FUNDAMENTAL CLASSES, DIAGONALS AND GYSIN MAPS

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One of the fundamental problems in the study of a concrete (closed) subscheme of a given (smooth) scheme X is the computation of its fundamental class in terms of given generators of the Chow ring of X.

The decisive role in the method described in the present paper plays the *diagonal* of the ambient scheme or, more precisely, its class in the corresponding Chow group of a fibre product.[2] As a matter of fact the class of a diagonal has already been used in intersection theory to the computation of fundamental classes ([F2]). In the situation of loc.cit., there is a vector bundle on the product of Flag bundles endowed with a section vanishing precisely on the diagonal. The top Chern class of this bundle is represented by the so called "top double Schubert polynomial" in the Chern roots of tautological vector bundles on the two Flag bundles. By applying divided differences to this polynomial, the author gets in loc.cit. polynomials representing the classes of other (i.e. higher dimensional) degeneracy loci in the product of flag bundles. (This generalizes a classical procedure discovered in the beginning of the seventies independently by Bernstein-Gelfand-Gelfand and Demazure: starting from the class of the point and applying divided differences one gets the class of a curve, then - the class of a surface etc.)

The procedure given below is of different nature. By using a desingularization of the subscheme whose class we want to compute and the diagonal of the ambient scheme, we replace the original problem by the one of computing the image of the class of the diagonal under an appropriate Gysin map. Moreover, since not always the diagonal is represented as the scheme of zeros of a section of a vector bundle (this seems to happen, e.g., for Flag bundles for other classical groups than SL_n), we give a recipe allowing to calculate the class of the diagonal of the fibre product with the help of Gysin maps.

We illustrate a uselfulness of the latter result on the example of Lagrangian and orthogonal Grassmannian and Flag bundles; the so obtained formulas are crucial for our study of the classes of degeneracy loci in [P-R].

The results of this preprint will appear as parts of two separate publications. Proposition 1 and Theorem 2 will be published in Section 5 of the paper: P. Pragacz, Symmetric polynomials and divided differences in formulas of intersection theory; to appear in "Parameter Spaces", Banach Center Publications **36** (1996). Theorems 6 and 8 will be

[1]The results of this paper have been obtained by the author during his recent stay at the Max-Planck-Institut für Mathematik. The author gratefully thanks the MPIfM for a generous hospitality.

[2]A discovery of this method is inspired by the construction used in the proof of the main formula in the paper by G. Kempf and D. Laksov [K-L].

published in the paper: P. Pragacz and J. Ratajski, Formulas for Lagrangian and orthogonal degeneracy loci; the $\widetilde{Q}$ - polynomials approach; to appear in Compositio Math. (1996).

The author thanks A. Lascoux and J.-Y. Thibon for valuable comments about Q'- and $\widetilde{Q}$-polynomials.

We start with some recollection on intersection theory (see [F1] for details). Recall that if $D \subset X$ is a (closed) subscheme then $[D] \in A_*(X)$ is the class of the fundamental cycle associated with D, i.e., if $D = D_1 \cup \ldots \cup D_n$ is a minimal decomposition into irreducible components then

$$[D] = \textstyle\sum_{i=1}^{n} (length\ \mathcal{O}_{D,D_i})[D_i],$$

where $\mathcal{O}_{D,D_i}$ is the local ring of D along D_i. Recall also that if $f : X \to Y$ is a proper morphism then it induces a morphism of abelian groups $f_* : A_*(X) \to A_*(Y)$ such that $f_*[V] = deg(f|_V)[f(V)]$ if $dim\, f(V) = dim\, V$ and 0 - otherwise. In particular, if f establishes a birational isomorphism of V and $f(V)$ then $f_*[V] = [f(V)]$. If X and Y are nonsingular then a morphism $f : X \to Y$ induces a ring homomorphism $f^* : A^*(Y) \to A^*(X)$.

Let S be a smooth scheme (over a field) and let $\pi : X \to S$ be a smooth morphism of schemes. Suppose that $D \subset X$ is a (closed) subscheme whose class is to be computed. Let $p : Z \to S$ be a proper smooth morphism and $\alpha : Z \to D$ – a proper birational map of S-schemes. Consider a commutative diagram:

$$\begin{tikzcd}
\Delta \arrow[r, hook] & X \times_S X & \\
& X \times_S Z \arrow[u, "1 \times \alpha"] \arrow[d, "p_1"] \arrow[r, shift right, "p_2"'] & Z \arrow[l, shift right, "\sigma"'] \arrow[dl, "\alpha"'] \arrow[d, "p"] \\
D \arrow[r, hook] & X \arrow[r, "\pi"'] & S
\end{tikzcd}$$

Here p_1 and p_2 are the projections, the section σ (of p_2) equals $id \times_S \alpha$ and Δ is the diagonal in the fibre product $X \times_S X$.

Proposition 1. *Suppose that the class of the diagonal Δ in $A^*(X \times_S X)$ is $[\Delta] = \sum pr_1^*(x_i) \cdot pr_2^*(y_i)$ where $pr_i : X \times_S X \to X$ are the projections and $x_i, y_i \in A^*(X)$. Then, in $A^*(X)$,*

$$[D] = \sum_i x_i \cdot \big(\pi^*\, p_*\, \alpha^*(y_i)\big).$$

Proof. By the assumption $[D] = \alpha_*([Z])$. Since $\alpha = p_1 \circ \sigma$, we have $\alpha_*([Z]) = (p_1)_*[\sigma(Z)]$. Now, the key observation is that, in the scheme-theoretic sense, one has the equality

$\sigma(Z) = (1 \times \alpha)^{-1}(\Delta)$. Since $\Delta \cong X$ is smooth, this implies $[\sigma(Z)] = (1 \times \alpha)^*([\Delta])$ (see Lemma 9 in [K-L]). We then have:

$$\begin{aligned}
[D] &= (p_1)_*\Big([\sigma(Z)]\Big) = (p_1)_*\Big((1 \times \alpha)^*([\Delta])\Big) \\
&= (p_1)_*\Big((1 \times \alpha)^*\big(\sum_i pr_1^*(x_i) \cdot pr_2^*(y_i)\big)\Big) \\
&= (p_1)_*\Big(\big(\sum_i p_1^*(x_i) \cdot p_2^*(\alpha^*(y_i))\big)\Big) \\
&= \sum_i x_i \cdot \big((p_1)_* \, p_2^* \, \alpha^*(y_i)\big) \\
&= \sum_i x_i \cdot \big(\pi^* \, p_* \, \alpha^*(y_i)\big),
\end{aligned}$$

where the last equality follows from the above fibre product diagram and [F1, Proposition 1.7]. □

The next result shows how one can compute the fundamental class of the diagonal $[\Delta] \in A^*(X \times_S X)$.

Theorem 2. *Let S be as above and $\pi : X \to S$ be a proper smooth morphism such that π^* makes $A^*(X)$ a free $A^*(S)$-module; $A^*(X) = \oplus_{\alpha \in \Lambda} A^*(S) \cdot a_\alpha$, where $a_\alpha \in A^{n_\alpha}(X)$ and $A^*(X) = \oplus_{\beta \in \Lambda} A^*(S) \cdot b_\beta$, where $b_\beta \in A^{m_\beta}(X)$. Suppose that for any α there is a unique $\beta = \alpha'$ such that $n_\alpha + m_{\alpha'} = dim\ X - dim\ S$ and $\pi_*(a_\alpha \cdot b_{\alpha'}) \neq 0$ (assume $\pi_*(a_\alpha \cdot b_{\alpha'}) = 1$). Moreover, denoting by $pr_i : X \times_S X \to X$ $(i = 1, 2)$ the projections, suppose that the homomorphism $A^*(X) \otimes_{A^*(S)} A^*(X) \to A^*(X \times_S X)$, defined by $g \otimes h \mapsto pr_1^*(g) \cdot pr_2^*(h)$, is an isomorphism. Then*

(i) The class of the diagonal Δ in $X \times_S X$ equals $\big[\Delta\big] = \sum_{\alpha,\beta} d_{\alpha\beta} a_\alpha \otimes b_\beta$, where, for any α, β, $d_{\alpha\beta} = P_{\alpha\beta}(\{\pi_*(a_\kappa \cdot b_\lambda)\})$ for some polynomial $P_{\alpha\beta} \in \mathbb{Z}[\{x_{\kappa\lambda}\}]$.

(ii) The following conditions are equivalent:

a) One has $\pi_*(a_\alpha \cdot b_{\beta'}) = \delta_{\alpha,\beta}$, the Kronecker delta.

b) The class of the diagonal $\Delta \subset X \times_S X$ equals $\big[\Delta\big] = \sum_\alpha a_\alpha \otimes b_{\alpha'}$.

Proof. Denote by $\delta : X \to X \times_S X$, $\delta' : X \to X \times_K X$ (the Cartesian product) the diagonal embeddings and by γ the morphism $\pi \times_S \pi : X \times_S X \to S$. For $g, h \in A^*(X)$ we have

$$\pi_*(g \cdot h) = \pi_*\Big((\delta')^*(g \times h)\Big) = \pi_*\Big(\delta^*(g \otimes h)\Big) = \gamma_* \delta_*\Big(\delta^*(g \otimes h)\Big) = \gamma_*\Big([\Delta] \cdot (g \otimes h)\Big),$$

where all the equalities follow from the theory in [F1, Chap.8], taking into account, for the second one, the commutative diagram

$$\begin{array}{ccc}
X \times_S X & \hookrightarrow & X \times X \\
\nwarrow_{\delta} & & \nearrow_{\delta'} \\
& X &
\end{array}$$

and, for the third one, the equality $\pi = \gamma \circ \delta$. Hence, writing $[\Delta] = \sum d_{\mu\nu} b_\mu \otimes a_\nu$, we get

$$(*) \qquad \begin{aligned} \pi_*(a_\alpha \cdot b_\beta) &= \gamma_*([\Delta] \cdot (a_\alpha \otimes b_\beta)) = (\pi_* \otimes \pi_*)\Big(\big(\sum d_{\mu\nu} b_\mu \otimes a_\nu\big) \cdot (a_\alpha \otimes b_\beta)\Big) \\ &= \sum_{\mu,\nu} d_{\mu\nu} \pi_*(b_\mu \cdot a_\alpha) \cdot \pi_*(a_\nu \cdot b_\beta). \end{aligned}$$

(i) By the assumption and (*) we get

$$(**) \qquad d_{\alpha\beta} = \pi_*(b_{\alpha'} \cdot a_{\beta'}) - \sum_{\mu \neq \alpha, \nu \neq \beta} d_{\mu\nu} \pi_*(a_\mu \cdot b_{\alpha'}) \cdot \pi_*(b_\nu \cdot a_{\beta'}).$$

where the degree of $d_{\mu\nu} \in A^*(S)$ such that $\pi_*(a_\mu \cdot b_{\alpha'}) \cdot \pi_*(b_\nu \cdot a_{\beta'}) \neq 0$ and $\mu \neq \alpha$ or $\nu \neq \beta$, is smaller than the degree of $d_{\alpha\beta}$. The assertion now follows by induction on the degree of $d_{\alpha\beta}$.

(ii) a) $\Rightarrow$ b) : By virtue of a), Equation (**) now reads $\pi_*(b_{\alpha'} \cdot a_{\beta'}) = d_{\alpha\beta}$ and immediately implies b).

b) $\Rightarrow$ a) : Without loss of generality we can assume that Λ is endowed with a linear ordering $\prec$ compatible with codimension, i.e. $n_{\alpha_1} < n_{\alpha_2} \Rightarrow \alpha_1 \prec \alpha_2$, $m_{\beta_1} < m_{\beta_2} \Rightarrow \beta_1 \prec \beta_2$ and such that $\alpha_1 \prec \alpha_2 \Rightarrow \alpha_2' \prec \alpha_1'$. The rows and columns of the matrices below are ordered using the ordering $\prec$. Write $x_{\alpha\beta} = \pi_*(a_\alpha \cdot b_\beta)$. By virtue of b), Equation (*) gives us the following system of equations:

$$x_{\alpha\beta} = \textstyle\sum_\mu x_{\alpha\mu} x_{\mu'\beta},$$

where $\alpha, \beta \in \Lambda$. Note that the antidiagonal of the matrix $M := (x_{\alpha\beta})_{\alpha,\beta\in\Lambda}$ is indexed by $\{(\alpha, \alpha') | \alpha \in \Lambda\}$. The assumption implies that this antidiagonal consists of units. Moreover, because of dimension reasons and the assumption again, we know that the entries above the diagonal are zero. Let P be the permutation matrix corresponding to the bijection $\alpha \mapsto \alpha'$ of Λ. The above system of equations is rewritten in the matrix form as:

$$MP = MP \cdot MP.$$

Then MP as a (lower) triangular matrix with the units on the diagonal, must be the identity matrix. Hence $M = P^{-1}$ and this implies a). $\square$

Remark 3. A standard situation when the theorem can be applied is when $\pi : X \to S$ is a Zariski locally trivial fibration and $\{a_\alpha\}$, $\{b_\beta\}$ restrict to bases of the Chow ring of a fiber F which are dual under the Poincaré duality map : $(a, b) \mapsto \int_F a \cdot b$.

Example 4. **a)** Let $\pi : \mathcal{G} = G^q(E) \to X$ be the Grassmannian bundle parametrizing q-quotients of of a vector bundle E of rank n on X. Write $r = n - q$. Let

$$0 \to R \to E_{\mathcal{G}} \to Q \to 0$$

be the tautological exact sequence of vector bundles on $\mathcal{G}$. It is easy to see the diagonal in $\mathcal{G}_1 \times_X \mathcal{G}_2$, where $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}$, is given (in the scheme-theoretic sense) by the vanishing of the entries of a matrix of the homomorphism $R_{\mathcal{G}_1} \to E_{\mathcal{G}_1} = E_{\mathcal{G}_2} \to Q_{\mathcal{G}_2}$. Hence, by the theorem and a formula for the top Chern class of the tensor product (see [L]), we have that $\pi_*\big(s_I(Q)\cdot s_{\overline{J}}(R^\vee)\big) = \delta_{I,J}$ for partitions $I, J \subset (r)^q$ [3], where $\overline{J}$ is the partition whose Ferrers' diagram complements the one of $J^\sim$ in the rectangle $(q)^r$.[4] Equivalently, $\pi_*\big(s_I(Q)\cdot s_{(r)^q/J}(-R)\big) = \delta_{I,J}$. This is coherent with a well-known description of the Gysin map associated with π.

b) Let now $\tau : Fl(E) \to X$ be the Flag bundle parametrizing the complete flags of (sub)bundles of E. In a similar way, using the calculation of the class of the diagonal from [F2, Proposition 7.5] via the top Chern class of a suitable vector bundle, one reproves the following equality from [L-S]. For permutations $\mu, \nu \in S_n$,

$$\tau_*\Big(\mathfrak{S}_\mu(A)\cdot \mathfrak{S}_{\nu\omega}(-a_n, -a_{n-1}, \dots, -a_1)\Big) = \delta_{\mu,\nu}.$$

where $\mathfrak{S}_\mu(A)$ is the Schubert polynomial (see loc.cit. where this polynomial is denoted by $X_\mu(A)$) associated with the permutation μ and the sequence of the Chern roots $A = (a_1, \dots, a_n)$ of E.

We pass now to the situation where the diagonal seems not to be the zero subscheme of a section of a vector bundle; we will investigate Lagrangian Grassmannian- and Flag bundles parametrizing respectively top dimensional Lagrangian subspaces and flags of Lagrangian subbundles of successive ranks $1, 2, \dots, n$ of a vector bundle V of rank $2n$ endowed with a nondegenerate symplectic form. For what concerns the notation and elementary properties of these schemes, we refer the reader to [P-R].

We need also from loc.cit. the *$\widetilde{Q}$-polynomials* - a family of symmetric polynomials invented and studied in loc.cit.. Let us recall briefly their definition and give a Pieri-type theorem for them. For more about the properties of $\widetilde{Q}$-polynomials, we refer the reader to loc.cit..

Let $X = (x_1, x_2, \dots)$ be a sequence of independent variables. Denote by X_n the subsequence $(x_1, \dots, x_n)$. We set $\widetilde{Q}_i(X_n)$ to be the i-th elementary symmetric polynomial $e_i(X_n)$ in X_n. Given two nonnegative integers i, j we define

$$\widetilde{Q}_{i,j}(X_n) = \widetilde{Q}_i(X_n)\widetilde{Q}_j(X_n) + 2\sum_{p=1}^{j}(-1)^p \widetilde{Q}_{i+p}(X_n)\widetilde{Q}_{j-p}(X_n).$$

Finally, for any (i.e. not necessary strict) partition $I = (i_1 \geqslant i_2 \geqslant \dots \geqslant i_k \geqslant 0)$, with even k (by putting $i_k = 0$ if necessary), we set $\widetilde{Q}_I(X_n)$ to be the Pfaffian of the antisymmetric matrix with $\widetilde{Q}_{i_p,i_q}(X_n)$ on the (p,q)-place, $1 \leqslant p < q \leqslant k$.

Invoking the raising operators R_{ij} ([Mcd, I.1]) the above definition is rewritten

$$\widetilde{Q}_I(X_n) = \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}} e_I(X_n),$$

[3] All the notions (as well as the notation) concerning partitions follow here [P-R].

[4] Given a vector bundle F and a partition $I = (i_1 \geqslant \dots \geqslant i_k \geqslant 0)$, we denote by $s_I(F)$ the *Schur polynomial* equal to $Det\big[s_{i_p-p+q}(F)\big]_{1\leqslant p,q\leqslant k}$, where $s_i(F)$ is the i-th complete symmetric polynomial applied to the Chern roots of F.

where $e_I(X_n)$ is the product of the elementary symmetric polynomials in X_n associated with the parts of I.

Lemma 5. *Let $I = (i_1, ..., i_k)$ be a strict partition of length k. Then*

$$\widetilde{Q}_I(X_n) \cdot \widetilde{Q}_r(X_n) = \sum 2^{m(I,r;J)} \widetilde{Q}_J(X_n),$$

where the sum is over all partitions (i.e. not necessary strict) $J \supset I$ such that $|J| = |I| + r$ and J/I is a horizontal strip. Moreover, $m(I, r; J) = card\{1 \le p \le k |\ j_{p+1} < i_p < j_p\}$ or, equivalently, it is expressed as the number of connected components of the strip J/I not meeting the first column.

Proof. Let after [L-L-T], $Q'_I(X_n; q)$ denote the Hall–Littlewood polynomial $Q_I(Y; q)$ (see [Mcd, III]) where the alphabet Y is equal to $X_n/(1 - q)$ (in the sense of λ-rings). Using raising operators R_{ij} ([Mcd, I.1]), we have

$$Q'_I(X_n; q) = \prod_{i<j} (1 - qR_{ij})^{-1} s_I(X_n).$$

Specialize $q = -1$ and invoke the well known Jacobi–Trudi formula (see, e.g., [Mcd, I.3]):

$$s_I(X_n) = \prod_{i<j} (1 - R_{ij}) h_I(X_n),$$

where $h_I(X_n)$ is the product of complete homogeneous polynomials in X_n associated with the parts of I. We have

$$Q'_I(X_n; -1) = \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}\ h_I(X_n).$$

Therefore, denoting by ω the Young duality-involution we get $\widetilde{Q}_I(X_n) = \omega\big(Q'_I(X_n; -1)\big)$.

The required assertion now follows by an appropriate specialization of the Pieri-type formula for Hall–Littlewood polynomials ([Mcd, III.3.(3.8)]). □

We now state the following "orthogonality" theorem. Given a vector bundle E and a partition I, we denote by $\widetilde{Q}_I E$ the polynomial $\widetilde{Q}_I(X_n)$ with $e_i(X_n)$ replaced by $c_i(E)$.

Theorem 6. *For $\pi : LG_nV \to X$ and any strict partitions I, J $(\subset \rho_n)$,*

$$\pi_*(\widetilde{Q}_I R^\vee \cdot \widetilde{Q}_J R^\vee) = \delta_{I, \rho_n \smallsetminus J}.$$

Here, R is the tautological (sub)bundle on LG_nV and $\delta_{.,.}$ is the Kronecker delta.

Proof. Let $X_n = (x_1, ..., x_n)$ be a sequence of variables. We know, after [P-R, Proposition 5.8][5] that π_* is induced by the operator $\nabla = \partial_{(\overline{n}, \overline{n-1}, \ldots, \overline{1})} : \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$.

[5] A correction: the Chern roots $q_1, \ldots, q_n$ of $R^\vee$ should be replaced, in the formula of the proposition, by the ones of R.

We show that the operator ∇ satisfies the following formula for any strict partitions I, J ($\subset \rho_n$):

$$\nabla\big(\widetilde{Q}_I(X_n^\vee)\cdot\widetilde{Q}_J(X_n^\vee)\big)=\delta_{I,\rho_n\setminus J}.$$

Observe that for the degree reasons $\nabla(\widetilde{Q}_I\cdot\widetilde{Q}_J)=0$ for $|I|+|J|<n(n+1)/2$ (here and in the rest of the proof, $\widetilde{Q}_I=\widetilde{Q}_I(X_n^\vee)$). Also, because of the universality of the formula for π_* (see [loc.cit., Theorem 5.10]), we know by [loc.cit., Lemma 2.3] and Theorem 2 that for $|I|+|J|=n(n+1)/2$, $\nabla(\widetilde{Q}_I\cdot\widetilde{Q}_J)=0$ unless $J=\rho_n\setminus I$, when $\nabla(\widetilde{Q}_I\cdot\widetilde{Q}_J)=1$. So it remains to show that for $|I|+|J|>n(n+1)/2$, $\nabla(\widetilde{Q}_I\cdot\widetilde{Q}_J)=0$. The proof is by double induction whose first parameter is $l(I)$ and the second one is i_l where $l=l(I)$ (i.e. the shortest part of I).

Assume first that $I=(i)$ and use a Pieri-type formula from Lemma 5. A general partition J' indexing the R.H.S of the formula in Lemma 5 stems from J by adding a horizontal strip of length i. Since $|J|+i>n(n+1)/2$, the only possibility for getting $\nabla(\widetilde{Q}_{J'})\neq 0$ is the following one (use [loc.cit., Theorem 5.10]): there exist two equal parts p in J' such that after factorizing $\widetilde{Q}_{p,p}$ from $\widetilde{Q}_{J'}$ ([loc.cit., Proposition 4.3]) we obtain $\widetilde{Q}_{\rho_n}$ (recall that $\widetilde{Q}_{p,p}$ is a scalar w.r.t. ∇). But $l(J')\le l(J)+1\le n+1$, so after factorization the length of the so-obtained partition is not greater than $n-1$, i.e. this partition is not ρ_n.

To perform the induction step write $I'=(i_1,...,i_{l-1})$ and $r=i_l$ where we assume that $l=l(I)\ge 2$. Using the Pieri-type formula again, we have:

$$\widetilde{Q}_I\cdot\widetilde{Q}_J=(\widetilde{Q}_{I'}\cdot\widetilde{Q}_r)\cdot\widetilde{Q}_J-\Big(\sum_M 2^{m(I',r;M)}\widetilde{Q}_M\Big)\cdot\widetilde{Q}_J=\widetilde{Q}_{I'}\cdot(\widetilde{Q}_J\cdot\widetilde{Q}_r)-\Big(\sum_M 2^{m(I',r;M)}\widetilde{Q}_M\Big)\cdot\widetilde{Q}_J$$

$$=\widetilde{Q}_{I'}\cdot\Big(\sum_N 2^{m(J,r;N)}\widetilde{Q}_N\Big)-\Big(\sum_M 2^{m(I',r;M)}\widetilde{Q}_M\Big)\cdot\widetilde{Q}_J.$$

Here M runs over all partitions different from I which contain I' with M/I' being a horizontal strip of length r. Observe that either $l(M)<l(I)$ or $l(M)=l(I)$ but $m_l<i_l=r$, so we can apply the induction assumption to M. The partitions M and N can have equal parts; if so, using the factorization property, we write:

$$\widetilde{Q}_M=\widetilde{Q}_{p_1,p_1}\cdot\ldots\cdot\widetilde{Q}_{p_s,p_s}\cdot\widetilde{Q}_{M_1}\quad\text{and}\quad\widetilde{Q}_N=\widetilde{Q}_{q_1,q_1}\cdot\ldots\cdot\widetilde{Q}_{q_t,q_t}\cdot\widetilde{Q}_{N_1},$$

where M_1, N_1 are strict partitions and $p_1>\ldots>p_s$, $q_1>\ldots>q_t$ are positive integers. Using the induction assumption or because of the degree reasons we see that the only possibility to get in the first sum a summand (corresponding to N) which is not anihilated by ∇ is: after adding to J a horizontal strip of length r and factorizing all pairs of equal rows, we obtain the partition $N_1=\rho_n\setminus I'$. Similarly, the only possibility to get in the second sum a summand (corresponding to M) which is not anihilated by ∇ is: after adding to I' a horizontal strip of length r and factorizing all pairs of equal rows, we obtain the partition $M_1=\rho_n\setminus J$.

Therefore to conclude the proof it is sufficient to define, for a fixed pair of strict partitions I', J and fixed positive integers r and $p_.$: $p_1>\ldots>p_s$, a bijection between the sets of partitions:

$\mathcal{N} = \{\ N \mid N \supset J;\ N/J$ is a horizontal strip of length r; N has exactly s parts occuring twice, equal to $p.$; after subtraction from N the parts $p.$ one obtains $\rho_n \setminus I'\}$

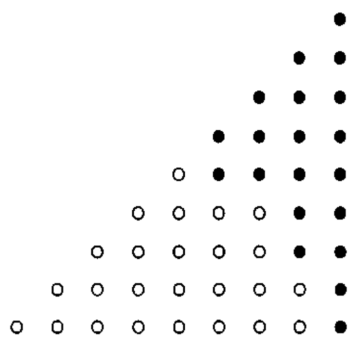
and

$\mathcal{M} = \{\ M \mid M \supset I'; M/I'$ is a horizontal strip of length r; M has exactly s parts occuring twice, equal to $p.$; after subtraction from M the parts $p.$ one obtains $\rho_n \setminus J\}$

which preserves the cardinality of the connected components of the strip, not meeting the first column (compare the Pieri-type formula used).

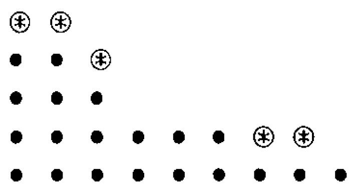
In order to define the bijection $\Phi : \mathcal{N} \to \mathcal{M}$ we first invoke the diagramatic presentation of the ρ_n-complementary partition from [P]: for example $n = 9$, $I = (9,6,3,2)$, $\rho_9 \setminus I = (8,7,5,4,1)$,

Fig.1



(the collection of "•" gives the shifted diagram of I (appropriately placed); the collection of "o" gives the shifted diagram of $\rho_9 \setminus I$). The map $\Phi : \mathcal{N} \to \mathcal{M}$ is defined as follows. Having an element $N \in \mathcal{N}$, i.e. a strict partition J with an added horizontal strip of length r, e.g. $J = (9,6,3,2), r = 5, N = (9,8,3,3,2), s = 1,\ p. : 3$ (and $I' = (7,6,5,4,3,1)$):

Fig.2



we remove the s bottom rows in all pairs of equal rows (in the example, the third row) and place the shift of the so-obtained diagram as in Fig.1 to get the diagram $\widehat{N}$, say. In our example we get the diagram in Fig.3 :

Fig.3

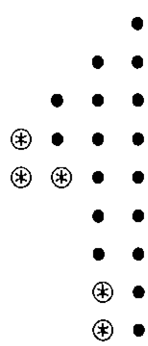
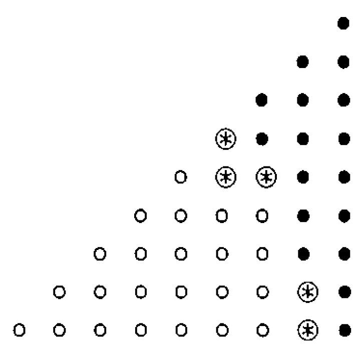


Fig.4



(We know, by the definition of $\mathcal{N}$, that if we would also remove from $\widehat{N}$ the remaining parts of lengths $p.$ then the resulting partition will be $\rho_n \smallsetminus I'$. We preserve these parts, however, because we need them for the construction of $\Phi(N)$.) Then we construct the complement of the so-obtained diagram in ρ_n. In our example, using "o" to visualize the complementary diagram we get the diagram in Fig.4. By reshifting the so-obtained complementary diagram plus *the same* horizontal strip (now added to this complementary diagram) - call it $\Phi(N)_0$, and inserting s rows of lengths $p.$, we get the needed partition $\Phi(N)$. Observe that :

1) Since at the last stage we have inserted rows of lengths $p.$, $\Phi(N)$ consists of the diagram I' with an added horizontal strip of length r.

2) $\Phi(N)$ has exactly s parts occuring twice, equal to $p.$ (apart from the parts inserted at the last stage, the remaining s parts are the rows whose the rightmost boxes are precisely the lowest boxes of the rows of length $p.$ in $\widehat{N}$).

3) After removing from $\Phi(N)$ the $2s$ parts equal to $p.$, we get $\rho_n \smallsetminus J$ (this is the same as the removing from $\Phi(N)_0$ the s parts equal to $p.$ - but $\Phi(N)_0$ minus s parts equal to $p.$ complements precisely J in ρ_n).

Therefore $\Phi(N) \in \mathcal{M}$. Also, the cardinality of the connected components of the strip not meeting the first column is preserved by Φ. In our example, we obtain

$$\text{Fig.5} \qquad \begin{array}{cccccccc}
\circledast & & & & & & & \\
\circ & \circledast & \circledast & & & & & \\
\circ & \circ & \circ & & & & & \\
\circ & \circ & \circ & \circ & & & & \\
\circ & \circ & \circ & \circ & \circ & & & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circledast & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circledast
\end{array}$$

i.e. $\Phi(N) = (8,7,5,4,3,3,1)$.

Let us now define, by reversing the roles of J and I', the map $\Psi : \mathcal{M} \to \mathcal{N}$. If we define, by a complete analogy to the above, the partitions $\widehat{M}$ and $\Psi(M)_0$, then we have $\widehat{N} = \Psi(M)_0$ and $\Phi(N)_0 = \widehat{M}$; and clearly $\Psi \circ \Phi = id_{\mathcal{N}}$ and $\Phi \circ \Psi = id_{\mathcal{M}}$.

This proves the orthogonality theorem. $\square$

The theorem, combined with Theorem 2, gives a transparent proof of the key Proposition 2.5 in [P-R].

Corollary 7. *Let $\tau : LFl(V) \to X$ be the flag bundle parametrizing flags of isotropic subbundles of V of successive ranks $1, 2, \ldots, n$. Let $A = (a_1, \ldots, a_n)$ be the sequence of the Chern roots of the tautological flag on $LFl(V)$. Then for strict partitions $I, J \subset \rho_n$ and $\mu, \nu \in S_n$, one has*

$$\tau_*\Big(\widetilde{Q}_I(A) \cdot \widetilde{Q}_{\rho_n \smallsetminus J}(A) \cdot \mathfrak{S}_\mu(A) \cdot \mathfrak{S}_{\nu\omega}(-a_n, -a_{n-1}, \ldots, a_1)\Big) = \delta_{I,J} \cdot \delta_{\mu,\nu}.$$

Proof. The assertion follows from Theorem 6, the following factorization:

$$\tau : LFl(V) \xrightarrow{\omega} LG_n V \xrightarrow{\pi} X,$$

where ω is the Flag bundle $Fl(R) \to LG_nV$ parametrizing complete flags of the tautological bundle R on LG_nV, and Example 4 b). $\square$

Similar results (and their proofs) hold for vector bundles endowed with orthogonal forms. Set $P_I(E) := 2^{-l(I)}Q_I(E)$ for a vector bundle E and a partition I. Denote by $\pi : OG_nV \to X$ (resp. $OG'_nV \to X$ and $OG''_nV \to X$) the Grassmannians parametrizing isotropic subbundles of rank n of a bundle V of rank $2n+1$ (resp. $2n$) endowed with a nondegenerate orthogonal form.[6] One has the following result.

Theorem 8. *(i) For $\pi : OG_nV \to X$ ($dimV = 2n+1$) and any strict partitions I, J ($\subset \rho_n$),*

$$\pi_*(\widetilde{P}_I R^\vee \cdot \widetilde{P}_J R^\vee) = \delta_{I,\rho_n \smallsetminus J}.$$

(ii) For $\pi : OG'_nV \to X$ (resp. $OG''_nV \to X$), and any strict partitions I, J ($\subset \rho_{n-1}$),

$$\pi_*(\widetilde{P}_I R^\vee \cdot \widetilde{P}_J R^\vee) = \delta_{I,\rho_{n-1} \smallsetminus J}.$$

Here, R is the tautological (sub)bundle on the corresponding orthogonal Grassmannian and $\delta_{.,.}$ is the Kronecker delta.

An obvious analog of Corollary 7 is left, in this case, to the interested reader.

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[6] In the orthogonal case, one must assume that the form is hyperbolic, i.e., V has an isotropic subbundle of rank n. Moreover, recall that if $rankV = 2n$ then the corresponding orthogonal Grassmannian breaks up into a sum of two disjoint schemes (see, e.g., [P-R] for more on that).