Logarithmic De Rham complexes and vanishing theorems

Hélène Esnault[1] and Eckart Viehweg[2]

[1] Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3, Federal Republic of Germany
Supported by "Deutsche Forschungsgemeinschaft/
Heisenberg Programm"

[2] Universität - Gesamthochschule - Essen
FB6, Mathematik
Universitätsstr. 3
D-4300 Essen 1, Federal Republic of Germany

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Classically the vanishing of cohomology groups of a compact complex Kähler manifold X with values in certain locally free sheaves M is proved by studying positivity properties of the curvature form of a differentiable connection on M compatible with the complex structure of X (e.g. [7]). If the Chern classes of M are non trivial, the connection is neither holomorphic nor integrable. Therefore, trying to replace the differentiable connection with non trivial curvature by an integrable holomorphic connection ∇, one has to choose a "boundary divisor" D and to allow ∇ to have poles along D. We will always assume D to be a normal crossing divisor and ∇ to have at most logarithmic poles along D. Since ∇ is non singular and integrable on $U = X - D$, the positivity properties have to be replaced by topological properties of U together with conditions on the boundary behaviour of (M,∇).

This point of view is supported by a construction due to J.L. Verdier and independently to the first author (and probably to many others, too), describing the Atiyah class and Chern classes of M in terms of the restriction of ∇ to the boundary D. For the reader's convenience we give the exact statement and the proof in appendix B at the end of this article.

Because of the restriction made on the poles of ∇ along D one has at disposal the theory of P. Deligne on differential

equations with regular singular points ([3]) and in fact his Lecture Notes was the main source of inspiration of our work:
Let V be the local constant system on U defined by sections of $\mathcal{M}_{|U}$, flat with respect to ∇ and $j:U \hookrightarrow X$ be the inclusion. $(\mathcal{M},\nabla)$ is equipped with it's logarithmic De Rham complex $DR_D\mathcal{M} = \Omega^{\bullet}\langle D\rangle \otimes_{\mathcal{O}_X} \mathcal{M}$, which is over U quasi-isomorphic to V. If the monodromies of V around the components of D do not have 1 as eigenvalue the complexes $j_! V$, $Rj_* V$ and $DR_D\mathcal{M}$ are all quasi isomorphic and the hypercohomology of $DR_D\mathcal{M}$ is the same as the cohomology or as the cohomology with compact support of U with values in V. The spectral sequence $E_1(\mathcal{M})$ associated to the "filtration bête" of the logarithmic De Rham complex describes the hypercohomology in terms of the cohomology of the coherent $\mathcal{O}_X$ - modules $\Omega^p\langle D\rangle \otimes \mathcal{M}$. If in addition the spectral sequence degenerates in E_1, topological vanishing theorems on U imply global coherent vanishing theorems on X.

In general it is quite difficult to decide when the spectral sequence degenerates (see (2.6)). Using Deligne's theory of mixed Hodge structures ([4]), this is true however for sheaves arising from finite covers of X, branched along D. The main examples of sheaves arising by this construction are invertible sheaves $\mathcal{L}^{-1}$, where $\mathcal{L}$ is ample or more generally related to integral parts of effective $\mathbb{Q}$-divisors with support in D. If $U = X - D$ is affine, the degeneration of $E_1(L^{-1})$ implies immediately a general global vanishing theorem containing as special cases:

- Kodaira-Nakano's vanishing theorem (2.10)
- Bogomolov-Sommese's vanishing theorem (2.11) and
- Grauert-Riemenschneider's vanishing theorem as well as its generalization due to Y. Kawamata and the second author ((2.12) and (2.13)).

If one drops the assumption on U , the degeneration of $E_1(M)$ implies the vanishing of certain natural restriction maps in cohomology.
Applied to the sheaves L^{-1} considered above one obtains the vanishing of the restriction maps of twisted differential forms in the cohomology ((3.2) and (3.3)). Especially one gets an improvement of the Kollâr-Tankeev vanishing theorems (3.5) as a direct interpretation of the degeneration of the spectral sequence.

In § 1 we recall properties of sheaves with logarithmic connections and their De Rham complexes. The condition that the monodromies of V do not have 1 as eigenvalue implies that the minimal and the maximal extensions of V coincide, as we prove in (1.6).
This also follows from a much stronger statement proved in appendix A to § 1. This result - superfluous for the vanising theorems considered - says that the Verdier dual of

the complex $DR_D\mathcal{M}$ is quasi-isomorphic to $DR_D(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X) \otimes \mathcal{O}_X(-D_{red}))$. This duality is quite similar to the corresponding duality for $\mathcal{D}$-modules, which is one of the key-points in the proof of the Riemann-Hilbert correspondence for $\mathcal{D}$-modules (Z. Mebkhout, M.Kashiwara, see for example [2]). We hope that the appendix explains why (2.6) is true and that it casts some light on the statements of § 2. We believe that the language of $\mathcal{O}_X$-coherent logarithmic $\mathcal{D}$-modules is a quite adequate tool in algebraic geometry, and we hope that the duality is useful for different purposes as well.

In § 2 we give the cohomological interpretation of (1.6), provided the spectral sequence $E_1(\mathcal{M})$ degenerates and $U = X - D$ is affine. We discuss examples where all three assumptions hold and state and prove the vanishing theorems mentioned.

§ 3 contains the applications so the cohomology of restriction maps, useful if U is not affine. The main observation is that the conditions posed on the monodromy of V imply that the residue maps obtained from ∇ are surjective on each component of D and can be identified with the natural restriction map.

After finishing a first version of this paper we learned that our approach towards global vanishing theorems is close in spirit to methods used by J. Kollár [10] and related to results by M. Saito (see also (2.6)).

The theory of $\mathcal{D}$-modules and perverse sheaves, even if they do not appear explicitly, strongly inspired this article. We had a small seminar on those subjects at the MPI at Bonn and we thank our friends for their active participation which certainly helped us to understand the point of view explained here.

We like to thank P. Deligne for a general and helpful conversation, M. Saito for answering questions on $\mathcal{D}$-modules and K. Timmerscheidt for answering those on the analytic part of the classical Hodge theory.
J.L. Verdier considered independently and for different purposes geometric applications of logarithmic $\mathcal{D}$-modules. It is a pleasure for both of us to thank him for telling us about his ideas.

§ 1 Logarithmic De Rham complexes and extensions of local constant systems

In this section we recall the basic properties of sheaves with logarithmic connections, their monodromy and the extensions of the corresponding local constant systems, as developed by Deligne in [3]. The reader mainly interested in the proof of the global vanishing theorems is invited to read just up to (1.5) and then to read § 2.

(1.1) Throughout this article X denotes either a proper algebraic manifold over $\mathbb{C}$ or a compact complex analytic manifold of dimension n. $\mathcal{O}_X$ denotes the sheaf of algebraic functions or the sheaf of analytic functions in the second case. In § 1 – starting from (1.2) – and in appendix A we have to restrict ourselves to the complex analytic case.

Let $D = \sum_{i=1}^{s} \nu_i D_i$ be an effective normal crossing divisor on X, i.e. an effective divisor whose components are non singular and meet transversally. We write $j: U = X - D_{red} \to X$ for the open embedding. We consider locally free sheaves M of $\mathcal{O}_X$ - modules endowed with a holomorphic connection ∇ with logarithmic poles along D

$$\nabla: M \longrightarrow \Omega_X^1\langle D\rangle \otimes M$$

as defined by Deligne ([3], II § 3). Such a pair (M,∇) will be called a <u>logarithmic connection</u> along D. It induces

$$\nabla_p : \Omega_X^p\langle D\rangle \otimes M \to \Omega_X^{p+1}\langle D\rangle \otimes M$$

by the rule $\nabla_p(\omega \otimes m) = d\omega \otimes m + (-1)^p \cdot \omega \wedge \nabla m$. We assume ∇ to be integrable, i.e.: $\nabla_p \circ \nabla_{p+1} = 0$. The complex $\Omega_X^{\bullet}\langle D\rangle \otimes M$ obtained is denoted by $DR_D M$ and called the <u>logarithmic De Rham complex</u> of (M,∇).

(1.2) From now on all sheaves are considered in the classical topology and $\mathcal{O}_X$ denotes the sheaf of holomorphic functions.

The Riemann-Hilbert correspondence, proved by Deligne [3] says:

a) The complex $DR_D M|_U$ is exact at $p > 0$ and the flat sections form a local constant system $V = \operatorname{Ker} \nabla_{|U}$. In other words, the inclusion $V \to DR_D M|_U$ is a quasi-isomorphism of complexes.

b) For any local constant system V on U denote $M_U = \mathcal{O}_U \otimes_{\mathbb{C}} V$ and $\nabla_U = d \otimes_{\mathbb{C}} \mathrm{id} : M_U \to \Omega_U^1 \otimes_{\mathbb{C}} V = \Omega_U^1 \otimes_{\mathcal{O}_X} M_U$. Then there exists a locally free sheaf M on X and a connection $\nabla : M \to \Omega_X^1\langle D\rangle \otimes M$ with logarithmic poles, extending (M_U, ∇_U).

c) The extension (M,∇) in b) depends on the choice of the logarithm of the monodromy: If one chooses Γ_i such that $\exp(-2 \cdot \sqrt{-1} \cdot \pi \cdot \Gamma_i) = \gamma_i$ is the monodromy of V around D_i and such that the real part of the eigenvalues of Γ_i lies in $[0,1[$, then (M,∇) is uniquely determined and characterized by $\operatorname{Res}_i(\nabla) = \Gamma_i$. Here $\operatorname{res}_i : \Omega_X^1\langle D\rangle \otimes M \to \mathcal{O}_{D_i} \otimes M$ is the Poincaré residue and $\operatorname{Res}_i(\nabla) = \operatorname{res}_i \circ \nabla$ is the residue of ∇

along D_i (see [3], II; 5.3 and 5.4). In this case we call $(\mathcal{M},\nabla)$ the canonical extension of $\mathbb{V}$.

d) The canonical extension is compatible with covers in the following sense:
For any cover $\tau: X' \to X$ ramified only along D, with X' smooth, the inverse image of the canonical extension is contained in the canonical extension of $\tau^{-1}\mathbb{V}$. Moreover, if $(\mathcal{M}_1,\nabla_1)$ is any other extension of $\mathbb{V}$ with this property, then the canonical extension $\mathcal{M}$ contains $\mathcal{M}_1$.

e) $\mathbb{V}$ being uniquely determined by $(\mathcal{M},\nabla)$ we will omit refering to $\mathbb{V}$ and talk about the monodromy of $(\mathcal{M},\nabla)$ and will call $(\mathcal{M},\nabla)$ the canonical extension if it is the canonical extension of $\mathrm{Ker}\ \nabla|_U$.

(1.3) If $(\mathcal{M},\nabla)$ is a logarithmic connection along D and $B = \sum_{i=1}^{s} b_i D_i$ any divisor supported in D, the Leibnitz rule for ∇ implies that ∇ induces a connection ∇^B of $\mathcal{M}(B) = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(B)$, logarithmic along D. The residues of the connections along D_i are related by

$$\mathrm{Res}_i(\nabla^B) = \mathrm{Res}_i(\nabla) - b_i \cdot \mathrm{id}\ .$$

Hence, if $\mathrm{Res}_i(\nabla)$ has no integer as eigenvalue, then $\mathrm{Res}_i(\nabla^B)$ can not have integers as eigenvalues either. Moreover, if B is effective and if $\mathrm{Res}_i(\nabla)$ has no strictly positive integer as eigenvalue, the same is true for $\mathrm{Res}_i(\nabla^B)$. By [3], II; 3.13 and 3.14 the property that no strictly positive integer occurs as eigenvalue of the residues of the connection ∇ implies that $DR_D M$ and $Rj_*(DR_D M_{|U})$ are quasi-isomorphic. Therefore one obtains:

(1.4) Lemma. Let (M,∇) be a logarithmic connection along D and $V = \text{Ker}(\nabla_{|U})$. Let B be any divisor supported on D. Assume that one of the following conditions is satisfied:

a) (M,∇) is the canonical extension (or - more generally - for all i, $\text{Res}_i(\nabla)$ has no strictly positive integer as eigenvalue) and B is effective.

b) For $i = 1 \dots s$, $\text{Res}_i(\nabla)$ has no integers as eigenvalue.

Then $DR_D M$, $DR_D M(B)$, $Rj_*(DR_D M_{|U})$ and Rj_*V are quasi-isomorphic.

As well known, (1.4) together with Serre's vanishing theorem imply the topological vanishing theorem (needed in § 2):

(1.5) Corollary. If U is an affine manifold of dimension n and V a local constant system on U, then

$$H^k(U,V) = 0 \quad \text{for} \quad k > n.$$

If moreover the mondoromies γ_i of V around D_i (for $i = 1 \dots s$) do not have 1 as eigen value, then

$$H^k(U,V) = 0 \quad \text{for} \quad k \neq n.$$

Proof: We may choose X to be a projective compactification satisfying the assumptions made in (1.1), $(\mathcal{M},\nabla)$ to be the canonical extension of V and B to be a very ample effective divisor supported in D. Replacing B by some multiple we may assume that

$$H^q(X,\Omega_X^p\langle D\rangle \otimes \mathcal{M}(B)) = 0 \quad \text{for} \quad q > 0.$$

Looking at the spectral sequence associated to $DR_D\mathcal{M}(B)$ with the "filtration bête" (see [3], 1.4) and converging to

$$H^k(U,V) = \mathbb{H}^k(X,Rj_*V) = \mathbb{H}^k(X,DR_D\mathcal{M}(B))$$

one obtains

$$H^k(U,V) = 0 \quad \text{for} \quad k > n.$$

Assume now that the γ_i do not have 1 as eigenvalue. If one chooses B to be a sufficiently high multiple of an ample divisor, we have

$$H^q(X,\Omega_X^p\langle D\rangle \otimes \mathcal{M}^\vee(B - D_{red})) = 0 \quad \text{for} \quad q > 0.$$

Here $\mathcal{M}^\vee$ is the $\mathcal{O}_X$ - module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{O}_X)$. By Serre-duality one has

$$H^q(X,\omega_X \otimes (\Omega_X^p\langle D\rangle\otimes \mathcal{M}^\vee(B - D_{red}))^\vee) =$$
$$= H^q(X,\Omega_X^{n-p}\langle D\rangle \otimes \mathcal{M}(-B)) = 0$$

for $q < n$. By (1.4.6) the same spectral sequence as considered above converges to $H^k(U,V)$ and we obtain $H^k(U,V) = 0$ for $k < n$.

The second part of (1.5) is not surprising. We will see below the assumptions made imply that $H^k(X,Rj_*V)$ is the same as the cohomology with compact support $H^k_c(U,V)$. Rj_*V is "the maximal extension" of V to X and its cohomology sheaf in degree zero, j_*V, contains the "minimal extension" $j_!V$, i.e. the sheaf of abelian groups obtained by extending V by zero. The functor $j_!$ is exact and $H^k_c(U,V) = H^k(X,j_!V)$.

(1.6) Lemma. Let V be a local constant system on U such that the monodromies γ_i of V around D_i (for $i = 1 \ldots s$) do not have 1 as eigenvalue. Then $j_!V$ and Rj_*V are quasi-isomorphic. Especially $j_!V = j_*V$ and

$$R^q j_*V = 0 \quad \text{for} \quad q > 0.$$

Proof: Since we have a natural morphism $j_!V \to Rj_*V$, it is enough to prove (1.6) locally.

Let W be a small neighbourhood of a point on D. We have to show that $R^p j_*V(W) = H^p(U \cap W,V) = 0$ for all p. If $0 \to V' \to V \to V'' \to 0$ is an exact sequence of local systems on U and if V' and V'' have no cohomology on $U \cap W$, the same holds for V.

Choosing W small enough we may assume that
$W = \prod_{j=1}^{n} \Delta_j$ and $U \cap W = \prod_{j=1}^{r} \Delta_j^* \times \prod_{j=r+1}^{n} \Delta_j$ where Δ_j is a small disk and Δ_j^* the punctured disk. Since the monodromies γ_j around the components of $D \cap W$ commute we can find a local subsystem V' of V stable by the γ_j such that the cokernel is a local system of lower rank. By induction on the rank of V we are reduced to the case $\mathrm{rk}V = 1$.

We may write $V = p_1^{-1}V_1 \otimes \ldots \otimes p_r^{-1}V_r$ where $p_j : U \cap W \to \Delta_j^*$ is the j-th projection and V_j the local constant system on Δ_j^* corresponding to the representation of γ_j on a one dimensional vector space L.

By the Künneth formula we just have to show that $H^k(\Delta_j^*, V_j) = 0$ for $k = 0$ and $k = 1$. We may replace Δ_j^* by its boundary $S^1 = \partial\Delta_j^*$ and we parametrize S^1 by $e^{2i\pi \cdot t}$, $t \in \mathbb{R}$.
Take $U_1 = \{e^{2i\pi\cdot t}, \ t \in]0,1[\}$ and
$U_2 = \{e^{2i\pi\cdot s}, \ s \in]-\frac{1}{2}, \frac{1}{2}[\}$ as cover of S^1.

Then U_1, U_2 and the two connected components W^+ and W^- of $U_1 \cap U_2$ are simply connected. The coordinate change from U_1 to U_2 is

$$W^+ \amalg W^- \longrightarrow W^+ \amalg W^-$$

$$t \longmapsto s = \begin{cases} t & \text{if } t \in W^+ \\ t-1 & \text{if } t \in W^- \end{cases} .$$

The Čech cohomology with values in V_j is computed by the

the cohomology of the complex

$$0 \to L_{U_1} \times L_{U_2} \xrightarrow{d} L_{W+} \times L_{W-} \longrightarrow 0$$

$$(\ell_1, \ell_2) \longrightarrow (\ell_1 - \ell_2, \ell_1 - \gamma_j \ell_2) .$$

However, if $\gamma_j \neq 1$, d is an isomorphism.

Lemma (1.6) gives the following improvement of (1.4, b)

(1.7) Corollary. Let (M,Δ) be a logarithmic connection along D and $V = \mathrm{Ker}\,(\nabla_{|U})$. If $\mathrm{Res}_i(\nabla)$ has no integer as eigenvalue (for $i = 1\ldots s$) then $DR_D(M)$, $j_!(DR_D(M)_{|U})$, Rj_*V and $j_!V$ are quasi-isomorphic.

Remark: In Appendix A we will consider for any logarithmic connection (M,∇) the dual connection on $M^\vee = \mathcal{H}om_{\mathcal{O}_X}(M,\mathcal{O}_X)$ and the logarithmic connection $(M^\vee(-D_{red}), \nabla^\vee)$. If $\mathbb{D}$ denotes the Verdier-duality functor, we will see that

$$\mathbb{D}\,(DR_D M) = DR_D M^\vee(-D_{red}) .$$

Since $\mathbb{D}(Rj_* V) = j_! V^\vee$, for $V^\vee = \mathcal{H}om_{\mathbb{C}}(V,\mathbb{C})$, we obtain from (1.4,a):

(1.8) If (M,∇) is the canonical extension, then $DR_D M^\vee(-D_{red})$ is quasi isomorphic to $j_! V^\vee$.

Moreover the duality together with (1.4 , b) gives another proof of (1.6). In fact, if V has no monodromy with 1 as eigenvalue, the same is true for $V^{\vee}$. Hence $Rj_*V \cong DR_D M$ and $Rj_*V^{\vee} \cong DR_D M^{\vee}(-D_{red})$ and we obtain the quasi-isomorphisms

$$Rj_*V \cong DR_D M \cong \mathbb{D}\, DR_D M^{\vee}(-D_{red}) \cong \mathbb{D}\, Rj_*V^{\vee} = j_!V.$$

Appendix A: Duality for logarithmic De Rham complexes

We keep the notations and assumptions introduced in (1.1), except that X is a (not necessarily compact) analytic manifold and - to simplify the notation - that D is reduced.

(A.1) Let $D^b_c(X)$ be the derived category of bounded complexes of $\mathbb{C}$-sheaves with constructible cohomology. The Verdier dual is given by the functor

$$\mathbb{D} : D^b_c(X) \longrightarrow D^b_c(X)$$
$$F^\bullet \longmapsto \mathbb{D}(F^\bullet) = R\, \mathcal{H}om_{\mathbb{C}}(F^\bullet, \mathbb{C}_X) .$$

For an $\mathcal{O}_X$ - module M we write $M^\vee = \mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)$ and, if (M,∇) is a logarithmic connection along D, $\nabla^\vee$ denotes the dual connection. The main result of this appendix is:

(A.2) <u>Proposition.</u> <u>In</u> $D^b_c(X)$ <u>one has</u>

$$DR_D M \cong \mathbb{D}(DR_D M^\vee(-D))$$

The arguments needed to prove (A.2) are quite similar to a proof of the corresponding statement for D_X - modules, due to J. Bernstein ([2], §5). We recall some notations from the theory of $\mathcal{D}_X$ - modules. Details can be found in [2]:

(A.3) $\mathcal{D}_X$ denotes the sheaf of holomorphic differential operators on X and $\mathcal{D}_X\langle -D\rangle$ the subalgebra of $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and $T_X\langle -D\rangle = (\Omega^1_X\langle D\rangle)^v$ the sheaf of vectorfields preserving $\mathcal{O}_X(-D)$.

Locally we choose a parameter system of X such that D is given by $x_1 \cdot \ldots \cdot x_r = 0$.

Let $\partial_1, \ldots, \partial_n$ be the vectorfields orthogonal to $x_1, \ldots, x_n$ and define

$$\delta_i = \begin{cases} x_i \cdot \partial_i & \text{for } 1 \leq i \leq r \\ \partial_i & \text{for } r+1 \leq i \leq n . \end{cases}$$

δ_i is dual to $\frac{dx_i}{x_i}$ $(1 \leq i \leq r)$ or dx_i $(r+1 \leq i \leq n)$, and $T_X\langle -D\rangle$ is generated by $\delta_1, \ldots, \delta_n$.

The logarithmic connection ∇ on M gives M the structure of a left $\mathcal{D}_X\langle -D\rangle$ module and for $m \in M$,

$$\nabla m = \sum_{i=1}^{r} \delta_i . m \frac{dx_i}{x_i} + \sum_{i=r+1}^{u} \delta_i \cdot m \, dx_i .$$

(A.4) <u>Claim. Let A and B be two left $\mathcal{D}_X\langle -D\rangle$- modules. Then</u>

a) $Hom_{\mathcal{D}_X\langle -D\rangle}(A,B) \cong Hom_{\mathcal{D}_X\langle -D\rangle}(A(D), B(D))$

b) <u>One has an isomorphism</u>

$$Hom_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X, Hom_{\mathcal{O}_X}(A,B)) \xrightarrow{\sim} Hom_{\mathcal{D}_X\langle -D\rangle}(A,B)$$

<u>given by</u> $\varphi \longmapsto \varphi(1)$.

<u>Proof:</u> a) If $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of $\mathcal{D}_X\langle -D\rangle$ modules we define $\varphi' : \mathcal{A}(D) \to \mathcal{B}(D)$ by $\varphi'\left(\frac{a}{x_1 \cdot \ldots \cdot x_r}\right) = \frac{1}{x_1 \cdot \ldots \cdot x_r} \cdot \varphi(a)$. This is clearly $\mathcal{O}_X$-linear and we just have to verify the compatibility with δ_i.

If $i \leq r$ then $\delta_i\left(\frac{a}{x_1 \cdot \ldots \cdot x_r}\right) = \frac{\delta_i a}{x_i \cdot \ldots \cdot x_r} - \frac{a}{x_1 \cdot \ldots \cdot x_r}$

and $\varphi'\left(\delta_i \frac{a}{x_1 \cdot \ldots \cdot x_r}\right) = \varphi'\left(\frac{\delta_i a}{x_1 \cdot \ldots \cdot x_r}\right) - \varphi'\left(\frac{a}{x_1 \cdot \ldots \cdot x_r}\right)$

$$= \frac{\delta_i \varphi(a)}{x_1 \cdot \ldots \cdot x_r} - \frac{\varphi(a)}{x_1 \cdot \ldots \cdot x_r} = \delta_i \varphi'\left(\frac{a}{x_1 \cdot \ldots \cdot x_r}\right)$$

The case $i > r$ is similar.

Conversely, if $\varphi' \in \mathcal{H}om_{\mathcal{D}\langle -D\rangle}(\mathcal{A}(D), \mathcal{B}(D))$ we obtain $\varphi \in \mathcal{H}om_{\mathcal{D}\langle -D\rangle}(\mathcal{A}, \mathcal{B})$ by restriction.

b) As for connections the $\mathcal{D}_X\langle -D\rangle$ - module structure on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B})$ is given by $(\delta_i \Psi)(a) = \delta_i(\Psi(a)) - \Psi(\delta_i a)$ for $\Psi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B})$. The morphism $\varphi(1)$ is $\mathcal{O}_X$ - linear and $\delta_i(\varphi(1)(a)) - \varphi(1)(\delta_i a) = (\delta_i(\varphi(1)))(a) = (\varphi(\delta_i 1))(a) = 0$. Hence $\varphi(1)$ is $\mathcal{D}_X\langle -D\rangle$ linear. On the other hand, if $\eta \in \mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{A}, \mathcal{B})$ we define $\varphi \in \mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B}))$ by $\varphi(\lambda) = \lambda \cdot \eta$. In fact, φ is $\mathcal{D}_X\langle -D\rangle$ linear since $\varphi(\delta_i \lambda)(a) = (\delta_i \lambda) \cdot \eta(a) = \delta_i(\lambda \cdot \eta(a)) - \lambda \cdot \delta_i(\eta(a)) =$ $= \delta_i(\varphi(\lambda)(a)) - \varphi(\delta_i a)$ and $\varphi(1) = \eta$.

$\mathcal{O}_X$ has a locally free resolution as $\mathcal{D}_X\langle -D\rangle$- module by the Koszul complex

$$0 \to \mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \overset{n}{\Lambda}\, T_X\langle -D\rangle \xrightarrow{d^{n-1}} \mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \overset{n-1}{\Lambda}\, T_X\langle -D\rangle \xrightarrow{d^{n-2}} \cdots$$

$$\cdots \xrightarrow{d^1} \mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} T\langle -D\rangle \xrightarrow{d^0} \mathcal{D}_X\langle -D\rangle \to 0$$

where

$$d^p(\rho \otimes (\delta_{i_1} \wedge \ldots \wedge \delta_{i_{p+1}})) = \sum_{j=1}^{p+1} (-1)^{j-1} \rho\cdot\delta_{i_j} \otimes (\delta_{i_1} \wedge \ldots \wedge \hat{\delta}_{i_j} \wedge \ldots \wedge \delta_{i_{p+1}})$$

(A.5) <u>Claim</u>.

$$DR_D M \cong R\, Hom_{\mathcal{D}\langle -D\rangle}(\mathcal{O}_X, M) \cong Hom_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \overset{\bullet}{\Lambda} T_X\langle -D\rangle, M) \cong$$

$$\cong Hom_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \Lambda T_X\langle -D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(D), M(D))$$

<u>Proof:</u> The last quasi-isomorphism follows from (A.4. a)). The Koszulcomplex is a locally free resolution and therefore one obtains the secondquasi-isomorphism. Since

$$Hom_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \overset{p}{\Lambda}\, T_X\langle -D\rangle, M) \cong \Omega_X^p\langle D\rangle \otimes_{\mathcal{O}_X} M$$

we just have to verify that the differentials d_p of the third complex are the same as ∇_p. For simplicity we assume $p = 0$.

Let $m = \varphi(1)$ for $\varphi \in Hom_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle, M)$.

One has $d_0 m = \sum_{i=1}^{r} n_i \frac{dx_i}{x_i} + \sum_{i=r+1}^{n} n_i\, dx_i$ for

$n_i = (\varphi \circ d)(\delta_i) = \varphi(\delta_i) = \delta_i \varphi(1) = \delta_i m$. By definition of the $\mathcal{D}_X\langle -D\rangle$- module structure on M we have $d_0 m = \nabla m$.

(A.6) <u>Claim.</u> $DR_D M^\vee \cong R\,\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(M,\mathcal{O}_X)$

<u>Proof:</u> By taking $I^\bullet$ to be an injective resolution of $\mathcal{O}_X$ over $\mathcal{D}_X\langle -D\rangle$ we obtain $R\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(M,I^\bullet) = R\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(M,\mathcal{O}_X)$ and b (A.4,b) this is quasi-isomorphic to $\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X}(M,I^\bullet))$. Since M is locally free and $\mathcal{O}_X$ quasi-isomorphic to $I^\bullet$, $\mathcal{H}om_{\mathcal{O}_X}(M,I^\bullet) \cong M^\vee \otimes_{\mathcal{O}_X} I^\bullet$ is an injective resolution of $M^\vee$. In fact, $M^\vee \otimes_{\mathcal{O}_X} I^\bullet$ is locally a direct sum of copies of $I^\bullet$ and $M^\vee = \mathcal{H}om_{\mathcal{O}_X}(M,\mathcal{O}_X) \cong R\mathcal{H}om_{\mathcal{O}_X}(M,\mathcal{O}_X) \cong R\mathcal{H}om_{\mathcal{O}_X}(M,I^\bullet)$. Therefore $\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X}(M,I^\bullet)) = R\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,\mathcal{H}om_{\mathcal{O}_X}(M,I^\bullet)) =$ $=R\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{O}_X,M^\vee)$ and using (A.5) we are done.

(A.7) <u>There is a natural pairing, non degenerate over</u> U

$$DR_D M \overset{\mathbb{L}}{\otimes}_{\mathbb{C}} DR_D\, M^\vee(-D) \to \mathbb{C} \quad .$$

<u>Proof:</u> Again, let $I^\bullet$ be an injective resolution of $\mathcal{O}_X$ as $\mathcal{D}_X\langle -D\rangle$- module. Using (A.5) and (A.6) we obtain the pairing

$$\mathrm{DR}_D\mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathbb{C}} \mathrm{DR}_D\mathcal{M}^{\vee}(-D)$$

$$\wr|$$

$$\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \overset{\bullet}{\Lambda}\, T_X\langle -D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(D), \mathcal{M}(D)) \otimes_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{M}(D), I^{\bullet})$$

$$\downarrow$$

$$\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \overset{\bullet}{\Lambda}\, T_X\langle -D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(D), I^{\bullet})$$

$$\wr|$$

$$\mathcal{H}om_{\mathcal{D}_X\langle -D\rangle}(\mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \overset{\bullet}{\Lambda}\, T_X\langle -D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(D), \mathcal{O}_X)$$

The last sheaf is by scalar extension isomorphic to

$$\mathcal{H}om_{\mathcal{D}_X}((\mathcal{D}_X \otimes_{\mathcal{D}_X\langle -D\rangle} \mathcal{D}_X\langle -D\rangle) \otimes_{\mathcal{O}_X} \overset{\bullet}{\Lambda}\, T_X\langle -D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}(D),\ \mathcal{O}_X)$$

In fact, if φ is a $\mathcal{D}_X\langle -D\rangle$ linear morphism

$$\varphi : \mathcal{D}_X\langle -D\rangle \otimes_{\mathcal{O}_X} \overset{\bullet}{\Lambda}\, T_X\langle -D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \to \mathcal{O}_X$$

one can extend the operation of $\mathcal{D}_X\langle -D\rangle$ to $\mathcal{D}_X$ using the $\mathcal{O}_X$-linearity and writing $\partial_i = \frac{\delta_i}{x_i}$ for $i \leq r$.

The inclusion $\overset{\bullet}{\Lambda}\, T_X \to \overset{\bullet}{\Lambda} T_X\langle -D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}(D)$ gives a morphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{\bullet}{\Lambda}\, T_X\langle -D\rangle \otimes_{\mathcal{O}_X} \mathcal{O}_X(D), \mathcal{O}_X)$$

$$\downarrow$$

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \overset{\bullet}{\Lambda} T_X, \mathcal{O}_X) = R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{C}\ .$$

As a corollary we obtain:

(A.8) <u>There is a natural morphism, isomorphic over</u> U:

$$DR_D M \xrightarrow{\Phi} \mathbb{D}\, DR_D\, M^{\vee}(-D).$$

<u>Proof of (A.2):</u> If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of logarithmic connections along D and the morphism in (A.8) an isomorphism for M' and M'', it is an isomorphism for M as well.

Moreover the question whether Φ is an isomorphism can be answered locally.

As in [3], p. 83, one may assume that the coefficients of the connection in a neighbourhood of a given point are constant and that the connection matrix is triangular. We can find a subconnection M' of M, being of lower rank. Therefore we can argue by induction on the rank, and we may assume that the rank of M is one.

Choosing the neighbourhood small enough we may write $(X,D) \cong (X_1,D_1) \times (X_2,D_2)$ and M as $M = p_1^* M_1 \otimes p_2^* M_2$ where M_i is a rank one connection on X_i, logarithmic along D_i (See [3], p. 81). Then $DR_D M = p_1^* DR_{D_1} M_1 \otimes p_2^* DR_{D_2} M_2$ and since the Verdier duality is also compatible with products we are reduced to the case of curves:

Let X be a curve, $D \in X$ a point, given by $x = 0$, and M a rank one bundle whose connection has constant coefficients.

If the residue $\operatorname{Res} \nabla$ of M at D is given by multiplication with a , $\operatorname{Res} (\nabla^{v,-D})$ of $M^{v}(-D)$ is given by multiplication with $(1 - a)$. Hence changing the role of M and $M^{v}(-D)$ if necessary, we may assume that $1 - a \notin \mathbb{N} - \{0\}$. By [3], II, 3.14, $DR_D M^{v}(-D) \cong Rj_* V^{v}$ and $\mathbb{D}\, DR_D M^{v}(-D) \cong j_! V$ where $V = \operatorname{Ker} (\nabla_{|U})$. Therefore we just have to show that $-a \notin \mathbb{N}$ implies that $DR_D M = (0 \to M \to \Omega^1_X\langle D\rangle \otimes M \to 0)$ is quasi-isomorphic to $j_! V$.

Since $\operatorname{Res} \nabla : M \to M_{|D}$ is given by multiplication with $a \neq 0$, $\operatorname{Ker} \nabla \subset M(-D)$. Similarly , since $\operatorname{Res} (\nabla_{|M(-\ell\cdot D)}$ is given by multiplication with $a + \ell$ and is nontrivial for $\ell \geq 0$, one obtains

$$\operatorname{Ker} \nabla \subset \bigcap_{\ell\geq 0} M(-\ell\cdot D) \cap j_* V = j_! V .$$

On the other hand, if e is a generating section of M such that $\nabla(f\cdot e) = (x\cdot\partial + a)\, f\cdot e\, \frac{dx}{x}$ and

$$g\cdot e\cdot \frac{dx}{x} = \Big(\sum_\ell \lambda_\ell \cdot x^\ell\Big) \cdot e\, \frac{dx}{x} \in \Omega^1_X\langle D\rangle \otimes M \text{ , then } \sum_\ell \frac{\lambda\ell}{a+\ell}\; x^\ell$$

converges as well and

$$\nabla\Big(\Big(\sum_\ell \frac{\lambda_\ell}{a+\ell}\; x^\ell\Big)\cdot e\Big) = \Big(\sum_\ell \frac{\lambda_\ell}{a+\ell}\,(\ell . x^\ell + a . x^\ell)\Big) \cdot e\, \frac{dx}{x} = g\cdot e\cdot \frac{dx}{x} .$$

Hence ∇ is surjective and the quasi-isomorphism is established.

§ 2 The $E_1(M)$ -degeneration, applications to global vanishing theorems and examples

From now on we allow X to be algebraic over $\mathbb{C}$ or - as in § 1 - analytic. We keep the assumptions made in (1.1). Since we only deal with hypercohomology of logarithmic De Rham complexes over compact manifolds we can use GAGA theorems and switch from the algebraic case to the analytic case whenever it is necessary.

(2.1) On the logarithmic De Rham complex $DR_D M$ one considers the "filtration bête"

$$F^p: 0 \to \Omega_X^p\langle D\rangle \otimes M \to \Omega_X^{p+1}\langle D\rangle \otimes M \to \dots \to \Omega_X^n\langle D\rangle \otimes M$$

and the associated E_1-spectral sequence $(E_1^{pq}(M), d_1) = (H^q(X, \Omega_X^p\langle D\rangle \otimes M), H^q(\nabla))$, which converges to $\mathbb{H}^{p+q}(X, DR_D M)$ (see [4], 1.4).

By definition of a spectral sequence, the following two conditions are equivalent:

A) $\dim \mathbb{H}^k(X, DR_D M) = \sum_{p+q=k} \dim H^p(X, \Omega_X^p\langle D\rangle \otimes M)$

B) <u>The spectral sequence</u> $E_1^{pq}(M)$ <u>degenerates at</u> E_1.

<u>If</u> A <u>and</u> B <u>hold, we say that</u> (M, ∇) <u>satisfies "the</u> $E_1(M)$ <u>degeneration"</u>.

(2.2) <u>Main Lemma. Let</u> (M,∇) <u>be a logarithmic connection along</u> D <u>satisfying the</u> $E_1(M)$ <u>degeneration. Assume that</u> U <u>is affine. Then</u>

1) <u>if</u> $DR_D M$ <u>is quasi-isomorphic to</u> $Rj_* V$, <u>one has</u>
$H^q(X,\Omega^p_X\langle D\rangle\otimes M) = 0$ <u>for</u> $p + q > n$.

2) <u>if</u> $DR_D M$ <u>is quasi-isomorphic to</u> $j_! V$, <u>one has</u>
$H^q(X,\Omega^p_X\langle D\rangle\otimes M) = 0$ <u>for</u> $p + q < n$.

3) <u>if for</u> $i = 1 \dots s$ <u>the monodromy around</u> D_i <u>does not have 1 as eigenvalue, one has</u>
$H^q(X,\Omega^p_X\langle D\rangle\otimes M) = 0$ <u>for</u> $p + q \neq n$.

<u>Remark:</u> As we have seen in (1.4) the assumption in 1) is satisfied for the canonical extension or - more generally - if for all i, $Res_i(\nabla)$ has no strictly positive integer as eigenvalue. Correspondingly the assumption of 2) is satisfied if (for all i) $Res_i(\nabla)$ has no eigenvalue lying in $-\mathbb{N}$.

<u>Proof:</u> One just writes

$$\dim \mathbb{H}^k(X,DR_D M) = \bigoplus_{p+q=k} \dim H^q(X,\Omega^p_X\langle D\rangle\otimes M).$$

In case 1) or 3) (use (1.4,b)) this is nothing but $\dim \mathbb{H}^k(X,Rj_* V) = \dim H^k(U,V)$ and in case 2) this is $\dim \mathbb{H}^k(X,j_! V) = \dim H^k_c(U,V) = \dim H^{2n-k}(U,V^\vee)$ and the Main Lemma follows from the "topological vanishing" (1.5).

(2.3) <u>Remarks.</u> a) In fact, by a small modification of the arguments given, it is enough to assume that the conditions in 1), 2), 3) are satisfied along enough components of D, such that the complement remains affine. For example 3) could be replaced by

3') if for $i = 1,\dots,r$ the monodromy around D_i does not have 1 as eigenvalue and $X - \bigcup_{i=1}^{r} D_i$ is affine, then

$$H^q(X,\Omega_X^p\langle D\rangle\otimes M) = 0 \quad \text{for} \quad p+q \neq n .$$

b) If U is not affine, we still have the following result. Assume X to be a proper algebraic or compact Kähler manifold and assume that there exists a morphism $g:U\to W$ on a affine manifold W, such that the maximal fibre dimension of g is r. Using the Leray spectral sequence one obtains $H^k(U,V) = 0$ for $k > n + r$ and – under the additional assumption made in (2.2,3) one has $H^q(X,\Omega_X^p\langle D\rangle \otimes M) = 0$ for $p + q < n - r$ or $p + q > n + r$.

(2.4) Let Y be a normal manifold and $\pi:Y\to X$ be a Galois cover ramified only along the normal crossing divisor D. Let $\sigma:Z\to Y$ be a desingularization of Y such that $\sigma^{-1}\pi^{-1}D = \Delta$ is a normal crossing divisor too. Y has rational singularities (see [5]) and $\pi_*\mathcal{O}_Y$ is locally free. By [5], § 1, $R(\pi \circ \sigma)_* \Omega_Z^p\langle\Delta\rangle = \Omega_X^p\langle D\rangle \otimes_{\mathcal{O}_X} \pi_* \mathcal{O}_Y$.

The pull down of the Kähler differential $d: \mathcal{O}_Z \to \Omega_Z^1\langle\Delta\rangle$ induces a connection ∇' on $\pi_* \mathcal{O}_Y$ and $R(\pi \circ \sigma)_* DR_\Delta(\mathcal{O}_Z) = DR_D(\pi_* \mathcal{O}_Y)$.

The Galois group G operates on $\mathcal{O}_Y$ and $\pi_* \mathcal{O}_Y$. Let M be a direct summand, invariant under G. Then ∇' induces a logarithmic connection ∇ on M and $DR_D(M)$ is a summand of the complex $DR_D(\pi_* \mathcal{O}_Y)$. Hence (M, ∇) satisfies the $E_1(M)$ - degeneration, as $(\mathcal{O}_Z, d)$ satisfies the $E_1(\mathcal{O}_Z)$ - degeneration.

By Deligne's mixed Hodge theory for open varieties [3] this is true if X (and hence Z) is algebraic or Kähler or - more generally - if there exists a Kähler manifold X' and a bimeromorphic map $\tau: X' \to X$. In the last case we will say that X is <u>bimeromorphically dominated</u> by a Kähler manifold.

By definition (M, ∇) is the canonical extension. The local constant system V of flat (analytic) sections is given by a representation of $\pi_1(U)$ on a vector space L factorizing over G. The assumption made in (2.2,3) says that

(*) The ramification groups of a components D_i of D are mapped injectively to $\mathrm{Aut}(L)$. Alltogether we obtain:

(2.5) <u>Corollary.</u> <u>Let</u> (M, ∇) <u>be the logarithmic connection constructed above. Assume that</u> X <u>is a proper algebraic (or</u>

compact Moišezon) manifold and U *is affine, then* $H^q(X,\Omega_X^p\langle D\rangle \otimes M) = 0$ *for* $q + p > n$. *Moreover, if* (∇, M) *satisfies* (*), *then*

$$H^q(X,\Omega_X^p\langle D\rangle \otimes M) = 0 \quad \text{for } q + p \neq n.$$

(2.6) Remark:

Let V be a the local constant system on U given by a unitary representation of $\pi_1(U)$ and (M,∇) be the canonical extension. P. Deligne told us that S. Zucker's methods and results in [12] should extend to this case and prove the $E_1(M)$- degeneration. One has to regard V as a polarized variation of Hodge structures of weight (0,0). In fact Zucker studied in [12] the case of arbitrary variations of polarized Hodge structures, but he had to assume D to be a smooth divisor.

Recently E. Cattani, A. Kaplan and W. Schmidt on one side and M. Kashiwara on the other side generalized Zucker's results to the case of a normal crossing divisor. It seems that the extension considered by them is in the case of weight (0,0) the same as the canonical extension and that their results imply the $E_1(M)$ degeneration.

A "good" extension of variations of Hodge structures together with the degeneration of the corresponding spectral sequence might imply vanishing theorems for certain subquotients of the variations of Hodge structure. Some more precise questions can also be found in J. Kollár's paper [10], § 5.

(2.7) The simplest case of the covering construction given in (2.4) is that of a cyclic cover.

Let L be an invertible sheaf on X and $D = \sum_{i=1}^{s} \nu_i D_i$ be an effective normal crossing divisor, such that for some $N > 1$ one has $L^N = \mathcal{O}_X(D)$. Define for $0 \le j \le N-1$ the sheaves $L^{(j)} = L^j(-[\frac{j \cdot D}{N}])$ where [ ] denotes the integral part of the $\mathbb{Q}$-divisor $\frac{j \cdot D}{N}$ (see [5] or [11]). Let $L \to X$ and $L^N \to X$ be the line bundles corresponding to L and L^N and $\eta: L \to L^N$ the map obtained by taking the N-th power. Let $s: X \to L^N$ be the section corresponding to D and Y the normalization of $\eta^{-1}((s(X))$. The cover $\pi: Y \to X$ obtained is a cyclic cover, ramified over D. It is the came cover constructed in [5] or [11] as normalization of $Spec\,(\bigoplus_{j=0}^{N-1} L^{-j})$. One has $\pi_* \mathcal{O}_Z = \bigoplus_{j=0}^{N-1} L^{(j)^{-1}}$ and the sheaves $L^{(j)^{-1}}$ correspond to the different sheaves of eigen spaces.

By the construction of (2.4) the sheaves $L^{(j)^{-1}}$ are endowed with a natural logarithmic connection along D. It can locally be described in the following way:

If t^{-1} is a local generator of L and $f = x_1^{\nu_1} \cdot \ldots \cdot x_r^{\nu_r}$ a local equation for D, one has $t^N = f$. A local generator of $L^{(j)^{-1}}$ is given by $\sigma_j = t^j \cdot x_1^{-[\frac{j \cdot \nu_1}{N}]} \cdot \ldots \cdot x_r^{-[\frac{j \cdot \nu_r}{N}]}$.

One has $\nabla(\sigma_j) = \sigma_j \cdot (j\frac{dt}{t} - \sum_{i=1}^{r} [\frac{j \cdot \nu_1}{N}] \frac{dx_i}{x_i}) = \sigma_j \cdot (\sum_{i=1}^{r} (\frac{j \cdot \nu_i}{N} - [\frac{j \cdot \nu_i}{N}]) \frac{dx_i}{x_i})$.

The condition (*) of (2.4), saying that the monodromy of $(\mathcal{L}^{(j)},\nabla)$ does not have 1 as eigenvalue means exactly that $\frac{j\cdot\nu_i}{N} \notin \mathbb{Z}$ for $i = 1 \dots s$.

Rewriting (2.5) in this case one obtains:

(2.8) Global vanishing theorem for integral parts of $\mathbb{Q}$-divisors.

Let X be a proper algebraic (or compact Moišezon) manifold and U affine. Let $\mathcal{L}$ be an invertible sheaf and $\mathcal{L}^N = \mathcal{O}_X(D)$. Then

1) for $0 \leq j \leq N-1$ and $p+q>n$ one has $H^q(X,\Omega_X^p\langle D\rangle \otimes \mathcal{L}^{(j)^{-1}}) = 0$.

2) if moreover, for some j, $1 \leq j \leq N-1$, and for all i, one has $\frac{i\cdot\nu_j}{N} \notin \mathbb{Z}$, then $H^q(X,\Omega_X^p\langle D\rangle\otimes \mathcal{L}^{(j)^{-1}}) = 0$ for $p+q \neq n$.

(2.9) Remarks:

1) Let $D' = \Sigma D_i$, where the sum is taken over all components D_i with $\frac{j\cdot\nu_i}{N} \in \mathbb{Z}$. Then $\mathcal{L}^{(j)}(-D_{red}) = \mathcal{L}^{(N-j)^{-1}}(-D')$. Using Serre duality one obtains in (2.8.1) the vanishing of $H^q(X,\Omega_X^p\langle D\rangle \otimes \mathcal{L}^{(N-j)^{-1}}(-D'))$ for $p+q<n$.

2) Using (2.3,a) it is in (2.8,2) again sufficient to ask for the condition "$\frac{i\cdot\nu_j}{N} \notin \mathbb{Z}$" for "enough"

components of D. Moreover - as remarked in (2.3,6) - one can weaken the condition "U affine" and obtains still the vanishing of some cohomology groups.

3) Replacing L by L^j and N by $j \cdot N$ we may always assume that the sheaf considered is of the form $L^{(1)}$. Moreover, since $L^{(1)}$ does not change if we replace D by $D - N \cdot D_i$ and L by $L(-D_i)$ for some i with $\nu_i \geq N$, we can as well assume that all $0 < \nu_i < N$. In this case the assumption of (2.8,2)) is satisfied for the new divisor D. However, if from the beginning $\frac{i \cdot \nu_j}{N} \notin \mathbb{Z}$, D_{red} does not change.

At the end of this section we want to show how to obtain from (2.8) several of the classical vanishing theorems.

(2.10) Kodaira-Nakano-vanishing theorem. (see for example [7])

Let X be a projective manifold and L be an invertible ample sheaf. Then $H^q(X, \Omega_X^p \otimes L^{-1}) = 0$ for $p + q < n$.

Proof: For some $N > 1$ we can find a smooth very ample divisor D such that $L^N = \mathcal{O}_X(D)$. One has an exact sequence $0 \to \Omega_X^p \to \Omega_X^p\langle D\rangle \to \Omega_D^{p-1} \to 0$ and a long exact sequence

$$\dots \to H^{q-1}(X, \Omega_X^p\langle D\rangle \otimes L^{-1}) \to H^{q-1}(D, \Omega_D^{p-1} \otimes L^{-1}) \to H^q(X, \Omega_X^p \otimes L^{-1}) \to$$

$$\to H^q(X, \Omega_X^p\langle D\rangle \otimes L^{-1}) \to \dots$$

By construction $U = X - D_{red}$ is affine and (2.8.2) implies

$$H^{q-1}(D,\Omega_D^{p-1} \otimes L^{-1}) \cong H^q(X,\Omega_X^p \otimes L^{-1})$$

for $q + p < n$ (or $q + p > n + 1$).
The sheaf $L_{|D}$ is ample and - by induction on the dimension - we may assume that $H^{q-1}(D,\Omega_D^{p-1} \otimes L^{-1}) = 0$ for $p + q \leq n$.

(2.11) Bogomolov-Sommese-Vanishing theorem.
(see for example [11])

Let X be a proper algebraic (or compact Moišezon) manifold. L an invertible sheaf with $\kappa(L) = n$. Then $H^0(X,\Omega_X^p \otimes L^{-1}) = 0$ for $p < n$.

Proof: The statement is compatible with blowing up. Therefore we may assume X to be projective. Since $\kappa(L) = n$ we can find $N > 1$, a very ample sheaf H and an effective divisor B such that $L^N = H(B)$. Let $\sigma: X' \to X$ be an embedded desingularization of B and $-E$ a relative ample divisor, E supported in the exceptional locus of σ. Replacing N by $\nu \cdot N$ and H by H^ν we may assume that $H' = \sigma^* H(-E)$ is very ample and for $L' = \sigma^* L$ we have an effective normal crossing divisor $B' = \sigma^*(B) + E$ with $L'^N = H'(B')$. Hence we may assume that from the beginning B was a normal crossing divisor. Of course, in order to prove (2.11) we may replace L be a smaller sheaf and hence we can also assume that the multiplicities of all components of B are strictly smaller than N. Let H be a general divisor of $\mathcal{H}$. Then $D = H + B$ is a normal crossing divisor. As in (2.9) we have $\mathcal{L}^{(1)} = \mathcal{L}$. Since $\mathcal{H}$ is very ample $U = X - D_{red}$ is affine and (2.11) follows from (2.8.2).

(2.12) <u>The vanishing theorem for numerically effective sheaves.</u>

(see [8] or [11]). <u>Let</u> X <u>be a proper algebraic (or compact Moišezon) manifold</u>, $\mathcal{L}$ <u>a numerically effective invertible sheaf</u> (i.e $\deg(\mathcal{L}|_C) \geq 0$ <u>for all curves</u> $C \subset X$) <u>and</u> $c_1(\mathcal{L})^n > 0$. <u>Then</u> $H^q(X, \mathcal{L}^{-1}) = 0$ <u>for</u> $q < n$.

<u>Proof:</u> Again (2.12) is compatible with blowing up and we may assume X to be projective. For numerically effective sheaves the condition $c_1(\mathcal{L})^n > 0$ is equivalent to $\kappa(\mathcal{L}) = n$ (the proof is quite simple; see for example [11]). As in the proof of (2.11) we can find - after blowing up again - an ample sheaf $\mathcal{H}$ and a normal crossing divisor B such that $\mathcal{L}^N = \mathcal{H}(B)$. Since $\mathcal{L}$ is numerically effective $\mathcal{H} \otimes \mathcal{L}^\nu$ is ample for all $\nu \geq 0$. Replacing N by $N + \nu$, we may assume that N is larger than the multiplicities of the components of B and- replacing $N, \mathcal{L}, \mathcal{H}, B$ by $\mu \cdot N, \mathcal{L}^\mu, \mathcal{H}^\mu, \mu \cdot B$ - that $\mathcal{H}$ is a very ample. Let H be a general divisor of $\mathcal{H}$ and $D = B + H$. Then $\mathcal{L}^{(1)} = \mathcal{L}$, $U = X - D_{red} = (X - H_{red}) - B_{red}$ is affine and (2.12) follows from (2.8.2).

(2.12) can be generalized to $\mathbb{Q}$-divisors. The most general form is

(2.13) Theorem. (see [8],[11]or[5])
Let X be a proper algebraic (or compact Moišezon) manifold, $\mathcal{L}$ an invertible sheaf and C an effective normal crossing divisor such that for some $N > 1$ $\mathcal{L}^N(-C)$ is numerically effective. If for some $j < N$ the "$\mathcal{L}$-dimension"

$$\kappa(\mathcal{L}^j(-[\frac{j\cdot C}{N}])) = n, \text{ then } H^q(X,\mathcal{L}^{-j}([\frac{j\cdot C}{N}])) = 0 \text{ for } q<n.$$

The proof is similar to (2.12): If $\sigma:X' \to X$ is a blowing up, such that $\sigma^*C = C'$ is again a normal crossing divisor, then $R\sigma_*\mathcal{O}_{X'}([\frac{j\cdot C'}{N}]) = \mathcal{O}_X([\frac{j\cdot C}{N}])$. This follows from the fact that the cover Y of X constructed in (2.7) has at most rational singularities, or from elementary local calculations (see [11]). Hence the statement of (2.13) is compatible with blowing up.

If we allow "fractional powers of sheaves", one has $\mathcal{L}^j(-[\frac{j\cdot C}{N}]) = (\mathcal{L}^N(-C))^{\frac{j}{N}} \otimes \mathcal{O}(\frac{j}{N} C - [\frac{j\cdot C}{N}])$. Hence the assumption says that we can find (after replacing N by some high multiple) a subdivisor C' of C such that $[\frac{j\cdot C}{N}] = [\frac{j\cdot(C-C')}{N}]$ and such that $\mathcal{L}^N(-C + C')$ contains an ample sheaf $\mathcal{H}$. After blowing up we may assume that $\mathcal{L}^N(-(C- C')) = \mathcal{H}(B)$ where $B + C$ is a normal crossing divisor. Replacing $\mathcal{H}$ by $\mathcal{H} \otimes \mathcal{L}^{\nu\cdot N}(-C)$ we can increase N without changing the multiplicity of the components of B. Alltogether we are reduced to the case that $\mathcal{L}^N = \mathcal{O}(D)$ where $D = H + B + (C - C')$ is a normal crossing divisor, H is ample and $[\frac{j\cdot D}{N}] = [\frac{j\cdot C}{N}]$. Now (2.13) follows from (2.8,2).

(2.14) Remark: a) It seems surprising that the vanishing theorems (2.11) for $q = 0$ and (2.12) for $p = 0$ are more general than (2.10). However, it is well known, that (2.10) is no longer true, if one replaces the condition "$\mathcal{L}$ ample" by "$\kappa(\mathcal{L}) = n$ and $\mathcal{L}^\mu$ generated by global sections for some $\mu > 0$". In this case one could still choose a normal crossing divisor D with small multiplicities, such that $\mathcal{L}^N = \mathcal{O}(D)$ and such that $U = X - D_{red}$ is affine. One obtains the vanishing of $H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{L}^{-1})$ for $q + p \neq n$, but the induction used in the proof of (2.10) breaks down, since for some components D_i, $\kappa(\mathcal{L}|_{D_i})$ might be too small.

b) The proof of (2.12) and (2.13) in [11] used Hodge duality to reduce the vanishing of cohomology of invertible sheaves to the Bogomolov-Sommese vanishing theorem. In the approach described here, both follow from the same statement, the E_1-degeneration of the spectral sequence associated to the Hodge filtration, and one does not use the Hodge duality.

§ 3 Applications to the vanishing of the cohomology of morphisms

We keep the notations and assumptions made in (1.1) and (2.1). Whereas in § 2 we just considered the dimension of $E_1^{pq}(\mathcal{M})$ for a logarithmic connection $\mathcal{M}$, we will now regard the differentials d_p of the spectral sequence.

(3.1) As usual [i] denotes the shift operator for complexes. Hence $F^p[p]$ is the complex starting with $\Omega_X^p\langle D\rangle \otimes \mathcal{M}$ in degree zero and - if $\mathcal{F}$ is any complex - one has $\mathbb{H}^k(\mathcal{F}) = \mathbb{H}^{k+i}(\mathcal{F}[-i])$.

The differential

$$d_1 : H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}) \to H^q(X, \Omega_X^{p+1}\langle D\rangle \otimes \mathcal{M}) = \mathbb{H}^{q+1}(X, F^{p+1}/F^{p+2}[p]),$$

is the connecting morphism of

$$0 \to F^{p+1}/F^{p+2}[p] \to F^p/F^{p+2}[p] \to \Omega_X^p\langle D\rangle \otimes \mathcal{M} \to 0.$$

Hence $d_1 = 0$ implies that $\mathbb{H}^q(X, F^p/F^{p+2}[p]) \to H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M})$ is surjective and in this case d_2 is the connecting morphism of

$$0 \to F^{p+2}/F^{p+3}[p] \to F^p/F^{p+3}[p] \to F^p/F^{p+2}[p] \to 0.$$

If $d_2 = 0$ one gets a surjection

$$\mathbb{H}^q(X, F^p/F^{p+3}[p]) \to H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M})$$

and repeating this construction long enough one finds the wellknown equivalence of the following two conditions:

A) <u>For all</u> p,q <u>the connecting morphisms</u>

$$\delta_p : H^q(X, \Omega_X^p\langle D\rangle \otimes \mathcal{M}) \to \mathbb{H}^q(X, F^{p+1}[p+1]) = \mathbb{H}^{q+1}(X, F^{p+1}[p]) \text{ \underline{of}}$$

$$0 \to F^{p+1}[p] \to F^p[p] \to \Omega_X^p\langle D\rangle \otimes \mathcal{M} \to 0$$

<u>are zero</u>.

B) $(\mathcal{M}, \nabla)$ <u>satisfies</u> $E_1(\mathcal{M})$ <u>degeneration.</u>

Of course δ_p is induced by $\Omega_X^p\langle D\rangle \otimes \mathcal{M} \xrightarrow{\nabla} F^{p+1}[p+1]$.

Under the additional condition that $DR_D\mathcal{M}$ is quasi-isomorphic to $j_!V$, where V denotes as usual the flat (analytic) sections of $\mathcal{M}$, the $E_1(\mathcal{M})$ degeneration can be interpretated in a more geometric way. In the Lemma below part 1) and 3) use the whole vanishing of δ , whereas 2) follows from the vanishing of d_1.

(3.2) <u>Main Lemma</u>: <u>Let</u> $(\mathcal{M}, \nabla)$ <u>be a logarithmic connection satisfying</u> $E_1(\mathcal{M})$ <u>degeneration. Assume that the monodromies of</u> $(\mathcal{M}, \nabla)$ <u>around the components</u> D_i <u>of</u> D <u>do not have</u> 1 <u>as an eigenvalue.</u>

1) Then for any effective divisor B with $B_{red} \le D_{red}$, and all $q \ge 0$, the morphism, induced by restriction of M to B, $H^q(R^0): H^q(X,M) \to H^q(B,M|_B)$ is zero.

2) Let C be a smooth subdivisor of D_{red} and $D' = D_{red} - C$. Then for all $q \ge 0$ and $p \ge 0$ the morphism, induced by restriction of differentials,

$$H^q(R^p): H^q(X,\Omega_X^p\langle D'\rangle \otimes M) \to H^q(C,\Omega_C^p\langle D'\cap C\rangle \otimes M)$$

is zero. Especially, if D is smooth, the map $H^q(X,\Omega_X^p \otimes M) \to H^q(D,\Omega_D^p \otimes M)$ is zero:

3) Then for all $q \ge 0$ and $p \ge 0$ the morphism, induced by the connection ∇,

$$H^q(\nabla): H^q(X,\Omega_X^p\langle D\rangle \otimes M) \to H^q(X,\nabla(\Omega_X^p\langle D\rangle \otimes M))$$

is zero.

Proof: 1) By (1.4,b) $DR_D M$ and $DR_D M(-B)$ are quasi-isomorphic. By (3.1, A) the morphism $\delta_0: H^q(X,M) \to \mathbb{H}^q(X,F^1[1])$ is zero. Hence in the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^q(X,DR_D M(-B)) & \to & H^q(X,M(-B)) \\ \downarrow\beta & & \downarrow\gamma \\ \mathbb{H}^q(X,DR_D M) & \xrightarrow{\alpha} & H^q(X,M) \end{array}.$$

β is an isomorphism and α surjective. Therefore γ is also surjective.

2) By assumption $\mathrm{Res}_C(\nabla)$ can not have zero as eigenvalue and this just means ([3], p. 78) that the composition $\mathrm{Res}_C(\nabla): M \xrightarrow{\nabla} \Omega^1_X\langle D\rangle \otimes M \xrightarrow{\mathrm{res}} \mathcal{O}_C \otimes M$ is surjective. Hence one has

$$\begin{array}{ccc} \Omega^p_X\langle D'\rangle \otimes M & \xrightarrow{\nabla} & \Omega^{p+1}_X\langle D\rangle \otimes M \\ {\scriptstyle R^p}\downarrow & \searrow^{\gamma} & \downarrow{\scriptstyle \mathrm{res}} \\ \Omega^p_C\langle D'\cap C\rangle \otimes M & \xrightarrow{\cong} & \Omega^p_C\langle D'\cap C\rangle \otimes M \end{array}$$

$H^q(\nabla) = 0$ implies that $H^q(\gamma)$ and $H^q(R^p)$ are both zero.

3) We have a quasi-isomorphism (1.7) $j_! V \to F^0 = DR_D M$ and therefore $\nabla(\Omega^p_X\langle D\rangle \otimes M) \to F^{p+1}$ is a quasi-isomorphism for $p \geq 0$. Hence 3) is just saying that δ_p in (3.1,A) is zero.

Applying (3.2 ,1 and 2) to invertible sheaves arising from cyclic covers of X (2.7) we obtain:

(3.3) Relative vanishing theorem for integral parts of $\mathbb{Q}$-divisors.

Let X be a proper algebraic manifold or a compact analytic manifold which is bimeromorphically dominated by a Kähler manifold. Let L be an invertible sheaf on X, D be an effective normal crossing divisor and $L^N = \mathcal{O}_X(D)$ for some $N > 1$. Let $1 \leq j \leq N-1$.

1) Let B be an effective divisor supported in $\mathrm{supp}\,(j\cdot D - N \cdot [\frac{j\cdot D}{N}])$. Then the maps

$$H^0(R^0): H^q(X, L^{(j)^{-1}}) \to H^q(B, L^{(j)^{-1}}|_B)$$

are zero for all $q \geq 0$.

2) Let C be a smooth subdivisor of

$$D_{red} \cap \mathrm{supp}\,(j \cdot D - N \cdot [\frac{j\cdot D}{N}]) \quad \text{and} \quad D' = D_{red} - C.$$

Then the maps

$$H^q(R^p): H^q(X, \Omega^p_X\langle D'\rangle \otimes L^{(j)^{-1}}) \to H^q(C, \Omega^p_C\langle D' \cap C\rangle \otimes L^{(j)^{-1}})$$

are zero for all $p \geq 0$ and $q \geq 0$.

(3.4) Remark: As described in (2.9,3) one may rephrase (3.3) in the following way.

Assume that for an effective normal crossing divisor D one has, $L^N = \mathcal{O}(D)$, where N is larger than the multiplicities of the components of D, and let B be any divisor supported in D_{red}. Then the maps $H^q(X, L^{-1}) \to H^q(B, L^{-1}|_B)$ are zero for all $q \geq 0$.

If C *is a smooth subdivisor of* D_{red} *, then the maps*

$$H^q(X,\Omega_X^p\langle D'\rangle \otimes \mathcal{L}^{-1}) \to H^q(C,\Omega_C^p\langle D'\cap C\rangle \otimes \mathcal{L}^{-1})$$

are zero for all $p,q \geq 0$.

(3.5) *Corollary.* (Kollár , [9], 2.2). *Let* X *be as in* (3.3), $\mathcal{L}$ *an invertible sheaf, such that some power of* $\mathcal{L}$ *is generated by it's global sections, and* B *an effective divisor, such that* $\mathcal{O}_X(B)$ *is contained in a power of* $\mathcal{L}$. *Then the restriction maps* $H^q(X,\mathcal{L}^{-1}) \to H^q(B,\mathcal{L}^{-1}|_B)$ *are zero for all* $q \geq 0$.

Proof: We choose D' such that $\mathcal{O}_X(D' + B) = \mathcal{L}^{\mu}$.
In order to show that $H^q(X,\mathcal{L}^{-1}(-B)) \to H^q(X,\mathcal{L}^{-1})$ is surjective, we may replace X be a blowing up and thereby we may assume $B + D'$ to be a normal crossing divisor. By assumption $\mathcal{L}^{\nu}$ is generated by it's global sections for some $\nu >> 0$ and one finds a smooth divisor D'' such that $D = B + D' + D''$ is a normal crossing divisor. Choosing ν large enough one may assume that the multiplicities of the components of D are smaller than $N = \mu + \nu$ and obtains (3.5) from (3.3,1) and (3.4,1).

In (3.2,2) and correspondingly in (3.3,2) one can weaken the hypothesis "C smooth" to "C reduced". However, in this case we just get that the natural map

$$H^q(\tilde{R}):H^q(X,\Omega^p_X\langle D'\rangle \otimes M) \to H^q(C,\Omega^p_{\tilde{C}}\langle\tilde{D}\rangle \otimes M)$$

is zero, where $\tilde{C}$ is the normalization of C and $\tilde{D}$ the pullback of the one by one intersections of D to $\tilde{C}$.

Of course the map we are really interested in is

$$H^q(R):H^q(X,\Omega^p_X\langle D'\rangle \otimes M) \to H^q(C,\Omega^p_C\langle D'\rangle\otimes M).$$

The only cases where we know that $H^q(\tilde{R}) = 0$ implies $H^q(R) = 0$ are the trivial one, $q = 0$, or the case $p = 0$, handled in (3.2,1) by different methods.

In [6] 1.1 we proved (3.3,2) for $q = 0$ by direct calcuation, and - similarly to the global case (see (2.14,6)) - we used Hodge duality to obtain the $p = 0$ case. Finally we used the strict compatiblity of the restriction map with the Hodge and the weight filtration ([4], 8.2.7) to show that for $p = 0$ $H^q(\tilde{R}) = 0$ implies $H^1(R) = 0$ (see [6], 1.6).

If one tries to consider more complicated restriction maps, the picture is even worse and the interpretation of the morphisms nearly impossible. Nevertheless, we will try in the last part of this chapter to use (3.2,3) to obtain some generalizations of (3.2,1) and (3.2,2).

We assume in the sequel that D is reduced.

The idea of the constructions following is quite simple. We try to find $\mathcal{O}_X$-modules (or complexes) N^p and K^p and an $\mathcal{O}_X$-linear map $\gamma : N^p \to K^p$ which fits into a commutative diagram

$$\begin{array}{ccccc}
\Omega_X^p\langle D\rangle \otimes M & \xrightarrow{\nabla} & \nabla(\Omega_X^p\langle D\rangle \otimes M) & \hookrightarrow & \Omega_X^{p+1}\langle D\rangle \otimes M \\
\alpha \uparrow & & \downarrow \beta & & \\
K^p & \xrightarrow[\gamma]{} & K^p & &
\end{array}$$

of $\mathbb{C}_X$ sheaves. Then $H^q(\nabla) = 0$ implies $H^q(\gamma) = 0$.

(3.6). The sheaves N^p will be given by the weight filtration (see [4]) W_k of $\Omega_X^p\langle D\rangle$ where $W_k(\Omega_X^p\langle D\rangle) = \Omega_X^k\langle D\rangle \wedge \Omega_X^{p-k}$. We denote by C_k^p the quotient sheaf $\Omega_X^p\langle D\rangle \otimes M / W_k(\Omega_X^p\langle D\rangle) \otimes M$ and by K_k^p the image of

$$\nabla(\Omega_X^{p-1}\langle D\rangle \otimes M) \quad \text{in} \quad C_k^p .$$

By the Leibnitz rule one has

$$\nabla(W_k(\Omega_X^p\langle D\rangle \otimes M) \subset W_{k+1}(\Omega_X^{p+1}\langle D\rangle) \otimes M ,$$

and ∇ induces a map

$$\nabla' : C_k^p \to C_{k+1}^{p+1} \quad \text{such that} \quad K_k^p \subseteq \mathit{Ker}\,(\nabla') .$$

In general ∇' is not $\mathcal{O}_X$-linear and K_k^p is not an $\mathcal{O}_X$-module. Applying again the Leibnitz rule we obtain an $\mathcal{O}_X$-linear map

$$\mathrm{Res}_k^{p-1}(\nabla) : W_k(\Omega_X^{p-1}\langle D\rangle) \otimes M \xrightarrow{\nabla} \Omega_X^p\langle D\rangle \otimes M \longrightarrow C_k^p$$

and $Im(\mathrm{Res}_k^{p-1}(\nabla)) \subseteq K_k^p$.

(3.7) Denote by $D^{[s]}$ the normalization of the s by s intersections of the components D_i of D and by D^{s+1} the normal crossing divisor on $D^{[s]}$ obtained by pulling back the $(s+1)$ by $(s+1)$ intersections of the components of D. One has an inclusion

$$C_k^p \hookrightarrow \Omega_{D^{[k+1]}}^{p-k-1} \langle D^{k+2}\rangle \otimes M$$

given locally at a point on D = zero set of $x_1 \cdot \ldots \cdot x_r = 0$ by

$$\frac{dx_{i_1}}{x_{i_1}} \wedge \ldots \wedge \frac{dx_{i_p}}{x_{i_p}} \otimes m \longmapsto \oplus \pm \bigwedge_{i \in J} \frac{dx_{i_j}}{x_{i_j}} \otimes m \Big|_{\{x_{i_\ell}=0,\ \ell \notin J,\ \ell=1\ldots r\}}$$

where $1 \le i_1 < \ldots < i_p \le r$, and where the direct sum is taken over all subsets $J \subseteq \{1,\ldots,r\}$ or $r - k - 1$ elements, and the signs are given by the usual rule.

If $\Gamma_i : M \xrightarrow{\nabla} \Omega^1_X\langle D\rangle \otimes M \xrightarrow{res_i} M|D_i$ denotes the residue of ∇ along D_i then, for example, $Res_k^{p-1}(\nabla)$ maps $\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_k}{x_k} \wedge \varphi \otimes m$ to $\bigoplus_{i>k} \pm \varphi \wedge \Gamma_i(m)$. Especially, if the monodromies of (M,∇) around the components D_i of D do not have one as eigenvalue, then Γ_i is surjective as well as $Res_k^{p-1}(\nabla)$ at the general points of the components of $D^{[k+1]}$. Moreover $Res_k^{p-1}(\nabla)$ factors in the following way

$$W_k(\Omega_X^{p-1}\langle D\rangle) \otimes M \to W_k(\Omega_X^{p-1}\langle D\rangle) \otimes M / W_{k-1}(\Omega_X^{p-1}\langle D\rangle) \otimes M \cong$$

$$\cong \Omega^{p-k-1}_{D^{[k]}} \otimes M \to \Omega^{p-k-1}_{\widetilde{D^{k+1}}} \otimes M \xrightarrow{\rho} \Omega^{p-k-1}_{D^{[k+1]}} \otimes M \hookrightarrow C_k^p .$$

Here $\widetilde{D^{k+1}}$ is the normalization of D^{k+1} and ρ is mapping $\varphi \otimes m$ to the alternating sum of the possible restrictions $\varphi \otimes \Gamma_i(m)$.

By the $E_1(M)$ degeneration we obtain.

(3.8) <u>Claim:</u> $H^q(Res_h^{p-1}(\nabla)) : H^q(X, W_k(\Omega_X^{p-1}\langle D\rangle) \otimes M) \to$

$$\to H^q(X, \Omega^{p-k-1}_{D^{[k+1]}}\langle D^{k+2}\rangle \otimes M)$$

<u>is the zero map.</u>

Of course, the $\mathcal{O}_X$-linear map $Res_k^{p-1}(\nabla)$ depends on the residues of the connection and the only case where one can find an isomorphism α of $\Omega^{p-k-1}_{D^{[k+1]}}\langle D^{k+2}\rangle \otimes M$

such that $\alpha \cdot \mathrm{Res}_k^{p-1}(\nabla)$ does not, is for $k = 0$.
In fact (3.2.3) implies a stronger statement:

(3.9) $\quad H^q(\mathrm{Res}_k^{p-1}(\nabla)) : H^q(X, W_k(\Omega_X^{p-1}\langle D\rangle) \otimes \mathcal{M}) \to H^q(X, \mathcal{K}_k^p)$

is the zero map, where $\mathcal{K}_k^p = \mathcal{K}er \ (\nabla' : \mathcal{C}_k^p \to \mathcal{C}_{k+1}^{p+1})$.

However, both sheaves, $\mathcal{K}_k^p$ and $\mathcal{C}_k^p$ are quite difficult to describe.

For $\mathcal{C}_k^p$, at least, we have a reasonable filtration.
If $\mathcal{W}_\ell$ denotes the image of $W_\ell(\Omega_X^p\langle D\rangle) \otimes \mathcal{M}$ in $\mathcal{C}_k^p$, one obtains a filtration $0 = \mathcal{W}_k \subset \mathcal{W}_{k+1} \subset \ldots \subset \mathcal{W}_p = \mathcal{C}_k^p$ such that $\mathcal{W}_\ell / \mathcal{W}_{\ell-1} = \Omega_{D^{[\ell]}}^{p-\ell} \otimes \mathcal{M}$.

For $k = p-1$, one obtains $\mathcal{C}_{p-1}^p = \mathcal{O}_{D^{[p]}} \otimes \mathcal{M}$. However, $\nabla' : \mathcal{M}|_{D^{[p]}} \to \mathcal{M}|_{D^{[p+1]}} = \mathcal{C}_p^{p+1}$ is given by the alternating sum of the Γ_i, considered as an isomorphism of $\mathcal{M}|_{D_i}$.

Define $\gamma^{[p]} : \mathcal{M}|_{D^{[p]}} \to \mathcal{M}|_{D^{[p]}}$ to be the automorphism given by $\Gamma_{i_1} \circ \Gamma_{i_2} \circ \ldots \circ \Gamma_{i_p}$ on $\mathcal{M}|_{D_{i_1} \cap \ldots \cap D_{i_p}}$. Since ∇ is integrable $\gamma^{[p]}$ is independent of the numbering of the components. One obtains a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}|_{D^{[p]}} & \xrightarrow{\ \nabla'\ } & \mathcal{M}|_{D^{[p+1]}} \\
\downarrow \gamma^{[p]} & & \downarrow \gamma^{[p+1]} \\
\mathcal{M}|_{D^{[p]}} & \xrightarrow[\varepsilon^p \otimes \mathrm{id}_\mu]{} & \mathcal{M}|_{D^{[p+1]}}
\end{array}$$

where ε^p is the usual map $\mathcal{O}_{D^{[p]}} \to \mathcal{O}_{D^{[p+1]}}$. Hence $\gamma^{[p]}$ maps K^p_{p-1} to $\mathcal{K}er\,(\varepsilon^p) \otimes M = \mathcal{I}m(\varepsilon^{p-1}) \otimes M$.
Locally, if D is the zero set of $x_1 \cdot \ldots \cdot x_r$,
$\gamma^{[p]^{-1}} \circ \mathrm{Res}^p_{p-1}(\nabla)$ maps $\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_{p-1}}{x_{p-1}} \otimes m$ to

$$\bigoplus_{i \geq p} \pm \gamma^{[p]^{-1}} \circ \Gamma_i(m)\,\big|_{D_1 \cap \ldots \cap D_{p-1} \cap D_i} =$$

$$= \bigoplus_{i \geq p} \pm \Gamma_1^{-1} \circ \ldots \circ \Gamma_{p-1}^{-1}(m)\,\big|_{D_1 \cap \ldots \cap D_{p-1} \cap D_i} .$$

Hence we obtain

(3.10) <u>Claim.</u> <u>Keeping the assumptions made in (3.2) and the notations introduced above the map</u>

$$H^q(\gamma^{[p]^{-1}} \circ \mathrm{Res}^p_{p-1}(\nabla)) : H^q(X, \Omega^{p-1}_X \langle D \rangle \otimes M) \to$$

$$\to H^q(X, \mathcal{K}er\,(\varepsilon^p) \otimes M)$$

<u>is zero.</u>

For $p = 1$ and $D = B$ this is the same as (3.2,1). For $p > 1$ the map $\gamma^{[p]^{-1}} \circ \mathrm{Res}^p_{p-1}(\nabla)$ depends on ∇. Of course, we can apply (3.10) to the situation of invertible sheaves coming from cyclic covers (as in (3.3)). In this case, one can give a more explicite description of the morphism considered.

Appendix B Chern classes and logarithmic connections

Let (M,∇) be a connection on a proper algebraic or compact analytic manifold X with logarithmic poles along a normal crossing divisor D.

As we have seen in § 2 the classical positivity conditions on a C^{∞} curvature matrix of a differentiable connection on M can be replaced by conditions on the residues of ∇ along the components of D, if one is interested in vanishing theorems of Kodaira-Nakano type.

In this appendix we want to show how to define the Chern classes of M using the logarithmic connection ∇ .

This is a second example indicating that both, the theory of C^{∞}-connections without singularities but with non trivial curvature matrix and the theory of holomorphic integrable connections with logarithmic singularities can be applied in a quite similar way in algebraic geometry.

The computation of the Chern classes and the Atiyah class described here was done independently by J.L. Verdier and the first author about one year ago.

Let $\tilde{D}$ be the normalization of D and $\mathrm{Res}:\Omega^1_X\langle D\rangle\otimes M \to \mathcal{O}_{\tilde{D}}\otimes M$ be the Poincaré residue. The element

$\Gamma = \mathrm{Res} \circ \nabla \in \mathrm{Hom}_{\mathcal{O}_X}(M, M_{|\tilde{D}})$ is mapped under the connecting morphism of the exact sequence

$$0 \to \Omega^1_X \otimes M \to \Omega^1\langle D\rangle \otimes M \to \mathcal{O}_D \otimes M \to 0$$

to an element $\gamma \in \mathrm{Ext}^1_{\mathcal{O}_X}(M, \Omega^1_X \otimes M)$.

(B.1) Proposition. <u>$-\gamma$ is the Atiyah class of</u> M.

<u>Proof:</u> The Ativah class is constructed in the following way (see [1]): Let J be the ideal sheaf of the diagonal $X \hookrightarrow X \times X$. The differentials are $\Omega^1_X = J/J^2$ and the first order jets of $\mathcal{O}_X$ are given by $P^1 = \mathcal{O}_{X\times X}/J^2$. So P^1 is endowed with a left $\mathcal{O}_X$-module structure, for which the exact sequence

$$0 \to \Omega^1_X \to P^1 \to \mathcal{O}_X \to 0$$

splits. However, P^1 carries also a right module structure, and one uses it to define $P^1(M) = P^1 \otimes_{\mathcal{O}_X} M$. Then $P^1(M)$ is endowed with a left module structure, as well as its submodule $\Omega^1_X \otimes_{\mathcal{O}_X} M$.

The sequence

$$0 \to \Omega^1_X \otimes_{\mathcal{O}_X} M \to P^1(M) \to M \to 0$$

defines a class $\gamma_A \in \mathrm{Ext}^1_{\mathcal{O}_X}(M, \Omega^1_X \otimes_{\mathcal{O}_X} M)$, the Atiyah class.

One has the map of first order principle parts
$j^1 : \mathcal{M} \to P^1(\mathcal{M})$ with $j^1(m) = (1 \otimes 1) \otimes m$.

Similarly to the construction of the jet bundle we define the sheaf of logarithmic jets as the (right and left) $\mathcal{O}_X$-submodule of the sheaf of rational functions of $(X \times X)/J^2$, which is generated by P^1 and $\Omega^1_X\langle D\rangle$. In other words, P^1_D is obtained from P^1 by adding locally $x_i \otimes \frac{1}{x_i}$ and $\frac{1}{x_i} \otimes x_i$, where x_i is a local equation of D_i.
We define in the same way $\Omega^1_X\langle D\rangle \otimes \mathcal{M}$, $P^1_D(\mathcal{M})$ and the exact sequence of left $\mathcal{O}_X$-modules

$$0 \to \Omega^1_X\langle D\rangle \otimes \mathcal{M} \to P^1_D(\mathcal{M}) \to \mathcal{M} \to 0 .$$

Define $s : \mathcal{M} \to P^1_D(\mathcal{M})$ by $s(m) = j^1(m) - \nabla(m)$. s is a $\mathcal{O}_X$-splitting ([3], p.2).

Consider the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^1_X \otimes \mathcal{M} & \longrightarrow & P^1(\mathcal{M}) & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^1_X\langle D\rangle \otimes \mathcal{M} & \longrightarrow & P^1_D(\mathcal{M}) & \underset{s}{\rightleftarrows} & \mathcal{M} & \longrightarrow & \\
 & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \pi} & & & & \\
 & & \mathcal{O}_{\tilde{D}} \otimes \mathcal{M} & = & \mathcal{O}_{\tilde{D}} \otimes \mathcal{M} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since $j^1(m) \in P^1(M)$ one has $\pi \circ s = -\Gamma = \mathrm{Res} \circ \nabla \in Hom_{\mathcal{O}_X}(M, M_{|\tilde{D}})$.
Since the exact sequence in the middle of the diagram splits, the image of $-\Gamma$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(M, \Omega^1_X \otimes M)$ is the Atiyah class and $\gamma \doteq -\gamma_A$.

Atiyah himself explained how to use the Atiyah class to compute the Chern classes (i.e.: the symmetric functions of the Chern roots).

Usually one gives the formula for the Newton classes N_p (i.e. the sum over the p-th powers of the Chern roots) and obtains the Chern classes by the interchange formulas.

(B.2) <u>Corollary</u>. ([1], Prop. 13) <u>Let</u> $\Gamma_i = \mathrm{Res}_i \circ \nabla \in \mathrm{Hom}_{\mathcal{O}_X}(M, M_{|D_i})$ <u>and</u> $[D_i]$ <u>the class of</u> D_i <u>in</u> $H^1(X, \Omega^1_X)$. <u>Then</u>

$$N_p(M) = (-1)^p \sum_{\alpha_1+\ldots+\alpha_s=p} \binom{p}{\alpha} \mathrm{Tr}\,(\Gamma_1^{\alpha_1} \circ \ldots \circ \Gamma_s^{\alpha_s}) \cdot [D_1]^{\alpha_1} \cdot \ldots \cdot [D_s]^{\alpha_s}.$$

<u>Especially</u> $C_1(M) = N_1(M) = -\sum_{i=1}^{s} \mathrm{Tr}(\Gamma_i) \cdot [D_i]$.

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