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Abstract. The Hurwitz space is the moduli space of pairs (X, f) where X is a compact Riemann surface and f is a meromorphic function on it. We study the Laplace operator $\Delta^{|df|^2}$ of the flat singular Riemannian manifold $(X, |df|^2)$. We define a regularized determinant for $\Delta^{|df|^2}$ and study it as a functional on Hurwitz space. We prove that this functional is related to a system of PDE which admits explicit integration. This leads to an explicit expression for the determinant of the Laplace operator in terms of the basic objects on the underlying Riemann surface (the prime form, theta-functions, the canonical meromorphic bidifferential) and the divisor of the meromorphic differential df. The proof has several parts that can be of independent interest. As an important intermediate result we prove a decomposition formula of the type of Burghelea-Friedlander-Kappeler for the determinant of the Laplace operator on flat surfaces with conical singularities and Euclidean or conical ends. We introduce and study the so-called S-matrix, $S(\lambda)$, of a surface with conical singularities and relate its behaviour at $\lambda = 0$ with the so-called Schiffer projective connection on the Riemann surface X. We also prove variational formulas for eigenvalues of the Laplace operator of a compact surface with conical singularities when the latter move.

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1 Introduction

1.1 General part

Study of the determinants of Laplacians on Riemann surfaces is motivated by the needs of quantum field theory (in connection with various partition functions) and geometric analysis (in particular, in connection with Sarnak program, [33]). The explicit expressions for the determinant of the Laplacian in the metric of constant negative curvature ([7]) and in the Arakelov metric (obtained in [2] in relation to so-called bosonization formulas from the string theory) for compact Riemann surfaces of genus $g > 1$ are among the most beautiful and deep results of the subject. According to Sarnak program, these determinants (which are functions on the moduli space of Riemann surfaces) can be used to study the geometry of the moduli space via the methods of Morse theory. In particular, their behavior at the boundary of the moduli space is of great importance and was intensively studied (see, e. g., [39], [38]).

It seems very interesting to consider the case which is in a certain sense opposite to the case of the metric of constant curvature: instead of distributing the curvature uniformly along the Riemann surface X one can concentrate it at a finite set $\{P_1, \dots, P_M\} \subset X$. This leads to a flat metric $\mathbf{m}$ on X with conical singularities at P_k. Determinant of Laplacians for various classes of flat metrics of this type were introduced and studied on the formal level (via path integrals) by physicists ([35],[40], [18], [3]) and certain explicit expressions for them were produced (see, e. g., [35], [40]).

One of the main goals is to study such determinants from the point of view of the spectral theory of self-adjoint operators and perturbation theory (as it was done in the mathematical literature for the determinants of the Laplacians in smooth metrics (in particular, those two mentioned above), see, e. g., Fay's book [9] for complete compendium and consistent exposition) using the standard definition of the determinant via the ζ-function of the corresponding Laplace operator

$$\ln \det\Delta^{\mathbf{m}} = -\zeta'_{\Delta^{\mathbf{m}}}(0)\,, \tag{1.1}$$

$$\zeta_{\Delta^{\mathbf{m}}}(s) = \sum_j \frac{1}{\lambda_j^s}\,, \tag{1.2}$$

where, in the latter expression the sum is extended over all non-zero eigenvalues of $\Delta^{\mathbf{m}}$.

Let us say from the very beginning that the Laplace operator $\Delta^{\mathbf{m}}$ (with natural domain consisting of smooth functions on X supported outside the conical points P_k) is not essentially self-adjoint. This fact is never mentioned by the physicists who have been working with these determinants and it is not clear whether this issue is addressed or not. Comparing the determinants of the different self-adjoint extensions of $\Delta^{\mathbf{m}}$ leads to a nice application of Birman-Krein theory and is done in [15] (see also the references therein). In what follows we chose once and forever the Friedrichs extension of $\Delta^{\mathbf{m}}$, all our results refer only to the case of this particular self-adjoint extension.

In [23] it was found an explicit expression for the determinant of (the Friedrichs extension of) the Laplace operator corresponding to flat conical metric $\mathbf{m}$ with *trivial holonomy*. Any metric of this type can be represented in the form $|\omega|^2$, where ω is a holomorphic one-form on X, zeros of ω of multiplicity ℓ are the conical points of the metric $|\omega|^2$ with conical angle $2\pi(\ell + 1)$. The moduli space of pairs (X, ω), where X is a compact Riemann surface and ω is a holomorphic one-form on X is stratified according to multiplicities of ω (see [27]). In [23] it was proved that on each stratum of the moduli space of holomorphic differentials the ratio $\frac{\det\Delta^{|\omega|^2}}{\det\Im\mathbf{B}}$, where $\mathbf{B}$ is the matrix of b-periods of the Riemann surface X, coincides with the modulus square of a holomorphic function τ on the stratum. This holomorphic function τ (the so-called Bergman tau-function on the space of holomorphic differentials) admits explicit expression through theta-functions, prime-forms and the divisor of the holomorphic one-form ω. In the case $g = 1$ the holomorphic one-forms have no zeroes, the metric $|\omega|^2$ is smooth and the corresponding result coincides with the classical Ray-Singer formula for the determinant of the Laplacian on an elliptic curve with flat conformal metric.

In [21] it was found a comparison formula (an analog of classical Polyakov formula) relating determinants of the Laplacians in two conformally equivalent flat conical metrics, this lead to the generalization of the results of [23] to the case of arbitrary flat conformal metrics with conical singularities.

Together with determinant of the Laplacians in flat conical metrics given by the modulus square of the holomorphic one form (these metric have finite volume and the spectra of the corresponding self-adjoint Laplacians are discrete) in physical literature appear determinants of the Laplacians corresponding to flat metrics $|\omega|^2$, where ω is now a *meromorphic* one form on X. Depending on the order of the poles of ω, the corresponding non compact Riemannian manifold $(X, |\omega|^2)$ of the infinite volume has cylindrical, Euclidean, or conical ends. The spectrum of the corresponding Laplace operator is continuous (with possible embedded eigenvalues, say, in case of cylindrical ends) and the Ray-Singer regularization of the determinant (1.1, 1.2) is no longer applicable. The way to regularize such determinants is, in principle, also well-known (see, e. g., [30]): considering the Laplacian Δ as a perturbation of some properly chosen "free" operator $\mathring{\Delta}$, one introduces the relative determinant $\det(\Delta, \mathring{\Delta})$ in terms of the so-called relative ζ-function

$$\zeta(s; \Delta, \mathring{\Delta}) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-\Delta t} - e^{-\mathring{\Delta} t}) t^{s-1}\, dt, \tag{1.3}$$

where a suitable regularization of the integral is used (being understood in the conventional sense the integral is usually divergent for any value of s).

Following this approach, in [16] we studied the regularized determinants

$$\det(\Delta, \mathring{\Delta}) = e^{-\zeta'(0;\Delta,\mathring{\Delta})}$$

of the Laplacians on the so-called Mandelstam diagrams - the flat surfaces with cylindrical ends (more precisely, Riemann surfaces X with the metric $|\omega|^2$, where ω is a meromorphic one-form on X with simple poles such that all the periods of ω are pure imaginary and all the residues of ω at the poles are real).

In the present paper we consider determinants of the Laplacian corresponding to flat metrics with even wilder singularities: the corresponding Riemannian manifold has Euclidean (i. e. isometric to a vicinity of the point at infinity of the Euclidean plane) or even conical ends (i. e. isometric to a vicinity of the point at infinity of a straight cone). These metrics are given as the modulus square of the differential of an arbitrary meromorphic function f on a compact Riemann surface X. The moduli space of pairs (X, f) is called the Hurwitz space $\mathcal{H}$. We define and study the regularized determinant of the Laplace operator corresponding to the metric $|df|^2$ as a functional on $\mathcal{H}$. The main result of the work is an explicit formula for this determinant.

It should be noted that such determinants for the first time appeared in [40], [3] (see also [18]), although no attempt was made to define them rigorously.

1.2 Results and organization of the paper

Let X be a Riemann surface and f be a meromorphic function $f : X \to \mathbb{P}^1$. We start by recalling that the metric $|df|^2$ gives to X the structure of a (non-compact) flat Riemannian manifold with conical singularities and conical (or Euclidean) ends. The conical singularities are located at the critical points $(P_m)_{1\le m\le M}$ of f. The moduli space of (equivalence classes) of such pairs (X, f) is known as Hurwitz space $\mathcal{H}$ and the the critical values $(z_m := f(P_k))_{1\le m\le M}$ where P_k locally parametrizes $\mathcal{H}$. Given such a Riemannian manifold $(X, |df|^2)$, we introduce the reference manifold $(\mathring{X}, \mathring{\mathbf{m}})$. This reference manifold can be seen as the disjoint union of the complete cones corresponding to the ends of $(X, |df|^2)$ and we will denote the Laplace operator $\Delta^{\mathring{\mathbf{m}}} := \mathring{\Delta}$.

The first part of the paper aims at defining the relative zeta-regularized determinant $\det^*_\zeta(\Delta^{|df|^2}, \mathring{\Delta})$ and proving a version of the Burghelea-Friedlander-Kappeler (BFK in what follows) gluing formula (see [4]). This new BFK type formula is a generalization of the Hassell-Zelditch formula for the determinant of the Laplacian in exterior domains [14] and we rely heavily on ideas from [6, 5, 14].

In order to obtain the gluing formula, we have to cut X along some hypersurface Σ. This decomposes X into a compact part X^- and the conical/Euclidean ends X_+. The latter is isometric to the reference surface $\mathring{X}$ with a compact part $\mathring{X}_-$. removed. There is some latitude in choosing the initial Σ. We choose our Σ by first specifying some large R and then choosing one circle in each conical end whose radius depend on R and on the cone angle of the conical end - see definition 3. As expected the gluing formula then involves the Neumann jump operator $\mathcal{N}$ on Σ and reads as follows.

Theorem 1. *Fix R large enough, we have*

$$\det^*_\zeta(\Delta^{|df|^2}, \mathring{\Delta}) = C \det^*_\zeta \mathcal{N} \cdot \det_\zeta \Delta^D_-$$

where $\mathcal{N}$, Δ^D_- depend on R. The constant C depends on R but not on the moduli parameters $z_1, \dots z_M$ as long as the corresponding critical points P_m do not approach Σ.

We should note that the proof of the gluing formula actually holds for a more general class of metric (see Remark 1).

Let us now sketch the different steps leading to this Theorem. First we start form the BFK gluing formula for $\det_\zeta(\Delta - \lambda, \mathring{\Delta} - \lambda)$ obtained in [6] for negative (regular) values of the spectral parameter λ. In order to obtain a gluing formula for $\det^*_\zeta(\Delta, \mathring{\Delta})$ (i.e. at the bottom of the continuous spectrum of Δ and $\mathring{\Delta}$), we study the behaviour of all ingredients in the gluing formula for $\det_\zeta(\Delta - \lambda, \mathring{\Delta} - \lambda)$ as $\lambda \to 0-$ and then pass to the limit. As usual, this essentially reduces to derivation of asymptotics as $\lambda \to 0-$ for the zeta regularized determinant of the Neumann jump operator and for the spectral shift function of the pair $(\Delta, \mathring{\Delta})$. In principle both asymptotics were obtained in [5] for Schroedinger type operators on manifolds with conical ends. Unfortunately those asymptotics cannot be used for our purposes because the asymptotic for the Neumann jump operator contains an unspecified constant and the asymptotic for the spectral shift function is not sufficiently sharp. We demonstrate that at least in our setting (no potential) the methods of [5] can be improved to specify the constant and to obtain a sufficiently sharp asymptotic of the spectral shift function as needed for the proof of our BFK formula. Once these asymptotics are obtained, we follow the lines of [14] in our study of the behaviour of $\det_\zeta(\Delta - \lambda, \mathring{\Delta} - \lambda)$ as $\lambda \to 0-$ and also in definition of $\det^*_\zeta(\Delta, \mathring{\Delta})$.

Using this BFK formula, we prove (as it was done in similar situations in [21], [16]) that the variations of the determinant of the Laplacian with respect to the moduli parameters z_k remain the same if we replace the metric $\mathbf{m} = |df|^2$ of infinite volume by a metric $\tilde{\mathbf{m}}$ of finite volume, where $\tilde{\mathbf{m}}$ coincides with $\mathbf{m}$ outside vicinities of the poles of f and with some standard nonsingular metric of finite volume inside these vicinities. The aim of the second part of the paper is thus to study the zeta-regularized determinant of this new metric $\tilde{\mathbf{m}}$ and its variation with respect to moduli parameters.

It turns out that these variations are conveniently expressed using the so-called S-matrix so we start the second part of the paper by introducing this object and deriving several of its properties. We think that the S-matrix is an important characteristic of a compact Riemann surface X equipped with a conformal metric $\tilde{\mathbf{m}}$ with conical singularity. It is defined in analogy both with scattering situations and the general theory of boundary triples (see [13]). Of particular interest here will be the fact that the value at $\lambda = 0$ can be expressed using the Schiffer projective connection of the Riemann surface X.

We will continue by studying the moduli variations of the zeta-regularized determinant of $\Delta^{\tilde{\mathbf{m}}}$. We will here use the Kato-Rellich perturbation theory to compute the variation of individual eigenvalue branches and then a contour argument similar to the one used [15] to get the variational formula for the determinant. This formula will be naturally expressed using $S(0)$ and hence using the Schiffer projective connection. Writing it into an invariant form, we will obtain the following theorem.

Theorem 2. *Let P_m be a zero of the meromorphic differential df of multiplicity l_m and let $z_m = f(P_m)$ be the corresponding critical value of f. Let also $x_m = (z - z_m)^{\frac{1}{l_m+1}}$ be the distinguished local parameter in a vicinity of P_m. Let $\tilde{\mathbf{m}}$ be the metric $|df|^2$ in which the conical ends have been smoothed. Then*

$$\partial_{z_m} \ln \frac{\det_\zeta^*(\Delta^{\tilde{\mathbf{m}}})}{\det \Im \mathbb{B}} = -\frac{1}{12\pi i} \oint_{P_m} \frac{S_B - S_f}{df} \tag{1.4}$$

where S_B is the Bergman projective connection, $S_f = \frac{f'''f' - \frac{3}{2}(f'')^2}{(f')^2}$ is the Schwarzian derivative, and $\mathbb{B}$ is the matrix of b-periods.

In this theorem we can replace $\det_\zeta^*(\Delta^{\tilde{\mathbf{m}}})$ by $\det^*(\Delta^{|df|^2}, \mathring{\Delta})$ since we have proved before that the moduli variations of both function coincide.

The system of PDE for $\det \Delta^{\tilde{\mathbf{m}}}$ that appears Theorem 2 is the governing system for the Bergman tau-function on the Hurwitz space (introduced and studied in [20], [23], [25], [24]). The latter system was explicitly integrated in [22] and in §5 we remind this result (unfortunately, technically involved). This leads to the following explicit formula for $\det_\zeta^*(\Delta, \mathring{\Delta})$.

Theorem 3. *Let (X, f) be an element of the Hurwitz space $\mathcal{H}(M, N)$ and let $\tau(X, f)$ be given by expressions (6.10,6.9, 6.8). There is the following explicit expression for the regularized relative determinant of the Laplacian $\Delta^{|df|^2}$ on the Riemann surface X:*

$$\det_\zeta(\Delta^{|df|^2}, \mathring{\Delta}) = C \det \Im \mathbb{B} \, |\tau|^2 \,, \tag{1.5}$$

where C is a constant that depends only on the connected component of the space $\mathcal{H}(M, N)$ containing the element (X, f).

We finish the paper with two illustrating examples in genus 0, deriving the formulas for the determinant of the Laplacian on the space of polynomials of degree N and on the space of rational functions with three simple poles.

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2 The regularized determinant as a functional on Hurwitz space and a BFK gluing formula

The Hurwitz space is the space of equivalence classes of pairs (X, f) where X is a compact Riemann surface and f is a meromorphic function defined on X. We start by recalling how a flat singular metric $\mathbf{m}$ on X may be associated to such a (X, f). Our aim is to define a regularized determinant and to prove a BFK-type gluing formula. However, since the metric $\mathbf{m}$ has conical singularities and non compact conical ends this will require several steps. First, we will work at non-positive energies (i.e. values of the spectral parameter λ such that $\lambda^2 \in \mathbb{C} \setminus [0, \infty)$). In that case, defining the relative determinant and the BFK-gluing formula follow from [6]. We will then derive estimates for the determinant of the Dirichlet-to-Neumann operator when λ approaches 0. The methods are very close to those of [5] and following [14] these estimates will allow us to define a zeta-regularized determinant at the energy 0. It will remain to study various limits to prove the gluing formula. We will end this section by using the gluing formula to prove that we can compactify X in such a way, that locally, the moduli variations remain the same.

2.1 The flat Laplacian of an element in Hurwitz space

We will be dealing with conical singularities and conical ends. These are defined in the following way.

Definition 1.

- *For any $\ell \in \mathbb{N}$ the Euclidean cone of total angle $2\ell\pi$ is the Riemannian manifold $(\mathbb{C}, |\ell y^{\ell-1} dy|^2)$.*
- *A point P in a Riemannian manifold will be a conical singularity of angle $2\ell\pi$ if there is a neighbourhood of P that is isometric to the set $\big(\{|y| < \varepsilon\}, |\ell y^{\ell-1} dy|^2\big)$ for some positive ε.*
- *A open set $\Omega \subset X$ of a Riemannian manifold $(X, \mathbf{m})$ such that $(\Omega, \mathbf{m})$ is isometric to $\big(\{|y| > R\}, |\ell y^{\ell-1} dy|^2\big)$ for some positive R will be called a conical end of angle $2\ell\pi$ (Euclidean end if $\ell = 1$).*

Let $(X, \mathbf{m})$ be a Riemannian manifold such that the metric is flat with a finite number of conical singularities and conical ends. We will denote by $\Delta^{\mathbf{m}}$ the self-adjoint operator which is obtained by the Friedrichs procedure starting from the (non-negative) Laplace operator on smooth functions that vanish near the conical singularities.

Let f be a meromorphic function on a compact Riemann surface X of genus $g \geq 0$ or, what is the same, a ramified covering of the Riemann sphere

$$f : X \to \mathbb{P}^1 \tag{2.1}$$

Two coverings $f_1 : X_1 \to P^1$ and $f_2 : X_2 \to P^1$ are called equivalent if there exists a biholomorhic map $g : X_1 \to X_2$ such that $f_1 = f_2 \circ g$.

The following construction is standard, we recall it for the convenience of the reader.

The critical points, P_m, $m = 1, \ldots, M$, of the function f (i. e. those points for which $df(P_m) = 0$) are the ramification points of the covering, the points $z_m = f(P_m)$ are called the critical values. The ramification index of the covering at the point P_m equals to $\ell_m + 1$, where ℓ_m is the order of the zero of the one-form df at P_m. Denote by $\infty_1, \ldots, \infty_K$ the poles of f, and let $k_1, \ldots, k_K$ be their multiplicities.

Then the covering (2.1) has degree $N = k_1 + \ldots k_K$ and the following Riemann-Hurwitz formula holds:

$$\sum_{m=1}^{M} \ell_m - \sum_{j=1}^{K} (k_j + 1) = 2g - 2\,,$$

where g is the genus of X.

Pick some regular value $z_0 \in \mathbb{P}^1$ and draw on $\mathbb{P}^1$ the segments $I_0 := [z_0, \infty]$, $I_m = [z_0, z_m]$, $m = 1, \ldots, M$. It may happen that some segment is repeated several times if different critical points take the same critical value. We may also choose z_0 such that all these segments have pairwise disjoint interiors. Denote by $L := \bigcup_{m=0}^{M} I_m$ the union of these segments and observe that $\mathbb{P}^1 \setminus L$ contains only regular values of f. It follows that $X \setminus f^{-1}(L)$, the complement of the preimage of L by f in X has N connected components. By construction f is a biholomorphic map from each of these connected components onto $\mathbb{P}^1 \setminus L$. We denote these connected components by $\mathbb{C}_n$, $n = 1 \ldots N$ and call them the sheets of the covering. Each $\mathbb{C}_n$ can be seen as a copy of the complex plane equipped with the cuts provided by L.

On each sheet, the metric $|dz|^2$ lifted from the base $\mathbb{P}^1$ to the covering space coincides with the metric $|df|^2$ and the Riemannian manifold $(X, |df|^2)$ is thus obtained by gluing N copies of a Euclidean plane $(\mathbb{C}, |dz|^2)$ with a system of non-intersecting cuts, one of which extends to infinity.

For each critical point P_m of ramification index $\ell_m + 1$, we obtain a $\ell_m + 1$-cycle γ_j obtained by looking in which order the sheets are following one another when making a small loop around P_m. It follows that each P_m, is a conical singularity of angle $2\pi(\ell_m + 1)$.

For each z_m, $m \neq 0$ we obtain a permutation in S_N by composing the cycles for each critical point in $f^{-1}(z_m)$. We thus obtain M' permutations $\sigma_{m'}$, $m' = 1 \ldots M'$ where M' is the number of different critical values. Each z_m is thus associated with one cycle in one of the permutations $\sigma_{m'}$, $m' = 1 \ldots M'$.

In the same manner we obtain a permutation σ_0 by looking at the preimage of a large loop that surrounds z_0 (or equivalently, a small loop around ∞ in the base $\mathbb{P}^1$). This permutation describes the structure at infinity of the Riemannian manifold $(X, |df|^2)$: each fixed point of σ_0 corresponds to a flat Euclidean end and a cycle of length k to a conical end of angle $2k\pi$. A pole in f of order k corresponds to a conical end of angle $2k\pi$ (and therefore a Euclidean end for a simple pole).

The flat structure on $(X, |df|^2)$ is completely characterized by the positions of the critical values z_m, $m = 1 \ldots M$ and by the permutations $\sigma_{m'}$, $m' = 0 \ldots M'$. Conversely, starting from $M'+1$ permutations of S_N, and M' distinct points $w_1,\dots w_{m'}$ in $\mathbb{C}$, we construct the sequence $z_0,\dots z_M$ by choosing a distinct point z_0 and, for $m>0$, by repeating $w_{m'}$ as many times as there are disjoint cycles in $\sigma_{m'}$. We then glue the N sheets according to the scheme prescribed by the permutations. We obtain a (not necessarily connected) flat surface $(X,\mathbf{m})$ with conical singularities and conical ends.

It turns out that it is always possible to find a meromorphic function f from X to $\mathbb{P}^1$ such that $(X,\mathbf{m})$ is isometric to $(X,|df|^2)$.

Introduce the Hurwitz space $\mathcal{H}(N,M)$ of equivalence classes of coverings $f:X\to P^1$ of degree N with M ramification points of (fixed) indices $\ell_1+1,\ \dots,\ \ell_M+1$ and l poles of (fixed) multiplicities $k_1,\dots,k_\ell$; $k_1+\dots,k_K=N$. The space $\mathcal{H}(N,M)$ is a complex manifold of dimension M (see [11], we notice here that it may have more than one connected components) and the critical values $z_1,\dots,z_M$ can be taken as local coordinates on $\mathcal{H}(N,M)$.

If all the critical points of the maps f are simple, then the corresponding Hurwitz space is usually denoted by $H_{g,N}(k_1,\dots,k_K)$ and is known to be connected (see [31]).

Definition 2. *We will refer to the coordinates $z_1,\dots,z_M$ as moduli.*

From the flat metric point of view, moving z_m can be easily easily realized by excising a small ball around P_m, then move P_m inside this ball. Since the boundary of the ball does not change we can then glue back the new ball in the surface.

For such a Riemannian manifold $(X,|df|^2)$ we define a reference manifold $(\mathring{X},\mathring{\mathbf{m}})$ which is obtained in the following way. Take the N sheets with cuts that correspond to X, and in the gluing scheme of X, keep σ_0 and replace all the permutations $\sigma_{m'}, m'>0$ by the identity. It can be easily seen that $(\mathring{X},\mathring{\mathbf{m}})$ actually consists in the disjoint union of the cones that correspond to the conical ends of X and the tip of each cone is now located above z_0.

The Laplacian $\Delta:=\Delta^{|df|^2}$ can be considered as a perturbation of the free Laplacian $\mathring{\Delta}:=\Delta^{\mathring{\mathbf{m}}}$ acting in the space $L_2((\mathbb{C}\setminus L)^N)$ or equivalently $L^2(\mathring{X})$. The perturbation is basically reduced to the change of the domain of the unbounded operator: when we make slits on $\mathring{X}$ and glue them according to a certain gluing scheme, it induces boundary conditions on the sides of the cuts. The regularized determinant of Δ will then be defined in terms of the relative zeta function (1.3) as a regularized relative determinant $\det^*_\zeta(\Delta,\mathring{\Delta})$.

The main goal of this work is to study the relative determinant $\det^*_\zeta(\Delta^{|df|^2},\mathring{\Delta})$ as a functional on the space $H(N,M)$.

2.2 Relative Determinant and BFK gluing formula for negative energies

Let X be a compact Riemann surface and let f be a nonconstant meromorphic function on X. Introduce the flat singular metric $\mathbf{m}=|df|^2$ on X. As it is explained in the previous section, the flat singular Riemannian manifold $(X,\mathbf{m})$ has conical points (at the zeros, $P_1,\dots,P_M$, of the differential df) and conical ends of angle $2\pi k_j$ at the poles, $\infty_1,\dots,\infty_K$, of f where k_j is the order of the corresponding pole. Let Δ be the (Friedrichs) Laplace operator of $(X,\mathbf{m})$.

We also let $(\mathring{X},\mathring{\mathbf{m}})$ be the reference unperturbed setting $\mathring{\Delta}$ the associated Laplace operator. We recall that $(\mathring{X},\mathring{\mathbf{m}})=\bigcup_{j=1}^K(\mathbb{C},|k_jy_j^{k_j-1}dy_j|^2)$.

Since $(X, \mathbf{m})$ and $(\mathring{X}, \mathring{\mathbf{m}})$ are isometric outside a compact region the methods and results of [5] apply.

For $R > 0$ large enough, there is a subset $X_+(R) \subset X$ that is isometric to

$$\cup_{j=1}^{K} \{y_j \in \mathbb{C}_j : |y_j| \geq R^{1/k_j}\} \subset \mathring{X}. \tag{2.2}$$

Definition 3. *We denote by Σ_R the boundary of the region $X_+(R)$. It is the union of K circles $\{y \in \mathbb{C} : |y| = R^{1/k_j}\}$ on X.*

Observe that R will be chosen at the very beginning of the construction and will then be fixed. In order to make the notations lighter, we will drop the reference to R and simply write $\Sigma,\ X_+$.

We represent X in the form

$$X = X_- \cup_\Sigma X_+,$$

where $X_- = X \setminus (X_+ \cup \Sigma)$.

Following [6] we first define the external Dirichlet-to-Neumann operator. We work in each conical end $\{y_j \in \mathbb{C}_j : |y_j| \geq R^{1/k_j}\}$ of X_+ separately and drop the index j for readibility. We introduce coordinates (r, φ) where $r = |y|^k \in [R, \infty)$ and $\varphi = \arg y \in (-\pi, \pi]$. We have

$$g = dr^2 + k^2 r^2\, d\varphi^2, \quad \Delta = r^{-2}\big((r\partial_r)^2 + k^{-2}\partial_\varphi^2\big).$$

Separation of variables shows that for $\lambda \in \mathbb{C} \setminus \{0\}$ with $\Im\lambda \geq 0$ the exterior Dirichlet problem

$$(\Delta - \lambda^2)u(\lambda) = 0 \text{ on } X_+, \quad u(\lambda) = f \text{ on } \Sigma, \tag{2.3}$$

has a unique solution of the form

$$u(r, \varphi; \lambda) = \sum_{n=-\infty}^{\infty} C_n \frac{H^{(1)}_{\nu_n}(\lambda r)}{H^{(1)}_{\nu_n}(\lambda R)} e^{in\varphi}, \quad \nu_n = \frac{|n|}{kR},$$

where $f \in C^\infty(\Sigma)$, $C_n = (2\pi)^{-1}\int_{-\pi}^{\pi} f(\varphi)e^{-in\varphi}\, d\varphi$, and $H^{(1)}_n$ is the Hankel function. This solution is in $L^2(X_+)$ if $\Im\lambda > 0$. If $\Im\lambda = 0$, it is the unique outgoing solution that satisfies the Sommerfeld radiation condition

$$\sqrt{r}\big(\partial_r u(\lambda) - i\lambda r u(\lambda)\big) \to 0 \text{ as } r \to \infty.$$

The external Dirichlet-to-Neumann operator on Σ acts by the formula

$$\mathcal{N}_+(\lambda)f = -\partial_r u(\lambda) \restriction_{r=R}. \tag{2.4}$$

Thus $\psi_n(\varphi) = (\operatorname{Vol}\Sigma)^{-1/2}e^{in\varphi}$ are eigenfunctions of $\mathcal{N}_+(\lambda)$ with $\|\psi_n\|_{L^2(\Sigma)} = 1$, and

$$\mu_n(\lambda) = \mu_{-n}(\lambda) = -\frac{\partial_r H^{(1)}_{\nu_n}(\lambda r) \restriction_{r=R}}{H^{(1)}_{\nu_n}(\lambda R)} \tag{2.5}$$

are the corresponding eigenvalues (if $\Im\lambda \geq 0$, $\lambda \neq 0$).

We can also identify Σ and X^+ as subsets of $\mathring{X}$ and, in the same manner we have $\mathring{X} = \mathring{X}_- \cup_\Sigma \mathring{X}_+$.

Let $\Delta^D_\pm$ be the Friedrichs extensions of the Dirichlet Laplace operator in $L^2(X_\pm)$. We denote by $\Delta^D := \Delta^D_- \oplus \Delta^D_+$ the Friedrichs Laplace operator on $L^2(X)$ with Dirichlet boundary condition on Σ. Similarly, we define $\mathring{\Delta}^D_\pm$ and $\mathring{\Delta}^D$.

The spectrum $\operatorname{spec}(\Delta^D_-)$ of the positive self-adjoint operator Δ^D_- is discrete. For any $\lambda^2 \in \mathbb{C} \setminus \sigma(\Delta^D_-)$ and $f \in H^1(\Sigma)$ there exists a unique solution $u(\lambda) \in H^{3/2}(X_-)$ to the Dirichlet problem

$$(\Delta - \lambda^2)u(\lambda) = 0 \text{ on } X_- \setminus \Sigma, \quad u(\lambda) = f \text{ on } \Sigma, \tag{2.6}$$

such that

$$u(\lambda) = \tilde{f} - (\Delta^D_- - \lambda^2)^{-1}(\Delta - \lambda^2)\tilde{f}, \tag{2.7}$$

where $\tilde{f} \in H^{3/2}(X_-)$ is a continuation of f and

$$(\Delta^D_- - \lambda^2)^{-1} : H^{-1/2}(X_-) \to H^{3/2}(X_-)$$

is a holomorphic function of $\lambda^2 \in \mathbb{C} \setminus \sigma(\Delta^D_-)$; here $\|u; H^s(X_-)\| = \|(\Delta^D_-)^{s/2}u; L^2(X_-)\|$. The Dirichlet-to-Neumann operator $\mathcal{N}_-(\lambda)$ on Σ acts by the formula

$$\mathcal{N}_-(\lambda)f = \partial_r u(\lambda) \restriction_{r=R},$$

where f is the right hand side in (2.6) and $u(\lambda)$ is defined by (2.7). The function $\lambda^2 \mapsto \mathcal{N}_-(\lambda) \in \mathcal{B}(H^1(\Sigma), L^2(\Sigma))$ is holomorphic in $\mathbb{C} \setminus \sigma(\Delta^D_-)$; here and elsewhere $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ stands for the space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$.

Finally, we introduce the Neumann jump operator

$$\mathcal{N}(\lambda) = \mathcal{N}_+(\lambda) + \mathcal{N}_-(\lambda),$$

which is a first order elliptic classical pseudodifferential operator on Σ with the principal symbol $2|\xi|$.

For $\lambda^2 \le 0$ the operator $\mathcal{N}(\lambda)$ is formally self-adjoint and nonnegative, it is positive if $\lambda^2 < 0$, and $\ker \mathcal{N}(0) = \{c \in \mathbb{C}\}$ (see e.g. [5, Sec. 3.3] for details). (Note that in Theorem 4 the operator $\mathcal{N}(0)$ is denoted by $\mathcal{N}$.) Let $\lambda^2 < 0$. The function $\zeta(s) = \operatorname{Tr} \mathcal{N}(\lambda)^{-s}$ is holomorphic in $\{s \in \mathbb{C} : \Re s > 1\}$ and admits a meromorphic continuation to $\mathbb{C}$ with no pole at zero; we set $\det_\zeta \mathcal{N}(\lambda) = e^{-\zeta'(0)}$.

It is known (see [6, Theorem 2.2]) that the difference

$$(\Delta + 1)^{-1} - (\Delta^D + 1)^{-1}$$

is in the trace class. By the Krein theorem, see e.g. [37, Chapter 8.9] or [6, Theorem 3.3], there exists a spectral shift function $\xi(\cdot\,; \Delta, \Delta^D) \in L^1(\mathbb{R}_+, (1 + \lambda^2)^{-2}\lambda\, d\lambda)$ such that

$$\operatorname{Tr}\big((\Delta + 1)^{-1} - (\Delta^D + 1)^{-1}\big) = -\int_0^\infty \xi(\lambda\,; \Delta, \Delta^D)(1 + \lambda^2)^{-2} 2\lambda\, d\lambda. \tag{2.8}$$

Moreover, the following representation is valid

$$\operatorname{Tr}\big(e^{-t\Delta} - e^{-t\Delta^D}\big) = -t \int_0^\infty e^{-t\lambda^2} \xi(\lambda\,; \Delta, \Delta^D) 2\lambda\, d\lambda, \tag{2.9}$$

which implies that the left hand side in (2.9) is absolutely bounded uniformly in $t > \epsilon > 0$. The heat trace asymptotic

$$\operatorname{Tr}\big(e^{-t\Delta} - e^{-t\Delta^D}\big) \sim \sum_{j \ge -2} a_j t^{j/2}, \quad t \to 0+, \tag{2.10}$$

can be obtained in a usual way, see e.g. [16, Lemma 4]. Thus for $\lambda^2 < 0$ the relative zeta function given by

$$\zeta(s; \Delta - \lambda^2, \Delta^D - \lambda^2) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{\lambda^2 t} \operatorname{Tr}(e^{-t\Delta} - e^{-t\Delta^D})\, dt$$

is defined for $\Re s > 1$ and continues meromorphically to the complex plane with no pole at $s = 0$ by the usual argument. The relative determinant is defined to be

$$\det_\zeta(\Delta - \lambda^2, \Delta^D - \lambda^2) = e^{-\zeta'(0; \Delta - \lambda^2, \Delta^D - \lambda^2)}.$$

By [6, Theorem 4.2] we have the gluing formula

$$\det_\zeta(\Delta - \lambda^2, \Delta^D - \lambda^2) = \det_\zeta \mathcal{N}(\lambda), \quad \lambda^2 < 0. \tag{2.11}$$

(Although only smooth manifolds are considered in [6], it is fairly straightforward to see that the argument in [6] remains valid for (2.11) as far as we consider only Friederichs extensions and there are no conical points on Σ.)

All the constructions above can also be done for $(\mathring{X}, \mathring{\mathbf{m}})$. Thus similarly to (2.11) we have

$$\det_\zeta(\mathring{\Delta} - \lambda^2, \mathring{\Delta}^D - \lambda^2) = \det \mathring{\mathcal{N}}_\zeta(\lambda), \quad \lambda^2 < 0. \tag{2.12}$$

Observe that since all operators can be seen as acting on $L^2(\mathring{X})$ we have

$$\begin{aligned} e^{-t\Delta} - e^{-t\mathring{\Delta}} &= \left(e^{-t\Delta} - e^{-t\Delta^D}\right) - \left(e^{-t\mathring{\Delta}} - e^{-t\mathring{\Delta}^D}\right) \\ &\quad + \left(e^{-t\Delta^D} - e^{-t\mathring{\Delta}^D}\right) \\ &= \left(e^{-t\Delta} - e^{-t\Delta^D}\right) - \left(e^{-t\mathring{\Delta}} - e^{-t\mathring{\Delta}^D}\right) \\ &\quad + \left(e^{-t\Delta_-^D} - e^{-t\mathring{\Delta}_-^D}\right), \end{aligned}$$

where for the last line we have used that $\Delta^D = \Delta_-^D \oplus \Delta_+^D$, $\mathring{\Delta}^D = \mathring{\Delta}_-^D \oplus \Delta_+^D$ since X_+ and $\mathring{X}_+$ are isometric.

It follows that we can take the trace of both sides and thus define the following relative zeta function for $\Re s > 1$

$$\zeta(s; \Delta - \lambda^2, \mathring{\Delta} - \lambda^2) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{\lambda^2 t} \operatorname{Tr}(e^{-t\Delta} - e^{-t\mathring{\Delta}})\, dt, \quad \lambda^2 < 0.$$

Moreover, we obtain the relation

$$\begin{aligned} \zeta(s; \Delta - \lambda^2, \mathring{\Delta} - \lambda^2) =\ &\zeta(s; \Delta - \lambda^2, \mathring{\Delta}^D - \lambda^2) - \zeta(s; \mathring{\Delta} - \lambda^2, \mathring{\Delta}^D - \lambda^2) \\ &+ \zeta(s, \Delta_-^D - \lambda^2) - \zeta(s, \mathring{\Delta}_-^D - \lambda^2). \end{aligned}$$

All the functions continue meromorphically to the complex plane with no pole at 0. Passing to the determinant, we obtain

$$\frac{\det_\zeta(\Delta - \lambda^2, \Delta^D - \lambda^2)}{\det_\zeta(\mathring{\Delta} - \lambda^2, \mathring{\Delta}^D - \lambda^2)} = \frac{\det_\zeta(\Delta - \lambda^2, \mathring{\Delta} - \lambda^2) \det_\zeta(\mathring{\Delta}_-^D - \lambda^2)}{\det_\zeta(\Delta_-^D - \lambda^2)}.$$

Thus dividing (2.11) by (2.12) we obtain

$$\frac{\det_\zeta(\Delta - \lambda^2, \mathring{\Delta} - \lambda^2) \det_\zeta(\mathring{\Delta}_-^D - \lambda^2)}{\det_\zeta(\Delta_-^D - \lambda^2)} = \frac{\det_\zeta \mathcal{N}(\lambda)}{\det_\zeta \mathring{\mathcal{N}}(\lambda)}, \quad \lambda^2 < 0, \tag{2.13}$$

where $\mathring{\mathcal{N}}(\lambda)$ and $\mathring{\Delta}_-^D$ are moduli independent.

In order to take the limit $\lambda^2 \to 0-$ in (2.13), we will need the asymptotic behavior of all the ingredients in the latter equation. We start with $\det_\zeta \mathcal{N}(\lambda)$.

2.3 Asymptotic of $\det_\zeta \mathcal{N}(\lambda)$ as $|\lambda| \to 0+$, $\Im\lambda \geq 0$

In this section we follow closely [5] in which a similar problem is studied.

First we need to understand the behavior of the internal and external Dirichlet-to-Neumann operators. Since the internal Dirichlet Laplace operator Δ_-^D is positive, there is no problem in letting λ go to 0 in the definition of $\mathcal{N}_-(\lambda)$.

Concerning the external Dirichlet-to-Neumann operator, using again separation of variables in each conical end, we see that the problem

$$(\Delta - \lambda^2)u(\lambda) = 0 \text{ on } X_+, \quad u(\lambda) = f \text{ on } \Sigma, \tag{2.14}$$

in the case $\lambda = 0$, has a unique solution that satisfies the Sommerfeld radiation condition. This solution is given by

$$u(r, \varphi; 0) = \sum_{n=-\infty}^{\infty} C_n \Big(\frac{R}{r}\Big)^{\nu_n} e^{in\varphi} \tag{2.15}$$

where $f = \sum C_n e^{in\varphi}$ and we recall that $\nu_n = \frac{|n|}{kR}$ and that we have set $\psi_n(\phi) = (2\pi)^{-\frac{1}{2}} e^{in\phi}$. The external Dirichlet to Neumann $\mathcal{N}_+(0)$ is obtained by applying $-\partial_r$ to this solution and clearly, $\{|n|/(kR^2), \psi_n\}_{n=-\infty}^{\infty}$ is a complete set of the eigenvalues and orthonormal eigenfunctions of the operator $\mathcal{N}_+(0)$.

Remark 1. *Note that thanks to the special choice of the lower bound on y in* (2.2) *the eigenvalue $\mu_0(\lambda)$ of $\mathcal{N}_+(\lambda)$ corresponding to the constant eigenfunction ψ_0 does not depend on k. This will be important in our proof of the BFK gluing formula in the case $K \geq 1$ if $k_j \neq k_i$ for some $i, j = 1, \dots, K$.*

It is convenient to present the argument in the case where $K = 1$ so that X has only one conical end. We will explain afterwards how the proof is modified for $K > 1$.

2.3.1 The case $K = 1$

In the series (2.15) only the terms with $\nu_n > 1$ are in $L^2(X_+)$. As a result, in a neighborhood of zero, properties of $\lambda \mapsto \mathcal{N}_+(\lambda)$ on the eigenspaces of $\mathcal{N}_+(0)$ corresponding to the eigenvalues $|n|/(kR^2) > 1/R$ and $|n|/(kR^2) \leq 1/R$ are essentially different. Consider the spectral projector $\mathbf{P} = \sum_{0 \leq n \leq kR} P_n$ of $\mathcal{N}_+(0)$ on the interval $[0, 1/R]$; here

$$P_0 = (\cdot, \psi_0)_{L^2(\Sigma)}; \quad P_n = (\cdot, \psi_n)_{L^2(\Sigma)}\psi_n + (\cdot, \psi_{-n})_{L^2(\Sigma)}\psi_{-n}.$$

Lemma 1 (see [5, Prop 4.5])**.** *We have*

$$\mathcal{N}_+(\lambda)\big(\mathrm{Id} - \mathbf{P}\big) = \Psi(\lambda^2) + \mathcal{L}(\lambda^2),$$

where $\Psi(z)$ is an elliptic pseudo-differential operator of order 1 which is a holomorphic function of z in a neighbourhood of zero and $\mathcal{L}(z)$ is an operator with smooth integral kernel which is a C^1 function of z in a neighbourhood of zero with $\Im z \geq 0$.

Recall that the eigenfunctions ψ_n of $\mathcal{N}_+(\lambda)$ do not depend on λ and we have

$$\mathcal{N}_+(\lambda)\mathbf{P}f = \sum_{0 \leq n \leq kR} \mu_n(\lambda) P_n$$

where the μ_n have been defined in (2.5).

The eigenvalues of $\mathcal{N}_+(0)$ on $[0,1/R]$ are the limits of $\mu_n(\lambda)$. As $|\lambda|\to 0+$, $\Im\lambda\geq 0$, the formula (2.5) and properties of the Hankel functions (see [1]) imply that

$$\mu_0(\lambda)=-\frac{1}{R\ln\lambda}\Big(1-\Big(\ln\frac{R}{2}+\frac{\pi\gamma}{2}-i\frac{\pi}{2}\Big)\frac{1}{\ln\lambda}+O\Big(\frac{1}{(\ln\lambda)^2}\Big)\Big), \tag{2.16}$$

where γ is the Euler's constant, and

$$\mu_n(\lambda)=|n|/(kR^2)+O(\lambda^\epsilon),\quad 0<|n|<kR, \tag{2.17}$$

with some $\epsilon>0$.

We show the following proposition.

Proposition 1. *Assume $(X,|df|^2)$ has only one conical end then, for any R large enough we have, as $|\lambda|\to 0+$, $\Im\lambda\geq 0$,*

$$\det_\zeta\mathcal{N}(\lambda)=-\frac{1}{R\ln\lambda}\det{}^*_\zeta\mathcal{N}(0)\Big(1-\Big(\ln\frac{R}{2}+\frac{\pi\gamma}{2}-i\frac{\pi}{2}\Big)\frac{1}{\ln\lambda}+O\Big(\frac{1}{(\ln\lambda)^2}\Big)\Big),$$

where $\det_\zeta\mathcal{N}(\lambda)$ is the zeta regularized determinant of $\mathcal{N}(\lambda)$, and $\det^*_\zeta\mathcal{N}(0)$ is the zeta regularized determinant of $\mathcal{N}(0)$ with zero eigenvalue excluded.

Proof. Due to the representation $L^2(\Sigma)=\ker P_0\oplus\ker(\mathbf{P}-P_0)\oplus\ker(\mathrm{Id}-\mathbf{P})$ we have

$$\mathcal{N}(\lambda)=\begin{pmatrix}\mathcal{N}_{0,0}(\lambda)&\mathcal{N}_{0,1}(\lambda)&\mathcal{N}_{0,2}(\lambda)\\ \mathcal{N}_{1,0}(\lambda)&\mathcal{N}_{1,1}(\lambda)&\mathcal{N}_{1,2}(\lambda)\\ \mathcal{N}_{2,0}(\lambda)&\mathcal{N}_{2,1}(\lambda)&\mathcal{N}_{2,2}(\lambda)\end{pmatrix}, \tag{2.18}$$

where $\mathcal{N}_{i,j}(\lambda)=\mathcal{P}_i\mathcal{N}(\lambda)\mathcal{P}_j$ with $\mathcal{P}_0=P_0$, $\mathcal{P}_1=\mathbf{P}-P_0$, and $\mathcal{P}_2=\mathrm{Id}-\mathbf{P}$.

The operator $\mathcal{N}_{2,2}(0)$ is invertible and therefore $\det_\zeta\mathcal{N}_{2,2}(0)\neq 0$. Note that $\mathcal{N}_{2,2}(\lambda)=\mathcal{P}_2\mathcal{N}_-(\lambda)\mathcal{P}_2+\mathcal{P}_2\mathcal{N}_+(\lambda)\mathcal{P}_2$, where $\mathcal{N}_+(\lambda)\mathcal{P}_2=\mathcal{N}_+(\lambda)(\mathrm{Id}-\mathbf{P})$ is the same as in Lemma 1 and $\mathcal{N}_-(\lambda)$ is a holomorphic function of λ^2 in a small neighbourhood of zero. This implies that

$$\det_\zeta\mathcal{N}_{2,2}(\lambda)-\det_\zeta\mathcal{N}_{2,2}(0)=o(1),\quad |\lambda|\to 0+,\Im\lambda\geq 0. \tag{2.19}$$

Thanks to Lemma 1 we also have

$$\begin{aligned}\|\mathcal{N}_{2,2}(\lambda)-\mathcal{N}_{2,2}(0);\mathcal{B}(H^1(\Sigma),L^2(\Sigma))\|&=O(\lambda^2),\\ \|\partial_\lambda\mathcal{N}_{2,2}(\lambda)-\partial_\lambda\mathcal{N}_{2,2}(0);\mathcal{B}(H^1(\Sigma),L^2(\Sigma))\|&=O(\lambda).\end{aligned} \tag{2.20}$$

In order to refine (2.19), we estimate the absolute value of $\partial_\lambda\ln\det_\zeta\mathcal{N}_{2,2}(\lambda)$. Since $\partial_\lambda\mathcal{N}_{2,2}(\lambda)$ and $\mathcal{N}^{-1}_{2,2}(\lambda)$ are pseudodifferential operators of order -1, the operator

$$\mathcal{N}^{-1}_{2,2}(\lambda)\partial_\lambda\mathcal{N}_{2,2}(\lambda)$$

is in the trace class, and hence

$$\partial_\lambda\ln\det_\zeta\mathcal{N}_{2,2}(\lambda)=\mathrm{Tr}\big\{\mathcal{N}^{-1}_{2,2}(\lambda)\partial_\lambda\mathcal{N}_{2,2}(\lambda)\big\}, \tag{2.21}$$

see [4, 10]. The first estimate in (2.20) and the Neumann series for $\mathcal{N}^{-1}_{2,2}(\lambda)$ give

$$\begin{aligned}\mathcal{N}^{-1}_{2,2}(\lambda)&=\big(\mathrm{Id}+L(\lambda)\big)\mathcal{N}^{-1}_{2,2}(0),\quad \|L(\lambda);\mathcal{B}(H^1(\Sigma))\|=O(\lambda^2);\\ \mathcal{N}^{-1}_{2,2}(\lambda)&=\mathcal{N}^{-1}_{2,2}(0)\big(\mathrm{Id}+R(\lambda)\big),\quad \|R(\lambda);\mathcal{B}(L^2(\Sigma))\|=O(\lambda^2).\end{aligned} \tag{2.22}$$

As a consequence of (2.21), (2.22), and (2.20) we get

$$\begin{aligned}
|\partial_\lambda \ln \det_\zeta \mathcal{N}_{2,2}(\lambda)| &= |\operatorname{Tr}\{\mathcal{N}_{2,2}^{-1}(\lambda)\partial_\lambda \mathcal{N}_{2,2}(\lambda)\}| \\
&\le \|\mathcal{N}_{2,2}(\lambda)(\operatorname{Id}+L(\lambda))\mathcal{N}_{2,2}^{-2}(0)(\operatorname{Id}+R(\lambda))\partial_\lambda \mathcal{N}_{2,2}(\lambda)\|_1 \\
&\le \|\mathcal{N}_{2,2}(\lambda); \mathcal{B}(H^1(\Sigma), L^2(\Sigma))\|\|\operatorname{Id}+L(\lambda); \mathcal{B}(H^1(\Sigma))\|\|\mathcal{N}_{2,2}^{-2}(0)\|_1 \\
&\times \|\operatorname{Id}-R(\lambda); \mathcal{B}(L^2(\Sigma))\|\|\partial_\lambda \mathcal{N}_{2,2}(\lambda); \mathcal{B}(H^1(\Sigma), L^2(\Sigma))\| = O(1),
\end{aligned}$$

where $\|\cdot\|_1$ is the trace norm. This together with (2.19) implies $|\partial_\lambda \det_\zeta \mathcal{N}_{2,2}(\lambda)| = O(1)$. Now, as a refinement of (2.19), we obtain

$$\det_\zeta \mathcal{N}_{2,2}(\lambda) - \det_\zeta \mathcal{N}_{2,2}(0) = O(\lambda).$$

This together with (2.18) implies

$$\begin{aligned}
\det_\zeta \mathcal{N}(\lambda) &= \det_{\mathrm{Fr}} \begin{pmatrix} \mathcal{N}_{0,0}(\lambda) & \mathcal{N}_{0,1}(\lambda) & \mathcal{N}_{0,2}(\lambda)\mathcal{N}_{2,2}(\lambda)^{-1} \\ \mathcal{N}_{1,0}(\lambda) & \mathcal{N}_{1,1}(\lambda) & \mathcal{N}_{1,2}(\lambda)\mathcal{N}_{2,2}(\lambda)^{-1} \\ \mathcal{N}_{2,0}(\lambda) & \mathcal{N}_{2,1}(\lambda) & \operatorname{Id} \end{pmatrix} \det_\zeta \begin{pmatrix} \operatorname{Id} & 0 & 0 \\ 0 & \operatorname{Id} & 0 \\ 0 & 0 & \mathcal{N}_{2,2}(\lambda) \end{pmatrix} \\
&= \det_{\mathrm{Fr}} \begin{pmatrix} \mathcal{N}_{0,0}(\lambda) & \mathcal{N}_{0,1}(\lambda) & \mathcal{N}_{0,2}(\lambda)\mathcal{N}_{2,2}(\lambda)^{-1} \\ \mathcal{N}_{1,0}(\lambda) & \mathcal{N}_{1,1}(\lambda) & \mathcal{N}_{1,2}(\lambda)\mathcal{N}_{2,2}(\lambda)^{-1} \\ \mathcal{N}_{2,0}(\lambda) & \mathcal{N}_{2,1}(\lambda) & \operatorname{Id} \end{pmatrix} \big(\det_\zeta \mathcal{N}_{2,2}(0)\big)\big(1+O(\lambda)\big);
\end{aligned} \tag{2.23}$$

see [26] for the first equality. On the next step we rely on the estimate

$$|\det_{\mathrm{Fr}}(\operatorname{Id}+A) - \det_{\mathrm{Fr}}(\operatorname{Id}+B)| \le \|A-B\|_1 e^{\|A\|_1+\|B\|_1+1}, \tag{2.24}$$

see [34, f-la (3.7), and references therein], for

$$\operatorname{Id}+A = \begin{pmatrix} \mathcal{N}_{0,0}(\lambda) & 0 & 0 \\ 0 & \mathcal{N}_{1,1}(0) & \mathcal{N}_{1,2}(0)\mathcal{N}_{2,2}^{-1}(0) \\ 0 & \mathcal{N}_{2,1}(0) & \operatorname{Id} \end{pmatrix},$$

$$\operatorname{Id}+B = \begin{pmatrix} \mathcal{N}_{0,0}(\lambda) & \mathcal{N}_{0,1}(\lambda) & \mathcal{N}_{0,2}(\lambda)\mathcal{N}_{2,2}(\lambda)^{-1} \\ \mathcal{N}_{1,0}(\lambda) & \mathcal{N}_{1,1}(\lambda) & \mathcal{N}_{1,2}(\lambda)\mathcal{N}_{2,2}(\lambda)^{-1} \\ \mathcal{N}_{2,0}(\lambda) & \mathcal{N}_{2,1}(\lambda) & \operatorname{Id} \end{pmatrix}.$$

Since $\mathcal{N}_-(0)$ is a selfadjoint operator in $L^2(\Sigma)$ and $\ker \mathcal{N}_-(0) = \{c \in \mathbb{C}\}$, we have $\mathcal{N}_-(0)P_0 = P_0\mathcal{N}_-(0) = 0$. Then thanks to

$$P_i\big(\mathcal{N}(\lambda) - \mathcal{N}(0)\big)P_j = \delta_{ij}\big(\mu_j(\lambda) - \mu_j(0)\big) + P_i\big(\mathcal{N}_-(\lambda) - \mathcal{N}_-(0)\big)P_j, \quad i,j \in [0, 1/R],$$

where δ_{ij} is the Kronecker delta function, and

$$P_j\big(\mathcal{N}(\lambda) - \mathcal{N}(0)\big)(\operatorname{Id}-\mathbf{P}) = P_j\big(\mathcal{N}_-(\lambda) - \mathcal{N}_-(0)\big)(\operatorname{Id}-\mathbf{P}), \quad j \in [0, 1/R],$$

together with (2.17) and (2.22), we obtain $\|A-B\|_1 = O(\lambda^\epsilon)$ with some $\epsilon > 0$. From (2.23) and (2.24) we get

$$\det_\zeta \mathcal{N}(\lambda) = \det_{\mathrm{Fr}} \begin{pmatrix} \mathcal{N}_{0,0}(\lambda) & 0 & 0 \\ 0 & \mathcal{N}_{1,1}(0) & \mathcal{N}_{1,2}(0)\mathcal{N}_{2,2}(0)^{-1} \\ 0 & \mathcal{N}_{2,1}(0) & \operatorname{Id} \end{pmatrix} \det_\zeta \mathcal{N}_{2,2}(0)(1+O(\lambda^\epsilon)). \tag{2.25}$$

It remains to note that

$$\mathcal{N}_{0,0}(\lambda) = (\mu_0(\lambda) + P_0(\mathcal{N}_-(\lambda) - \mathcal{N}_-(0))P_0 = (\mu_0(\lambda) + O(\lambda^2))P_0,$$

$$\det{}^*_\zeta \mathcal{N}(0) = \det{}_{\mathrm{Fr}} \begin{pmatrix} \mathcal{N}_{1,1}(0) & \mathcal{N}_{1,2}(0)\mathcal{N}_{2,2}(0)^{-1} \\ \mathcal{N}_{2,1}(0) & \mathrm{Id} \end{pmatrix} \det{}_\zeta \mathcal{N}_{2,2}(0).$$

This together with (2.25) and (2.16) completes the proof. □

Corollary 1. *The spectral shift function ξ in* (2.9) *satisfies*

$$\xi(\lambda\,;\Delta,\Delta^D) = (\ln\lambda^2)^{-1} + O\big((\ln\lambda)^{-2}\big), \quad \lambda \to 0+\,. \tag{2.26}$$

Proof. By [6, Theorem 3.5] we have $\xi(\lambda) = \pi^{-1}\operatorname{Arg}\det\mathcal{N}(\sqrt{\lambda^2+i0})$ as $\lambda^2 \to 0+$, where $\operatorname{Arg} z \in (-\pi,\pi]$, and $\xi(\lambda) = 0$ if $\lambda^2 < 0$. Calculation of the argument in the asymptotic obtained in Propositon 1 gives (2.26). □

2.3.2 The case $K > 1$

Let us outline the changes in Proposition 1 and Corollary 1 needed in the case $K > 1$. Now we have $\mathcal{N}_+(\lambda) = \oplus_{j=1}^K \mathcal{N}_+^{(j)}(\lambda)$, where each $\mathcal{N}_+^{(j)}(\lambda)$ is defined on the circle $\{y \in \mathbb{C}_j : |y| = R^{1/k_j}\}$ as in (2.4). The first eigenvalue of $\mathcal{N}_+^{(j)}(\lambda)$ is $\mu_0(\lambda)$ and the corresponding eigenspace consists of constant functions on the circle. As a consequence, in the estimate (2.25) the eigenvectors in $\mathcal{N}_+$ with eigenvalue $\mu_0(\lambda)$ contribute at the order $O(\frac{1}{\ln\lambda})$ instead of $O(\lambda^\epsilon)$ and this is not good enough for our purpose.

We thus introduce $\mathbf{P_0}$ the orthogonal projection onto the eigenspace of $\mathcal{N}_+(\lambda)$ corresponding to $\mu_0(\lambda)$ (note that $\mathbf{P_0}$ does not depend on λ, and that $\operatorname{rank}\mathbf{P_0} = \mathbf{K}$). Observe that we have $\ker\mathcal{N}(0) \subset \ker(\mathrm{Id} - \mathbf{P_0})$. We repeat the argument of Proposition 1, where $\mathcal{P}_0$ is now the orthogonal projection onto $\ker\mathcal{N}(0) = \{c \in \mathbb{C}\}$, $\mathcal{P}_1 = (\mathrm{Id} - \mathcal{P}_0)\mathbf{P}$ where $\mathbf{P}$ is the spectral projection of $\mathcal{N}_+(0)$ on the interval $[0, 1/R]$, and $\mathcal{P}_2 = (\mathrm{Id} - \mathbf{P})$. Clearly, $\mathcal{N}_+(\lambda)\mathcal{P}_0 = \mu_0(\lambda)\mathcal{P}_0$ and $\mathcal{N}_+(\lambda)(\mathrm{Id} - \mathcal{P}_0) = (\mathrm{Id} - \mathcal{P}_0)\mathcal{N}_+(\lambda)$.

The same argument as in the case $K = 1$ leads to

$$\begin{aligned}\det{}_\zeta \mathcal{N}(\lambda) = -\frac{1}{R\ln\lambda} \det{}_{\mathrm{Fr}} &\begin{pmatrix} \mathcal{N}_{1,1}(0) + \mu_0(\lambda)\mathbf{P_0}(\mathrm{Id} - \mathcal{P}_0) & \mathcal{N}_{1,2}(0)\mathcal{N}_{2,2}(0)^{-1} \\ \mathcal{N}_{2,1}(0) & \mathrm{Id} \end{pmatrix} \\ &\times \det{}_\zeta \mathcal{N}_{2,2}(0)\Big(1 - \Big(\ln\frac{R}{2} + \frac{\pi\gamma}{2} - i\frac{\pi}{2}\Big)\frac{1}{\ln\lambda} + O\Big(\frac{1}{(\ln\lambda)^2}\Big)\Big).\end{aligned}$$

(Note that in the case $K = 1$ we have $\mathbf{P_0} = \mathcal{P}_0$ and the term $\mu_0(\lambda)\mathbf{P_0}(\mathrm{Id} - \mathcal{P}_0)$ does not appear.) This together with (2.16) and (2.24) gives

$$\begin{aligned}\det{}_\zeta \mathcal{N}(\lambda) = -\frac{1}{R\ln\lambda} \det{}_{\mathrm{Fr}} &\begin{pmatrix} \mathcal{N}_{1,1}(0) - \frac{1}{R\ln\lambda}\mathbf{P_0}(\mathrm{Id} - \mathcal{P}_0) & \mathcal{N}_{1,2}(0)\mathcal{N}_{2,2}(0)^{-1} \\ \mathcal{N}_{2,1}(0) & \mathrm{Id} \end{pmatrix} \\ &\times \det{}_\zeta \mathcal{N}_{2,2}(0)\Big(1 - \Big(\ln\frac{R}{2} + \frac{\pi\gamma}{2} - i\frac{\pi}{2}\Big)\frac{1}{\ln\lambda} + O\Big(\frac{1}{(\ln\lambda)^2}\Big)\Big).\end{aligned}$$

Here the Fredholm determinant is a holomorphic function of the parameter $\tau := \frac{1}{R\ln\lambda}$, we have

$$\det{}_\zeta \mathcal{N}(\lambda) = -\frac{1}{R\ln\lambda} \det{}^*_\zeta \mathcal{N}(0)\Big(1 - \Big(C + \ln\frac{R}{2} + \frac{\pi\gamma}{2} - i\frac{\pi}{2}\Big)\frac{1}{\ln\lambda} + O\Big(\frac{1}{(\ln\lambda)^2}\Big)\Big) \tag{2.27}$$

with some constant $C = C(R)$.

We observe that C must be real since $\det_\zeta \mathcal{N}(\lambda)$ is positive for $\lambda \in \mathbb{C}$, $\operatorname{Arg}\lambda = \pi/2$ (as $\mathcal{N}(\lambda)$ is a positive self-adjoint operator for those values of λ). Thus C does not influence the calculation of the argument in the asymptotic of $\det_\zeta \mathcal{N}(\lambda)$ and Corollary 1 remains valid for $K > 1$. We will use Corollary 1 to define a relative determinant of $(\Delta, \mathring{\Delta})$ at the energy 0.

2.4 The relative determinant and the gluing formula at energy 0

In this section we prove the following Theorem.

Theorem 4. *The gluing formula*

$$\det_\zeta^*(\Delta, \mathring{\Delta}) = C \det_\zeta^* \mathcal{N} \cdot \det_\zeta \Delta_-^D$$

is valid, where $\mathcal{N}$, Δ_-^D *depend on* R. *The constant* C *depends on* R *but not on the moduli parameters* $z_1, \dots z_M$.

Observe that this theorem first requires a definition for the left-hand side of the equality. Once this is done, we will let λ go to zero in (2.13) and study the limit of both sides.

As before the case $K = 1$ is simpler than the general one. We will present the proof for this case first. The case $K > 1$ is more technically involved but the arguments we need can be adapted from [14].

2.4.1 The case $K = 1$

In this case, the definition of $\det_\zeta^*(\Delta, \mathring{\Delta})$ is rather straightforward, since, for $K = 1$, the following definition for the relative zeta function makes sense.

$$\zeta(s; \Delta, \mathring{\Delta}) = \left(\int_0^1 + \int_1^\infty\right) \frac{t^{s-1}}{\Gamma(s)} \operatorname{Tr}\big(e^{-t\Delta} - e^{-t\mathring{\Delta}}\big)\, dt. \tag{2.28}$$

Indeed, the first integral defines an analytic in $\Re s > 1$ function that has a meromorphic continuation to $\mathbb{C}$ with no pole at zero by the usual argument based on short time heat trace asymptotic

$$\operatorname{Tr}\big(e^{-t\Delta} - e^{-t\mathring{\Delta}}\big) \sim \sum_{j \geq -2} a_j t^{j/2}, \quad t \to 0+. \tag{2.29}$$

For the second integral we need the long time heat trace behaviour given by the following lemma

Lemma 2. *Assume that* $K = 1$. *Then*

$$\operatorname{Tr}\big(e^{-t\Delta} - e^{-t\mathring{\Delta}}\big) = O\big((\ln t)^{-2}\big) \text{ as } t \to +\infty.$$

Proof. Since $\Delta_+^D \equiv \mathring{\Delta}_+^D$ and the operators Δ_-^D, $\mathring{\Delta}_-^D$ are Dirichlet Laplacians on compact manifolds, we have

$$\begin{aligned}
\operatorname{Tr}\big(e^{-t\Delta} - e^{-t\mathring{\Delta}}\big) = {} & \operatorname{Tr}\big(e^{-t\Delta} - e^{-t\Delta_-^D \oplus \Delta_+^D}\big) - \operatorname{Tr}\big(e^{-t\mathring{\Delta}} - e^{-t\mathring{\Delta}_-^D \oplus \mathring{\Delta}_+^D}\big) \\
& + \operatorname{Tr} e^{-t\Delta_-^D} - \operatorname{Tr} e^{-t\mathring{\Delta}_-^D} \\
= {} & -t \int_0^\infty e^{-t\lambda^2}(\xi(\lambda\,;\, \Delta, \Delta^D) - \xi(\lambda\,;\, \mathring{\Delta}, \mathring{\Delta}^D)) 2\lambda\, d\lambda + O(e^{-t\delta}), \quad t \to +\infty,
\end{aligned} \tag{2.30}$$

where $\delta > 0$ is the smallest eigenvalue in the spectra of Δ_-^D and $\mathring{\Delta}_-^D$. (In (2.30) we also used (2.9) for Δ and $\mathring{\Delta}$.) As a consequence of Corollary 1 (which is also valid for $\mathring{\xi}$ in the case $K = 1$) we have

$$\xi(\lambda\,;\,\Delta,\Delta^D) - \xi(\lambda,\mathring{\Delta},\mathring{\Delta}^D) = O\big((\ln\lambda)^{-2}\big), \quad \lambda \to 0+\,.$$

This together with (2.30) implies the assertion; see e.g. [17, Theorem 1.7] for details. □

As a consequence, the second integral in (2.28) defines a holomorphic in $\Re s < 0$ function that has a continuous in $\Re s \le 0$ derivative. Thus $\zeta(s;\Delta,\mathring{\Delta})$ is a meromorphic function in $\Re s < 0$ and $\zeta'(s;\Delta,\mathring{\Delta})$ tends to a certain limit $\zeta'(0;\Delta,\mathring{\Delta})$ as $s \to 0-$. The relative zeta regularized determinant is defined to be

$$\det_\zeta(\Delta,\mathring{\Delta}) = e^{-\zeta'(0;\Delta,\mathring{\Delta})}. \tag{2.31}$$

We now prove the gluing formula in the case $K = 1$. First observe that, by Proposition 1 (applied also to $\mathring{\mathcal{N}}(\lambda)$) we have

$$\frac{\det_\zeta \mathcal{N}(\lambda)}{\det_\zeta \mathring{\mathcal{N}}(\lambda)} \to \frac{\det_\zeta^* \mathcal{N}(0)}{\det_\zeta^* \mathring{\mathcal{N}}(0)} \text{ as } \lambda^2 \to 0-,\ K = 1. \tag{2.32}$$

The limit $\lambda \to 0$ is then addressed by the

Proposition 2. *In the case $K = 1$ we have*

$$\det_\zeta(\Delta - \lambda^2, \mathring{\Delta} - \lambda^2) \to \det_\zeta(\Delta,\mathring{\Delta}) \text{ as } \lambda^2 \to 0-,$$

where the determinant $\det_\zeta(\Delta,\mathring{\Delta})$ *has been defined in* (2.31).

Proof. Let us write the relative zeta function in the form

$$\zeta(s;\Delta - \lambda^2, \mathring{\Delta} - \lambda^2) = \left(\int_0^1 + \int_1^\infty\right) \frac{t^{s-1}e^{t\lambda^2}}{\Gamma(s)} \operatorname{Tr}\big(e^{-t\Delta} - e^{-t\mathring{\Delta}}\big)\, dt.$$

Thanks to (2.10) the first integral converges for $\Re s > 1$ uniformly in $\lambda \le 0$ and has a meromorphic continuation to $\mathbb{C}$ with no pole at zero (by the usual argument based on the short time heat trace asymptotic (2.29)). Due to Lemma 2 the second integral defines a holomorphic in $\Re s < 0$ and continuous in $\lambda^2 \le 0$ and $\Re s \le 0$ function. Moreover, as $1/\Gamma(s)$ has a first order zero at $s = 0$, Lemma 2 also implies that the first derivative with respect to s of the second integral is also continuous in $\lambda^2 \le 0$ and $\Re s \le 0$. Thus we obtain

$$\zeta'(0;\Delta - \lambda^2, \mathring{\Delta} - \lambda^2) \to \zeta'(0;\Delta,\mathring{\Delta}), \quad \lambda^2 \to 0-\,.$$

where $\zeta'(0;\Delta,\mathring{\Delta})$ is defined using (2.28). □

Proof of Theorem 4 in the case $K = 1$. We pass to the limit as $\lambda^2 \to 0-$ in (2.13). Since Δ_-^D is positive, we have $\det_\zeta(\Delta_-^D - \lambda^2) \to \det_\zeta \Delta_-^D$ as $\lambda^2 \to 0-$, and the same is true for $\mathring{\Delta}_-^D$. Thanks to (2.32) and Proposition 2 we obtain

$$\frac{\det_\zeta(\Delta,\mathring{\Delta})\det_\zeta \mathring{\Delta}_-^D}{\det_\zeta \Delta_-^D} = \frac{\det_\zeta^* \mathcal{N}(0)}{\det_\zeta^* \mathring{\mathcal{N}}(0)},$$

which proves Theorem 4, where $\mathcal{N}(0)$ is denoted by $\mathcal{N}$ and the constant

$$C = \big(\det_\zeta \mathring{\Delta}_-^D \det_\zeta^* \mathring{\mathcal{N}}(0)\big)^{-1}$$

is moduli independent. □

2.4.2 The case $K>1$

In the case $K>1$ we have $\mathring{\mathcal{N}}(\lambda)=\oplus_{j=1}^K\mathring{\mathcal{N}}^{(j)}(\lambda)$, where $\mathring{\mathcal{N}}^{(j)}(\lambda)$ is the Neumann jump operator on the circle $\{y\in\mathbb{C}_j:|y|=R^{1/k_j}\}$ located on the infinite cone $(\mathbb{C}_j,|dy^{k_j}|^2)$. We have

$$\det_\zeta\mathring{\mathcal{N}}(\lambda)=\prod_{j=1}^K\det_\zeta\mathring{\mathcal{N}}^{(j)}(\lambda),\quad \det{}^*_\zeta\mathring{\mathcal{N}}(0)=\prod_{j=1}^K\det{}^*_\zeta\mathring{\mathcal{N}}^{(j)}(0).$$

We apply Proposition 1 to each $\det_\zeta\mathring{\mathcal{N}}^{(j)}(\lambda)$, $j=1,\dots,K$ and get

$$\det_\zeta\mathring{\mathcal{N}}(\lambda)=(-R\ln\lambda)^{-K}\det{}^*_\zeta\mathcal{N}(0)\Big(1-\Big(\ln\frac{R}{2}+\frac{\pi\gamma}{2}-i\frac{\pi}{2}\Big)\frac{K}{\ln\lambda}+O\Big(\frac{1}{(\ln\lambda)^2}\Big)\Big),\tag{2.33}$$

as $|\lambda|\to 0+$, $\Im\lambda\geq 0$.

Thanks to the relation $\xi(\lambda\,;\mathring{\Delta},\mathring{\Delta}^D)=\pi^{-1}\operatorname{Arg}\det\mathring{\mathcal{N}}(\sqrt{\lambda^2+i0})$ as $\lambda^2\to 0+$, calculation of the argument in (2.33) leads to

$$\xi(\lambda\,;\mathring{\Delta},\mathring{\Delta}^D)\;=\;K(\ln\lambda^2)^{-1}+O((\ln\lambda)^{-2}),\quad \lambda\to 0+,$$

where $\xi(\cdot\,;\mathring{\Delta},\mathring{\Delta}^D)\in L^1(\mathbb{R}_+,(1+\lambda^2)^{-2}\,d\lambda^2)$ is the spectral shift function satisfying

$$\operatorname{Tr}\big((\mathring{\Delta}+1)^{-1}-(\mathring{\Delta}^D+1)^{-1}\big)=-\int_0^\infty\xi(\lambda\,;\mathring{\Delta},\mathring{\Delta}^D)(1+\lambda^2)^{-2}\,d\lambda^2;$$

cf. Corollary 1. This together with Corollary 1 gives

$$\xi(\lambda,\Delta,\Delta^D)-\xi(\lambda,\mathring{\Delta},\mathring{\Delta}^D)=-(K-1)(\ln\lambda^2)^{-1}+O((\ln\lambda)^{-2}),\quad \lambda\to 0+\,.\tag{2.34}$$

Besides, Proposition 1 together with (2.33) implies that

$$\Big(\ln\frac{i}{\lambda}\Big)^{1-K}\frac{\det_\zeta\mathcal{N}(\lambda)}{\det_\zeta\mathring{\mathcal{N}}(\lambda)}\to R^{\ell-1}\frac{\det{}^*_\zeta\mathcal{N}(0)}{\det{}^*_\zeta\mathring{\mathcal{N}}(0)}\text{ as }\lambda^2\to 0-,\ K\geq 1.\tag{2.35}$$

Recall that for $\lambda^2<0$ the relative zeta function is defined as the meromorphic continuation of

$$\zeta(s;\Delta-\lambda^2,\mathring{\Delta}-\lambda^2)=\int_0^\infty\frac{t^{s-1}e^{t\lambda^2}}{\Gamma(s)}\operatorname{Tr}\big(e^{-t\Delta}-e^{-t\mathring{\Delta}}\big)\,dt\tag{2.36}$$

from $\Re s>1$.

We have

$$\operatorname{Tr}\big(e^{-t\Delta}-e^{-t\mathring{\Delta}}\big)=\operatorname{Tr}\big(e^{-t\Delta}-e^{-t\Delta^D_-\oplus\Delta^D_+}\big)-\operatorname{Tr}\big(e^{-t\mathring{\Delta}}-e^{-t\mathring{\Delta}^D_-\oplus\mathring{\Delta}^D_+}\big)+\operatorname{Tr}e^{-t\Delta^D_-}-\operatorname{Tr}e^{-t\mathring{\Delta}^D_-}.$$

(Now the short time asymptotic (2.29) is a consequence of (2.10) and similar short time asymptotics for $\operatorname{Tr}\big(e^{-t\mathring{\Delta}}-e^{-t\mathring{\Delta}^D}\big)$, $\operatorname{Tr}e^{-t\Delta^D_-}$, and $\operatorname{Tr}e^{-t\mathring{\Delta}^D_-}$.) Let $N(\lambda)=\sum_{j:\lambda_j^2\leq\lambda^2}\dim\ker(\Delta^D_--\lambda_j^2)$, where λ_j^2 are the eigenvalues of Δ^D_-, be the counting function of Δ^D_-. Similarly, let $\mathring{N}(\lambda)$ be the counting function of $\mathring{\Delta}^D_-$. Then

$$\operatorname{Tr}(e^{-t\Delta^D_-}-e^{-t\mathring{\Delta}^D_-})=t\int_0^\infty e^{-t\lambda^2}\Big(N(\lambda)-\mathring{N}(\lambda)\Big)2\lambda\,d\lambda$$

and

$$\xi(\lambda;\Delta\,;\,\mathring{\Delta})=\xi(\lambda\,;\,\Delta,\Delta^D)-\xi(\lambda\,;\,\mathring{\Delta},\mathring{\Delta}^D)-N(\lambda)+\mathring{N}(\lambda)$$

is the spectral shift function for the pair $(\Delta,\mathring{\Delta})$ such that

$$\operatorname{Tr}\bigl(e^{-t\Delta}-e^{-t\mathring{\Delta}}\bigr)=-t\int_0^\infty e^{-t\lambda^2}\xi(\lambda;\Delta,\mathring{\Delta})2\lambda\,d\lambda.$$

Since the operators Δ^D_- and $\mathring{\Delta}^D_-$ are positive, from (2.34) it follows the asymptotic

$$\xi(\lambda;\Delta,\mathring{\Delta})=-(K-1)(\ln\lambda^2)^{-1}+O((\ln\lambda)^{-2}),\quad \lambda\to 0+\,. \tag{2.37}$$

Introduce a cutoff function $\chi\in C^\infty(\mathbb{R})$ such that $\chi(\mu)=1$ for $\mu<1/2$ and $\chi(\mu)=0$ for $\mu>3/4$. Following the scheme in [14] we write

$$\operatorname{Tr}\bigl(e^{-t\Delta}-e^{-t\mathring{\Delta}}\bigr)=e_1(t)+e_2(t),$$

where

$$\begin{aligned}e_1(t)&=-t\int_0^\infty e^{-t\mu^2}\chi(\mu)\xi(\mu;\Delta,\mathring{\Delta})2\mu\,d\mu,\\ e_2(t)&=-t\int_0^\infty e^{-t\mu^2}(1-\chi(\mu))\xi(\mu;\Delta,\mathring{\Delta})2\mu\,d\mu;\end{aligned}$$

cf. (2.9). Note that e_2 is exponentially decreasing as $t\to+\infty$. Thanks to the short time asymptotic (2.29) and smoothness of e_1 at $t=0$, we see that $e_2(t)$ has a short time asymptotic of the same form. Therefore for $\lambda^2\le 0$ the holomorphic in $\Re s>1$ zeta function

$$\zeta_2(s;\lambda^2)=\int_0^\infty\frac{t^{s-1}e^{t\lambda^2}}{\Gamma(s)}e_2(t)\,dt$$

continues as a meromorphic function to $\mathbb{C}$ with no pole at zero and $\zeta_2'(0,\lambda^2)\to\zeta_2'(0,0)$ as $\lambda^2\to 0-$ by the usual argument.

We are now in position to define the regularized determinant $\det^*_\zeta(\Delta,\mathring{\Delta})$ We start from $\zeta(s;\Delta,\mathring{\Delta})=\zeta_1(s;0)+\zeta_2(s;0)$, where $\zeta_1(s;0)$ is defined by

$$\zeta_1(s;0)=\int_0^\infty\frac{t^{s-1}}{\Gamma(s)}e_1(t)\,dt=s\int_0^\infty(-\mu^2)^{-s-1}\chi(\mu)\xi(\lambda;\Delta,\mathring{\Delta})2\mu\,d\mu. \tag{2.38}$$

From this expression and (2.34) one can easily see that $\zeta_1(s;0)$ is a holomorphic function in $\Re s<0$, and we already know that $\zeta_2(s;0)$is a meromorphic function of $s\in\mathbb{C}$ with no pole at 0.

The asymptotic behaviour of $\zeta(s;0)$ near $s=0$ is given by the following proposition.

Proposition 3. *Set*

$$\begin{aligned}Z_0':=&\zeta_2'(0;0)+(K-1)\lim_{\delta\to 0+}\left(\int_\delta^\infty\frac{\chi(\mu)d\mu}{\mu\ln\mu}+\ln\ln\frac{1}{\delta}\right)\\ &-2\int_0^\infty\mu^{-1}\chi(\mu)\bigl(\xi(\mu;\Delta,\mathring{\Delta})-(K-1)(\ln\mu^2)^{-1}\bigr)\,d\mu+(K-1)(\gamma+\ln 2).\end{aligned}$$

When $s\to 0-$ we have

$$\zeta(s;\Delta,\mathring{\Delta})=\zeta_2(0;0)+s(K-1)\ln(-s)+sZ_0'+o(s).$$

We define $\det^*_\zeta(\Delta,\mathring{\Delta}):=e^{-Z_0'}$.

Proof. Since $\zeta_2(s;0)$ is a meromorphic function of s with no pole at zero, as $s \to 0-$ we have

$$\zeta(s;\Delta,\mathring{\Delta}) = \zeta_1(s;0) + \zeta_2(0;0) + s\zeta_2'(0;0) + O(s^2).$$

It remains to study the behaviour of

$$\zeta_1(s;0) = s\int_0^\infty (-\mu^2)^{-s-1}\chi(\mu)\xi(\mu;\Delta,\mathring{\Delta})2\mu\,d\mu$$

as $s \to 0-$. We represent the last integral as a sum of two integrals. Due to (2.37) the first integral

$$s\int_0^\infty (-\mu^2)^{-s-1}\chi(\mu)\big(\xi(\mu;\Delta,\mathring{\Delta}) + (K-1)(\ln\mu^2)^{-1}\big)2\mu\,d\mu$$

converges uniformly in $s \le 0$ and thus gives the contribution

$$-s\int_0^\infty 2\mu^{-1}\chi(\mu)\big(\xi(\mu;\Delta,\mathring{\Delta}) + (K-1)(\ln\mu^2)^{-1}\big)\,d\mu$$

into the expansion of $\zeta(s;\Delta,\mathring{\Delta})$. For the second integral we have

$$s(1-K)\int_0^\infty (-\mu^2)^{-s-1}\chi(\mu)(K-1)(\ln\mu^2)^{-1})2\mu\,d\mu =$$
$$s(1-K)\left(-\ln(-s) + \gamma + \ln 2 - \lim_{\delta\to 0+}\left(\int_\delta^\infty \frac{\chi(\mu)d\mu}{\mu\ln\mu} + \ln\ln\frac{1}{\delta}\right) + o(1)\right);$$

see [14, p.13]. □

The proof of the gluing formula will also require that we understand the limit when λ goes to 0. At this stage we have

$$\zeta(s;\Delta-\lambda^2,\mathring{\Delta}-\lambda^2) = \zeta_1(s;\lambda^2) + \zeta_2(s;\lambda^2),$$

where only properties of the zeta function

$$\zeta_1(s;\lambda^2) = \int_0^\infty \frac{t^{s-1}e^{t\lambda^2}}{\Gamma(s)}e_1(t)\,dt = s\int_0^\infty (\lambda^2-\mu^2)^{-s-1}\chi(\mu)\xi(\lambda;\Delta,\mathring{\Delta})2\mu\,d\mu \qquad (2.39)$$

remain unknown. Notice that the last integrand is compactly supported and therefore the integral converges uniformly near $s=0$ for fixed $\lambda^2 < 0$ due to (2.37). We get

$$\zeta_1'(0;\lambda^2) = \int_0^\infty (\lambda^2-\mu^2)^{-1}\chi(\mu)\xi(\mu;\Delta,\mathring{\Delta})2\mu\,d\mu. \qquad (2.40)$$

Proposition 4. *As $\lambda^2 \to 0-$ we have*

$$\zeta'(0;\Delta-\lambda^2,\mathring{\Delta}-\lambda^2) = \ln\left(\ln\frac{i}{\lambda}\right)^{1-K} + (K-1)\lim_{\delta\to 0+}\left(\int_\delta^\infty \frac{\chi(\mu)\,d\mu}{\mu\ln\mu} + \ln\ln\frac{1}{\delta}\right)$$
$$-2\int_0^\infty \mu^{-1}\chi(\mu)\Big(\xi(\mu;\Delta,\mathring{\Delta}) + (K-1)(\ln\mu^2)^{-1}\Big)\,d\mu + \zeta_2'(0;0) + o(1).$$

Proof. We only need to study the behaviour of $\zeta'(0;\lambda^2)$ in (2.40) as $\lambda^2 \to 0-$. Thanks to (2.37) the integral

$$\int_0^\infty (\lambda^2-\mu^2)^{-1}\chi(\mu)\Big(\xi(\mu;\Delta,\mathring{\Delta})+(K-1)(\ln\mu^2)^{-1}\Big)2\mu\,d\mu$$

converges uniformly in $\lambda^2 \le 0$ and thus tends to

$$-\int_0^\infty 2\mu^{-1}\chi(\mu)\Big(\xi(\mu;\Delta,\mathring{\Delta})+(K-1)(\ln\mu^2)^{-1}\Big)\,d\mu$$

as $\lambda^2 \to 0-$. It remains to note that

$$\int_0^\infty (\lambda^2-\mu^2)^{-1}\chi(\mu)(\ln\mu^2)^{-1}2\mu\,d\mu = \ln\ln\frac{i}{\lambda} - \lim_{\delta\to 0+}\left(\int_\delta^\infty \frac{\chi(\mu)\,d\mu}{\mu\ln\mu}+\ln\ln\frac{1}{\delta}\right)+o(1);$$

as $\lambda^2 \to 0-$; see [14, p. 12 and appendix]. □

Note that by definition of $\det^*_\zeta(\Delta,\mathring{\Delta}) = e^{-Z_0'}$ we have

$$\zeta'(0;\Delta-\lambda^2,\mathring{\Delta}-\lambda^2) = \ln\left(\ln\frac{i}{\lambda}\right)^{1-\ell} - \big(-Z_0'-(\ell-1)(\gamma+\ln 2)\big)+o(1), \quad \lambda^2\to 0-,$$

see Prop. 4.

Proof of Theorem 4 in the general case. From Propositions 4 and 3 we immediately get

$$\left(\ln\frac{i}{\lambda}\right)^{1-K}\det_\zeta(\Delta-\lambda^2;\mathring{\Delta}-\lambda^2)\to e^{(1-K)(\gamma+\ln 2)}\det^*_\zeta(\Delta,\mathring{\Delta}), \quad \lambda^2\to 0-. \tag{2.41}$$

We pass in (2.13) to the limit as $\lambda^2 \to 0-$. Taking into account (2.35) and (2.41) we obtain

$$\frac{\det^*_\zeta(\Delta,\mathring{\Delta})\det_\zeta\mathring{\Delta}^D_-}{\det_\zeta\Delta^D_-} = \big(Re^{\gamma+\ln 2}\big)^{K-1}\frac{\det^*_\zeta\mathcal{N}(0)}{\det^*_\zeta\mathring{\mathcal{N}}(0)}.$$

This proves Theorem 4, where $\mathcal{N}\equiv\mathcal{N}(0)$ and the constant

$$C = \big(Re^{\gamma+\ln 2}\big)^{K-1}/\big(\det_\zeta\mathring{\Delta}^D_-\det^*_\zeta\mathring{\mathcal{N}}(0)\big)$$

is moduli independent. □

Remark 1. *The proof of the gluing formula holds verbatim for a more general class of metrics under the following two assumptions. First, the structure at infinity is given by the union of K conical/Euclidean ends. Second, we have to assume nothing bad happens with the Laplace operator Δ^D_- of the compact part. In particular we have to assume that it has a well-defined zeta-function that extends to the complex plane with no pole at* 0. *It works for instance if the metric is smooth in the compact part or, if it is flat with conical singularities and we have chosen the Friedrichs extension.*

2.5 Closing the Euclidean (conical) ends with the help of gluing formulas

Let R be a sufficiently large positive number such that all the critical values of the meromorphic function f lie in the ball $\{|z| < R\}$.

In the holomorphic local parameter $\eta_j = y^{-1/k_j}$ in a vicinity U_j ($|y_j| > R$) of the j-th conical end of the angle $2\pi k_j$ ($k_j \geq 1$) of the Riemannian manifold $(X, |df|^2)$ (i. e. a pole of f of order k_j) the metric $\mathbf{m} = |df|^2$ takes the form

$$\mathbf{m} = k_j^2 \frac{|d\eta_j|^2}{|\eta_j|^{2k_j+2}} \quad .$$

Let χ_j be a smooth function on $\mathbb{C}$ such that $\chi_j(\xi) = \chi_j(|\eta|)$, $|\chi_j(\eta)| \leq 1$, $\chi_j(\eta) = 0$ if $|\eta| > (R+1)^{-1/k_j}$, $\chi_j(\eta) = 1$ if $|\eta| < (R+2)^{-1/k_j}$. Introduce the metric $\tilde{\mathbf{m}}$ on X such that

$$\tilde{\mathbf{m}} = \begin{cases} \mathbf{m} \quad \text{for} \quad |z| < R \\ [1 + (|\eta_j|^{2k_j+2} - 1)\chi_j(\eta_j)]\mathbf{m} \quad \text{in} \quad U_j \, . \end{cases}$$

Since the (Friedrichs extension of) the Laplace operator $\Delta^{\tilde{\mathbf{m}}}$ has discrete spectrum and the corresponding operator ζ-function is regular at $s = 0$ (see e. g. [19]and references therein), one can define the determinant $\det^* \Delta^{\tilde{\mathbf{m}}}$ via usual Ray-Singer zeta regularization. Moreover, for this determinant the usual BFK gluing formula ([4], Theorem B^*) holds (under the condition that the contour cutting the surface X does not pass through the conical singularities of the metric $\tilde{\mathbf{m}}$). Applying this standard BFK gluing formula, we get

$$\ln \det{}_\zeta^* \Delta^{\tilde{\mathbf{m}}} = \ln C_0 + \ln \det{}_\zeta \Delta_-^D + \ln \det{}_\zeta^* \mathcal{N} + \ln \det \Delta_{ext}^{\tilde{\mathbf{m}}} \, , \tag{2.42}$$

where $\Delta_{ext}^{\tilde{\mathbf{m}}}$ is the operator of the Dirichlet problem for $\Delta^{\tilde{\mathbf{m}}}$ in the union $\cup_j U_j$. Using conformal invariance we see that $\mathcal{N}$ is the same as in Theorem 1 and C_0 is a moduli independent constant ($C_0 = \frac{\text{Area}(X, \tilde{\mathbf{m}})}{\text{length}(\Sigma)}$).

Now equation (2.42) and Theorem 1 imply the following proposition.

Proposition 1. *The relative zeta regularized determinant* $\det_\zeta^*(\Delta, \mathring{\Delta})$ *and the zeta-regularized determinant* $\det_\zeta \Delta^{\tilde{\mathbf{m}}}$ *has the same variations with respect to moduli i. e. one has*

$$\partial_{z_k} \ln \det{}_\zeta^*(\Delta, \mathring{\Delta}) = \partial_{z_k} \ln \det{}_\zeta \Delta^{\tilde{\mathbf{m}}} \tag{2.43}$$

for $k = 1, \dots, M$.

Thus, the relative determinant of the Laplacian on a noncompact surface $(X, \mathbf{m})$ with conical points and conical/Euclidean ends can be studied via consideration of the zeta-regularized determinant of Laplacian on a compact surface $(X, \tilde{\mathbf{m}})$ with conical points. The latter surface is flat everywhere except the conical singularities (whose positions vary when one changes the moduli $z_1, \cdot, z_M$) and smooth ends of nonzero curvature which remain unchanged.

In the next two sections we study some spectral properties of compact surfaces with conical points. The final goal is to derive the variational formulas for $\ln \det_\zeta \Delta^{\tilde{\mathbf{m}}}$.

3 S-matrix

In this section we introduce the so-called S-matrix and relate its behavior at $\lambda = 0$ with the Schiffer projective connection. The definition of the S-matrix is related to the general theory of boundary triplets (see [13] sect. 13) and more specifically to the general theory of self-adjoint extensions of elliptic operators in singular settings (see [32]). Here we will follow closely [15]. However, we should point out that some of the constants in the latter reference are badly taken into account and the normalization we use here is slightly different. The context is also slightly different.

It is convenient to introduce the S-matric in the following general setting

3.1 General Setting and normalizations

Let $(X, \tilde{\mathbf{m}})$ be a compact singular Riemannian surface (possibly with boundary). Let P be an interior point of X such that in a neighbourhood V of p the metric is isometric to a neighbourhood of the tip of the Euclidean cone of angle $2\ell\pi$.

We set $X_0 := X\backslash\{P\}$ and $X_\varepsilon := X\backslash B(p,\varepsilon)$. We also denote by γ_r the circle of radius r centered at p.

We will occasionnally use several different ways of parametrizing V.

- Polar coordinates $(r,\theta) \in (0, r_{\max}) \times \mathbb{R}/2\ell\pi\mathbb{Z}$,
- Local complex coordinate z. The 1-form dz is well-defined on $V\backslash\{p\}$ and extends to V to a holomorphic one form α with a zero of order $\ell-1$ at p. Note that it may not extend to the all of X.
- Distinguished complex parameter y such that $\alpha = \ell y^{\ell-1}dy$.

The Riemannian area element dS is Euclidean near p and the associated scalar product is

$$\langle u, v\rangle = \int_X u\overline{v}dS$$

We now want to consider the Laplace operator that is associated with $\tilde{\mathbf{m}}$. We assume that the only singularities in $\tilde{\mathbf{m}}$ are P and maybe some other conical singularities. We take Δ to be the Friedrichs extension of the Riemannian Laplace operator that is defined on functions that vanish near the singularities.

Remark 2. *Actually we don't really care about what extension we have chosen at the other singularities as long as Δ is a self-adjoint extension of the Laplace operator defined on functions smooth away of all kind of singularity and that at P, the Friedrichs extension has been chosen.*

By definition we set $H^2(X)$ to be the domain of Δ and by H^1 to be its form domain.We denote by Δ_0 the restriction of Δ to functions in $H^2(X)$ that vanish near p and by Δ_0^* its formal adjoint. By choice, the self-adjoint extension Δ corresponds to the Friedrichs extension of Δ_0. We will also denote by $H_0^2 := \operatorname{dom}\overline{\Delta_0}$. Near P, we have

$$\Delta_0^* = -4\partial_z\partial_{\overline{z}} = -4\left(\ell^2|y|^{2(\ell-1)}\right)^{-1}\partial_y\partial_{\overline{y}}.$$

We define

$$
\begin{aligned}
c_0 &= \frac{1}{2\sqrt{\ell\pi}} \\
c_\nu &= \frac{1}{2\sqrt{\nu\ell\pi}}.
\end{aligned}
\tag{3.1}
$$

and the following singular functions.

We first fix a cut-off function ρ such that ρ has support in $r \leq r_{\max}$ and is identically 1 near $r = 0$.

$$
\begin{aligned}
F^0(z) &= c_0 \ln(z\overline{z})\rho(z) \;=\; c_0 \ln(r^2)\rho(z), \\
F^a_\nu(z) &= c_\nu \overline{z}^{-\nu}\rho(z), \quad \nu = \frac{k}{\ell},\ 0 < k < \ell \\
F^h_\nu(z) &= c_\nu z^{-\nu}\rho(z), \quad \nu = \frac{k}{\ell},\ 0 < k < \ell.
\end{aligned}
$$

The h stands for holomorphic and a for antiholomorphic in agreement with the behaviour of the corresponding F near 0.

By separating variables near P it can be shown that any function in $\mathrm{dom}(\Delta_0^*)$ admits the following expression (cf. [19], [29])

$$
\begin{aligned}
u = {} & c_0\Lambda^0(u) + c_0\Lambda^{0,-}(u)F^0(z) \\
& + \sum_\nu c_\nu \Lambda^{h,-}_\nu(u) z^{-\nu} \;+\; c_\nu \Lambda^{a,-}_\nu(u)\overline{z}^{-\nu} \\
& + \sum_\nu c_\nu \Lambda^{h}_\nu(u) z^{\nu} \;+\; c_\nu \Lambda^{a}_\nu(u)\overline{z}^{-\nu} \\
& + u_0
\end{aligned}
\tag{3.2}
$$

where the Λ are linear functionals on $\mathrm{dom}(\Delta_0^*)$ that vanish on H_0^2 and u_0 is in $\mathrm{dom}(\Delta)$. Observe that we have

$$
\overline{\Lambda^a_\nu(u)} \;=\; \Lambda^h_\nu(\overline{u}).
$$

We oriente any circle around p positively so that Stokes' formula for a one-form ω in X_ε reads

$$
\int_{X_\varepsilon} d\omega \;=\; -\int_{\gamma_\varepsilon} \omega.
$$

We define Green's formula for $u, v \in \mathrm{dom}(\Delta^*)$ by

$$
\begin{aligned}
\mathcal{G}(u,v) &:= \langle \Delta^* u, v\rangle - \langle u, \Delta^* v\rangle \\
&= \int \Delta^* u \cdot \overline{v} - u \cdot \Delta^* \overline{v}\, dS \\
&= \lim_{\varepsilon \to 0} \int_{X_\varepsilon} \Delta^* u \cdot \overline{v} - u \cdot \Delta^* \overline{v}\, dS \\
&= \frac{2}{i} \lim_{\varepsilon \to 0} \int_{X_\varepsilon} d\left(\partial_z u\overline{v}\, dz + u\partial_{\overline{z}}\overline{v}\, d\overline{z}\right) \\
&= \lim_{\varepsilon \to 0} \frac{2}{i} \int_{\gamma_\varepsilon} \partial_z u\overline{v}\, dz + u\partial_{\overline{z}}\overline{v}\, d\overline{z}.
\end{aligned}
\tag{3.3}
$$

Since we have set c_0 and c_ν so that $\dfrac{2c_0^2}{i}\displaystyle\int_{\gamma_\varepsilon}\dfrac{dz}{z}=1$ and $\dfrac{2\nu c_\nu^2}{i}\displaystyle\int_{\gamma_\varepsilon}\dfrac{dz}{z}=1$, the application of Green's formula gives the following normalized expression

$$
\begin{aligned}
\mathcal{G}(u,v) = {} & \Lambda^{0,-}(u)\overline{\Lambda^0(v)} - \Lambda^0(u)\overline{\Lambda^{0,-}(v)} \\
& + \sum_\nu \Lambda_\nu^{h,-}(u)\overline{\Lambda_\nu^a(v)} - \Lambda_\nu^a(u)\overline{\Lambda_\nu^{h,-}(v)} \\
& + \sum_\nu \Lambda_\nu^{h}(u)\overline{\Lambda_\nu^{a,-}(v)} - \Lambda_\nu^{a,-}(u)\overline{\Lambda_\nu^{h}(v)}.
\end{aligned}
$$

The domain of the Friedrichs extension of Δ is characterized by requiring that all the coefficients with the superscript $-$ vanish and it follows from Green's formula that the linear functionals $\Lambda_\nu^{a,h,0}$ are continuous over $H^2(X)$ and supported at p. It also follows from the general theory that any linear functional Λ that is continuous over H^2 and supported at p can be written as a linear combination of the linear functionals Λ_0, Λ_ν^a and Λ_ν^h. Finally, we also have for H_0^2 in the following equivalent expression :

$$H_0^2 = \{u \in H^2, \mid \forall\nu,\ \Lambda_\nu^\sharp(u) = 0,\ \sharp = a, h\ \}.$$

3.2 Definition of the S-matrix

We follow closely [15] paying special attention to conjugations and normalizing constants.

In the following the symbols $\sharp$ and $\flat$ are to be substituted by 0, h or a. When the superscript is 0 the subscript ν is 0, when it is a or h, $\nu = \frac{j}{\ell}$ where j ranges from 1 to $\ell - 1$.

We define

$$
\begin{aligned}
f_\nu^\sharp(\cdot;\lambda) &= (\Delta_0^* - \lambda)F_\nu^\sharp \\
g_\nu^\sharp(\cdot;\lambda) &= -(\Delta-\lambda)^{-1} f_\nu^\sharp(\cdot;\lambda) \\
G_\nu^\sharp(\cdot;\lambda) &= F_\nu^\sharp + g_\nu^\sharp(\cdot;\lambda) \\
S_{\mu\nu}^{\sharp\flat} &= \Lambda_\mu^\sharp\left(g_\nu^\flat(\cdot;\lambda)\right)
\end{aligned}
\tag{3.4}
$$

Observe that by definition the g-functions belong to H^2 which makes the latter definition consistent when seeing $\Lambda_\mu^\sharp$ as a linear functional over H^2. Since $\Lambda_\mu^\sharp$ also makes sense as a linear functional over $\ker(\Delta^* - \lambda)$ we may also write

$$S_{\mu\nu}^{\sharp\flat} = \Lambda_\mu^\sharp\left(G_\nu^\flat(\cdot;\lambda)\right)$$

Remark 3. *The functions F, f, g depend on the initial choice of ρ but the linear functionals Λ and the functions G don't.*

The S-matrix is defined by blocks :

$$
S := \begin{pmatrix} S^{00} & S_{0\nu}^{0h} & S_{0\nu}^{0a} \\ S_{\mu 0}^{h0} & S_{\mu\nu}^{hh} & S_{\mu\nu}^{ha} \\ S_{\mu 0}^{a0} & S_{\mu\nu}^{ah} & S_{\mu\nu}^{aa} \end{pmatrix}
\tag{3.5}
$$

Remark 4. *When there are several conical points on the surface, there are several ways of defining a S-matrix depending on how many points we want to take into account. In [15], the S-matrix that is constructed takes all the conical points into account whereas here, only the conical point P is considered (even if there are other conical points on the surface). Thus the S-matrix that is constructed here is only a part on the one in [15].*

Applying Green's formula, we have that for any test function $u \in H^2$

$$\begin{aligned} \Lambda_\nu^\sharp(u) = & \ \mathcal{G}(u, \overline{F^\sharp}) \\ = & \int (\Delta - \lambda)u \cdot F_\nu^\sharp - u \cdot f_\nu^\sharp(\cdot;\lambda) dS \\ = & \int (\Delta - \lambda)u \cdot F_\nu^\sharp + u \cdot (\Delta - \lambda) g_\nu^\sharp(\cdot;\lambda) dS \\ = & \int (\Delta - \lambda)u \cdot G_\nu^\sharp \, dS \end{aligned} \tag{3.6}$$

Observe that $\overline{F^{0,a,h}} = F^{0,h,a}$ and we have used that both u and g are in H^2 and that Δ is real (i.e. commutes with complex conjugation) and self-adjoint.

Applying the precedent equation to the g-functions gives the following alternative expressions for the S-matrix coefficients (we omit the dependence with respect to λ):

$$\begin{aligned} S_{\mu\nu}^{\sharp\flat} &= \int G_\mu^\sharp \, (\Delta - \lambda) \, g_\nu^\flat \, dS \\ &= - \int G_\mu^\sharp f_\nu^\flat \, dS \end{aligned} \tag{3.7}$$

Remark 5. *The S-matrix allows the description of the elements of* $\ker(\Delta_0^* - \lambda)$ *in the following way. For any element $u \in dom(\Delta_0^*)$, denote by $L^\pm$ the collection of its coefficients $\Lambda_\nu^{\sharp;\pm}$ that describe the singular behaviour of u near p. We then have*

$$(\Delta_0^* - \lambda)u = 0 \Leftrightarrow L^+ = S(\lambda)L^-.$$

This is a formal analogy with a typical scattering situation. In the latter, plane waves have an incoming and outgoing part that are related through the scattering matrix. In our setting, solutions to the equation $(\Delta_0^* - \lambda)u$ play the role of planes wave, $L^\pm$ are their incoming and outgoing parts and the S-matrix then is the scattering *matrix.*

4 Basic properties of the S-matrix

4.0.1 Analyticity and complex conjugation

From the analyticity of the resolvent we get that the S-matrix depends analytically on λ.

From the expression

$$S_{\mu\nu}^{\sharp\flat}(\lambda) = - \int G_\mu^\sharp(\cdot;\lambda) f_\nu^\flat(\cdot;\lambda) \, dS$$

and the fact that

$$\begin{aligned} \overline{f_\nu^{a,h}(\cdot,\lambda)} &= f_\nu^{h,a}(\cdot;\overline{\lambda}) \\ \overline{G^{a,h}(\cdot;\lambda)} &= G_\nu^{h,a}(\cdot;\overline{\lambda}), \end{aligned}$$

we get the following identities :

$$\begin{aligned} \overline{S^{hh}(\lambda)} &= S^{aa}(\overline{\lambda}) \\ \overline{S^{ah}(\lambda)} &= S^{ha}(\overline{\lambda}). \end{aligned} \tag{4.1}$$

4.0.2 Behavior for λ going to $-\infty$

On the half-line, consider the equation

$$-u'' - \frac{1}{r}u' + \frac{\nu^2}{r^2}u = \lambda u.$$

Since $\nu \neq 0$, any solution to this equation has the following asymptotic behaviour near 0 :

$$u = a_- r^{-\nu} + a_+ r^{\nu} + o(r^{\nu}).$$

and the vector space of solutions that belongs to $L^2(rdr)$ is one-dimensional. We set $k_\nu(r;\lambda)$ to be the unique solution to this equation which is in $L^2(rdr)$ and that is normalized in such a way that

$$\begin{aligned} F_\nu^h(r,\theta) - k_\nu(r;\lambda)\exp(-i\nu\theta) &= O(r^\nu), \\ F_\nu^a(r,\theta) - k_\nu(r;\lambda)\exp(i\nu\theta) &= O(r^\nu) \end{aligned}$$

(i.e. we adjust the coefficient of $r^{-\nu}$ in $k_\nu^\sharp$ so that it coincides with the coefficient of $F_\nu^\sharp$). By inserting a cut-off ρ, we define

$$\begin{aligned} K_\nu^h(r,\theta;\lambda) &:= k_\nu(r;\lambda)\exp(-i\nu\theta)\rho(r) \\ K_\nu^a(r,\theta;\lambda) &:= k_\nu(r;\lambda)\exp(i\nu\theta)\rho(r) \end{aligned}$$

as a function on X.

We compute $R_\nu^\sharp(\cdot;\lambda) = (\Delta_0^* - \lambda)\, K_\nu^\sharp,\ \sharp = a, h.$

Lemma 1. *We have for $\sharp = a, h$*

$$G_\nu^\sharp(\cdot;\lambda) = K_\nu^\sharp - [\Delta - \lambda]^{-1} R_\nu^\sharp.$$

Proof. By construction it is straightforward that both sides of the equation are in $\ker(\Delta_0^* - \lambda)$ and by choice of normalization, both share the same singular behaviour. □

We define by $\kappa_\nu(\lambda)$ the coefficient of r^ν in the asymptotic expansion of $k_\nu(\cdot;\lambda)$.

Corollary 1. *When $\Re\lambda$ goes to $-\infty$ we have*

$$\begin{aligned} S^{hh}(\lambda) &= O(|\lambda|^{-\infty}) \\ S^{aa}(\lambda) &= O(|\lambda|^{-\infty}) \\ S^{ah}(\lambda) &= \mathrm{diag}(\kappa_\nu(\lambda)) + O(|\lambda|^{-\infty}) \\ S^{ha}(\lambda) &= \mathrm{diag}(\kappa_\nu(\lambda)) + O(|\lambda|^{-\infty}) \end{aligned}$$

Proof. Use the asymptotic expansion of Bessel functions to prove that $\|R_\nu^\flat\|_{L^2} = O(|\lambda|^{-\infty})$. This implies that $\Lambda_\nu^\sharp\left([\Delta - \lambda]^{-1} R_\nu^\flat\right) = O(|\lambda|^{-\infty})$. Thus all entries of the S-matrix are given by $\Lambda_\mu^\sharp(K_\nu^\flat)$ up to $O(|\lambda|^{-\infty})$. The first term is seen to be 0 except for the diagonal terms in S^{ah} or S^{ha} for which it is $\kappa_\nu(\lambda)$. □

Remark 6. *A different proof is given in [15] using the heat kernel.*

4.0.3 Differentiation with respect to λ

We denote by a dot the differentiation with respect to λ. Differentiating the defining equation for $g^\sharp_\nu$ we find that

$$\dot{g}^\flat_\nu = (\Delta - \lambda)^{-1} G^\flat_\nu$$

Thus we get

$$\begin{aligned}\dot{S}^{\sharp\flat}_{\mu\nu} &= \Lambda^\sharp_\mu \left(\dot{g}^\flat_\nu\right) \\ &= \int G^\sharp_\mu G^\flat_\nu \, dS.\end{aligned} \tag{4.2}$$

From this relation we get the following proposition.

Proposition 5. *For any λ, $S^{aa}(\lambda)$ and $S^{hh}(\lambda)$ are symmetric matrices and*

$${}^t S^{ah}(\lambda) = S^{ha}(\lambda). \tag{4.3}$$

Proof. The expression for $\dot{S}^{\sharp\flat}_{\mu\nu}$ yields that $\dot{S}^{aa}$ and $\dot{S}^{hh}$ are symmetric matrices. Since both tends to a symmetric matrix (actually 0) when λ goes to $-\infty$, the first part of the claim follows. In the same way we have

$$\dot{S}^{ah}_{\mu\nu} - \dot{S}^{ha}_{\nu\mu} = 0$$

Since S^{ah} and ${}^t S^{ha}$ tends to the same diagonal matrix for λ going to $-\infty$, the second part of the claim also follows. $\square$

Putting together the identities (4.1) and (4.3) we obtain that S^{ah} is hermitian for λ real and actually is an analytic family of hermitian matrices (meaning that $S^{ah}(\lambda) = \left(S^{ah}(\overline{\lambda})\right)^*$).

4.0.4 Behavior for λ going to 0

The matrix $S(\lambda)$ is well defined a priori only for λ in the resolvent set of Δ. However it is always possible to define the function $f^\sharp_\nu(\cdot\,;\,\lambda = 0)$. Whenever $\sharp \neq 0$ the latter function is in the range of Δ. We can thus find solutions $g^\sharp_\nu(\cdot;0)$ to the equation

$$\Delta g^*_\nu = -f^*_\nu(\cdot, 0).$$

The latter solutions are defined only up the addition of a constant. It follows that for $\sharp \neq 0$ and $\flat \neq 0$ the definition of $S^{\sharp\flat}_{\mu\nu}(0)$ makes sense and it can be seen that the following holds.

Proposition 2. *For $\sharp \neq 0$ and $\flat \neq 0$ the matrix-valued function $\lambda \mapsto S^{\sharp\flat}(\lambda)$ extends holomorphically to a neighbourhood of* 0*. Moreover $S^{\sharp\flat}(0)$ depends only on the conformal class of $\tilde{\mathbf{m}}$.*

Proof. For λ close to 0 we have

$$g^\sharp_\nu \;=\; \frac{1}{\lambda}\int_X f_\nu(\cdot\,,\,\lambda)\,dS \;+\; g^{\sharp,\perp}_\nu(\cdot\,,\,\lambda),$$

where $\int_X g_\nu^{\sharp,\perp} = 0$. Since $\lambda \mapsto \int_X f_\nu^\sharp(\lambda; 0) = 0$, is holomorphic and vanish at 0, we obtain that $\lambda \mapsto G_\nu^*$ can be holomorphically continued to a neighbourhood of 0. The first statement follows. The second statement follows by remarking that $G_\nu^\sharp$ is a function in $\mathrm{dom}(\Delta_0^*)$ such that $\Delta_0^* G_\nu^\sharp = 0$ and the singular behaviour near p is prescribed. Both conditions are conformally invariant so that if we change the metric in its conformal class, we may only change G by adding a constant. This will not affect the coefficients in the S-matrix we are considering here. $\square$

4.1 $S(0)$ and the Schiffer projective connection

Chose a marking for the Riemann surface X, i. e. the canonical basis $a_1, b_1, \dots, a_g, b_g$ of $H_1(X, \mathbf{Z})$. Let $\{v_1, \dots, v_g\}$ be the basis of holomorphic differentials on X normalized via

$$\int_{a_i} v_j = \delta_{ij} \,.$$

Then the matrix of b-periods of the marked Riemann surface X is defined via

$$\mathbf{B} = || \int_{b_i} v_j || \,.$$

Let $W(\,\cdot\,, \,\cdot\,)$ be the canonical meromorphic bidifferential on $X \times X$, with properties $W(P, Q) = W(Q, P)$,

$$\int_{a_i} W(\,\cdot\,, P) = 0$$

and

$$\int_{b_j} W(\,\cdot\,, P) = 2\pi i v_j(P)$$

The bidifferential W has the only double pole along the diagonal $P = Q$. In any holomorphic local parameter $x(P)$ one has the asymptotics

$$W(x(P), x(Q)) = \left(\frac{1}{(x(P) - x(Q))^2} + H(x(P), x(Q)) \right) dx(P) dx(Q), \tag{4.4}$$

$$H(x(P), x(Q)) = \frac{1}{6} S(x(P)) + O(x(P) - x(Q)),$$

as $Q \to P$, where $S_B(\cdot)$ is the Bergman projective connection.

The Schiffer bidifferential $\mathcal{S}(P, Q)$ is defined via

$$\mathcal{S}(P, Q) = W(P, Q) - \pi \sum_{i,j} (\Im \mathbf{B})^{-1}_{ij} v_i(P) v_j(Q)$$

The Schiffer projective connection, S_{Sch}, is defined via the asymptotic expansion

$$\mathcal{S}(x(P), x(Q)) = \left(\frac{1}{(x(P) - x(Q))^2} + \frac{1}{6} S_{Sch}(x(P)) + O(x(P) - x(Q)) \right) dx(P) dx(Q)$$

One has the equality

$$S_{Sch}(x) = S_B(x) - 6\pi \sum_{i,j} (\Im \mathbf{B})^{-1}_{ij} v_i(x) v_j(x) \,. \tag{4.5}$$

In contrast to the canonical meromorphic differential and the Bergman projective connection the Schiffer bidifferential and the Schiffer projective connection are independent of the marking of the Riemann surface X.

Introduce also the so-called Bergman kernel (which is in fact the Bergman reproducing kernel for holomorphic differentials on X) as

$$B(x,\bar{x}) = \sum_{ij} (\Im \mathbf{B})^{-1}_{ij} v_i(x)\overline{v_j(x)}\,.$$

Proposition 6. *Let X be a Riemann surface and let $\tilde{\mathbf{m}}$ be a conformal metric on X, suppose that $\tilde{\mathbf{m}}$ has a conical singularity of angle $2\ell\pi$ at p. Let also x be the distinguished local parameter for $\tilde{\mathbf{m}}$ near p. Then there is the following relation between the entries of the holomorphic-holomorphic part, $S^{hh}(0)$, of the S-matrix :*

$$\sum_{k=1}^{\ell-1} \frac{\sqrt{k(\ell-k)}}{m} S^{hh}_{\frac{\ell}{m}\,\frac{\ell-k}{m}}(0) = -\frac{1}{6\ell(\ell-2)!}\left(\frac{d}{dx}\right)^{\ell-2} S_{Sch}(x)\Big|_{x=0}. \tag{4.6}$$

Remark 7. *The same would hold true for a conical singularity of angle β with $2\pi(\ell - 1) < \beta \leq 2\pi\ell$.*

Remark 8. *Observe that the left-hand-side of equation (4.6) can be written using the indices $\mu, \nu = \frac{k}{\ell}$ as*

$$\sum_{\mu+\nu=1} \sqrt{\mu}\sqrt{\nu} S^{hh}_{\mu\nu}(0).$$

Proof. Introduce the following one forms Ω_k and Σ_k on X:

$$\Omega_k = -\frac{1}{(k-1)!}\left(\frac{d}{dx}\right)^{k-1} \frac{W(\,\cdot\,,x)}{dx}\Big|_{x=0} + \frac{2\pi i}{(k-1)!}\sum_{\alpha,\beta} (\Im\mathbf{B})^{-1}_{\alpha\beta}\left\{\Im v_\beta^{(k-1)}(0)\right\} v_\alpha(\cdot)$$

$$\Sigma_k = -i\frac{1}{(k-1)!}\left(\frac{d}{dx}\right)^{k-1} \frac{W(\,\cdot\,,x)}{dx}\Big|_{x=0} + \frac{2\pi i}{(k-1)!}\sum_{\alpha,\beta} (\Im\mathbf{B})^{-1}_{\alpha\beta}\left\{\Re v_\beta^{(k-1)}(0)\right\} v_\alpha(\cdot)\,,$$

where

$$v_\beta^{(k-1)}(0) := \left(\frac{d}{dx}\right)^{k-1} \frac{v_\beta(x)}{dx}\Big|_{x=0}.$$

All the periods of the differentials Ω_k and Σ_k are pure imaginary, therefore, one can correctly define the function f_k on X via

$$f_k(Q) = \Re\left\{\int_{P_0}^{Q} \Omega_k\right\} - i\Re\left\{\int_{P_0}^{Q} \Sigma_k\right\}$$

where P_0 is an arbitrary base point not coinciding with P. Clearly, f_k is harmonic in $X \setminus \{P\}$ and

$$f_k(x) = \frac{1}{x^k} + \text{const} + \sum_{j=1}^{\infty}(c_j x^j + d_j \bar{x}^j) \tag{4.7}$$

in a vicinity of P. One gets

$$c_l = -\frac{1}{l!(k-1)!}\partial_x^{l-1}\partial_y^{k-1} H(x,y)\Big|_{x=y=0} + \frac{\pi}{l!(k-1)!}\sum_{\alpha,\beta}(\Im\mathbf{B})^{-1}_{\alpha\beta}\, v_\beta^{(k-1)}(0) v_\alpha^{(l-1)}(0)$$

and

$$S^{hh}_{\frac{k}{m}\,\frac{l}{m}}(0) = \sqrt{\frac{l}{k}}\; c_l$$

This implies that

$$\sum_{k=1}^{m-1} \frac{\sqrt{k(m-k)}}{m} S^{hh}_{\frac{k}{m}\,\frac{m-k}{m}}(0) = -\frac{1}{m} \sum_{k=0}^{m-2} \frac{1}{k!(m-2-k)!} \partial_x^{m-2-k} \partial_y^k H(x,y)\Big|_{x=y=0} +$$

$$\pi \frac{1}{m(m-2)!} \left(\frac{d}{dx}\right)^{m-2} \sum_{\alpha,\beta} (\Im \mathbf{B})^{-1}_{\alpha\beta} \frac{v_\alpha(x) v_\beta(x)}{(dx)^2}\Big|_{x=0}.$$

Since

$$\frac{1}{6} S_B(x) = H(x,x) = \sum_{n=0}^{\infty} \frac{1}{n!} (\partial_x + \partial_y)^n H(x,y)\Big|_{x=y} x^n,$$

one has

$$\frac{1}{6} S_B^{(n)}(0) = \sum_{p=0}^{n} \frac{n!}{p!(n-p)!} \partial_x^p \partial_y^{n-p} H(0,0),$$

which implies the proposition. $\square$

Remark 9. *From (4.7) with $k=1$ it follows that for conical angles $2\pi < \beta \le 4\pi$*

$$\begin{pmatrix} S^{hh}(0) & S^{ha}(0) \\ S^{ah}(0) & S^{aa}(0) \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} S_{Sch}(0) & \overline{B(0,0)} \\ B(0,0) & -\frac{1}{6} \overline{S_{Sch}(0)} \end{pmatrix},$$

where the Schiffer projective connection and the Bergman kernel are calculated in the distinguished local parameter at P.

5 Variational formulas with respect to moduli

In this section we derive the variational formulas for $\ln \det \Delta^{\tilde{\mathbf{m}}}$. This derivation goes as follows. First, using Kato-Rellich theory (see [Kato]), we prove the variational formulas for the individual eigenvalues of the operator $\Delta^{\tilde{\mathbf{m}}}$. Using these formulas and the contour integral representation of the zeta-function of $\Delta^{\tilde{\mathbf{m}}}$, we express the variations of the value $\zeta'_{\Delta^{\tilde{\mathbf{m}}}}(0)$ with respect to the critical value z_k through a combination of the matrix elements of the S-matrix at the conical point P_k (the zero of the meromorphic differential df) of the metric $\tilde{\mathbf{m}}$. The latter combination is the one appearing in Proposition 6 and can be expressed through the Schiffer projective connection.

5.1 Variational formula for eigenvalues of $\Delta^{\tilde{\mathbf{m}}}$

Remark 10. *In this section we will use w for the moduli parameter and on the surface we will use the complex parameter z and (x,y) for the associated local cartesian coordinates (so that $z = x+iy$). We warn the reader that in the rest of the paper we use z_i as the moduli parameters and x as a local complex parameter on X.*

5.1.1 Moving conical points

Let $\tilde{\mathbf{m}}$ be a metric as constructed in section 2.5. Let P be one of its conical point. For $w \in \mathbb{C}$ we want to define a metric $\tilde{\mathbf{m}}_w$ obtained by moving P of the amount w. The following makes this construction precise.

Let $\mathbb{C}$ be the complex plane with pointed origin. We set $\tilde{X}_w$ to be the ℓ-fold covering of $\mathbb{C}$ with one ramification point at w so that $\tilde{X}_w$ can be identified with the Euclidean cone of total angle $\ell\pi$.

Fix a cutoff function ρ and define ϕ_w from $\mathbb{C}$ to itself by

$$\phi_w(z) = z \, + \, \rho(|z|)w.$$

For w small enough, this defines a family of smooth diffeomorphisms from $\mathbb{C}$ to itself that coincide with the identity outside the ball $\{z \mid \rho(|z|) \neq 0\}$. The cone $\tilde{X}_0$ can be obtained by gluing together ℓ copies of the plane after cutting along a fixed half-line d that emanates from the origin. The cone $\tilde{X}_w$ can then be obtained by gluing ℓ copies of $\mathbb{C}$ after cutting it along $\phi_w(d)$.

The function ϕ_w thus defines a family of smooth diffeomorphisms from $\tilde{X}_0$ onto $\tilde{X}_w$. Let g_w be the metric that is obtained on $\tilde{X}_0$ by pulling-back the Euclidean metric on $\tilde{X}_w$ by ϕ_w.

We write $w = a + ib$ and use the local cartesian coordinates $x + iy = z$ near P. We obtain for the metric $g_w = A(x,y;w)dx^2 + 2B(x,y;w)dxdy + C(x,y;w)dy^2$ the following expressions :

$$\begin{aligned} D\phi_w &= \begin{pmatrix} 1 + \frac{ax}{r}\rho_1'(r) & \frac{ay}{r}\rho_1'(r) \\ \frac{bx}{r}\rho_1'(r) & 1 + \frac{by}{r}\rho_1'(r) \end{pmatrix} \\ \begin{pmatrix} A & B \\ B & C \end{pmatrix} &= {}^t D\phi_w D\phi_w. \end{aligned} \tag{5.1}$$

It follows by inspection that the coefficients of g_w are polynomials in a, b. Observe that g_w coincide with g_0 outside a ball centered at p so that g_w can be smoothly extended by any Riemannian metric that coincide with the Euclidean one in a annulus centered at p. This allows us to define a metric g_w on our given setting X_0 that corresponds to some X_w that is obtained by fixing the exterior of a small ball centered at p and, in an even smaller ball by shifting the conical point of w.

We still denote the metric on X_0 by g_w. We denote by J_w the jacobian determinant of g_w, by q_w the Dirichlet energy quadratic form associated with g_w and by n_w the Riemannian L^2 scalar product.

We thus have the following expressions (for a real u that is supported near p)

$$\begin{aligned} q_w(u) &= \int_{X_0} \left[C(\partial_x u)^2 - 2B\partial_x u \partial_y v + A(\partial_y u)^2 \right] J_w^{-\frac{1}{2}} dxdy \\ n_w(u) &= \int u^2 J_w^{\frac{1}{2}} dxdy \end{aligned} \tag{5.2}$$

Observe that for u supported away of p then $q_w(u)$ and $n_w(u)$ do not depend of w.

In order to apply spectral perturbation theory, we will need the first order variations of $q_w(u)$ and $n_w(u)$. We record here the relevant Lemma.

Lemma 2. *For any λ and any u we have*

$$\begin{aligned}
\left[-\partial_w q + \lambda \partial_w n\right]_{w=0}(u) &= 2\int_{X_0} (\partial_z u)^2 \frac{z\rho'(r)}{r}\, dxdy + \frac{\lambda}{2}\int_{X_0} u^2 \frac{\overline{z}\rho'(r)}{r}\, dxdy \\
\left[-\partial_{\overline{w}} q + \lambda \partial_{\overline{w}} n\right]_{w=0}(u) &= 2\int_{X_0} (\partial_{\overline{z}} u)^2 \frac{\overline{z}\rho'(r)}{r}\, dxdy + \frac{\lambda}{2}\int_{X_0} u^2 \frac{z\rho'(r)}{r}\, dxdy
\end{aligned} \tag{5.3}$$

Proof. Denote by

$$\mathbb{G}_w := \begin{pmatrix} C & -B \\ -B & A \end{pmatrix}$$

so that we have

$$q_w(u) := \int_{X_0} {}^t\nabla u \mathbb{G} \nabla u \cdot J_w^{-\frac{1}{2}} dxdy.$$

Differentiating at $w = 0$, we obtain

$$\partial_w q_w(u) = \int_{X_0} {}^t\nabla u \cdot \left(\partial_w \mathbb{G} - \frac{1}{2}\partial_w J \mathbb{I}\right) \cdot \nabla u dS.$$

A straightforward computation yields

$$\partial_a \mathbb{G} - \frac{1}{2}\partial_a J\mathbb{I} = \begin{pmatrix} -\frac{x\rho'}{r} & -\frac{y\rho'}{r} \\ -\frac{y\rho'}{r} & \frac{x\rho'}{r} \end{pmatrix}$$

$$\partial_b \mathbb{G} - \frac{1}{2}\partial_b J\mathbb{I} = \begin{pmatrix} \frac{y\rho'}{r} & -\frac{x\rho'}{r} \\ -\frac{x\rho'}{r} & -\frac{y\rho'}{r} \end{pmatrix}$$

From this we find

$$\begin{aligned}
{}^t\nabla u \cdot \left(\partial_w \mathbb{G} - \frac{1}{2}\partial_w J\mathbb{I}\right) \cdot \nabla u &= {}^t\nabla u \cdot \begin{pmatrix} -\frac{z\rho'}{2r} & \frac{iz\rho'}{2r} \\ \frac{iz\rho'}{2r} & \frac{z\rho'}{2r} \end{pmatrix} \cdot \nabla u \\
&= -\frac{z\rho'}{2r}\left((\partial_x u)^2 - (\partial_y u)^2 - 2i\partial_x u \partial_y u\right) \\
&= -\frac{2z\rho'}{r}(\partial_z u)^2.
\end{aligned}$$

The other terms proceed in the same way. $\square$

5.1.2 Variational formulas for eigenvalues of $\tilde{\mathrm{m}}$

In this section, we compute variational formulas for the eigenvalues of $\tilde{\mathbf{m}}_w$. In order to do so we use the Kato-Rellich perturbation theory. Since we may only consider directional derivatives, we thus fix w and define $q_t = q_{tw}$ and $n_t = n_{tw}$.

It should be noticed that the family of metrics g_w is smooth in w but *not analytic* (see 5.1) but the map $t \mapsto q_{tw}$ is analytic in t.

The eigenvalue equation that gives the spectrum of q_t relatively to n_t is

$$q_t(u_t, v) = E_t n_t(u_t, v). \tag{5.4}$$

This problem is analytic in t so that the eigenvalues are organized into real-analytic branches (see [Kato]).

The first-order variation for the eigenbranch (E_t, u_t) is given by the following Feynman-Hellmann formula

$$\frac{dE}{dt} = \frac{dq}{dt}(u) - E\frac{dn}{dt}(u) \tag{5.5}$$

which is obtained by differentiating eq. (5.4) with v fixed and then evaluating at $v = u_t$.

Proposition 3. *Let r be small enough and set*

$$\begin{aligned} \partial_w E &= \frac{2}{i}\int_{\gamma_r} (\partial_z u)^2 dz - \frac{E}{4}u^2 d\overline{z} \\ \partial_{\overline{w}} E &= -\frac{2}{i}\int_{\gamma_r} (\partial_{\overline{z}} u)^2 d\overline{z} - \frac{E}{4}u^2 dz. \end{aligned} \tag{5.6}$$

Let (E_t, u_t) be an eigenbranch of $q_t := q_{tw}$ then $E' = \frac{d}{dt}_{|t=0} E_t$ is given by

$$E' = w\partial_w E + \overline{w}\partial_{\overline{w}} E,$$

where in the expression of $\partial_w E$ and $\partial_{\overline{w}} E$, $u = u_0$ the eigenvector of the eigenbranch (E_t, u_t) at $t = 0$.

Remark 11. *We remind the reader of one subtlety of perturbation theory (see [Kato, 28]). In case of a multiple eigenvalue E_0, for any family q_t there are several eigenbranches emanating from E_0, and the initial corresponding eigenvectors may actually depend of the chosen family. In particular the expressions $\partial_w E$ and $\partial_{\overline{w}} E$ also depends on the initial w that defines q_t. In other terms, for any direction w it is possible to organize the spectrum into eigenvalues branches but it may not be possible to organize the eigenvalues as functions that are differentiable with respect to w varying in the ball.*

Proof. We start with the one form

$$\omega_u = \rho(z) \cdot \left((\partial_z u)^2 dz - \frac{E}{4}u^2 d\overline{z}. \right)$$

Since (E, u) is an eigenpair of the Laplace operator we compute

$$d\omega_u = -\left[\frac{z\rho'}{2r}(\partial_z u)^2 + \frac{E}{4}\cdot\frac{\overline{z}\rho'}{2r}u^2\right] dz \wedge d\overline{z}.$$

We now use Stokes formula to obtain

$$\begin{aligned} \int_{\gamma_r} \omega_u &= -\int_X d\omega_u \\ &= \int_X \left[\frac{z\rho'}{2r}(\partial_z u)^2 + \frac{E}{4}\cdot\frac{\overline{z}\rho'}{2r}u^2\right] dz \wedge d\overline{z} \\ &= \frac{1}{2}\int_X \left[\frac{z\rho'}{r}(\partial_z u)^2 + \frac{E}{4}\cdot\frac{\overline{z}\rho'}{r}u^2\right] (-2idxdy). \end{aligned}$$

On the other hand, in (5.5) we use the formulas provided by Lemma 2 to obtain :

$$\partial_w E = -2\int_{X_0} (\partial_z u)^2 \frac{z\rho'(r)}{r}\, dxdy - \frac{E}{2}\int_{X_0} u^2 \frac{\overline{z}\rho'(r)}{r}\, dxdy \tag{5.7}$$

Comparing the two yields the first formula. The second one follows either from the same computation or using complex conjugation. □

Since ω_u is closed in $B(p, r_0) \setminus \{p\}$ we may let tend r to 0 in the preceding formulas. We thus obtained a formula for $\partial_w E$ that is expressed only through the asymptotic expansion of u near p.

Recall that by definition of the linear functionals $\Lambda_\nu^\sharp$, we have in the local coordinate z the following expansion near p

$$u(z) := c_0 \Lambda^0(u) + \sum_\nu c_\nu \Lambda_\nu^h(u) z^\nu + c_\nu \Lambda^a(u) \overline{z}^\nu \; + \; u_0.$$

with $u_0 \in H_0^2$. By letting r go to zero we obtain the following lemma.

Lemma 3. *Let $A = [a_{\mu\nu}]$ be the matrix defined by*

$$\begin{cases} a_{\mu\nu} = 4\pi \mu c_\mu \cdot \nu c_\nu & \textit{if } \mu + \nu = 1, \\ a_{\mu\nu} = 0 & \textit{otherwise.} \end{cases}$$

We have the alternative expression

$$\begin{aligned} \partial_w E &= \sum_{\mu+\nu=1} \Lambda_\nu^h(u) a_{\mu\nu} \Lambda_\mu^h(u) \\ \partial_{\overline{w}} E &= \sum_{\mu+\nu=1} \Lambda_\nu^a(u) a_{\mu\nu} \Lambda_\mu^a(u). \end{aligned} \tag{5.8}$$

Proof. We prove the formula for $\partial_w E$, the proof is the same for $\partial_{\overline{w}} E$. First observe that since u is bounded we have

$$\lim_{r \to 0} \int_{\gamma_r} u^2 d\overline{z} = 0.$$

Now, if u_0 is smooth and compactly supported away of p, using Stokes' formula, we have that for any $v \in H^2$

$$\int_{\gamma_r} \partial_z v \partial_z u_0 dz = \frac{1}{4} \int_{B_r} \Delta v \partial_z u_0 \; + \; \partial_z v \Delta u_0 \, dz \wedge d\overline{z}$$

By continuity, this equality persists for $u_0 \in H_0^2$. It follows that for any $u \in H^2$ and any $u_0 \in H_0^2$ we have

$$\lim_{r \to 0} \int_{\gamma_r} \partial_z u \partial_z u_0 dz = 0.$$

It follows that

$$\partial_w E = \lim_{r \to 0} \int_{\gamma_r} \left(\partial_z (u - u_0)\right)^2 \, dz.$$

By definition we have

$$u - u_0 \; = \; c_0 \Lambda^0(u) + \sum_\nu c_\nu \Lambda_\nu^h(u) z^\nu + c_\nu \Lambda^a(u) \overline{z}^\nu$$

so that the claim follows by a direct computation. □

Using the definition of Λ_ν^h and the fact that u is an eigenfunction we obtain.

Corollary 2. *For any $\lambda \in \mathbb{C} \setminus [0, \infty)$ the series $\sum_{E_n \in \mathrm{spec}(\Delta^{\tilde{\mathbf{m}}})} \partial_w E_n (E_n - \lambda)^{-2}$ is absolutely convergent and*

$$\sum_{E_n \in \mathrm{spec}(\Delta^{\tilde{\mathbf{m}}})} \frac{\partial E_n}{(E_n - \lambda)^2} \; = \; \mathrm{Tr} \left(A \frac{\partial S^{hh}}{\partial \lambda}(\lambda) \right).$$

Proof. To get the absolute convergence, it suffices to show that, for any ν

$$\sum_{E_n \in \mathrm{spec}(\Delta^{\tilde{\mathbf{m}}})} \left| \frac{\Lambda^h_\nu(u_n)}{E_n - \lambda} \right|^2 < \infty.$$

Since

$$\Lambda^h_\nu(u_n) = \int (\Delta - \lambda) u_n G^h_\nu = (E_n - \lambda)\langle G^h_\nu, u\rangle,$$

the claims follows by remarking that u is an orthonormal basis. By Plancherel formula, we then obtain

$$\sum_{E_n \in \mathrm{spec}(\Delta^{\tilde{\mathbf{m}}})} \frac{\partial_w E_n}{(E_n - \lambda)^2} = \sum_{\mu,\nu} a_{\mu\nu} \int_X G^h_\mu(x\,;\,\lambda) G^h_\nu(x\,;\,\lambda)\, dS.$$

We now remark that using (4.2)

$$\int_X G^h_\mu(x\,;\,\lambda) G^h_\nu(x\,;\,\lambda)\, dS = \frac{\partial_\lambda S^{hh}_{\mu\nu}}{\partial \lambda},$$

and A is a symmetric matrix. $\square$

Remark 12. *We have a similar formula, involving S^{aa} for $\sum_{E_n \in \mathrm{spec}(\Delta^{\tilde{\mathbf{m}}})} \frac{\partial E_n}{(E_n - \lambda)^2}$.*

5.2 Variational formula for $\zeta'(0\,;\,\Delta^{\tilde{\mathbf{m}}})$

We prove the following proposition.

Proposition 4. *Let $\tilde{\mathbf{m}}_w$ be the family of metrics defined above, S^{hh} be the holomorphic-holomorphic part of the corresponding S-matrix and A be the matrix defined in Lemma 3. We have*

$$\partial_w \zeta'(0\,;\,\Delta^{\tilde{\mathbf{m}}}) = \mathrm{Tr}\left(A S^{hh}(0)\right) = \sum_{\mu+\nu=1} \sqrt{\mu}\sqrt{\nu} S^{hh}_{\mu\nu}(0). \tag{5.9}$$

Proof. We start from the following integral representation of the zeta-function of the operator $\Delta^{\tilde{\mathbf{m}}} - \lambda$ through the trace of the second power of the resolvent:

$$s\zeta(s+1; \Delta^{\tilde{\mathbf{m}}} - \lambda) = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-s} \mathrm{Tr}\left((\Delta^{\tilde{\mathbf{m}}} - z)^{-2}\right) dz, \tag{5.10}$$

where Γ_λ is a contour connecting $-\infty + i\epsilon$ with $-\infty - i\epsilon$ and following the cut $(-\infty, \lambda)$ at the (sufficiently small) distance $\epsilon > 0$. Using Corollary 2, differentiation under the integral sign is legitimate and we obtain

$$s\partial_w \zeta(s+1, \Delta^{\tilde{\mathbf{m}}} - \lambda) = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-s} \sum_{E_n \in \mathrm{spec}(\Delta^{\tilde{\mathbf{m}}})} \frac{-2\partial_w E_n}{(E_n - z)^3}\, dz. \tag{5.11}$$

Using again Corollary 2, it is legitimate to make an integration by parts under the integral sign to get

$$s\partial_w \zeta(s+1, \Delta^{\tilde{\mathbf{m}}} - \lambda) = \frac{-s}{2\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-s-1} \sum_{E_n \in \mathrm{spec}(\Delta^{\tilde{\mathbf{m}}}} \frac{\partial_w E_n}{(E_n - z)^2}\, dz. \tag{5.12}$$

We can now divide by s, use Corollary 2 again and replace $s+1$ by s to finally obtain

$$\partial_w \zeta(s, \Delta^{\tilde{\mathbf{m}}} - \lambda) \;=\; \frac{-1}{2\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-s} \,\mathrm{Tr}\left(A \partial_z S^{hh}(z) \right) \, dz. \tag{5.13}$$

Using the behaviour at infinity of S^{hh} we can make an integration by parts again and obtain

$$\partial_w \zeta(s, \Delta^{\tilde{\mathbf{m}}} - \lambda) \;=\; \frac{-s}{2\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-s-1} \,\mathrm{Tr}\left(A S^{hh}(z) \right) \, dz. \tag{5.14}$$

Differentiating with respect to s and setting $s=0$ gives

$$\partial_w \zeta'(0, \Delta^{\tilde{\mathbf{m}}} - \lambda) \;=\; \frac{-1}{2\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-1} \,\mathrm{Tr}\left(A S^{hh}(z) \right) \, dz.$$

The claim follows by applying Cauchy's theorem. □

Now, using Proposition 1, the preceding Proposition and Proposition 6 we arrive at the following corollary.

Corollary 3. *Let P_m be a zero of the meromorphic differential df of multiplicity ℓ_m and let $z_m = f(P_m)$ be the corresponding critical value of f. Let also $x_m = (z - z_m)^{\frac{1}{\ell_m+1}}$ be the distinguished local parameter in a vicinity of P_m. Then*

$$\partial_{z_m} \ln \det{}^*_\zeta(\Delta, \mathring{\Delta}) = \frac{1}{6(\ell_m+1)(\ell_m-1)!} \left(\frac{d}{dx_m} \right)^{\ell_m - 1} S_{Sch}(x_m) \Big|_{x_m=0}. \tag{5.15}$$

6 Integration of the equations for $\ln \mathrm{Det}$ and explicit expressions for the τ-function

Let, as before, $\mathbb{B}$ be the matrix of b-periods of the Torelli marked Riemann surface X and let $\{v_\alpha\}_{\alpha=1,\dots,g}$ be the basis of the normalized holomorphic differentials on X. Using the Rauch formulas (see, e. g., [22], [23], [20]),

$$\partial_{z_m} \mathbb{B}_{\alpha\beta} = \oint_{P_m} \frac{v_\alpha v_\beta}{df}\,,$$

one immediately gets the relation

$$\partial_{z_m} \ln \det \Im \mathbb{B} = \frac{1}{2i} \mathrm{Tr}\left[(\partial_{z_m} \mathbb{B})(\Im \mathbb{B})^{-1} \right] = \frac{1}{2i} \oint_{P_m} \frac{\sum_{\alpha\beta} \Im \mathbb{B}^{-1}_{\alpha\beta} v_\alpha v_\beta}{df}\,, \tag{6.1}$$

where the contour integrals are taken over a small contour on X encircling the point P_m (in the positive direction).

Now, using relation (4.5), equations (6.1) and (5.15) together with elementary properties of the Schwarzian derivative (see, e. g. [36]), we arrive at the following version of Corollary 3 rewritten in the invariant form.

Theorem 5. *Let P_m be a zero of the meromorphic differential df of multiplicity l_m and let $z_m = f(P_m)$ be the corresponding critical value of f. Let also $x_m = (z - z_m)^{\frac{1}{l_m+1}}$ be the distinguished local parameter in a vicinity of P_m. Then*

$$\partial_{z_m} \ln \frac{\det^*_\zeta(\Delta, \mathring{\Delta})}{\det \Im \mathbb{B}} = -\frac{1}{12\pi i} \oint_{P_m} \frac{S_B - S_f}{df} \tag{6.2}$$

where S_B is the Bergman projective connection, $S_f = \frac{f'''f' - \frac{3}{2}(f'')^2}{(f')^2}$ is the Schwarzian derivative.

Remark 13. *Notice that the difference $S_B - S_f$ is a quadratic differential, therefore, under the contour integral in (6.2) stands a meromorphic one form.*

It should be noted that the right hand side of (6.2) depends holomorphically on moduli $z_1, \dots, z_M$ and, therefore, one has

$$\partial^2_{z_m \bar{z}_n} \ln \frac{\det^*_\zeta(\Delta, \mathring{\Delta})}{\det \Im \mathbb{B}} = 0\,.$$

This implies the relation

$$\det^*_\zeta(\Delta, \mathring{\Delta}) = C \det \Im \mathbb{B}\, |\tau|^2\,, \tag{6.3}$$

where τ is a holomorphic function of moduli $z_1, \dots, z_M$ (actually, τ is a holomorphic section of some holomorphic line bundle over the Hurwitz space, see [25] for further information, here restrict ourselves to local considerations: the reader may assume for simplicity that all happens in a small vicinity of the covering $f : X \to \mathbb{C}P^1$ in the Hurwitz space $H(M, N)$) subject to the system of PDE

$$\partial_{z_m} \ln \tau = -\frac{1}{12\pi i} \oint_{P_m} \frac{S_B - S_f}{df} \tag{6.4}$$

and C is a moduli independent constant.

System of PDE (6.4) first appeared in the context of the theory of isomonodromic deformations and Frobenius manifolds in [20] and [22], where, in particular, it was explicitly integrated. We remind these results in the next subsection.

6.1 Explicit expressions for τ

In this section we recall explicit formulas for the holomorphic solution, τ, of the system (6.4) derived in [22], [20] (see also [24] and [23] for alternative and more straightforward proofs). The result should be formulated separately for low genera $g = 0, 1$ and for higher genus $g > 1$. We start with the higher genus situation.

Let $g > 1$. Take a nonsingular odd theta characteristic δ and consider the corresponding theta function $\theta[\delta](t; \mathbb{B})$, where $t = (t_1, \dots, t_g) \in \mathbb{C}^g$. Put

$$\omega_\delta = \sum_{i=1}^{g} \frac{\partial \theta[\delta]}{\partial t_i}(0; \mathbb{B})\, \omega_i\,.$$

All zeroes of the holomorphic 1-differential ω_δ have even multiplicities, and $\sqrt{\omega_\delta}$ is a well-defined holomorphic spinor on X. Following Fay [9], consider the prime form [1]

$$E(x, y) = \frac{\theta[\delta]\left(\int_x^y v_1, \dots, \int_x^y v_g; \mathbb{B}\right)}{\sqrt{\omega_\delta}(x)\sqrt{\omega_\delta}(y)}. \tag{6.5}$$

[1]The prime form $E(x, y)$ is the canonical section of the line bundle on $X \times X$ associated with the diagonal divisor $\{x = y\} \subset X \times X$.

To make the integrals uniquely defined, we fix $2g$ simple closed loops in the homology classes a_i, b_i that cut X into a connected domain, and pick the integration paths that do not intersect the cuts. The sign of the square root is chosen so that $E(x,y) = \frac{\zeta(y)-\zeta(x)}{\sqrt{d\zeta(x)}\sqrt{d\zeta(y)}}(1+O((\zeta(y)-\zeta(x))^2))$ as $y \to x$, where ζ is a local parameter such that $d\zeta = \omega_\delta$.

We introduce local coordinates on X that we call *distinguished* with respect to f. Consider the divisor $(df) = \sum_k d_k\, p_k,\ p_k \in X,\ d_k \in \mathbb{Z}, d_k \neq 0$, of the meromorphic differential df. We take $z = f(x)$ as a local coordinate on $X - \bigcup_k p_k$, and

$$x_k = \begin{cases} (f(x) - f(p_k))^{\frac{1}{d_k+1}} & \text{if } d_k > 0, \\ f(x)^{\frac{1}{d_k+1}} & \text{if } d_k < 0, \end{cases} \tag{6.6}$$

near $p_k \in X$. In terms of these coordinates we have $E(x,y) = \frac{E(z(x),z(y))}{\sqrt{dz(x)}\sqrt{dz(y)}}$, and we define

$$\begin{aligned} E(z, p_k) &= \lim_{y \to p_k} E(z(x), z(y)) \sqrt{\frac{dz_k}{dz}(y)}, \\ E(p_k, p_l) &= \lim_{\substack{x \to p_k \\ y \to p_l}} E(z(x), z(y)) \sqrt{\frac{dx_k}{d\zeta}(x)} \sqrt{\frac{dx_l}{d\zeta}(y)} \,. \end{aligned}$$

Let $\mathcal{A}^x$ be the Abel map with the basepoint x, and let $K^x = (K_1^x, \dots, K_g^x)$ be the vector of Riemann constants

$$K_i^x = \frac{1}{2} + \frac{1}{2}\mathbb{B}_{ii} - \sum_{j \neq i} \int_{a_i} \left(v_i(y) \int_x^y v_j \right) dy \tag{6.7}$$

(as above, we assume that the integration paths do not intersect the cuts on X). Then we have $\mathcal{A}^x((df)) + 2K^x = \Omega Z + Z'$ for some $Z, Z' \in \mathbb{Z}^g$. One has the following expression for the holomorphic solution to (6.4)(see [22], here we follow the presentation of this result in [25]):

$$\tau(X,f) = \frac{\left(\left(\sum_{i=1}^g v_i(\zeta) \frac{\partial}{\partial t_i} \right)^g \theta(t;\mathbb{B}) \Big|_{t=K^\zeta} \right)^{2/3}}{e^{6^{-1}\pi\sqrt{-1}\langle \mathbb{B}Z + 4K^\zeta, Z\rangle}\ W(\zeta)^{2/3}} \frac{\prod_{k<l} E(p_k,p_l)^{6^{-1}d_kd_l}}{\prod_k E(\zeta, p_k)^{3^{-1}(g-1)d_k}} \,. \tag{6.8}$$

Here $\theta(t;\mathbb{B}) = \theta[0](t;\mathbb{B})$ is the Riemann theta function, $t = (t_1, \dots, t_g) \in \mathbb{C}^g$, and W is the Wronskian of the normalized holomorphic differentials $v_1, \dots, v_g$ on X; the expression in (6.8) is independent of $\zeta \in X$.

Let $g = 1$. Then the function $\tau(X,f)$ is given (see [20]) by the equation

$$\tau(X,f) = [\theta_1{}'(0\,|\,\mathbb{B})]^{2/3} \frac{\prod_{k=1}^{\ell} h_j^{(k_j+1)/12}}{\prod_{m=1}^{M} f_m^{l_m/12}} \ , \tag{6.9}$$

where $v(P)$ is the normalized Abelian differential on the elliptic Torelli marked curve X; $v(P) = f_m(x_m)dx_m$ near P_m, where $x_m = (z - z_m)^{1/(l_m+1)}$ is the distinguished local parameter near the zero, P_m of the differential df; $f_m \equiv f_m(0)$; $v(P) = h_j(\zeta_j)d\zeta_j$ as $P \to \infty_j$, $\zeta_j = z^{-1/k_j}$, where k_j is the multiplicity of the pole ∞_j of f, $h_k \equiv h_k(0)$; θ_1 is Jacobi theta-function.

Let $g = 0$ and let $U : X \to \mathbb{P}^1$ be a biholomorphic map such that $U(\infty_1) = \infty$ and $U(P) = (f(P))^{1/k_1} + o(1)$ as $P \to \infty_1$. Then (see [24])

$$\tau(X, f) = \frac{\prod_{j=2}^{\ell}\left(\frac{dU}{d\zeta_j}\big|_{\zeta_j=0}\right)^{(k_j+1)/12}}{\prod_{m=1}^{M}\left(\frac{dU}{dx_m}\big|_{x_m=0}\right)^{l_m/12}}. \tag{6.10}$$

Summarizing (6.3) and (6.10,6.9 6.8), we get the main result of the present paper.

Theorem 6. *Let (X, f) be an element of the Hurwitz space $H(M, N)$ and let $\tau(X, f)$ be given by expressions (6.10,6.9, 6.8). There is the following explicit expression for the regularized relative determinant of the Laplacian $\Delta^{|df|^2}$ on the Riemann surface X:*

$$\det{}_\zeta(\Delta^{|df|^2}, \mathring{\Delta}) = C \det\Im\mathbb{B}\, |\tau|^2\,, \tag{6.11}$$

where C is a constant dependent only on the connected component of the space $H(M, N)$ containing the element (X, f).

6.2 Examples in genus 0

We finish the paper with two simple and especially instructive examples of the calculation of the determinant of the Laplacian $\Delta^{|df|^2}$ in genus zero.

Example 1. Let p be a polynomial with $N - 1$ simple critical points $w_1, \dots, w_{N-1}$ and let the corresponding critical values be $z_1, \dots, z_{N-1}$ (or, what is the same, a ramified covering with $N - 1$ simple branch points and one branch point of multiplicity N over the point at infinity of the base. In other words p is an element of the Hurwitz space $H_{0,N}([1]^N)$ of meromorphic functions of degree N on the Riemann sphere $\mathbb{P}^1$ with a single pole of multiplicity N.

Let also w be the holomorphic coordinate on the cover $\mathbb{P}^1$ (more precisely, on $\mathbb{P}^1 \setminus \{\infty\}$) and z be the holomorphic coordinate on the base $\mathbb{P}^1$. One can assume that the leading coefficient of the polynomial $p(w)$ is equal to one.

Introduce the distinguished local parameter $x_k = \sqrt{z - z_k}$ at P_k. Then for $x_k \neq 0$ one has

$$\frac{dw}{dx_k} = \frac{1}{z'(w)} 2x_k = \frac{w - w_k}{p'(w) - p'(w_k)} \frac{2x_k}{w - w_k}$$

Passing to the limit $x_k \to 0$, one gets

$$2\left[w'(x_k)\Big|_{x_k=0}\right]^2 = \frac{1}{p''(w_k)}$$

Thus,

$$\tau = \prod_{k=1}^{N-1}\left[w'(x_k)\Big|_{x_k=0}\right]^{-\frac{1}{12}} = \left\{\prod_{k=1}^{N-1} p''(w_k)\right\}^{\frac{1}{24}} = \mathcal{R}\,(p', p'')^{\frac{1}{24}}, \tag{6.12}$$

where $\mathcal{R}\,(f, g)$ is the resultant of polynomials f and g (since the τ-function is defined up to multiplicative constant, the power of 2 is omitted) and

$$\det{}_\zeta(\Delta^{|dp|^2}, \mathring{\Delta}) = C|\mathcal{R}\,(p', p'')|^{\frac{1}{12}}.$$

Example 2.

Let r is a rational function with three simple poles (which can be assumed to coincide with $\infty, 0, 1$,

$$r(w) = aw - \frac{b}{w} - \frac{c}{w-1} + d;$$

i. e., r is an element of the Hurwitz space $H_{0,3}(1,1,1)$ of meromorphic functions on P^1 of degree three with three simple poles.

Introducing the local parameter $\zeta = \frac{1}{z}$ in vicinities of the poles $w = 0$ and $w = 1$ of the cover, one gets for $\zeta \neq 0$

$$w'(\zeta) = -\frac{1}{r'(w)} r^2(w)$$

and, say, for $w = 1$

$$w'(0) = -\lim_{w \to 1} \frac{1}{r'(w)} r^2(w) = -c\,.$$

Analogously, $w'(0) = -b$ at the pole $w = 0$. For the local parameter $\tilde{w} = w/a$ (for which $\tilde{w}(P) = z(P) + o(1)$ as P tends to the pole $w = \infty$ of the cover) one has $\tilde{w}'(0) = -b/a$ at the pole $w = 0$ and $\tilde{w}'(0) = -c/a$ at the pole $w = 1$. On the other hand writing $r'(w) = \frac{f}{g}$, where f and g are two polynomials and introducing the local parameter $x_k = \sqrt{z - z_k}$ near the critical point w_k of r $(k = 1, 2, 3, 4)$, one gets similarly to (6.12)

$$\tilde{w}'(x_k) = a^{-1} w'(x_k),$$

$$C \prod_{k=1}^{4} [w'(x_k)|_{x_k=0}]^2 = \prod_{k=1}^{4} \frac{1}{r''(w_k)} = \frac{\mathcal{R}(f,g)}{\mathcal{R}(f,f')}$$

Calculating the resultants, one gets

$$\tau^{24} = a^3 b^3 c^3 \mathfrak{M}(a,b,c)$$

where

$$\mathfrak{M}(a,b,c) = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2a + 3b^2c + 3c^2a + 3c^2b - 21abc$$

and

$$\det{}_\zeta(\Delta^{|dr|^2}, \mathring{\Delta}) = C|abc|^{1/4} |\mathfrak{M}(a,b,c)|^{1/12}\,.$$

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