We will use this Theorem only for $A_0 = k[[\hbar]]$.

Remark. This theorem has an analogue for Dedekind domains. Namely, the localization of a Dedekind domain at any prime ideal is a discrete valuation ring, this is a global version of it. The main example of a Dedekind domain is the coordinate ring of a non-singular affine curve. Suppose A_0 is a Dedekind domain. We say that a quadratic algebra over A_0 is Koszul if its localization at any prime ideal is Koszul. Then we can prove the theorem analogous to the above for A_0 a Dedekind domain. More generally, we can speak about sheaves of Koszul dual quadratic algebras. At the moment the author does not know any interesting example of such situation, but he does not doubt that these examples exist.

Remark. Leonid Positselski claims in [P] that he constructed the analogous theory for any A_0. The arguments in [P] are rather complicated comparably with ours'; for the readers' convenience, we gave here a more direct simple proof in the case of discrete valuation rings.

2 Tamarkin's approach to the Kontsevich formality

Here we overview the Tamarkin's proof of Kontsevich formality theorem. The main references are [T1] and [H], some variations which allow to avoid the using of the Etingof-Kazhdan quantization (but replace it by another transcendental construction) were made by Kontsevich in [K99].

2.1 Kontsevich formality

For any associative algebra A we denote by $\mathrm{Hoch}^\bullet(A)$ the cohomological Hochschild complex of A. When $A = C^\infty(M)$ is the algebra of smooth functions on a smooth manifold M, we consider some completed tensor powers, or the polydifferential part of the usual Hochschild complex (see, e.g., [K97]). Under this assumption, the Hochschild cohomology of $A = C^\infty(M)$ is equal to smooth polyvector fields $T_{poly}(M)$. More precisely, consider the following Hochschild-Kostant-Rosenberg map $\varphi\colon T_{poly}(M) \to \mathrm{Hoch}^\bullet(C^\infty(M))$:

$$\varphi(\gamma) = \{f_1 \otimes \cdots \otimes f_k \mapsto \frac{1}{k!}\gamma(df_1 \wedge \cdots \wedge df_k)\} \tag{24}$$

for γ a k-polyvector field. Then the Hochschild-Kostant-Rosenberg theorem is

Lemma. *1. For any polyvector field γ, the cochain $\varphi(\gamma)$ is a cocycle; this gives an isomorphism of (completed or polydifferential) Hochschild cohomology of $A = C^\infty(M)$ with $T_{poly}(M)$;*

2. the bracket induced on the Hochschild cohomology from the Gerstenhaber bracket coincides, via the map φ, with the Schouten-Nijenhuis bracket of polyvector fields.

See, e.g., [K97] for definition of the Gerstenhaber and Schouten-Nijenhuis brackets.

The second claim of the Lemma means that

$$[\varphi(\gamma_1), \varphi(\gamma_2)]_G = \varphi([\gamma_1, \gamma_2]_{SN}) + d_{\mathrm{Hoch}}\mathcal{U}_2(\gamma_1, \gamma_2) \tag{25}$$

for some $\mathcal{U}_2\colon \Lambda^2(T_{poly}(M)) \to \mathrm{Hoch}^\bullet(C^\infty(M))[-1]$ (we denoted by $[\ ]_G$ the Gerstenhaber bracket and by $[\ ,\ ]_{SN}$ the Schouten-Nijenhuis bracket).

In the case when $M = \mathbb{C}^d$ M.Kontsevich constructed in [K97] an L_∞ morphism $\mathcal{U}\colon T_{poly}(\mathbb{C}^d) \to \mathrm{Hoch}^\bullet(S(\mathbb{C}^{d*}))$ whose first Taylor component is the Hochschild-Kostant-Rosenberg map φ. (Here we consider *polynomial* polyvector fields, and there is no necessity to complete the Hochschild complex). The second Taylor component $\mathcal{U}_2$ should then satisfy (25), and so on. This result is called *the Kontsevich's formality theorem.*

(The result for a general manifold M can be deduced from this local statement, see [K97], Section 7).

The original Kontsevich's proof uses ideas of topological field theory, namely, the Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) model, see [AKSZ]. Therefore, some transcendental complex numbers, the "Feynmann integrals" of the theory, are involved into the construction. The Kontsevich's proof appeared in 1997.

One year later, in 1998, D.Tamarkin found in [T] another proof of the Kontsevich formality for $\mathbb{C}^d$, using absolutely different technique. In the rest of this Section we outline the Tamarkin's proof [T], [H] in the form we use it in the sequel.

2.2 The idea of the Tamarkin's proof

The main idea it to construct not only an L_∞ map from $T_{poly}(\mathbb{C}^d)$ to $\mathrm{Hoch}^\bullet(S(\mathbb{C}^{d*}))$ but to involve the entire structure on polyvector fields and the Hochschild complex. This is the structure of (homotopy) Gerstenhaber algebra. For example, on polyvector fields (on any manifold) one has two operations: the wedge product $\gamma_1 \wedge \gamma_1$ of degree 0, and the Lie bracket $[\gamma_1, \gamma_2]_{SN}$ of degree -1, and they are compatible as

$$[\gamma_1, \gamma_2 \wedge \gamma_3] = [\gamma_1, \gamma_2] \wedge \gamma_2 \pm \gamma_2 \wedge [\gamma_1, \gamma_3] \tag{26}$$

This is called a Gerstenhaber algebra. To consider $T_{poly}(\mathbb{C}^d)$ as a Gerstenhaber algebra simplifies the problem because of the following Lemma:

Lemma. *The polyvector fields $T_{poly}(\mathbb{C}^d)$ is rigid as a homotopy Gerstenhaber algebra. More precisely, any* $\mathrm{Aff}(\mathbb{C}^d)$*-equivariant deformation of $T_{poly}(\mathbb{C}^d)$ as a homotopy Gerstenhaber algebra is homotopically equivalent to trivial deformation.*

We should explain what these words mean, we do it in the next Subsections. Let us now explain how it helps to prove the Kontsevich's formality theorem.

It is true, and technically it is the hardest place in the proof, that there is a structure of homotopical Gerstenhaber algebra (see Section 2.3) on the Hochschild complex $\mathrm{Hoch}^\bullet(A)$ of any associative algebra A. It is non-trivial, because the cup-product of Hochschild cochains $\Psi_1 \cup \Psi_2$ and the Gerstenhaber bracket $[\Psi_1, \Psi_2]_G$ do not obey the compatibility (26), as it should be in a Gerstenhaber algebra. It obeys it only up to a homotopy, and to find explicitly this structure uses also either some integrals like in [K99], or Drinfeld's Knizhnik-Zamolodchikov iterated integrals, as in [T]. We discuss it in Section 2.5. Now suppose that this structure exists, such that the Lie and Commutative parts of this structure are equivalent to the Gerstenhaber bracket and the cup-product on Hochschild cochains.

Then, as usual in homotopical algebra, there exists a homotopical Gerstenhaber algebra structure on the cohomology, equivalent to this structure on the cochains (it is something like "Massey operations" by Merkulov and Kontsevich-Soibelman). This push-forwarded structure is uniquely defined up to homotopy.

Now we can consider this structure as a formal deformation of the classical pure Gerstenhaber algebra on $T_{poly}(\mathbb{C}^d)$. Indeed, we rescale the Taylor components of this structure, such that the weight of k-linear Taylor components is λ^{k-2}. This gives again a homotopical Gerstenhaber structure, which value at $\lambda = 0$ is the classical Gerstenhaber structure on polyvector fields, because of the compatibility with Lie and Commutative structure, and by the Hochschild-Kostant-Rosenberg theorem.

Now we apply the Lemma above. All steps of our construction are $\mathrm{Aff}(\mathbb{C}^d)$-invariant, therefore, the obtained deformation can be chosen $\mathrm{Aff}(\mathbb{C}^d)$-equivariant. Then the Lemma says that this deformation is trivial, and the two homotopical Gerstenhaber structures on $T_{poly}(\mathbb{C}^d)$ are in fact isomorphic. This implies the Kontsevich's formality in the stronger, Gerstenhaber algebra isomorphism, form.

2.3 Koszul operads

From our point of view, the Koszulness of an operad $\mathcal{P}$ is very important because in this case any $\mathcal{P}$-algebra A has "very economic" resolution which is free dg $\mathcal{P}$-algebra. In the case of the operad $\mathcal{P} = Assoc$, this "very economic" resolution is the Quillen's bar-cobar construction. Thereafter, we use this free resolution to compute the (truncated) deformation complex of A as $\mathcal{P}$-algebra. In the case of $\mathcal{P} = Assoc$ this deformation complex is the Hochschild cohomological complex of A without the zero degree term, that is $\mathrm{Hoch}^\bullet(A)/A$.

We will consider only operads of dg $\mathbb{C}$-vector spaces here, with one of the two possible symmetric monoidal structures. A *quadratic* operad generated by a vector space E over $\mathbb{C}$ with an action of the symmetric group Σ_2 of two variables, with a Σ_3-invariant space of relations $R \subset \mathrm{Ind}_{\Sigma_2}^{\Sigma_3} E \otimes E$ (here Σ_2 acts only on the second factor) is the quotient of the free operad $\mathcal{P}$ generated by $\mathcal{P}(2) = E$ by the space of relations $R \subset \mathcal{P}(3)$. The operads Lie, $Comm$, $Assoc$ are quadratic, as well as the Gerstenhaber and the Poisson operads. See [GK], Section 2.1 for more detail. For a quadratic operad $\mathcal{P}$ define the quadratic dual operad $\mathcal{P}^!$ as the quadratic operad generated by $\mathcal{P}^!(2) = E^*[1]$, with the space of relations R^* in $\mathrm{Ind}_{\Sigma_2}^{\Sigma_3} E^*[1] \otimes E^*[1]$ equal to the orthogonal complement to $R \subset \mathrm{Ind}_{\Sigma_2}^{\Sigma_3} E \otimes E$. Example: $Com^! = Lie[-1]$, $Assoc^! = Assoc[-1]$, $(\mathcal{P}^!)^! = \mathcal{P}$.

Let $\mathcal{P}$ be a general, not necessarily quadratic, operad. For simplicity, we suppose that all vector spaces $\mathcal{P}(n)$ of an operad $\mathcal{P}$ are finite-dimensional. Recall the construction of the bar-complex of $\mathcal{P}$, see [GK], Section 3.2. Denote the bar complex of $\mathcal{P}$ by $\mathbf{D}(\mathcal{P})$. Then one has: $\mathbf{D}(\mathbf{D}(\mathcal{P}))$ is quasi-isomorphic to $\mathcal{P}$ ([GK], Theorem 3.2.16). Let now $\mathcal{P}$ be a quadratic operad. Then the bar-complex $\mathbf{D}(\mathcal{P})$ is a negatively-graded dg operad whose 0-th cohomology is canonically the quadratic dual operad $\mathcal{P}^!$. A quadratic operad $\mathcal{P}$ is called *Koszul* if the bar-complex $\mathbf{D}(\mathcal{P})$ is a resolution of $\mathcal{P}^!$. In this case $\mathbf{D}(\mathcal{P}^!)$ gives a free resolution of the operad $\mathcal{P}$.

Example. The operads Lie, $Comm$, $Assoc$, the Gerstenhaber and the Poisson operads, are Koszul. See [GK], Section 4 for a proof.

Definition. Let $\mathcal{P}$ be a quadratic Koszul operad. A homotopy $\mathcal{P}$-algebra (or $\mathcal{P}_\infty$-algebra) is an algebra over the free dg operad $\mathbf{D}(\mathcal{P}^!)$.

We denote by $\mathcal{P}^*$ the cooperad dual to an operad $\mathcal{P}$, if all spaces $\mathcal{P}(n)$ are finite-dimensional. Let $\mathcal{P}$ be a Koszul operad. Then to define a $\mathcal{P}_\infty$-algebra structure on X is the same that to define a differential on the free coalgebra $\mathbb{F}^\vee_{\mathcal{P}^{!*}}(X)$ which is a coderivation of the coalgebra structure. Any $\mathcal{P}$ algebra is naturally a $\mathcal{P}_\infty$-algebra.

We denote by $\mathbb{F}_{\mathcal{P}}(V)$ the free algebra over the operad $\mathcal{P}$, and by $\mathbb{F}^\vee_{\mathcal{P}^*}$ the free coalgebra over the cooperad $\mathcal{P}^*$. Here we suppose that all spaces $\mathcal{P}(n)$ are finite-dimensional.

Recall the following statement [GK], Thm. 4.2.5:

Lemma. *Let $\mathcal{P}$ be a Koszul operad, and V a vector space. Let $X = \mathbb{F}_{\mathcal{P}(X)}$. Then the natural projection*

$$(\mathbb{F}^\vee_{\mathcal{P}^{!*}}(X)) \to V \tag{27}$$

is a quasi-isomorphism.

It follows from this statement that any $\mathcal{P}$-algebra A has the following free resolution $\mathcal{R}^\bullet(A)$:

$$\mathcal{R}^\bullet(A) = (\mathbb{F}_{\mathcal{P}}(\mathbb{F}^\vee_{\mathcal{P}^{!*}}(A), Q_1), Q_2) \tag{28}$$

with the natural differentials Q_1 and Q_2.

Now we define the *truncated deformation complex* of the $\mathcal{P}$-algebra A as $(\mathrm{Der}(\mathcal{R}^\bullet(A)), Q)$ where Q comes from the differential in $\mathcal{R}^\bullet(A)$. This deformation complex is naturally a dg Lie algebra with the Lie bracket of derivations. We have the following statement:

Proposition. *The truncated deformation functor associated with this dg Lie algebra governs the formal deformations of A as $\mathcal{P}_\infty$-algebra.*

Remark. The word "truncated" means that for the "full" deformation functor we should take the quotient modulo the inner derivations. Although, a map $X \to \mathrm{Der}(X)$ is not defined for an arbitrary operad. Our truncated deformation functor looks like the Hochschild cohomological complex of A without the degree 0 term A.

The following trick simplifies computations with the deformation complex.

Any coderivation of the coalgebra $(\mathbb{F}^\vee_{\mathcal{P}^{!*}}(A), Q_1)$ can be extended to a derivation of $\mathcal{R}^\bullet(A)$ by the Leibniz rule. We have the following theorem:

Theorem. *The natural inclusion*

$$\mathrm{Coder}(\mathbb{F}^\vee_{\mathcal{P}^{!*}}(A), Q_1) \to \mathrm{Der}(\mathcal{R}^\bullet(A)) \tag{29}$$

is a quasi-isomorphism of dg Lie algebras.

It follows from this Theorem and the Proposition above that the dg Lie algebra $\mathrm{Coder}(\mathbb{F}^{\vee}_{\mathcal{P}^{!*}}(A), Q_1)$ governs the formal deformation of the $\mathcal{P}_\infty$-structure on A. This, of course, can be seen more directly. Indeed, a $\mathcal{P}_\infty$ structure on A is a differential on $\mathbb{F}^{\vee}_{\mathcal{P}^{!*}}(A)$ making latter a dg coalgebra over $\mathcal{P}^!$. We have some distinguished differential Q_1 on it, arisen from the $\mathcal{P}$-algebra structure on A. When we deform it, it is replaced by $Q_\hbar = Q_1 + \hbar d_\hbar$ such that $(Q_1 + \hbar d_\hbar)^2 = 0$. In the first order in $\hbar$ we have the condition $[Q_1, d_\hbar] = 0$, where the zero square condition is the Maurer-Cartan equation in the corresponding dg Lie algebra.

2.4 The main computation in the Tamarkin's theory

Here we compute the deformation cohomology of $T_{poly}(V)$, V a complex vector space, as Gerstenhaber algebra. We prove here Lemma 2.2, and a more general statement.

We start with the following Lemma:

Lemma. *The Gerstenhaber operad G is Koszul. The Koszul dual operad $G^!$ is $G[-2]$.*

Proof. We know that $Lie^! = Comm[-1]$ and $Comm^! = Lie[-1]$. A structure of a Gerstenhaber algebra on W consists from compatible actions of $Comm$ and $Lie[1]$ on W. The quadratic dual to $Comm$ is $Lie[-1]$, and the quadratic dual to $Lie[1]$ is $Comm[-2]$. Therefore, the quadratic dual to G is $G[-2]$. The Koszulity of G is proven in [GJ], see also [GK] and [H]. $\square$

Theorem 2.3 gives us a way how to compute the deformation functor for formal deformations of $T_{poly}(V)$ as homotopy Gerstenhaber algebra. We take the free coalgebra $\mathbb{F}^{\vee}_{G[-2]^*}(T_{poly}(V))$ over the cooperad $G[-2]^*$ cogenerated by $T_{poly}(V)$. It is clear that

$$\mathbb{F}^{\vee}_{G^*[2]}(T_{poly}(V)) = S^{\bullet}((\mathbb{F}_{\mathrm{Lie}} T_{poly}(V)[1])[1]) \tag{30}$$

The product $\wedge\colon S^2 T_{poly}(V) \to T_{poly}(V)$ and the Lie bracket $[\ ,\ ]\colon \Lambda^2 T_{poly}(V) \to T_{poly}(V)[-1]$ define two *coderivations* of the Gerstenhaber coalgebra structure on $\mathbb{F}^{\vee}_{G^*[2]}(T_{poly}(V))$; denote them by δ_{Comm} and δ_{Lie}, correspondingly.

The truncated deformation complex of $T_{poly}(V)$, as of Gerstenhaber algebra, is then $\mathrm{Coder}^{\bullet}(\mathbb{F}^{\vee}_{G^*[2]}(T_{poly}(V)))$ endowed with the differential $d = ad(\delta_{\mathrm{Comm}}) + ad(\delta_{\mathrm{Lie}})$. We denote the two summands by d_{Comm} and d_{Lie}, correspondingly.

Theorem. *The deformation complex* $(\mathrm{Coder}(\mathbb{F}^{\vee}_{G^*[2]}(T_{poly}(V))), d_{\mathrm{Comm}} + d_{\mathrm{Lie}})$ *has the k-th cohomology equal to $T^{k+1}_{poly}(V)$ for any $k \geq -1$, all other cohomology vanishes. In particular, there are no* $\mathrm{Aff}(V)$*-invariant cohomology classes. (Recall that $T^k_{poly}(V)$ is the space of k-polyvector fields).*

Proof. Firstly we deal with the coderivations of a cofree coalgebra, and they are defined uniquely by their restrictions to cogenerators which may be arbitrary. Therefore, we need to compute the cohomology of the complex

$$\mathrm{Hom}_{\mathbb{C}}(S^{\bullet}((\mathbb{F}_{\mathrm{Lie}}T_{poly}(V)[1])[1]), T_{poly}(V))[2] \tag{31}$$

This is a bicomplex with the differentials d_{Comm} and d_{Lie}. We want to use the spectral sequence which computes firstly the cohomology of d_{Comm}. Let us prove that this spectral sequence converges to the cohomology of the complex.

Discuss the bigrading more carefully. We say that an element in $\mathrm{Free}_{\mathrm{Lie}}^{k_1} \cdot \mathrm{Free}_{\mathrm{Lie}}^{k_2} \cdot \dots \cdot \mathrm{Free}_{\mathrm{Lie}}^{k_s}$ has the bigrading (A, B) where

$$\begin{aligned} A &= -s + (k_1 + \dots + k_s) - 2 \\ B &= 2s + \deg \gamma_{\mathrm{out}} - \sum \deg \gamma_i \end{aligned} \tag{32}$$

where $\{\gamma_i\}$ are the input polyvector fields in the monomial, and γ_{out} is the output polyvector field.

The number s is ≥ 1 and the number A is ≥ 2. The differential d_{Comm} increases A by 1 and preserves B. The differential d_{Lie} increases B by 1 and preserves B. The only nonzero elements are in the gray area in Figure 1.

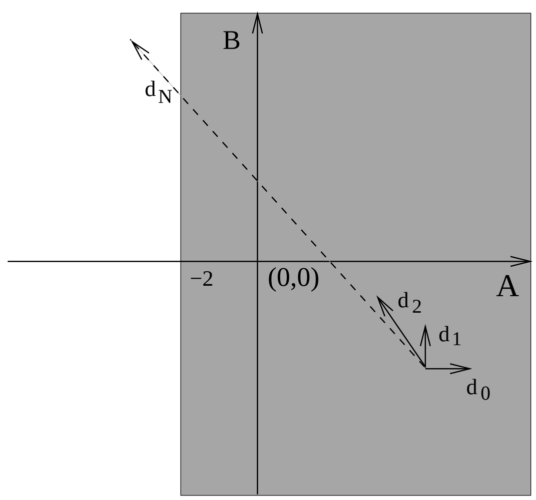

Figure 1: The differentials in the spectral sequence

The first differential d_{Comm} is the horizontal one. All sufficiently high differentials vanish for a fixed point on the plane. Therefore, the spectral sequence converges to the cohomology of the total complex.

Compute the first term of the spectral sequence. When we remember the only differential d_{Comm}, the deformation complex $\mathrm{Hom}_{\mathbb{C}}(S^\bullet((\mathrm{Free}_{\mathrm{Lie}}T_{poly}(V)[1])[1]), T_{poly}(V))[2]$ is a direct sum of complexes:

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{C}}(S^\bullet((\mathrm{Free}_{\mathrm{Lie}}T_{poly}(V)[1])[1]), T_{poly}(V))[2] = \\ &\bigoplus_{k\geq 1}(\mathrm{Hom}(S^k((\mathrm{Free}_{\mathrm{Lie}}T_{poly}(V)[1])[1]), T_{poly}(V)), d_{\mathrm{Comm}})[2]\end{aligned} \tag{33}$$

We claim that the cohomology of the complex $(S^k((\mathrm{Free}_{\mathrm{Lie}}T_{poly}(V)[1])[1]), d_{\mathrm{Comm}})$ is $T^k_{poly}(T^*[1]V)$. (Recall that the functions on $T^*[1]V$ is $T_{poly}(V)$).

To prove this claim, consider the first complex (for $k = 1$) $\mathrm{Hom}(\mathrm{Free}_{\mathrm{Lie}}(T_{poly}(V)[1]), T_{poly}(V))[1]$. This is the Harrison cohomology of the commutative algebra $T_{poly}(V)$ with coefficients in $T_{poly}(V)$. As the commutative algebra $T_{poly}(V)$ is free, it is equal to $\mathrm{Der}_{\mathrm{Comm}}T_{poly}(V)$.

Now, in general, by the Poicaré-Birkhoff-Witt theorem,

$$\begin{aligned}&\bigoplus_{k}\mathrm{Hom}(S^k((\mathrm{Free}_{\mathrm{Lie}}T_{poly}(V)[1])[1]), T_{poly}(V))[2] = \\ &(\mathrm{Hom}(\mathrm{Free}^\bullet_{\mathrm{Assoc}}(T_{poly}(V))[1], T_{poly}(V)), d_{\mathrm{Comm}})[1] = \\ &\mathrm{Hoch}^{\geq 1}(T_{poly}(V))[1]\end{aligned} \tag{34}$$

Therefore, $E_1^{p,q} = [T^{p+3}_{poly}(T[1]V)]^{p+q}$, $p \geq -2$, where the latter is the space of $(p+3)$-polyvector fields on a $\mathbb{Z}$-graded vector space of total degree $p+q$.

Compute the differential in the term E_1.

Consider the Lie bracket on $T_{poly}(V)$ as bivector in $[T^2(T[1]V)]^{-1}$, denote it by α. It is clear that the differential in E_1 is $ad(\alpha)$. The term E_2 then is $E_2^{-2,q} = T^{q+2}_{poly}V$, $q \geq 1$, and $E_2^{p,q} = 0$ for $p \neq -2$. The spectral sequence collapses at the term E_2 by dimensional reasons.

Theorem is proven. □

2.5 The final point: relation with the Etingof-Kazhdan quantization

The remaining part of the Tamarkin's proof of Kontsevich formality goes as follows.

One firstly proves the *Deligne conjecture* that there is a homotopy Gerstenhaber algebra structure on the Hochschild cohomological complex $\mathrm{Hoch}^\bullet(A)$ of any associative algebra such that it induces the Schouten bracket and the wedge product on the cohomology. This is the only transcendental step of the construction, this structure, as it is defined in [T], depends on a choice of Drinfeld associator [D].

We apply this fact for $A = S(V^*)$, V a vector space over $\mathbb{C}$. One can push-forward (given by the "Massey operations") this G_∞ structure from $\mathrm{Hoch}^\bullet(S(V^*))$ to its cohomology $T_{poly}(V)$. Then we get two G_∞ structures on $T_{poly}(V)$, the first is given from the Schouten bracket and the wedge product of polyvector fields, the second is the above pushforward. Moreover, one can introduce a formal parameter $\hbar$ to the pushforward, such that the original one is given when $\hbar = 1$. Then for $\hbar = 0$ we get the Schouten structure: it follows from the compatibility of the Deligne's conjecture G_∞ structure with the one on the cohomology. Thus we get a formal deformation of the classical Gerstenhaber algebra structure on $T_{poly}(V)$. This deformation is clearly $\mathrm{Aff}(V)$-invariant. By Theorem 2.4, infinitesimally all such deformations are trivial; therefore, they are trivial globally. This concludes the Tamarkin's proof.

In this Subsection we explain the Tamarkin's proof of the Deligne conjecture, based on the Etingof-Kazhdan quantization.

Recall the definitions of the dg operads B_∞ and B_{Lie}. By definition, a vector space X is an algebra over the operad B_∞ if there is a structure of a dg associative bialgebra on the cofree coalgebra $\mathbb{F}^\vee_{\mathrm{Assoc}}(X[1])$ such that the coalgebra structure coincides with the given one. This definition leads to the following data (see [H], Section 5 and [GJ], Section 5 for more detail):

(1) a differential $d\colon X[1]^{\otimes n} \to X[1]^{\otimes m}$ of degree 1, $m, n \geq 1$ being a differential of the free coalgebra structure is uniquely defined by the projections to the cogenerators. We denote them $m_n\colon X[1]^{\otimes n} \to X[2]$;

(2) the algebra structure, it is also given by the projection to $X[1]$. These are maps $m_{pq}\colon X[1]^{\otimes p} \otimes X[1]^{\otimes q} \to X[1]$, or $m_{pq}\colon X^{\otimes p} \otimes X^{\otimes q} \to X[1-p-q]$.

These data should define three series of equations: the first come from the associativity of $\{m_{pq}\}$, the second come from the fact that d is a derivation of the algebra structure, and the third comes from the condition $d^2 = 0$. These equations define a very complicated operad B_∞.

It is a remarkable and surprising result of Getzler-Jones [GJ], Section 5, that $X = \mathrm{Hoch}^\bullet(A)$, A an arbitrary associative algebra, is an algebra over the operad B_∞. This structure is defined as follows:

(1) m_1 is the Hochschild differential;

(2) m_2 is the cup-product on $\mathrm{Hoch}^\bullet(A)$;

(3) $m_i = 0$ for $i \geq 3$;

(4) $m_{1k}(f \otimes g_1 \otimes \cdots \otimes g_k)$ is the brace operation $f\{g_1, \dots, g_k\}$ defined below;

(5) $m_{ak} = 0$ for $a \geq 2$.

Now is the definition of the braces due to Getzler-Jones. It is better to describe it graphically, as is shown in Figure 2.

Let us emphasize again that it is a highly non-evident fact, proven by a direct computation, that in this way we make a B_∞ algebra structure on $\mathrm{Hoch}^\bullet(A)$.

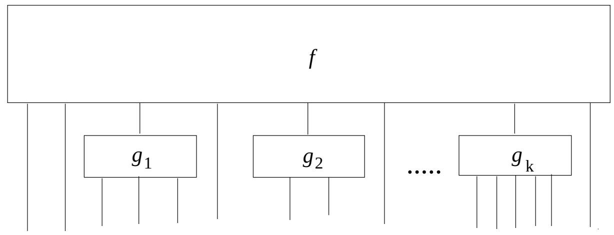

Figure 2: The brace operation: we insert $g_1, \dots, g_k$ into arguments of f, preserving the order of $g_1, \dots, g_k$, with the natural sign, and take the sum over all possible insertions

The role of this construction is that the cohomology operad of the dg operad B_∞ is equal to the Gerstenhaber operad G (probably, even to prove this fact we need the Etingof-Kazhdan quantization). So the idea is to prove that there is a quasi-isomorphism of operads $G \rightsquigarrow B_\infty$, and then we can consider $\mathrm{Hoch}^\bullet(A)$ as G_∞ algebra for any associative algebra A. The only trancendental step in the Tamarkin's construction is this quasi-isomorphism of operads $G \rightsquigarrow B_\infty$.

Technically it is done as follows. Introduce some operad B_{Lie} as follows. A vector space X is an algebra over the operad B_{Lie} if there is a dg Lie bialgebra structure on the free Lie coalgebra $\mathbb{F}^\vee_{\mathrm{Lie}}(X[1])$ generated by $X[1]$ such that the Lie coalgebra structure coincides with the given one.

The operad B_{Lie} is also quasi-isomorphic to the Gerstenhaber operad G. Moreover, there is a simple construction of a quasi-isomorphism $G_\infty \to B_{\mathrm{Lie}}$, as follows.

Let Y be an algebra over B_{Lie}. This means that there is a Lie dg bialgebra structure on the free coalgebra $\mathfrak{g} = \mathbb{F}^\vee_{\mathrm{Lie}}(Y[1])$. In particular, $\mathfrak{g}$ is a Lie algebra, and this defines a differential on the Lie chain complex $\mathbb{F}^\vee_{\mathrm{Com}}(\mathfrak{g}[1])$. Thus we get a differential on the free Gerstenhaber coalgebra $\mathbb{F}^\vee_{G^\vee}(Y[1])$ cogenerated by $Y[1]$, which by definition means that Y is a G_∞-algebra. This assignment is functorial, and therefore gives a map of operads $G_\infty \to B_{\mathrm{Lie}}$, which easily checked to be a quasi-isomorphism.

So, the conclusion is that the operad B_{Lie} can be connected to the Gerstenhaber operad in a very simple way, and now we should connect the operad B_∞ with the operad B_{Lie}.

This is given exactly by the Etingof-Kazhdan (de)quantization (see [T1] and [H], Section 7 for detail). The Etingof-Kazhdan dequantization is applied in a sense to $\mathbb{F}^\vee_{\mathrm{Assoc}}(\mathrm{Hoch}^\bullet(A)[1])$ which is an associative bialgebra by the Getzler-Jones braces' construction.

Remark. Let P be a Poisson algebra. Its Poisson complex $\mathrm{Pois}^\bullet(P)$ is defined as the dg space of coderivations of the free coalgebra over the dual cooperad $\mathcal{P}^{!*}[1]$ by the space $P[1]$. This space of coderivations is naturally equipped with a differential d_{Pois} arising from the Poisson bracket and the product on P. The author thinks that for any Poisson algebra P the Poisson complex $\mathrm{Pois}^\bullet(P)$ is an algebra over the operad

B_{Lie}. This structure is defined exactly by some generalization of the braces construction. Now, if $P = S(V^*)$ be the Poisson algebra with zero bracket, then by Getzler-Jones $\mathrm{Hoch}^\bullet(P)$ is a B_∞ algebra, and $\mathrm{Pois}^\bullet(P)$ is a B_{Lie} algebra. The author thinks that some the most natural Etingof-Kazhdan dequantization gives from the associative bialgebra $\mathbb{F}^\vee_{\mathrm{Assoc}}(\mathrm{Hoch}^\bullet(P)[1])$ the Lie bialgebra $\mathbb{F}^\vee_{\mathrm{Lie}}(\mathrm{Pois}^\bullet(P)[1])$. So far, the author does not know any direct proof of the last fact.

Appendix

Here we explain a construction of the Kontsevich formality morphism over $\mathbb{Q}$. The usual construction uses the Drinfeld's associator over $\mathbb{Q}$ and the Tamarkin's theory. This associator is not given by an explicit formula, it is constructed in [D] by proving that all associators over $\overline{\mathbb{Q}}$ form a torsor over the Grothendieck-Teichmüller group. The Knizhnik-Zamolodchikov associator gives an example of associator over $\mathbb{C}$; therefore, there exists an associator over $\overline{\mathbb{Q}}$. It proves that this torsor is trivial over $\overline{\mathbb{Q}}$. Then the torsor is trivial also over $\mathbb{Q}$, because the Grothendieck-Teichmüller group is pro-unipotent.

Here we propose a different proof, which seems to be more constructive. This approach seems to be new.

It follows from the previous results that if we succeed to construct a quasi-isomorphism of operads $B_\infty \to G$ over $\mathbb{Q}$, we will be done.

Consider the operad B_∞. It is a dg operad. All dg operads form a closed model category because they are algebras over some universal colored operad. In particular, there is a homotopy operad structure on the cohomology of B_∞, given by a kind of "Massey operations". This homotopy operad structure clearly is defined over $\mathbb{Q}$. To construct it explicitly, we should firstly split B_∞ as a complex into a direct sum of its cohomology and a contractible complex and, secondly, to contract this complex explicitly by a homotopy. It is clear that these two steps can be performed over $\mathbb{Q}$. Now we have two homotopy operad structures on G: one is the Gerstenhaber operad, and another one is given by the Massey operations. Moreover, there is a formal family of homotopy dg operads depending on $\hbar$ whose value at $\hbar = 1$ is the "Massey" homotopy operad structure on G, and whose value at $\hbar = 0$ is the Gerstenhaber operad.

We know from the Tamarkin's theory described above that this deformation is trivial over $\mathbb{C}$, because the "Massey" homotopy operad is quasi-isomorphic to B_∞ by construction, which is quasi-isomorphic over $\mathbb{C}$ to the Gerstenhaber operad by Section 2.5. We are going to prove that this formal deformation is trivial also over $\mathbb{Q}$.

For this it is sufficient to prove that infinitesimally this deformation is trivial over $\mathbb{Q}$ at each $0 \le \hbar \le 1$. Consider a resolution $\mathcal{R}^\bullet(G)$ of the Gerstenhaber operad over $\mathbb{Q}$; as G is a Koszul operad, we can take its Koszul resolution. Consider the truncated deformation complex $\mathbb{D}^+(G) = \mathrm{Der}(\mathcal{R}^\bullet(G))$. We need to prove that all infinitesimal deformations give trivial classes in $H^1(\mathbb{D}^+(G))$.

It is probably not true that $H^1(\mathbb{D}^+(G)) = 0$, it would be very unexpected, because Tamarkin imbeded in [T3] the Grothendieck-Teichmüller Lie algebra into $H^0(\mathbb{D}^+(G))$.

But we know that all classes are trivial in $H^1(\mathbb{D}^+(G), \mathbb{C})$ from the Tamarkin's theory. As the complex $\mathbb{D}^+(G)$ is defined over $\mathbb{Q}$, we have that the natural map

$$H^\bullet(\mathbb{D}^+(G), \mathbb{Q}) \hookrightarrow H^\bullet(\mathbb{D}^+(G) \otimes_{\mathbb{Q}} \mathbb{C}) = H^\bullet(\mathbb{D}^+(G), \mathbb{C}) \tag{35}$$

is an embedding.

Therefore, all our infinitesimal classes are trivial over $\mathbb{Q}$, and we get that the global formal deformation is trivial over $\mathbb{Q}$.

We think that this speculation is as explicit as it can give some explicit formulas for the Kontsevich's formality over $\mathbb{Q}$. We are going to describe it in detail in the sequel.

3 Two infinite-dimensional varieties (and a morphism between them)

In this Section we associate with each quasi-isomorphism of operads $\Theta\colon G_\infty \to B_\infty$ defined up to homotopy an L_∞ morphism $\mathcal{U}(\Theta)\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(V)$ (for any vector space V) defined up to homotopy. We show that this L_∞ morphism is given by a universal formula, that is by prediction of some weights to all Kontsevich graphs from [K97], but our graphs may contain simple loops. The image of the map $\Theta \mapsto \mathcal{U}(\Theta)$ gives that L_∞ morphisms for deformation quantizations associated with which we can prove our Koszul duality Theorem.

3.1 Few words about homotopy

Starting from now, we often use the word "homotopy" in the context like "homotopical map of dg operads" or "homotopical L_∞ morphisms". Here are some generalities on this.

The Quillen's formalism of closed model categories [Q] gives a tool for the inverting of quasi-isomorphisms in a non-abelian case. Let $\mathcal{O}$ be an operad. Consider the category $DGA(\mathcal{O})$ of dg algebras over $\mathcal{O}$. We want to construct a universal category in which the quasi-isomorphisms in $DGA(\mathcal{O})$ are invertible. This category can be constructed for any operad $\mathcal{O}$ and is called the homotopy category of $DGA(\mathcal{O})$, because the category $DGA(\mathcal{O})$ admits a closed model structure in which the weak equivalences coincide with the quasi-isomorphisms [H2]. There are several constructions of this category, but all them are equivalent due to the universal property with respect to the localization by quasi-isomorphisms. In Section 5 we recall a very explicit construction in the case when $\mathcal{O}$ is a Koszul operad.

Contrary, the dg algebras over a PROP do not form a closed model category (the Quillen's Axiom 0 that the category admits all finite limits and colimits fails in this case; for example we do not know what is a free algebra over a PROP). Therefore, for dg algebras over a PROP any construction of the homotopical category (to the best of our knowledge) is not known.

On the other hand, all dg operads form a closed model category as algebras over the universal colored operad, therefore, for dg operads the Quillen's construction works.

In the sequel we will skip some details concerning that homotopical maps of operads induce homotopical maps of dg algebras, avoiding to enlarge this already rather long paper.

Only what we need to know is that the homotopical category is unique, and in the final step we use a particular construction of it for dg Lie algebras in Section 5, appropriate for our needs.

3.2 The Kontsevich's variety $\mathfrak{K}$

The Kontsevich's variety $\mathfrak{K}$ is the variety of all *universal* L_∞ quasi-isomorphisms $T_{poly}(V) \to \mathrm{Hoch}^\bullet(S(V^*))$ defined for all vector spaces V. Any such universal L_∞ morphism is by definition given by prediction of some complex weights W_Γ to all Kontsevich graphs Γ from [K97] possibly with simple loops and not connected. These W_Γ are subject to some quadratic equations, arising from the L_∞ condition. The first Hochschild-Kostant-Rosenberg graph has the fixed weight, as in the Kontsevich's paper [K97]. This variety is not empty as is proven in [K97]. The homotopies acts by gauge action (see Section 5).

3.3 The Tamarkin's variety $\mathfrak{T}$

The Tamarkin's variety in our strict sense is formed from all quasi-isomorphisms of operads $G_\infty \to B_\infty$ which are identity on cohomology, modulo homotopies. As G_∞ is a free dg operad, any such map is uniquely defined by the generators $G[-2]$. So, a map of operads $G_\infty \to B_\infty$ is given by a map of vector spaces $G[-2] \to B_\infty$ subject to some quadratic relations arose from the compatibility with the differentials in the dg operads. This variety is not empty because we have constructed in Section 2, following [T1], such a particular quasi-isomorphism.

In a wider setting, one can consider Op_∞ maps of dg operads $G_\infty \to B_\infty$, but we do not do this.

3.4 A map $\mathfrak{X}\colon \mathfrak{T} \to \mathfrak{K}$

Suppose a point t of the Tamarkin's manifold $\mathfrak{K}$ is fixed. Then the Hochschild complex $\mathrm{Hoch}^\bullet(A)$ of any algebra A has a fixed structure of homotopy Gerstenhaber algebra (fixed modulo homotopy). Consider the case $A = S(V^*)$ for some vector space V. Then we get, as is explained in Section 2, two structures of G_∞ algebra on $T_{poly}(V)$ which are specifications of some formal deformation at $\hbar = 0$ and $\hbar = 1$. Then they should coincide, up to a homotopy, because the first cohomology $H^1(\mathrm{Coder}(\mathbb{F}^\vee_{G^*[2]}(T_{poly}(V)))) = T^2_{poly}(V)$ by Theorem 2.4, and there is no $\mathrm{Aff}(V)$-invariant classes (but our formal deformation is Aff-invariant).

Thus we get a map of G_∞ algebras $\mathfrak{X}^0(t)\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(S(V^*))$, where $T_{poly}(V)$ is considered with the standard Schouten-Nijenhuis Gerstenhaber structure, and the G_∞ structure on $\mathrm{Hoch}^\bullet(S(V^*))$ depends on the point $t \in \mathfrak{K}$. Then we restrict it to the Lie operad and get an L_∞ map $\mathfrak{X}(t)\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(S(V^*))$ which is an L_∞ morphism for the standard Lie structures on $T_{poly}(V)$ and $\mathrm{Hoch}^\bullet(S(V^*))$, and this L_∞ morphism is defined up to homotopy. This is the construction of the map $\mathfrak{X}$. It is, although, not proven yet that $\mathfrak{X}(t)$ is defined uniquely up to a homotopy.

Lemma. *For a fixed V, the L_∞ morphism $\mathfrak{X}(t)$ is uniquely defined up to a homotopy.*

Proof. Suppose there are two different L_∞ morphisms, for the same fixed G_∞ structure on $\mathrm{Hoch}^\bullet(S(V^*))$. Then we can get defined up to a homotopy G_∞ quasi-automorphism of $T_{poly}(V)$. It has the identity first Taylor component by the constructions (a point $t \in \mathfrak{T}$ is defined such we get the canonical Gerstenhaber structure on the cohomology of $\mathrm{Hoch}^\bullet(A)$ for any A). Therefore, the logarithm of this automorphism is well defined and gives a G_∞ derivation of $T_{poly}(V)$. Now we use the computation of Theorem 2.4 for 0-th cohomology: $H^0(\mathrm{Coder}(\mathbb{F}^\vee_{G^*[2]}(T_{poly}(V)))) = T^1_{poly}(V)$ is the vector fields. Again, there are no $\mathrm{Aff}(V)$-ivariant vector fields. Therefore, our derivation is inner. But then it is zero, because any inner derivation acts non-trivially on the first Taylor component which is fixed to be identity. Thus, we have proved that $\mathfrak{X}(t)$ is well-defined up to homotopy as G_∞ map, and therefore the same is true for its L_∞ part. $\square$

Now we prove the following almost evident corollary of the previous Lemma:

Theorem. *The L_∞ morphism $\mathfrak{X}(t)$ is a universal L_∞ morphism, that is, it is given by prediction of some weights to all Kontsevich graphs, possibly non-connected and with simple loops, and these weights up to a homotopy do not depend on the vector space V.*

Proof. Let $W \subset V$ be a subspace. Then we can decompose $V = W \oplus W^\perp$, and a G_∞ structure on $\mathrm{Hoch}^\bullet(S(V^*))$ defines a G_∞ structure on $\mathrm{Hoch}^\bullet(S(W^*))$. Clearly (because the G_∞ structures are $\mathfrak{gl}(V)$-invariant) it is, up to a homotopy, the structure on $\mathrm{Hoch}^\bullet(S(W^*))$ one gets from the map of operads $t\colon G_\infty \to B_\infty$. We have then two definitions of L_∞ morphisms $T_{poly}(W) \to \mathrm{Hoch}^\bullet(S(W^*))$: one is the direct $\mathfrak{X}(t)_W$, and the second one is the restriction to W of $\mathfrak{X}(t)_V$. They coincide up to a homotopy by the Lemma above, because the two G_∞ structures on $\mathrm{Hoch}^\bullet(S(W^*))$ are the same up to a homotopy. The remaining part of the Theorem (that the L_∞ morphism is given by a universal formula though Kontsevich graphs) follows from the $\mathfrak{gl}(V)$ invariance of it. $\square$

Is is not known if the map $t \mapsto \mathfrak{X}(t)$ is surjective, even when we allow t to be an Op_∞ map of dg operads $G_\infty \to B_\infty$. Our Main Theorem of this paper about the Koszul duality holds for the star-product obtained from any L_∞ morphism in the image of $\mathfrak{X}$, $\mathcal{U} = \mathfrak{X}(t)$, by the usual formula

$$f \star g = f \cdot g + \hbar \mathcal{U}_1(\alpha)(f \otimes g) + \frac{1}{2}\hbar^2 \mathcal{U}_2(\alpha, \alpha)(f \otimes g) + \dots \tag{36}$$

where α is a (quadratic) Poisson bivector field on V.

4 Koszul duality and dg categories

4.1 Some generalities on dg categories

We give here some basic definitions on dg categories. We define only the things we will directly use, see [Kel3] for much more detailed and sophisticated overview.

A *differential graded (dg) category* $\mathcal{A}$ over a field k is a category, in which the sets of morphisms $\mathcal{A}(X,Y)$ between any two objects $X,Y \in \mathrm{Ob}(\mathcal{A})$ are k-linear dg spaces (complexes of k-vector spaces) such that the compositions are defined as maps $\mathcal{A}(Y,Z)\otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z)$ for any $X,Y,Z \in \mathrm{Ob}(\mathcal{A})$ which are *maps of complexes*. In the last condition we regard $\mathcal{A}(Y,Z)\otimes\mathcal{A}(X,Y)$ as a complex with the differential defined by the Leibniz rule

$$d(f\otimes g) = (df)\otimes g + (-1)^{\deg f} f\otimes (dg) \tag{37}$$

It is clear that a differential graded category with one object is just a differential graded associative algebra. Then dg categories can be considered as "dg algebras with many objects".

For dg algebras we have a definition when a map $F\colon A^\bullet \to B^\bullet$ is a *quasi-isomorphism*: it means that the map F is a map of algebras and induces an isomorphism on cohomology. Such a map in general is not invertible, it can be inverted only as an A_∞ map.

What should be a definition of *a quasi-isomorphism for dg categories*?

We say that a functor $F\colon \mathcal{A}\to\mathcal{B}$ between two dg categories $\mathcal{A}$ and $\mathcal{B}$ is a *quasi-equivalence* if it is a functor, which is k-linear on morphisms (and, as such, preserves tensor compositions of morphisms) and induces an equivalence on the level of cohomology. The last condition means that for a dg category $\mathcal{A}$ we can consider the category $H^\bullet(\mathcal{A})$ with the same objects, and with $(H^\bullet\mathcal{A})(X,Y) = H^\bullet(\mathcal{A}(X,Y))$. Then a quasi-equivalence is not invertible in general, but it can be inverted as an A_∞ quasi-equivalence between two dg categories. We will not use this concept directly, and we refer to the reader to give the definition.

Now if we have a dg algebra, we know what is the cohomological Hochschild complex of it. It governs the A_∞ deformations of the dg algebra. It is possible to define the *Hochschild cohomological complex of a dg category*. This will be in a sense the main object of our study in this paper for some particular dg category, namely, for the B.Keller's dg category introduced in the next Subsection. Let us give the definition of it.

A first, it is the total *product* complex of a bicomplex. The vertical differential will be the inner differential appeared from the differential on $\mathcal{A}(X,Y)$ for any pair $X,Y\in\mathrm{Ob}\mathcal{A}$. The horizontal differential will an analog of the Hochschild cohomological differential. The columns have degrees ≥ 0. In degree 0 we have

$$\mathrm{Hoch}^{*0}(\mathcal{A}) = \prod_{X\in\mathrm{Ob}\mathcal{A}} \mathcal{A}(X,X) \tag{38}$$

and in degree $p \geq 1$

$$\mathrm{Hoch}^{*p}(\mathcal{A}) = \prod_{X_0,X_1,\dots,X_p\in\mathrm{Ob}\mathcal{A}} \mathrm{Hom}_k(\mathcal{A}(X_{p-1},X_p)\otimes\mathcal{A}(X_{p-2},X_{p-1})\otimes\cdots\otimes\mathcal{A}(X_0,X_1),\mathcal{A}(X_0,X_p)) \tag{39}$$

where the product is taken over all chains of objects $X_0, X_1, \dots, X_p \in \mathrm{Ob}\mathcal{A}$ of length $p+1$.

The Hochschild differential $d_{\mathrm{Hoch}}\colon \mathrm{Hoch}^{*p}(\mathcal{A}) \to \mathrm{Hoch}^{*,p+1}(\mathcal{A})$ is defined in the natural way. Let us note that even if a cochain $\Psi \in \mathrm{Hoch}^{*p}(\mathcal{A})$ is non-zero only for a single chain of objects $X_0, X_1, \dots, X_p$, its Hochschild differential $d_{\mathrm{Hoch}}\Psi$ in general is non-zero on many other chains of objects. Namely, at first it may be nonzero for on any chain

$$X_0, \dots, X_i, Y, X_{i+1}, \dots, X_p \text{ for } 0 \leq i \leq p-1 \tag{40}$$

such that there are nonzero compositions $\mathcal{A}(Y, X_{i+1}) \otimes \mathcal{A}(X_i, Y) \to \mathcal{A}(X_i, X_{i+1})$ (this is corresponded to the regular terms in the Hochschild differential), and on the chains

$$Z_-, X_0, \dots, X_p \tag{41}$$

and

$$X_0, \dots, X_p, Z_+ \tag{42}$$

such that there are non-zero compositions $\mathcal{A}(X_0, X_p) \otimes \mathcal{A}(Z_-, X_0) \to \mathcal{A}(Z_-, X_p)$ and $\mathcal{A}(X_p, Z_+) \otimes \mathcal{A}(X_0, X_p) \to \mathcal{A}(X_0, Z_+)$ (this is corresponded to the left and to the right boundary terms in the Hochschild differential, correspondingly).

The Hochschild cohomological complex of a dg category $\mathcal{A}$ is a dg Lie algebra with the direct generalization of the Gerstenhaber bracket. The solutions of the Maurer-Cartan equation of $\mathrm{Hoch}^\bullet(\mathcal{A}) \otimes k[[\hbar]]$ give the formal deformations of the dg category A as A_∞ category.

4.2 The B.Keller's dg category $\mathsf{cat}(A, B, K)$

Introduce here the main object of our story—the B.Keller's dg category $\mathsf{cat}(A, B, K)$. Here A and B are two dg associative algebras, and K is B-A-bimodule. The dg category $\mathcal{A} = \mathsf{cat}(A, B, K)$ has two objects, called say a and b, such that $\mathcal{A}(a, a) = A$, $\mathcal{A}(b, b) = B$, $\mathcal{A}(a, b) = 0$, $\mathcal{A}(b, a) = K$. Only what we need from K to define such a dg category is a structure on K of differential graded B-A-bimodule.

Consider in details the Hochschild complex of the category $\mathsf{cat}(A, B, K)$. It contains as subspaces $\mathrm{Hoch}^\bullet(A)$ and $\mathrm{Hoch}^\bullet(B)$, the usual Hochschild cohomological complexes of the algebras A and B, and also it contains the subspace

$$\mathrm{Hoch}^\bullet(B, K, A) = \sum_{m_1, m_2 \geq 0} \mathrm{Hom}(B^{\otimes m_1} \otimes K \otimes A^{\otimes m_2}, K)[-m_1 - m_2] \tag{43}$$

As a graded space,

$$\mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) = \mathrm{Hoch}^\bullet(A) \oplus \mathrm{Hoch}^\bullet(B) \oplus \mathrm{Hoch}^\bullet(B,K,A) \tag{44}$$

but certainly it is *not* a direct sum of *subcomplexes*. Namely, $\mathrm{Hoch}^\bullet(B,K,A)$ is a subcomplex of $\mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K))$, but $\mathrm{Hoch}^\bullet(A)$ and $\mathrm{Hoch}^\bullet(B)$ are *not*. There are well-defined projections $p_A\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(A)$ and $\mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(B)$.

The Hochschild component of the total differential acts like this:

$$\begin{tikzcd}
& X_3 \arrow[loop above, "d^K_{\mathrm{Hoch}}"] & \\
X_1 \arrow[ur, "d^{AK}_{\mathrm{Hoch}}"] \arrow[loop left, "d^A_{\mathrm{Hoch}}"] & & X_2 \arrow[ul, "d^{BK}_{\mathrm{Hoch}}"'] \arrow[loop right, "d^B_{\mathrm{Hoch}}"]
\end{tikzcd} \tag{45}$$

where $X_1 = \mathrm{Hoch}^\bullet(A)$, $X_2 = \mathrm{Hoch}^\bullet(B)$, $X_3 = \mathrm{Hoch}^\bullet(B,K,A)$.

In [Kel1], Bernhard Keller poses the following question: what is a sufficient condition on the triple (A,B,K) which would guarantee that the projections $p_A\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(A)$ and $\mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(B)$ are *quasi-isomorphisms of complexes*? (They are always maps of dg Lie algebras, it is clear). The answer is given as follows: it is enough if the following conditions are satisfied:

Consider the left action of B on K. It is a map of right A-modules, and we get a map $L_B^0\colon B \to \mathrm{Hom}_{mod-A}(K,K)$. We can also derive this map to a map $L_B\colon B \to \mathrm{RHom}_{mod-A}(K,K)$. Analogously, we define from the right A-action on K the map $R_A\colon A^{\mathrm{opp}} \to \mathrm{RHom}_{B-mod}(K,K)$.

Definition. Let A and B be two dg associative algebras, and let K be dg B-A-bimodule. We say that the triple (A,B,K) is a Keller's admissible triple if the maps

$$\begin{aligned} L_B &\colon B \to \mathrm{RHom}_{mod-A}(K,K) \\ R_A &\colon A^{\mathrm{opp}} \to \mathrm{RHom}_{B-mod}(K,K) \end{aligned} \tag{46}$$

are quasi-isomorphisms of algebras.

There are known two examples when the Keller's condition is satisfied:

(1) A is any dg associative algebra, and there is a map $\varphi\colon B \to A$ which is *a quasi-isomorphism* of algebras. We set $K = A$ with the tautological structure of right A-module on it, and with the left B-module structure given by the map φ;

(2) A is a quadratic Koszul algebra, $B = A^!$ is the Koszul dual algebra, and K is the Koszul complex of A considered as a B-A-bimodule.

The both statements are proven in [Kel1]. The theory developed in Section 1 makes the generalization of (2) for Koszul algebras over discrete valuation rings straightforward.

The following theorem was found and proven in [Kel1]:

Theorem. *Let (A, B, K) be a Keller's admissible triple. Then the natural projections $p_A\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A, B, K)) \to \mathrm{Hoch}^\bullet(A)$ and $p_B\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A, B, K)) \to \mathrm{Hoch}^\bullet(B)$ are quasi-isomorphisms of dg Lie algebras.*

Proof. Let $t\colon L^\bullet \to M^\bullet$ be a map of complexes. Recall that its cone $\mathrm{Cone}(t)$ is defined as $\mathrm{Cone}(t) = L^\bullet[1] \oplus M^\bullet$ with the differential given by matrix

$$d = \begin{pmatrix} d_{L[1]} & 0 \\ t[1] & d_M \end{pmatrix}$$

To prove that the map $t\colon L^\bullet \to M^\bullet$ is a quasi-isomorphism, it is equivalently than to prove that the cone $\mathrm{Cone}(t)$ is acyclic in all degrees.

Let us consider the cone $\mathrm{Cone}(p_A)$ where $p_A\colon \mathrm{Hoch}^\bullet(\mathsf{cat}) \to \mathrm{Hoch}^\bullet(A)$ is the natural projection. Let us prove that if the first condition of (46) is satisfied, the cone $\mathrm{Cone}(p_A)$ is acyclic.

We can regard $\mathrm{Cone}(p_A)$ as a bicomplex where the vertical differentials are the Hochschild differentials and the horizontal differential is $p_A[1]$. This bicomplex has two columns, therefore its spectral sequences converge. Compute firstly the differential $p_A[1]$. Then the term E_1 is the sum of $\mathrm{Hoch}^\bullet(B) \oplus \mathrm{Hoch}^\bullet(B, K, A)$, as a graded vector space. There are 3 components of the differential in E_1: the Hochschild differentials in $\mathrm{Hoch}^\bullet(B)$ and in $\mathrm{Hoch}^\bullet(B, K, A)$, and exactly the same differential $d^{BK}_{\mathrm{Hoch}}\colon \mathrm{Hoch}^\bullet(B) \to \mathrm{Hoch}^\bullet(B, K, A)[1]$, as in the Hochschild complex of the category $\mathrm{Hoch}^\bullet(\mathsf{cat}(A, B, K))$.

Compute firstly the cohomology of $\mathrm{Hoch}^\bullet(B, K, A)$ with the only Hochschild differential. One can write:

$$\mathrm{Hoch}^\bullet(B, K, A) = \mathrm{Hom}(T(B), \mathrm{Hom}(K \otimes T(A), K)) \tag{47}$$

with some differentials, where we denote by $T(V)$ the free associative algebra generated by V. More precisely, the term $\mathrm{Hom}_{\mathbb{C}}(K \otimes T(A), K)$ is equal to the complex $\mathrm{Hom}_{mod-A}(\mathrm{Bar}^\bullet_{mod-A}(K), K)$ of maps from the bar-resolution of K in the category of right A-modules to K. This is equal to $\mathrm{RHom}_{mod-A}(K, K)$, which is quasi-isomorphic to B by the first Keller's condition. But this is not all what we need–we also need to know that the left B-module structures on B and on $\mathrm{RHom}_{mod-A}(K, K)$ are the same. This is exactly guaranteed by the Keller's condition, which says that the quasi-isomorphism $B \to \mathrm{RHom}_{mod-A}(K, K)$ *is induced by the left action of* B *on* K.

Now we have two complexes, which are exactly the same, and are $\mathrm{Hoch}^\bullet(B)$, but there is also the component d^{BK}_{Hoch} from one to another. In other words, so far our complex is the cone of the identity map from $\mathrm{Hoch}^\bullet(B)$ to itself, and this cone is clearly acyclic.

We have proved that if the first Keller's condition is satisfied, the natural projection $p_A\colon \mathrm{Hoch}^\bullet(\mathsf{cat}) \to \mathrm{Hoch}^\bullet(A)$ is a quasi-isomorphism. If the second Keller's condition is satisfied, we conclude, analogously, that the projection $p_B\colon \mathrm{Hoch}^\bullet(\mathsf{cat}) \to \mathrm{Hoch}^\bullet(B)$ is a quasi-isomorphism. $\square$

B. Keller used this theorem in [Kel1] to show that in the two cases listed above when the Keller's conditions are satisfied, the Hochschild cohomological complexes of A and B are quasi-isomorphic as dg Lie algebras. In particular, this is true when A and B are Koszul dual algebras, the case of the most interest for us.

Remark. If A and B are Koszul dual algebras, but K is replaced by $\mathbb{C}$, the only cohomology of the Koszul complex, we still have the quasi-isomorphisms $B \to \mathrm{RHom}_{mod-A}(\mathbb{C},\mathbb{C})$ and $A^{\mathrm{opp}} \to \mathrm{RHom}_{B-mod}(\mathbb{C},\mathbb{C})$, *but these maps are not induced by the left (correspondingly, right) actions of B (correspondingly, A) on $\mathbb{C}$.* These actions define some stupid maps which are not quasi-isomorphisms. This example shows that the Keller's dg category in this case may be not quasi-equivalent (and it is really the case) to its homology dg category.

4.3 The maps p_A and p_B are maps of B_∞ algebras

Let A, B be two associative algebras, and let K be any $B-A$-bimodule, not necessarily satisfying the Keller's condition from Section 4.2. Then we have two projections $p_A\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(A)$ and $p_B\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(B)$. We know from Section 2 that the Hochschild complex $\mathrm{Hoch}^\bullet(A)$ of any associative algebra has the natural structure of B_∞ algebra by means of the Getzler-Jones' braces (see Figure 2). The same is true for $\mathrm{Hoch}^\bullet(\mathcal{C})$ where $\mathcal{C}$ is a dg category, which is established by the same braces' construction.

The following simple Lemma, due to Bernhard Keller [Kel1], is very important for our paper:

Lemma. *Let A, B be two associative algebras, and let K be a $B-A$-bimodule. Then the natural projections $p_A\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(A)$ and $p_B\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \to \mathrm{Hoch}^\bullet(B)$ are maps of B_∞ algebras.*

Proof. It is clear because the projections p_A and p_B are compatible with the braces, and with the cup-products. That is, they are compatible with the maps m_i and m_{ij} of the B_∞ structure, see Section 2.5. $\square$

4.4 We formulate a new version of the Main Theorem

Let $A = S(V^*)$, $B = \Lambda(V)$, and $K = K^\bullet(S(V^*))$. We know from Section 1.1 the the algebra $S(V^*)$ is Koszul, and its Koszul dual $A^! = \Lambda(V)$. Thus, we can apply Theorem 4.2 to the triple $(S(V^*), \Lambda(V), K^\bullet(S(V^*)))$. We have constructed a B_∞ algebra

$\mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K))$ for A,B,K as above, and the diagram

$$\begin{tikzcd} & \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ \mathrm{Hoch}^\bullet(A) & & \mathrm{Hoch}^\bullet(B) \end{tikzcd} \tag{48}$$

where the two right maps are maps of B_∞ algebras. Let now $t\colon G_\infty \to B_\infty$ be a point of the Tamarkin's manifold, see Section 3.3. Then the diagram (48) is a diagram of maps of G_∞ algebras, depending on $t \in \mathfrak{T}$.

Let now $\mathcal{U} = \mathfrak{X}^0(t)$ be the universal G_∞ morphism $\mathcal{G}_V\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(S(V^*))$ defined for all finite-dimensional (graded) vector spaces V, see Section 3.4. It depends on the point $t \in \mathfrak{T}$ and is defined up to a homotopy. Denote by $\mathcal{G}^S(t)$ and $\mathcal{G}^\Lambda(t)$ the specializations of this universal G_∞ morphism for the vector spaces V and $V^*[1]$, correspondingly. Identify $T_{poly}(V)$ with $T_{poly}(V^*[1])$ as in Section 0.1 of the Introduction. Then we have the following diagram of G_∞ maps:

$$\begin{tikzcd} & \mathrm{Hoch}^\bullet(A) & \\ T_{poly}(V) \arrow[ur, "\mathcal{G}^S(t)"] \arrow[dr, "\mathcal{G}^\Lambda(t)"'] & & \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \arrow[ul, "p_A"'] \arrow[dl, "p_B"] \\ & \mathrm{Hoch}^\bullet(B) & \end{tikzcd} \tag{49}$$

depending on $t \in \mathfrak{T}$.

Here and in Sections 6 we prove the following statement:

Theorem. *For any fixed $t \in \mathfrak{T}$, the diagram (49) is homotopically commutative, that is, it is commutative in the Quillen's homotopical category.*

Now restrict ourselves with the L_∞ component of the G_∞ maps. Then clearly the diagram remains to be homotopically commutative. We have the following

Corollary. (A new version of the Main Theorem) *For any $t \in \mathfrak{T}$, the diagram (49) defines a homotopically commutative diagram of L_∞ maps.*

We explain in Section 7 in detail why this Corollary implies the Main Theorem in our previous version, for Koszul duality in deformation quantization.

Now let us begin to prove the Theorem above.

Proof of Theorem (beginning): The proof is based on the following Key-Lemma:

Key-Lemma. *For any $t \in \mathfrak{T}$, the diagramm (49) defines a commutative diagram of isomorphisms maps on cohomology.*

We prove this Lemma in Section 6, and it will take some work.

Now let us explain how the Theorem follows from the Key-Lemma.

The diagram (49) is a diagram of G_∞ quasi-isomorphisms (the two left arrows clearly are quasi-isomorphisms, and the two right ones are by the Keller's Theorem proven in Section 4.2). We can uniquely up to a homotopy invert a G_∞ quasi-isomorphism. Then the map

$$\mathcal{G}(t) = (\mathcal{G}^\Lambda(t))^{-1} \circ p_B \circ p_A^{-1} \circ \mathcal{G}^S(t) \tag{50}$$

is uniquely defined, up to a homotopy, G_∞ quasi-automorphism of $T_{poly}(V)$. Now, by the Key-Lemma, its first Taylor component is the identity map. Then we can take the logarithm

$$\mathcal{D} = \log(\mathcal{G}) \tag{51}$$

which is a G_∞ derivation of $T_{poly}(V)$.

Now we continue as in the proof of Lemma 3.4. By the computation of $H^0(\mathrm{Coder}(\mathbb{F}^\vee_{G^*[2]}(T_{poly}(V)))) = T^1_{poly}(V)$ in Theorem 2.4 we conclude, that any $\mathrm{Aff}(V)$-equivariant G_∞ derivation $\mathcal{D}$ of $T_{poly}(V)$ is inner. Our $\mathcal{D}$ is clearly $\mathrm{Aff}(V)$-equivariant, because we never used a choice of basis in V in our constructions. Then we conclude that $\mathcal{D}$ is inner. But then, if it is nonzero, the exponent $\exp(\mathcal{D})$ should have the first Taylor component not equal to the identity. But again we know from the Key-Lemma that it is identity.

The Theorem is proven modulo the Key-Lemma which we prove in Section 6.

5 The homotopical category of dg algebras over a Koszul operad

Here we give, following [Sh3], a construction of the homotopy category, appropriate for our needs in the next Sections of this paper. Our emphasis here is how the homotopy relation reflects in the gauge equivalence condition for deformation quantization. We restrict ourselves with the case of the operad of Lie algebras because this is the only case we will use. The constructions for general Koszul operad are analogous.

Here we use the construction of Quillen homotopical category given in [Sh3]. In a sense, it is "the right cylinder homotopy relation". Recall here the definition.

5.1 The homotopy relation from [Sh3]

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two dg Lie algebras. Then there is a dg Lie algebra $\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)$ which is pro-nilpotent and such that the solutions of the Maurer-Cartan equation in $\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)^1$ are exactly the L_∞ morphisms from $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Then the zero degree component $\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)^0$ acts on the Maurer-Cartan solutions, as usual in deformation theory (the dg Lie algebra $\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)$ is pro-nilpotent), and this action gives a homotopy relation.

The dg Lie algebra $\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)$ is constructed as follows. As a dg vector space, it is

$$\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2) = \mathrm{Hom}(C_+(\mathfrak{g}_1, \mathbb{C}), \mathfrak{g}_2) \tag{52}$$

Here $C(\mathfrak{g}_1, \mathbb{C})$ is the chain complex of the dg Lie algebra $\mathfrak{g}_1$, it is naturally a counital dg coalgebra, and $C_+(\mathfrak{g}_1, \mathbb{C})$ is the kernel of the counit map.

Define now a Lie bracket on $\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)$. Let $\theta_1, \theta_2 \in \Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)$ be two elements. Their bracket $[\theta_1, \theta_2]$ is defined (up to a sign) as

$$C_+(\mathfrak{g}_1, \mathbb{C}) \xrightarrow{\Delta} C_+(\mathfrak{g}_1, \mathbb{C})^{\otimes 2} \xrightarrow{\theta_1 \otimes \theta_2} \mathfrak{g}_2 \otimes \mathfrak{g}_2 \xrightarrow{[,]} g_2 \tag{53}$$

where Δ is the coproduct in $C_+(\mathfrak{g}_1, \mathbb{C})$ and $[,]$ is the Lie bracket in $\mathfrak{g}_2$. It follows from the cocommutativity of Δ that in this way we get a Lie algebra.

An element F of degree 1 in $\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)$ is a collection of maps

$$\begin{aligned} &F_1 \colon \mathfrak{g}_1 \to \mathfrak{g}_2 \\ &F_2 \colon \Lambda^2(\mathfrak{g}_1) \to \mathfrak{g}_2[-1] \\ &F_3 \colon \Lambda^3(\mathfrak{g}_1) \to \mathfrak{g}_2[-2] \\ &\dots \end{aligned} \tag{54}$$

and the Maurer-Cartan equation $d_{\Bbbk} F + \frac{1}{2}[F, F]_{\Bbbk} = 0$ is the same that the collection $\{F_i\}$ are the Taylor components of an L_∞ map which we denote also by F. Note that the differential in $\Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)$ comes from 3 differentials: the both inner differentials in $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and from the chain differential in $C_+(\mathfrak{g}_1, \mathbb{C})$.

Now the solutions of the Maurer-Cartan equation form a quadric in $\mathfrak{g}^1$, and for any pro-nilpotent dg Lie algebra $\mathfrak{g}$, the component $\mathfrak{g}^0$ acts on (the pro-nilpotent completion of) this quadric by vector fields. Namely, each $X \in \mathfrak{g}^0$ defines a vector field

$$\frac{dF}{dt} = -dX + [X, F] \tag{55}$$

It can be directly checked that this vector field indeed preserves the quadric.

In our case, this vector field can be exponentiated to an action on the pro-nilpotent completion on $\Bbbk$. This action gives our homotopy relation on L_∞ morphisms.

5.2 Application to deformation quantization

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two dg Lie algebras, and let $\mathcal{F}^1, \mathcal{F}^2 \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be two homotopic in the sense of Section 2.4.1 L_∞ morphisms.

Let α be a solution of the Maurer-Cartan equation in $\mathfrak{g}_1$. Any L_∞ morphism $\mathcal{F} \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ gives a solution $\mathcal{F}_* \alpha$ of the Maurer-Cartan equation in $\mathfrak{g}_2$, by formula

$$\mathcal{F}_* \alpha = \mathcal{F}_1(\alpha) + \frac{1}{2}\mathcal{F}_2(\alpha \wedge \alpha) + \frac{1}{6}\mathcal{F}_3(\alpha \wedge \alpha \wedge \alpha) + \dots \tag{56}$$

(suppose that this infinite sum makes sense).

Then in our situation we have two solutions $\mathcal{F}^1_*\alpha$ and $\mathcal{F}^2_*\alpha$ of the Maurer-Cartan equation in $\mathfrak{g}^2$.

Lemma. *Suppose that all infinite sums (exponents) we need make sense in our situation. Suppose two L_∞ morphisms $\mathcal{F}_1, \mathcal{F}_2\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ are homotopic in the sense of Section 2.4.1, and suppose that α is a solution of the Maurer-Cartan equation in $\mathfrak{g}_1$. Then the two solutions $\mathcal{F}^1_*\alpha$ and $\mathcal{F}^2_*\alpha$ of the Maurer-Cartan equation in $\mathfrak{g}_2$ are gauge equivalent.*

Proof. Let $X \in \Bbbk(\mathfrak{g}_1, \mathfrak{g}_2)^0$ be the generator of the homotopy between $\mathcal{F}^1$ and $\mathcal{F}^2$. Define

$$X_*\alpha = X(\alpha) + \frac{1}{2}X(\alpha\wedge\alpha) + \frac{1}{6}X(\alpha\wedge\alpha\wedge\alpha) + \dots \tag{57}$$

Then $X_*\alpha \in (\mathfrak{g}_2)^0$. Consider the vector field on $(\mathfrak{g}_2)^1$:

$$\frac{dg}{dt} = -d(X_*\alpha) + [X_*\alpha, g] \tag{58}$$

Then the exponent of this vector field maps $\mathcal{F}^1 * \alpha$ to $\mathcal{F}^2_*\alpha$. $\square$

6 The main computation

Here we prove the Key-Lemma 4.4 which is only remains to conclude the proof of Theorem 4.4.

6.1

We are going to construct "the Hochschild-Kostant-Rosenberg map" $\varphi^{\mathsf{cat}}_{HKR}\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K))$ where $A = S(V^*)$, $B = \Lambda(V)$, and K is the Koszul complex of $S(V^*)$. At the final step of the compuation, we normalize the Koszul differential by $\dim V$, as follows:

$$d^{\mathrm{norm}}_{\mathrm{Koszul}} = \frac{1}{\dim V}\sum_{a=1}^{\dim V} x_a \frac{\partial}{\partial \xi_a} \tag{59}$$

However, in the computation below we suppose that the Koszul complex is not normalized. The normalized Koszul complex defines the equivalent Keller's category, so it is irrelevant.

Our Hochschild-Kostant-Rosenberg map $\varphi^{\mathsf{cat}}_{HKR}$ will make the following diagram commutative (up to a sign) on the cohomology:

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{Hoch}^\bullet(A) & \\
T_{poly}(V) \arrow[ur, "\mathcal{G}^S(t)"] \arrow[dr, "\mathcal{G}^\Lambda(t)"'] \arrow[rr, dotted, "\varphi^{\mathsf{cat}}_{HKR}"] & & \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K)) \arrow[ul, "p_A"'] \arrow[dl, "p_B"] \\
 & \mathrm{Hoch}^\bullet(B) &
\end{tikzcd} \tag{60}
$$

We did not specify the sign, but it does not make any problem.

In the computation below we use the graphical representation of the cochains in $\mathrm{Hom}(\Lambda(V)^{\otimes m} \otimes K \otimes S(V^*), K)$. The reader familiar with the Kontsevich's paper [K97] will immediately understand our graphical representation. (But for other readers, we define our cochains by the explicit formulas, see (61)-(63) below).

In our graphical cochains, we consider a circle with two fixed points, 0 and ∞. The arguments from $\Lambda(V)$ are placed on the left half of the circle, and the arguments from $S(V^*)$ are placed on the right half. Any arrow is the operator $\sum_{a=1}^{\dim V} \frac{\partial}{\partial \xi_a} \cdot \frac{\partial}{\partial x_a}$. In our convention, which coincides with the one in [K97], the start-point of any arrow "differentiates" the odd arguments, while the end-point differentiates the even arguments. We have one point inside the disc bounded by the circle, where we place our polyvector field γ. We use the notation $\gamma = \gamma^S \otimes \gamma^\Lambda$ (where γ^S and γ^Λ are the even and the odd coordinates of γ) and suppose that γ is homogeneous in the both x_i's and ξ_j's coordinates.

After this general remarks, let us start.

6.2 Some graph-complex

The problem of a construction of quasi-isomorphism $\varphi^{\mathsf{cat}}_{HKR}\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(\mathsf{cat})$ is rather non-trivial. Indeed, the usual Hochschild-Kostant-Rosenberg cochains $\varphi^S_{HKR}(\gamma) \in Hoch(S(V^*))$ and $\varphi^\Lambda_{HKR}(\gamma) \in \mathrm{Hoch}^\bullet(\Lambda(V))$ are not cocycles when considered as cochains in $\mathrm{Hoch}^\bullet(\mathsf{cat})$. Indeed, their boundaries have components which belong in $\mathrm{Hom}(K^\bullet \otimes S(V^*)^{\otimes m_1}, K^\bullet)$ and in $\mathrm{Hom}(\Lambda(V)^{\otimes m_1} \otimes K^\bullet, K^\bullet)$, correspondingly. Our map $\varphi^{\mathsf{cat}}_{HKR}$ contain as summand the both cochains φ^S_{HKR} and φ^Λ_{HKR}, and many other summands. These other summands are the cochains associated with the graphs F^0_{0,m_2} and $F^\infty_{m_1,0}$ shown in Figure below.

It is instructive to formulate the following Proposition in a bit more generality than we really need, for all graphs $F^0_{m_1,m_2}$ and $F^\infty_{m_1,m_2}$. Denote the corresponding maps Φ_Γ in $\mathrm{Hom}(\Lambda(V)^{\otimes m_1} \otimes K^\bullet \otimes S(V^*)^{\otimes m_2}, K^\bullet)$ by $F^\infty_{m_1,m_2}(\gamma)$, $F^0_{m_1,m_2}(\gamma)$, and $G_{m_1,m_2}(\gamma)$, where $\gamma \in T_{poly}(V)$. Suppose that γ is homogeneous in both x_i's and ξ's. As maps $T_{poly}(V) \to \mathrm{Hoch}^\bullet(\mathsf{cat})$ the maps $F^0_{m_1,m_2}$ and $F^\infty_{m_1,m_2}$ have degree 0, and the map G_{m_1,m_2} has degree 1.

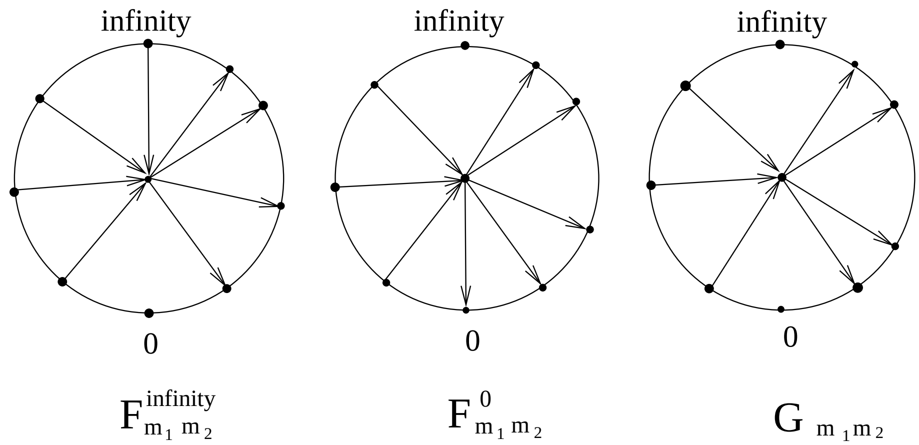

Figure 3: The cochains $F^{\infty}_{m_1,m_2}$, $F^0_{m_1,m_2}$, and G_{m_1,m_2} for $m_1 = 3, m_2 = 4$

We have the following explicit formulas for these maps:

$$
\begin{aligned}
&G_{m_1,m_2}(\gamma)(\lambda) = \frac{1}{n!}\frac{1}{m!} \sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{j_1,\dots,j_{m_2}=1}^{\dim V} \\
&\pm k \left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \cdots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\partial\xi_{j_1} \circ \cdots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times \\
&\times (\partial x_{i_1} \circ \cdots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \dots \partial x_{j_{m_2}}(f_{m_2})
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&F^0_{m_1,m_2}(\gamma)(\lambda) = \frac{1}{m_1!}(m_2+1)! \sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{a,j_1,\dots,j_{m_2}=1}^{\dim V} \\
&\pm \partial\mathbf{x_a}(k) \left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \cdots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\partial\xi_{\mathbf{a}} \circ \partial\xi_{j_1} \circ \cdots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times \\
&\times (\partial x_{i_1} \circ \cdots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \dots \partial x_{j_{m_2}}(f_{m_2})
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
&F^\infty_{m_1,m_2}(\gamma)(\lambda) = \frac{1}{(m_1+1)!}\frac{1}{m_2!} \sum_{b,i_1,..,i_{m_1}=1}^{\dim V} \sum_{j_1,\dots,j_{m_2}=1}^{\dim V} \\
&\pm k \left(\partial\xi_{\mathbf{b}} \left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \cdots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\partial\xi_{j_1} \circ \cdots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right)\right) \times \\
&\times (\partial\mathbf{x_b} \circ \partial x_{i_1} \circ \cdots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \dots \partial x_{j_{m_2}}(f_{m_2})
\end{aligned}
\tag{63}
$$

Here, as usual, we denote by $\{x_i\}$ some basis in V^*, and by $\{\xi_i\}$ the dual basis in $V[-1]$.

Let γ be a polynomial polyvector field in $T_{poly}(V)$, homogeneous in both x's and ξ's. Denote $\deg_S \gamma$ and $\deg_\Lambda \gamma$ the corresponding homogeneity degrees. (We have the Lie degree $\deg \gamma = \deg_\Lambda \gamma - 1$).

Denote d_{Hoch} and d_{Koszul} the Hochschild and Koszul components of the differential acting on $\mathrm{Hoch}^\bullet(\Lambda(V), K^\bullet, S(V^*)) \subset \mathrm{Hoch}^\bullet(\mathsf{cat})$.

Proposition. *Suppose $\sharp In_\Gamma(v) \le \deg_S \gamma$ and $\sharp Star(v) \le \deg_\Lambda \gamma$ for each separate graph Γ in the claims below, where v is the only vertex of the first type. Suppose that $F^0_{m,n}(\gamma)$ etc. means the sum over all orderings of the sets $Star(v)$ and $In(v)$ (see (10) and (11) in the definition of an admissible graph), that is, over all admissible graphs which are the same geometrically. (The sum should be taken with the appropriate signs depending naturally on the orderings). Then we have:*

(i) $d_{\mathrm{Hoch}} F^0_{m,n}(\gamma) = \pm G_{m,n+1}(\gamma)$,

(ii) $d_{\mathrm{Koszul}} F^0_{m,n}(\gamma) = \pm \dim V \cdot (\deg_\Lambda(\gamma) - n) \cdot G_{m,n}(\gamma)$,

(iii) $d_{\mathrm{Hoch}} F^\infty_{m,n}(\gamma) = \pm G_{m+1,n}(\gamma)$,

(iv) $d_{\mathrm{Koszul}} F^\infty_{m,n}(\gamma) = \pm \dim V \cdot (\deg_S(\gamma) - m) \cdot G_{m,n}(\gamma)$.

Proof. The proof of Proposition is just a straightforward computation. For convenience of the reader, we present it here in all details.

We give the proofs of (i) and (ii); the proofs of the second two statements are analogous.

Prove (i).

It would be instructive for the reader to recall before the proof the proof that the classical Hochschild-Kostant-Rosenberg $\varphi^S_{HKR}(\gamma)$ is a Hochschild cocycle in $\mathrm{Hoch}^\bullet(S(V^*))$ for any $\gamma \in T_{poly}(V)$. It goes as follows: we associate with a k-polyvector field γ the cochain $\varphi^S_{HKR}(\gamma) \in \mathrm{Hom}(S(V^*)^{\otimes k}, S(V^*))$ defined as

$$\varphi^S_{HKR}(\gamma)(f_1 \otimes \dots \otimes f_k) = \sum_{i_1,\dots,i_k=1}^{\dim V} \pm \gamma(dx_{i_1} \wedge \dots \wedge dx_{i_k}) \partial x_{i_1}(f_1) \dots \partial x_{i_k}(f_k) \tag{64}$$

The only nonzero terms may appear when all $i_1, \dots, i_k$ are different,and the sign $\pm$ is the sign of the permutation $(1, 2, \dots, k) \mapsto (i_1, i_2, \dots, i_k)$. The proof that $\varphi^S_{HKR}(\gamma)$ is a Hochschild cocycle just uses the Leibniz formula $\partial x_a(f_i f_{i+1}) = \partial x_a(f_i) f_{i+1} + f_i \partial x_a(f_{i+1})$ and the Hochschild coboundary formula

$$\begin{aligned} &d_{\mathrm{Hoch}}(\Psi)(f_1 \otimes \dots \otimes f_{k+1}) = \\ &f_1 \Psi(f_2 \otimes f_2 \otimes \dots \otimes f_{k+1}) - \\ &- \Psi((f_1 f_2) \otimes f_3 \otimes \dots) + \Psi(f_1 \otimes (f_2 f_3) \otimes \dots) \mp \dots \\ &+ (-1)^{k+1} \Psi(f_1 \otimes \dots \otimes f_k) f_{k+1} \end{aligned} \tag{65}$$

We see that the all terms will be mutually canceled. Now let us see when this kind of phenomenon may be destroyed in the coboundary of $F^0_{m,n}(\gamma)$. It is clear that any problem place is the marked point 0 at the boundary of the circle. Consider the sum of two "problematic" summands. This is

$$\pm F^0_{m,n}(\gamma)(\lambda_1\otimes\cdots\otimes\lambda_m\otimes(\lambda_{m+1}(k))\otimes f_n\otimes\cdots\otimes f_1)\mp F^0_{m,n}(\gamma)(\lambda_1\otimes\cdots\otimes\lambda_m\otimes((k)f_{n+1})\otimes f_n\otimes\cdots\otimes f_1) \tag{66}$$

Here we use the notation $\lambda(k)$ and $(k)f$ for the left action of $\Lambda(V)$ and for the right action of $S(V^*)$, correspondingly. These two summands give from (62)

$$\pm\partial x_a(\lambda_{m+1}(k)) = \pm\lambda_{m+1}(\partial x_a k) \tag{67}$$

which clearly is canceled with (a part of) the previous summand,

$$\mp F^0_{m,n}(\gamma)(\lambda_1\otimes\cdots\otimes(\lambda_m\lambda_{m+1})\otimes k\otimes f_n\otimes\cdots\otimes f_1) \tag{68}$$

So the first summand in (66) does not contribute to the answer. Contrary, the second summand gives the term

$$\partial x_a(k\cdot f_{n+1}) = \partial x_a(k)\cdot f_{n+1} + k\cdot\partial x_a(f_{n+1}) \tag{69}$$

The first summand in (69) is canceled with the one of two summands in $F^0_{m,n}(\gamma)(\lambda_1\otimes\cdots\otimes\lambda_m\otimes k\otimes(f_{n+1}\cdot f_n)\otimes\cdots\otimes f_1)$. The second summand in (69) is not canceled with an other summand, and it gives the only term which contributes to the answer. This term clearly gives $G_{m,n+1}(\gamma)$.

Prove (ii).

We need to compute

$$\begin{aligned}
&\sum_{i_1,..,i_{m_1}=1}^{\dim V}\ \sum_{a,j_1,\dots,j_{m_2}=1}^{\dim V} \pm\mathbf{d}_{\mathrm{Koszul}}\\
&\{\partial\mathbf{x_a}(\mathbf{k})\left(\lambda\wedge\partial\xi_{i_1}(\lambda_1)\wedge\cdots\wedge\partial\xi_{i_{m_1}}(\lambda_{m_1})\wedge(\partial\xi_{\mathbf{a}}\circ\partial\xi_{j_1}\circ\cdots\circ\partial\xi_{j_{m_2}}(\gamma^\Lambda))\right)\times\\
&\times(\partial x_{i_1}\circ\cdots\circ\partial x_{i_{m_1}})(\gamma^S)\cdot\partial x_{j_1}(f_1)\dots\partial x_{j_{m_2}}(f_{m_2})\}\\
&\mp\sum_{i_1,..,i_{m_1}=1}^{\dim V}\ \sum_{a,j_1,\dots,j_{m_2}=1}^{\dim V}\\
&\pm\partial\mathbf{x_a}(\mathbf{d}_{\mathrm{Koszul}}\mathbf{k})\left(\lambda\wedge\partial\xi_{i_1}(\lambda_1)\wedge\cdots\wedge\partial\xi_{i_{m_1}}(\lambda_{m_1})\wedge(\partial\xi_{\mathbf{a}}\circ\partial\xi_{j_1}\circ\cdots\circ\partial\xi_{j_{m_2}}(\gamma^\Lambda))\right)\times\\
&\times(\partial x_{i_1}\circ\cdots\circ\partial x_{i_{m_1}})(\gamma^S)\cdot\partial x_{j_1}(f_1)\dots\partial x_{j_{m_2}}(f_{m_2})
\end{aligned}\tag{70}$$

We have:

$$d_{\mathrm{Koszul}}k = \sum_{p=1}^{\dim V} x_p\partial\xi_p \tag{71}$$

Then (70) is equal to

$$\sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{a,j_1,...,j_{m_2}=1}^{\dim V}$$
$$\pm \mathbf{x_p}\partial\xi_\mathbf{p}\{\partial\mathbf{x_a}(\mathbf{k}) \left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \cdots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\partial\xi_\mathbf{a} \circ \partial\xi_{j_1} \circ \cdots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times$$
$$\times (\partial x_{i_1} \circ \cdots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \ldots \partial x_{j_{m_2}}(f_{m_2})\}$$
$$\mp \sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{a,j_1,...,j_{m_2}=1}^{\dim V}$$
$$\pm \partial\mathbf{x_a}(\mathbf{x_p}\partial\xi_\mathbf{p}\mathbf{k}) \left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \cdots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\partial\xi_\mathbf{a} \circ \partial\xi_{j_1} \circ \cdots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times$$
$$\times (\partial x_{i_1} \circ \cdots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \ldots \partial x_{j_{m_2}}(f_{m_2}) \tag{72}$$

where the summation over p is assumed. Clearly (72) is $A + B$ where

$$\mathbf{A} =$$
$$\sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{a,j_1,...,j_{m_2}=1}^{\dim V}$$
$$\pm \mathbf{x_p}[\partial\xi_p\{\partial\mathbf{x_a}(\mathbf{k}) \left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \cdots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\partial\xi_\mathbf{a} \circ \partial\xi_{j_1} \circ \cdots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times$$
$$\times (\partial x_{i_1} \circ \cdots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \ldots \partial x_{j_{m_2}}(f_{m_2})\}$$
$$\mp \sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{a,j_1,...,j_{m_2}=1}^{\dim V}$$
$$\pm (\partial\xi_\mathbf{p}\partial\mathbf{x_a}\mathbf{k}) \left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \cdots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\partial\xi_\mathbf{a} \circ \partial\xi_{j_1} \circ \cdots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times$$
$$\times (\partial x_{i_1} \circ \cdots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \ldots \partial x_{j_{m_2}}(f_{m_2})] \tag{73}$$

and

$$\mathbf{B} =$$
$$\mp \dim \mathbf{V} \sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{a,j_1,...,j_{m_2}=1}^{\dim V} \tag{74}$$
$$\pm (\partial\xi_\mathbf{a}\mathbf{k}) \left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \cdots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\partial\xi_\mathbf{a} \circ \partial\xi_{j_1} \circ \cdots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times$$
$$\times (\partial x_{i_1} \circ \cdots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \ldots \partial x_{j_{m_2}}(f_{m_2})$$

In the last equation the symbol δ_{ap} appears when we take the commutator $[\partial x_a, x_p]$ in the second summand of (72), which gives the factor $\dim V$ and summation only over a in B. We continue for A and B separately.

Let us start with B. We have:

$$
\begin{aligned}
&B = \\
&\mp \dim V \sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{a,j_1,\dots,j_{m_2}=1}^{\dim V} \\
&\pm \mathbf{k}\left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \dots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge (\xi_{\mathbf{a}}\partial\xi_{\mathbf{a}} \circ \partial\xi_{j_1} \circ \dots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times \\
&\times (\partial x_{i_1} \circ \dots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \dots \partial x_{j_{m_2}}(f_{m_2}) = \\
&\mp \dim V \cdot (\deg_\Lambda(\gamma) - m_2) \times \qquad (75) \\
&\times \sum_{i_1,..,i_{m_1}=1}^{\dim V} \sum_{a,j_1,\dots,j_{m_2}=1}^{\dim V} \\
&\pm \mathbf{k}\left(\lambda \wedge \partial\xi_{i_1}(\lambda_1) \wedge \dots \wedge \partial\xi_{i_{m_1}}(\lambda_{m_1}) \wedge \partial\xi_{j_1} \circ \dots \circ \partial\xi_{j_{m_2}}(\gamma^\Lambda))\right) \times \\
&\times (\partial x_{i_1} \circ \dots \circ \partial x_{i_{m_1}})(\gamma^S) \cdot \partial x_{j_1}(f_1) \dots \partial x_{j_{m_2}}(f_{m_2}) = \\
&= \pm \dim V \cdot (\deg_\Lambda(\gamma) - m_2) \cdot G_{m,n}(\gamma)
\end{aligned}
$$

Now turn back to the computation of A. Clearly (up to the sign, but the signs always work for us) that $A = 0$. Indeed, schematically the formula (73) for A looks like $\partial\xi_p(k(\lambda \wedge T)) - (\partial\xi_p(k))(\lambda \wedge T)$ for some $T \in \Lambda(V)$. If we define $k'(\lambda) = k(\lambda \wedge T)$ we need to compute

$$(\partial\xi_p(k'))(\lambda) - (\partial\xi_p(k))(\lambda \wedge T) \tag{76}$$

But $(\partial\xi_p(k'))(\lambda) = k'(\xi_p \wedge \lambda) = k(\xi_p \wedge \lambda \wedge T)$. Now we see that the two summands in (76) are equal.

We have proved the statements (i) and (ii) of the Proposition. The proofs of (iii) and (iv) are analogous. $\square$

6.3 Construction of the Hochschild-Kostant-Rosenberg map $\varphi^{\mathsf{cat}}_{HKR}$

Now we have everything we need to construct the map $\varphi^{\mathsf{cat}}_{HKR}\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(\mathsf{cat})$. Of course, it would be better to specify the signs in the Proposition above; however, we will see that the construction below does not depend seriously on these signs.

Suppose that $\deg_S \gamma = m$, $\deg_\Lambda \gamma = n$. Start with $\widetilde{\varphi}^S_{HKR}(\gamma) \in \mathrm{Hoch}^\bullet(S(V^*))$, which is by definition the Hochschild-Kopstant-Rosenberg cochain without division by the $n!$. It total differential in $\mathrm{Hoch}^\bullet(\mathsf{cat})$ is $d_{\mathrm{tot}}\widetilde{\varphi}^S_{HKR}(\gamma) = (\pm)G_{0,n}(\gamma)$. From now on, we will suppose that the all signs in Proposition above are " + ", if some of them are " − ", the formula will be the same up to some signs. So suppose that $d_{\mathrm{tot}}\widetilde{\varphi}^S_{HKR}(\gamma) = G_{0,n}(\gamma)$ with sign +. We know from statement (i) of the Proposition that $d_{\mathrm{Hoch}}F^0_{0,n-1}(\gamma) = G_{,n}(\gamma)$, the same cochain. Therefore, $d_{\mathrm{Hoch}}(\widetilde{\varphi}^S_{HKR}(\gamma) - F^0_{0,n-1}(\gamma)) = 0$. But then $\widetilde{\varphi}^S_{HKR}(\gamma) - F^0_{0,n-1}(\gamma)$ has a non-trivial Koszul differential which can be found by Proposition (ii). We have: $d_{\mathrm{Koszul}}(\widetilde{\varphi}^S_{HKR}(\gamma) - F^0_{0,n-1}(\gamma)) = d_{\mathrm{Koszul}}(F^0_{0,n-1}(\gamma)) = \dim V \cdot G_{0,n-1}(\gamma)$. Now we want to kill this coboundary by the Hochschild differential. We have: $d_{\mathrm{Hoch}}(\dim V \cdot F^0_{0,n-2}(\gamma)) = \dim V \cdot G_{0,n-1}(\gamma)$. Continuing in this way, we find that (we omit γ at each term):

$$\begin{aligned}&d_{\mathrm{tot}}(\widetilde{\varphi}^S_{HKR} - F^0_{0,n-1} + \dim V \cdot F^0_{0,n-2} - \dots + \dots + (-1)^n (n-1)! \dim^{n-1} V \cdot F^0_{0,0}) \\ &= (-1)^n n! \dim^n V G_{0,0}\end{aligned} \tag{77}$$

But we can start also with $\widetilde{\varphi}^\Lambda_{HKR}(\gamma)$, and finally get also $G_{0,0}$ with some multiplicity. More precisely, we have:

$$\begin{aligned}&d_{\mathrm{tot}}(\widetilde{\varphi}^\Lambda_{HKR} - F^\infty_{m-1,0} + \dim V \cdot F^\infty_{m-2,0} - \dots + (-1)^m (m-1)! \dim^{m-1} V \cdot F^\infty_{0,0}) \\ &= (-1)^m m! \dim^m V G_{0,0}\end{aligned} \tag{78}$$

We finally set:

$$\begin{aligned}\varphi^{\mathsf{cat}}_{HKR} =&(-1)^n \frac{1}{n! \dim^n V} \left(\widetilde{\varphi}^S_{HKR} + \sum_{i=1}^n (-1)^i (i-1)! \dim^{i-1} V F^0_{0,n-i} \right) - \\ &(-1)^m \frac{1}{m! \dim^m V} \left(\widetilde{\varphi}^\Lambda_{HKR} + \sum_{j=1}^m (-1)^j (j-1)! \dim^{j-1} V F^\infty_{m-j,0} \right)\end{aligned} \tag{79}$$

It is a cocycle in the Hochschild cohomological complex $\mathrm{Hoch}^\bullet(\mathsf{cat})$:

$$d_{\mathrm{tot}} \varphi^{\mathsf{cat}}_{HKR}(\gamma) = 0 \tag{80}$$

for any $\gamma \in T_{poly}(V)$.

We can prove the following

Theorem. *The map $\varphi^{\mathsf{cat}}_{HKR} \colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(\mathsf{cat})$ is a quasi-isomorphism of complexes. When we use the normalized Koszul differential instead of the usual one (so, it has the same effect as to set $\dim V = 1$ in the formula above), the map $\varphi^{\mathsf{cat}}_{HKR}$ makes the diagram (60) commutative (up to a non-essential sign) on the level of cohomology.*

Proof. The second statement is clear. The first one (that $\varphi^{\mathsf{cat}}_{HKR}$ is a quasi-isomorphism of complexes) follows from the second one and from Theorem 4.2 which says that the maps p_A and p_B are quasi-isomorphisms in our case. □

Key-Lemma 4.4 is proven. □

Theorem 4.4 is proven. □

7 Proof of the Main Theorem

First of all, we formulate the Main Theorem exactly in the form we will prove it here.

7.1 The final formulation of the Main Theorem

Theorem. (Main Theorem, final form) *Suppose $t\colon G_\infty \to B_\infty$ is a quasi-isomorphism of operads, and let $\mathcal{U}_V = \mathfrak{X}(t)_V\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(S(V^*))$ be the corresponding L_∞ map, defined uniquely up to homotopy (see Section 3). Let α be a quadratic Poisson bivector on V, and let $\mathcal{D}(\alpha)$ be the corresponding quadratic Poisson bivector on $V^*[1]$. Denote by $S(V^*)_\hbar$ and $\Lambda(V)_\hbar$ the corresponding deformation quantizations of $S(V^*)\otimes\mathbb{C}[[\hbar]]$ and $\Lambda(V)\otimes\mathbb{C}[[\hbar]]$ given by*

$$f\star g = f\cdot g + \hbar\cdot\mathcal{U}_1(\alpha)(f\otimes g) + \frac{1}{2}\hbar^2\cdot\mathcal{U}_2(\alpha\wedge\alpha)(f\otimes g)+\dots \tag{81}$$

Then the algebras $S(V^*)_\hbar$ and $\Lambda(V)_\hbar$ are graded (where $\deg\hbar=0$, $degx_i=1$ for all i) and quadratic. Also, they are Koszul as algebras over the discrete valuation ring $\mathbb{C}[[\hbar]]$, see Section 1. Moreover, they are Koszul dual to each other.

We prove the Theorem throughout this Section.

7.2 An elementary Lemma

We start with the following simple statement:

Lemma. *(1) Suppose $K_\hbar$ is a free $\mathbb{C}[[\hbar]]$-module, which is also a left (or right) $\mathbb{C}[[\hbar]]$-linear module over an algebra $A_\hbar$ which is supposed to be also free as $\mathbb{C}[[\hbar]]$-module. Then if the specialization $K_{\hbar=0}$ is a free module over the specialization $A_{\hbar=0}$, $K_\hbar$ is a free left (right) $A_\hbar$-module;*

(2) suppose $K^\bullet_\hbar$ is a complex of free $\mathbb{C}[[\hbar]]$-modules ($\deg\hbar=0$) with $\mathbb{C}[[\hbar]]$-linear differential. Suppose that the i-th cohomology (for some i) of the specialization $K^\bullet_{\hbar=0}$ is zero. Then the i-th cohomology of $K^\bullet_\hbar$ is also zero.

Proof. The both statements are standard; let us recall the proofs for convenience of the reader.

(1): Suppose the contrary, then for some $k_i(\hbar)\in K_\hbar$ and some $a_i(\hbar)\in A_\hbar$ one has $\sum_i a_i(\hbar)\cdot k_i(\hbar)=0$. Let N be the minimal power of $\hbar$ in the equation. Then we can divide the equation over $\hbar^N$ and the equation still holds, because the both $A_\hbar$ and $K_\hbar$ are free $\mathbb{C}[[\hbar]]$-modules. Then we reduce over $\hbar$ and get a nontrivial linear equation for the $A_{\hbar=0}$-module $K_{\hbar=0}$ which contradicts to the assumption.

(2): Let $k_i(\hbar)$ be an i-cicycle in $K^\bullet_\hbar$, we should prove that it is a coboundary. Suppose $\hbar^N$ is the minimal power of $\hbar$ in $k_i(\hbar)$, then we divide over $\hbar^N$. We get again a cocycle, because the differential is $\mathbb{C}[[\hbar]]$-linear and $K^\bullet_\hbar$ is a free $\mathbb{C}[[\hbar]]$-module. Denote this new cocycle again by $k_i(\hbar)$. Then its zero degree in $\hbar$ term is a cocycle in the reduced complex $K_{\hbar=0}$ and we can kill it by some coboundary. Then substract and divide over minimal power of $\hbar$, ans so on. □

7.3 The algebras $S(V^*)_\hbar$ and $\Lambda(V)_\hbar$ are Koszul

We start to prove the Theorem. Prove firstly that the algebras $S(V^*)_\hbar$ and $\Lambda(V)_\hbar$ are graded quadratic and Koszul. The first statement is proven analogously to the speculation in Section 0.2. The difference that here in a universal deformation quantization we may have more general graphs than in the Kontsevich's quantization, namely non-connected graphs and graphs with simple loops. But it does not change the proof.

Let us prove that these algebras are Koszul. Consider the case of $S(V^*)_\hbar$, the proof for $\Lambda(V)_\hbar$ is analogous.

By Lemma 1.2.5, it is necessary to prove that the Koszul complex $K^\bullet_\hbar = \big(S(V^*)_\hbar \otimes_{\mathbb{C}[[\hbar]]} \mathrm{Hom}_{\mathbb{C}[[\hbar]]}(S(V^*)^!, \mathbb{C}[[\hbar]]), d_{\text{Koszul}}\big)$ is acyclic in all degrees except degree 0. The complex $K^\bullet_\hbar$ is clearly a complex of free $\mathbb{C}[[\hbar]]$-modules with a $\mathbb{C}[[\hbar]]$-linear differential. We are in situation of Lemma 7.2(2), because the specialization at $\hbar = 0$ gives clearly the Koszul complex for the usual algebra $S(V^*)$ which is known to be acyclic. We are done.

7.4 We continue to prove the Main Theorem

Now we prove the only non-trivial part of the Theorem, that the algebras $S(V^*)_\hbar$ and $\Lambda(V)_\hbar$ are Koszul dual.

Consider the diagram (49). It is a diagram of G_∞ quasi-isomorphisms which is known to be homotopically commutative, see Theorem 4.4. Then we can construct a G_∞ quasi-isomorphism $\mathcal{F}\colon T_{poly}(V) \to \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K))$ dividing the diagram into two commutative triangles. Restrict $\mathcal{F}$ to its L_∞ part. Then we get an L_∞ quasi-isomorphism $\mathcal{F}\colon T_{poly}(V)\colon \mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K))$. Here $A = S(V^*)\otimes\mathbb{C}[[\hbar]]$, $B = \Lambda(V)\otimes\mathbb{C}[[\hbar]]$, etc.

Then this L_∞ map $\mathcal{F}$ attaches to the Maurer-Cartan solution $\alpha \in T_{poly}(V)$ (our quadratic Poisson bivector field) a solution of the Maurer-Cartan equation in $\mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K))$, by formula

$$\mathcal{F}_*(\alpha) = \hbar\mathcal{F}_1(\alpha) + \frac{1}{2}\hbar^2\mathcal{F}_2(\alpha\wedge\alpha) + \dots \tag{82}$$

What a solution of the Maurer-Cartan equation in $\mathrm{Hoch}^\bullet(\mathsf{cat}(A,B,K))$ means im more direct terms?

It consists from the following data:

(i) A deformation quantization $A_\hbar$ of the algebra $A = S(V^*)\otimes\mathbb{C}[[\hbar]]$;

(ii) a deformation quantization $B_\hbar$ of the algebra $B = \Lambda(V)\otimes\mathbb{C}[[\hbar]]$;

(iii) a deformed differential on the Koszul complex $K^\bullet(S(V^*))\otimes\mathbb{C}[[\hbar]]$, we denote the deformed complex by $K^\bullet_\hbar$;

(iv) a structure of a $B_\hbar$-$A_\hbar$-bimodule on $K^\bullet_\hbar$.

The crucial point is the following Lemma:

Lemma. *The algebra $A_\hbar$ is gauge equivalent (and therefore isomorphic) to the algebra $S(V^*)_\hbar$ from Section 7.1, and the algebra $B_\hbar$ is gauge equivalent to $\Lambda(V)_\hbar$.*

Proof. It follows from the commutativity of the diagram (49), and from Lemma 5.2. □

7.5 We finish to prove the Main Theorem

From Lemma 7.4, it is enough to prove that the quadratic graded algebras $A_\hbar$ and $B_\hbar$ are Koszul dual to each other. For this (because the both algebras are Koszul) it is enough to prove that $B_\hbar = A_\hbar^!$. Let us prove it.

The complex $K_\hbar$ is a complex of $B_\hbar$-$A_\hbar$ modules. As complex of $A_\hbar$-modules, it is free by Lemma 7.2(1). By Lemma 7.2(2), it is a free $A_\hbar$-resolution of the module $\mathbb{C}[[\hbar]]$. Therefore, we can use $K_\hbar$ for the computation of the Koszul dual algebra:

$$(A_\hbar)^! = \mathrm{RHom}_{Mod-A_\hbar}(K_\hbar, K_\hbar) \tag{83}$$

On the other hand, from the bimodule structure (see (iv) in the list in Section 7.4), we have an *algebra* homomorphism

$$B_\hbar \to \mathrm{RHom}_{Mod-A_\hbar}(K_\hbar, K_\hbar) \tag{84}$$

We only need to prove that it is an isomorphism. It again follows from the facts that the both sides are free $\mathbb{C}[[\hbar]]$-modules (for the l.h.s. it is clear, for the r.h.s. it follows from (83)), and that the specialization of (84) at $\hbar = 0$ is an isomorphism.

Theorem 7.1 is proven. □

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