Spectra of 'spaces' represented by abelian categories.

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Introduction

Spectral theory of abelian categories was started by P. Gabriel in early sixties [Gab]. Elements of the Gabriel's spectrum are isomorphism classes of indecomposable injectives. If R is a commutative noetherian ring, the Gabriel's spectrum of the category of R-modules is naturally isomorphic to the prime spectrum of the ring R. More generally, the Gabriel's spectrum of the category of quasi-coherent sheaves on a noetherian scheme is isomorphic to the underlying space of the scheme [Gab, Ch. VI, Theorem 1].

The Gabriel's spectrum was the only spectrum of an abelian category for more than a quarter of century (except for it was rediscovered a couple of times). Then three other spectra were assigned to an abelian category C_X: the spectrum $\mathbf{Spec}(X)$ (see [R1] and [R, Ch. 3]), the *S-spectrum* $\mathbf{Spec}^-(X)$, and the complete spectrum $\mathbf{Spec}^1(X)$ [R, Ch.6]. These spectra make sense without noetherian hypothesis. If C_X is the category of modules over an arbitrary commutative unital ring R, then $\mathbf{Spec}(X)$ is naturally isomorphic to the prime spectrum of R. More generally, if C_X is the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme, then $\mathbf{Spec}(X)$ is isomorphic to the underlying space of the scheme. This fact is obtained in [R4] as a consequence of a theorem on local properties of the spectrum $\mathbf{Spec}(X)$. In Section 9 of this work, we prove a refined version of this theorem. For an arbitrary abelian category C_X, isomorphism classes of simple objects of C_X correspond to closed points of $\mathbf{Spec}(X)$.

The S-spectrum is larger than $\mathbf{Spec}(X)$ – there is a natural embedding of $\mathbf{Spec}(X)$ into $\mathbf{Spec}^-(X)$. If C_X is a locally noetherian Grothendieck category (or, more generally, a Grothendieck category with a Gabriel-Krull dimension), then the S-spectrum is naturally isomorphic to the Gabriel spectrum. An advantage of the S-spectrum is that it makes sense for abelian categories without injective objects, like, for example, the category of coherent sheaves on a scheme, or the category of finitely generated modules over a ring.

One of practical merits of the complete spectrum is that it contains the S-spectrum and is functorial with respect to arbitrary open immersions, while the S-spectrum is functorial only with respect to open immersion having direct image functor. As a result, the underlying space of a scheme is naturally embedded into the complete spectrum of the category of quasi-coherent sheaves on the scheme.

In [R5], several new spectra were defined which make sense for 'spaces' represented by (not necessarily abelian) categories, in particular, the spectra of 'spaces' represented by triangulated categories. Finally, a natural, very general construction of spectra was proposed in [R6]. This construction is tested here on the spectra $\mathbf{Spec}(X)$, $\mathbf{Spec}^-(X)$, and $\mathbf{Spec}^1(X)$; that is we use the construction to study these spectra, as well as some other spectra which appeared in [R5] or suggested by the construction of [R6].

The main thrust of the paper is establishing local properties of the spectra which are needed to study the spectra of non-affine noncommutative schemes, and are crucial for reconstruction problems. These local properties are also used in computations of spectra and applications of noncommutative local algebra and algebraic geometry to representation theory [R7].

Section 1 contains the necessary preliminaries on 'spaces' and morphisms of 'spaces', topologizing, thick, and Serre subcategories, and the spectra $\mathbf{Spec}(X)$, $\mathbf{Spec}^-(X)$, and $\mathbf{Spec}^1(X)$ mentioned above. Section 2 presents, for the reader's convenience a short exposition of the main construction of [R6] whose applications are studied in this work. Sections 3 and 4 are dedicated to the spectrum $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ introduced in [R6, 2.3.5] and its counterpart $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ suggested by the general construction of Section 2. The natural decompositions of these spectra allow to recover the spectrum $\mathbf{Spec}(X)$ and see some of its important properties unnoticed earlier. In Section 5, we look at the complete spectrum and the S-spectrum. The general pattern invoke another spectrum, $\mathbf{Spec}_{\rtimes}(X)$, which is a natural extension of $\mathbf{Spec}(X)$ and plays a similar role. In Section 7 we study functorial properties of these spectra. In Section 8, we apply general pattern developed in the previous sections to the spectra related with exact localizations. We start with the thick spectrum introduced in [R5] and recover other spectra via its canonical decomposition.

In Section 9, we study local properties of the spectra with respect to finite covers. The results we obtain here are stronger than the results in this direction obtained in [R4]. The refinements are due to different arguments making use of a simple algebra of the monoid of topologizing subcategories which is sketched in Appendix 1.

Section 10 is dedicated to the spectra related to reflective topologizing categories and their local properties. The material of this section is entirely new. The general pattern of Section 2 produce a pair spectra – $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{c}}(X)$, together with a canonical morphism between them, which, for these particular spectra, turns out to be an isomorphism. For an arbitrary abelian category C_X, there is a natural embedding $\mathbf{Spec}(X) \hookrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$. If C_X is the category of quasi-coherent sheaves on a quasi-compact (noncommutative) scheme, then $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}(X)$ coincide.

Some time ago, O. Gabber constructed an example showing that the spectrum $\mathbf{Spec}(X)$ is not sufficient to recover the underlying topological space of an arbitrary non-quasi-compact (commutative) scheme from the category C_X of quasi-coherent sheaves on the scheme. He observed that the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$, on the other hand, suffices to reconstruct the underlying space in his example which gave a ground to the conjecture that it might suffice in the general case, under some mild restrictions. We show that this conjecture is, indeed, true. It follows from the local property of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ (or, ruther, of its counterpart $\mathbf{Spec}^1_{\mathfrak{c}}(X)$) with respect to infinite covers, which we formulate and prove in 10.6. If C_X is the category of quasi-coherent sheave on a scheme $\mathbf{X}$, then $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ endowed with the Zariski topology (which is defined in terms of topologizing subcategories) is naturally isomorphic to the underlying topological space of the scheme $\mathbf{X}$, under the condition that $\mathbf{X}$ admits an affine cover $\{U_i \hookrightarrow \mathbf{X} \mid i \in J\}$ such that each immersion $U_i \hookrightarrow \mathbf{X}$ has a direct image functor. The latter condition holds if the scheme is quasi-separated.

In Appendix 1, we relate topologies on spectra with some natural 'topological' structures on the monoid of topologizing subcategories. Besides, this appendix contains facts (mostly borrowed from [R4]) which are used in the main body of the paper, expecially in Sections 9 and 10. Appendix 2 contains some complements on (differences between) spectra $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ and $\mathbf{Spec}(X)$. It is a complement to Sections 3 and 4. We consider $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ and $\mathbf{Spec}(X)$ as topological spaces with respect to the topology associated with topologizing subcategories and establish some general properties of these spaces. Appendix 3 contains some observations on supports of objects and the Krull filtrations. In Appendix 4, we apply the results of Section 9 to compare closed points of the S-spectrum $\mathbf{Spec}^-(X)$ and $\mathbf{Spec}(X)$. Closed points of $\mathbf{Spec}(X)$ play a special role due to their significance for representation theory and algebraic geometry. The S-spectrum is oftenly easier to compute than $\mathbf{Spec}(X)$ due to its better functorial properties. We show that, although the S-spectrum is, usually, considerably larger than $\mathbf{Spec}(X)$, their closed points are in natural bijective correspondence in many (if not all) cases of interest.

Section 10 of this work was inspired by a conversation with O. Gabber who directed my attention to the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. I am happy to thank him for that and for other helpful remarks and discussions. A part of this text was written during my visiting the Max Planck Institut für Mathematik in Bonn in Summer of 2004. I would like to thank the Institute for hospitality and always excellent working atmosphere. The work was partially supported by the NSF grant DMS-0070921.

1. Preliminaries.

1.1. 'Spaces' and morphisms of 'spaces'. 'Spaces' here are spaces of noncommutative algebraic geometry. In the simplest (or the most abstract) setting, they are represented by categories. Morphisms of 'spaces' are functors regarded as inverse image functors. We denote by C_X the category representing a 'space' X and by f^* a functor $C_Y \longrightarrow C_X$ representing a morphism $X \xrightarrow{f} Y$. Formally, 'spaces' are objects of the category Cat^{op} opposite to the category Cat. The *bicategory of 'spaces'* is the bicategory Cat^{op}. The *category of 'spaces'* is the category $|Cat|^o$ having same objects as Cat^{op}. Morphisms from X to Y are isomorphism classes of (inverse image) functors $C_Y \longrightarrow C_X$.

The 'spaces' of this work are represented by abelian categories and morphisms have additive inverse image functors.

1.2. Localizations and conservative morphisms. Let X be a 'space' and Σ a family of arrows of the category C_X. We denote by $\Sigma^{-1}X$ the object of $|Cat|^o$ such that the corresponding category coincides with (the standard realization of) the category of fractions ot C_X for Σ (cf. [GZ], 1.1): $C_{\Sigma^{-1}X} = \Sigma^{-1}C_X$. We call $\Sigma^{-1}X$ the *'space' of fractions of X for Σ*. The canonical *localization functor* $C_X \xrightarrow{q^*_\Sigma} \Sigma^{-1}C_X$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$.

For any morphism $f: X \longrightarrow Y$ in $|Cat|^o$, we denote by Σ_f the family of all morphisms s of the category C_Y such that $f^*(s)$ is invertible (notice that Σ_f does not depend on the choice of an inverse image functor f^*). Thanks to the universal property of localizations, f^* is represented as the composition of the localization functor $p^*_f = p^*_{\Sigma_f} : C_Y \longrightarrow \Sigma_f^{-1}C_Y$ and a uniquely determined functor $f^*_{\mathfrak{c}} : \Sigma_f^{-1}C_Y \longrightarrow C_X$. In other words, $f = p_f \circ f_{\mathfrak{c}}$ for a uniquely determined morphism $X \xrightarrow{f_{\mathfrak{c}}} \Sigma_f^{-1}Y$.

A morphism $X \xrightarrow{f} Y$ is called *conservative* if Σ_f consists of isomorphisms, or, equivalently, p_f is an isomorphism. A morphism $X \xrightarrow{f} Y$ is called a *localization* if $f_{\mathfrak{c}}$ is an isomorphism, i.e. the functor $f_{\mathfrak{c}}^*$ is an equivalence of categories.

Thus, $f = p_f \circ f_{\mathfrak{c}}$ is a decomposition of a morphism f into a localization and a conservative morphism.

1.3. Topologizing subcategories of an abelian category. Fix an abelian category C_X. A full subcategory $\mathbb{T}$ of C_X is called *topologizing* if it is closed under finite coproducts and subquotients taken in C_X. We denote by $\mathfrak{T}(X)$ the preorder (with respect to the inclusions) of topologizing subcategories of the category C_X.

The Gabriel product, $\mathbb{S} \bullet \mathbb{T}$, of the pair of subcategories $\mathbb{S}$, $\mathbb{T}$ of C_X is the full subcategory of C_X spanned by all objects M of C_X such that there exists an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $L \in Ob\mathbb{T}$ and $N \in Ob\mathbb{S}$. It follows that $0 \bullet \mathbb{T} = \mathbb{T} = \mathbb{T} \bullet 0$ for any strictly full subcategory $\mathbb{T}$. The Gabriel product of two topologizing subcategories is a topologizing subcategory, and its restriction to topologizing categories is associative; i.e. topologizing subcategories form a monoid with respect to the Gabriel multiplication.

Note that the canonical *dualization* isomorphism $\mathcal{T}(X) \xrightarrow{\sim} \mathcal{T}(X^o)$, $\mathcal{S} \longmapsto \mathcal{S}^{op}$, is an inti-isomorphism of monoids $(\mathcal{T}(X), \bullet) \longrightarrow (\mathcal{T}(X^o), \bullet)$, that is $(\mathbb{T} \bullet \mathbb{S})^{op} = \mathbb{S}^{op} \bullet \mathbb{T}^{op}$.

1.3.1. The preorder $\succ$. For any two objects, M and N, of C_X, we write $M \succ N$ if N is a subquotient of a finite coproduct of copies of M.

Let $[M]$ denote the smallest topologizing subcategory containing the object M. One can show that $M \succ N$ iff $[N] \subseteq [M]$. This gives a description of $[M]$: its objects are those $L \in ObC_X$ for which $M \succ L$.

1.3.2. Lemma. *The smallest topologizing subcategory of C_X containing a family of objects $\mathcal{S}$ coincides with $\bigcup_{N \in \mathcal{S}_\Sigma} [N]$, where $\mathcal{S}_\Sigma$ denotes the family of all finite coproducts of objects of $\mathcal{S}$.*

Proof. Clearly, $\bigcup_{N \in \mathcal{S}_\Sigma} [N]$ is contained in every topologizing subcategory containing the family $\mathcal{S}$. It is closed under taking subquotients, because each $[N]$ has this property. It is closed under finite coproducts, because if $N_1, N_2 \in \mathcal{S}_\Sigma$ and $N_i \succ M_i$, $i = 1, 2$, then $N_1 \oplus N_2 \succ M_1 \oplus M_2$. ∎

For any subcategory (or a class of objects) $\mathcal{S}$, we denote by $[\mathcal{S}]$ the smallest topologizing subcategory containing $\mathcal{S}$.

1.3.3. Lemma. *Let $\mathbb{T}$, $\mathbb{S}$ be topologizing subcategories of an abelian category C_X.*

(a) The smallest topologizing subcategory, $[\mathbb{T}, \mathbb{S}]$, of C_X containing $\mathbb{T}$ and $\mathbb{S}$ coincides with

$$\bigcup_{(L,M) \in Ob(\mathbb{S} \times \mathbb{T})} [L \oplus M]. \qquad (1)$$

(b) Every $\mathbb{S}$-torsion free object of $[\mathbb{T}, \mathbb{S}]$ belongs to $\mathbb{T}$.

Proof. (a) The assertion follows from 1.3.2.

(b) It follows from the definition of the Gabriel product that every $\mathbb{S}$-torsion free object of $\mathbb{T} \bullet \mathbb{S}$ belongs to $\mathbb{T}$, and $[\mathbb{T}, \mathbb{S}] \subseteq \mathbb{T} \bullet \mathbb{S}$, hence the assertion. ■

We leave to the reader the formulation of the analogous assertion for an arbitrary family of topologizing subcategories.

1.4. Thick subcategories. A topologizing subcategory $\mathbb{T}$ of the category C_X is called *thick* if $\mathbb{T}\bullet\mathbb{T} = \mathbb{T}$. In other words, a full subcategory $\mathbb{T}$ of C_X is thick iff the following condition holds: the object M in an exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ belongs to the subcategory $\mathbb{T}$ iff M' and M'' are objects of $\mathbb{T}$.

We denote by $\mathfrak{Th}(X)$ the preorder of thick subcategories of C_X. Given a thick subcategory $\mathcal{T}$ of C_X, we shall denote by $X/\mathcal{T}$ the *quotient 'space'* defined by $C_{X/\mathcal{T}} = C_X/\mathcal{T}$.

1.5. Serre subcategories. We recall the notion of a *Serre subcategory* of an abelian category as it is defined in [R, III.2.3.2]. Let $\mathbb{T}$ be a subcategory of C_X. We denote by $\mathbb{T}^-$ the full subcategory of C_X generated by all objects L of C_X such that any nonzero subquotient of L has a nonzero subobject which belongs to $\mathbb{T}$. This construction has the following properties [R, III.2.3.2.1]:

(a) The subcategory $\mathbb{T}^-$ is thick.

(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.

(c) $\mathbb{T} \subseteq \mathbb{T}^-$ iff any subquotient of an object of $\mathbb{T}$ is isomorphic to an object of $\mathbb{T}$.

A subcategory $\mathbb{T}$ of C_X is called a *Serre subcategory* if $\mathbb{T}^- = \mathbb{T}$.

1.5.1. Note. (a) One can see that for any subcategory $\mathbb{T}$ of the category C_X, the associated Serre subcategory $\mathbb{T}^-$ is the largest topologizing subcategory of C_X such that every its nonzero object has a nonzero subobject from $\mathbb{T}$.

(b) It follows from this description that if the subcategory $\mathbb{T}$ is closed under taking quotient objects (e.g. $\mathbb{T}$ is a topologizing subcategory), then $\mathbb{T}^-$ is the largest topologizing subcategory of C_X which has the zero intersection with $\mathbb{T}^\perp$. Here $\mathbb{T}^\perp$ denotes, as usually, the *right orthogonal* to $\mathbb{T}$, i.e. the full subcategory of C_X generated by all objects M such that $C_X(L, M) = 0$ for any $L \in Ob\mathbb{T}$.

1.5.2. The property (sup) and coreflective subcategories. Recall that the category C_X has the property (sup) if for any ascending chain, Ω, of subobjects of an object M, the supremum of Ω exists, and for any subobject L of M, the natural morphism $sup(N \cap L \mid N \in \Omega) \longrightarrow (sup\Omega) \cap L$ is an isomorphism.

Recall that a subcategory $\mathbb{S}$ of C_X is called *coreflective* if the inclusion functor $\mathbb{S} \hookrightarrow C_X$ has a right adjoint. In other words, the subcategory $\mathbb{S}$ is coreflective iff every object of C_X has a biggest subobject which belongs to $\mathbb{S}$.

1.5.3. Lemma. *Any coreflective thick subcategory of an abelian category C_X is a Serre subcategory. If C_X has the property (sup), then any Serre subcategory of C_X is coreflective.*

Proof. See [R, III.2.4.4]. ■

1.5.4. Note. If C_X is a category with small coproducts, then a thick subcategory of C_X is coreflective iff it is closed under small coproducts (taken in C_X).

1.5.5. Proposition. *Let C_X be a category with the property (sup). Then for every thick subcategory $\mathbb{T}$ of C_X, every object of $\mathbb{T}^-$ is the supremum of its subobjects which belong to $\mathbb{T}$.*

Proof. Since C_X has the property (sup), the supremums of objects from $\mathbb{T}$ form a thick coreflective, hence Serre, subcategory. Therefore it coincides with $\mathbb{T}^-$. $\blacksquare$

1.6. The spectrum Spec(X). We denote by $Spec(X)$ the family of all nonzero objects M of the category C_X such that $[L] = [M]$ (or, equivalently, $L \succ M$) for any nonzero subobject L of M. In other words, a nonzero object M of C_X belongs to $Spec(X)$ iff it is equivalent to any of its nonzero subobjects.

In particular, every simple object of the category C_X belongs to $Spec(X)$.

The spectrum $\mathbf{Spec}(X)$ of the 'space' X is the family of topologizing subcategories $\{[M] \mid M \in Spec(X)\}$ endowed with the *specialization* preorder $\supseteq$.

Let $\tau^{\succ}$ denote the topology on $\mathbf{Spec}(X)$ associated with the specialization preorder: the closure of $W \subseteq \mathbf{Spec}(X)$ consists of all $[M]$ such that $[M] \subseteq [M']$ for some $[M'] \in W$.

1.6.1. Proposition. *(a) The inclusion $Simple(X) \hookrightarrow Spec(X)$ induces an embedding of the set of isomorphism classes of simple objects of C_X into the set of closed points of $(\mathbf{Spec}(X), \tau^{\succ})$.*

(b) If the category C_X has enough objects of finite type, then this injection is a bijection, i.e. every closed point of $(Spec(X), \tau^{\succ})$ is of the form $[M]$ for some simple object M of the category C_X.

Proof. (a) If M is a simple object, then $Ob[M]$ consists of all objects isomorphic to coproducts of finite number of copies of M. In particular, if M and N are simple objects, then $[M] \subseteq [N]$ iff $M \simeq N$.

(b) Having enough objects of finite type means that every object of C_X is the supremum of its subobjects of finite type. By a standard argument, this property implies that every nonzero object of C_X has a simple quotient. Suppose, $P \in Spec(X)$ is such that $[P]$ is a closed point. Let M be a simple quotient of P. Since $P \succ M$ and $[P]$ is closed, $M \succ P$, i.e. $[M] = [P]$. $\blacksquare$

Notice that the notion of a simple object of an abelian category is selfdual, i.e. $Simple(X) = Simple(X^o)$, where X^o is the dual 'space' defined by $C_{X^o} = C_X^{op}$. In particular, the map $M \longmapsto [M]$ induces an embedding of isomorphism classes of simple objects of C_X into the intersection $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$.

1.6.2. Proposition. *If the category C_X has enough objects of finite type, then the set of closed points of $\mathbf{Spec}(X)$ coincides with $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$.*

Proof. Since every nonzero object of C_X has a nonzero subobject of finite type, $\mathbf{Spec}(X)$ consists of $[M]$ such that M is of finite type and belongs to $Spec(X)$. On the other hand, if M is of finite type and $[M]$ belongs to $\mathbf{Spec}(X^o)$, then $[M] = [M_1]$, where M_1 is a simple quotient of M. Hence the assertion. $\blacksquare$

1.7. Local 'spaces'. A nonzero object P of an abelian category C_X is called *quasi-final* if $N \succ P$ for every nonzero object N of C_X. It follows that any quasi-final object of C_X belongs to $Spec(X)$ and all quasi-final objects are equivalent to each other.

We call the 'space' X (and the category C_X) *local* if C_X has a quasi-final object.

If X is local, then $\mathbf{Spec}(X)$ has the (unique) smallest element which coincides with the smallest nonzero topologizing subcategory of C_X and is a unique closed point of the spectrum (in the topology associated with the preorder $\supseteq$).

1.7.1. Proposition. *Let X be local, and let the category C_X have simple objects. Then all simple objects of C_X are isomorphic to each other, and every quasi-final object of C_X is isomorphic to a direct sum of a finite number of copies of a simple object.*

Proof. In fact, if M is a simple object in C_X, then $[M]$ is a closed point of $\mathbf{Spec}(X)$. If X is local, this closed point is unique. Therefore, M is a quasi-final object and every object of $[M]$ is a finite coproduct of copies of M (see the argument of 1.6.1). ∎

1.8. The complete spectrum and the S-spectrum. The *complete spectrum* of X is the preorder (with respect to $\subseteq$) of all thick subcategories $\mathcal{P}$ of the category C_X such that $X/\mathcal{P}$ is a local 'space'. We denote it by $\mathbf{Spec}^1(X)$.

The *S-spectrum* of X is defined as the subpreorder of $\mathbf{Spec}^1(X)$ formed by Serre subcategories, i.e. $\mathbf{Spec}^-(X) = \mathbf{Spec}^1(X) \bigcap \mathfrak{Se}(X) = \{\mathcal{P} \in \mathbf{Spec}^1(X) \mid \mathcal{P} = \mathcal{P}^-\}$.

For any $M \in ObC_X$, we denote by $\langle M \rangle$ the full subcategory of C_X generated by all objects N such that $N \not\succ M$. Notice that $[M] \subseteq [L]$ iff $\langle M \rangle \subseteq \langle L \rangle$.

1.8.1. Proposition. *The map $[P] \longmapsto \langle P \rangle$ induces a monomorphism of preorders* $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-(X)$.

Proof. See [R, III.2.3.3 and III.3.3.1]. ∎

1.8.2. The S-spectrum and the Gabriel spectrum. We call a 'space' X *locally noetherian* if C_X is a locally noetherian abelian category. It is argued in [R, Ch. VI] that if X is locally noetherian, then the elements of the S-spectrum are in bijective correspondence with the set of isomorphism classes of indecomposable injectives of the category C_X. In other words, $\mathbf{Spec}^-(X)$ is isomorphic to the Gabriel spectrum of the category C_X.

1.8.2. Remark. If $C_X = R-mod$, where R is a commutative noetherian ring, then the Gabriel spectrum of C_X (hence $\mathbf{Spec}^-(X)$) is isomorphic to the prime spectrum of the ring R [Gab]. If R is a non-noetherian commutative ring, $\mathbf{Spec}^-(X)$ might be much bigger than the prime spectrum of R, while $\mathbf{Spec}(X)$ is naturally isomorphic to the prime spectrum of R (cf. [R], Ch.3).

2. A general construction of spectra.

Fix a category $\mathfrak{H}$. Let $\mathfrak{H}_0$ denote the full subcategory of $\mathfrak{H}$ whose objects are initial objects of $\mathfrak{H}$. Thus, $\mathfrak{H}_0$ is either empty, or a groupoid. Let $\mathfrak{H}^1$ denote the full subcategory of $\mathfrak{H}$ defined by $Ob\mathfrak{H}^1 = Ob\mathfrak{H} - Ob\mathfrak{H}_0$.

2.1. Definition. We call $\mathfrak{H}$ *local* if the category $\mathfrak{H}^1$ has an initial object.

2.1.1. Note. It follows that if $\mathfrak{H}$ is local, than $\mathfrak{H}$ has initial objects, i.e. $\mathfrak{H}_0 \neq \emptyset$.

2.2. The spectrum $\mathfrak{Spec}^1(\mathfrak{H})$. We denote by $\mathfrak{Spec}^1(\mathfrak{H})$ the full subcategory of the category $\mathfrak{H}$ generated by all $x \in Ob\mathfrak{H}$ such that the category $x\backslash\mathfrak{H}$ is local. We call $\mathfrak{Spec}^1(\mathfrak{H})$ the *local spectrum* of $\mathfrak{H}$.

In other words, an object x of $\mathfrak{H}$ belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ iff there exists an object $x^\star$ of $\mathfrak{H}$ and an arrow $x \xrightarrow{\gamma_x} x^\star$ such that γ_x is not an isomorphism and if $x \xrightarrow{f} y$ is not an isomorphism, then there exists a unique arrow $x^\star \xrightarrow{\bar{f}} y$ such that $f = \bar{f}\circ\gamma_x$. The morphism $x \xrightarrow{\gamma_x} x^\star$ (in particular, the object $x^\star$) is determined by these conditions uniquely up to isomorphism.

2.2.1. Note. It follows from this definition and 2.1.1 that $\mathfrak{H}$ is local iff it has initial objects and they belong to $\mathfrak{Spec}^1(\mathfrak{H})$.

2.2.2. Functorial properties. Let $\mathfrak{H} \xrightarrow{F} \widetilde{\mathfrak{H}}$ be a functor. For any $x \in Ob\mathfrak{H}$, the functor F induces a functor $x\backslash\mathfrak{H} \xrightarrow{F_x} F(x)\backslash\widetilde{\mathfrak{H}}$. Suppose that the functor F is such that F_x is an equivalence of categories for every $x \in Ob\mathfrak{H}$. Then F induces a functor $\mathfrak{Spec}^1(\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(\widetilde{\mathfrak{H}})$.

A typical example is the functor

$$y\backslash\mathfrak{H} \xrightarrow{f_\star} z\backslash\mathfrak{H}, \quad (y, y \xrightarrow{g} v) \longmapsto (z, z \xrightarrow{gf} v),$$

corresponding to a morphism $z \xrightarrow{f} y$, or the canonical functor $y\backslash\mathfrak{H} \longrightarrow \mathfrak{H}$.

2.3. Supports. For any $x \in Ob\mathfrak{H}$, we denote by $\mathfrak{Supp}_{\mathfrak{H}}(x)$ the full subcategory of $\mathfrak{H}$ generated by all $y \in Ob\mathfrak{H}$ such that $\mathfrak{H}(x,y) = \emptyset$. We call $\mathfrak{Supp}_{\mathfrak{H}}(x)$ the *support of x in $\mathfrak{H}$*.

2.3.1. Proposition. *(a) For any two objects, x and y, of the category $\mathfrak{H}$, there exists an arrow $x \to y$ iff $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$.*

(b) Let $\{x_i \mid i \in J\}$ be a set of objects of $\mathfrak{H}$ such that there exists a coproduct, $\coprod_{i\in J} x_i$. Then

$$\mathfrak{Supp}_{\mathfrak{H}}\big(\coprod_{i\in J} x_i\big) = \bigcup_{i\in J} \mathfrak{Supp}_{\mathfrak{H}}(x_i). \tag{1}$$

Proof. (a) If there exists a morphism $x \longrightarrow y$ and $\mathfrak{H}(x,z) = \emptyset$, then, obviously, $\mathfrak{H}(y,z) = \emptyset$, hence $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$.

If $\mathfrak{H}(x,y) = \emptyset$, i.e. $y \in \mathfrak{Supp}_{\mathfrak{H}}(x)$, then, since $y \notin Ob\mathfrak{Supp}_{\mathfrak{H}}(y)$, the inclusion $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$ does not hold.

(b) Since $\mathfrak{H}(\coprod_{i\in J} x_i, z) \simeq \prod_{i\in J}\mathfrak{H}(x_i,z)$, $\mathfrak{H}(\coprod_{i\in J} x_i, z) = \emptyset$ iff $\mathfrak{H}(x_i,z) = \emptyset$ for some $i \in J$, whence the equality (1). ■

2.3.2. Support in $\mathfrak{Spec}^1(\mathfrak{H})$. For any $x \in Ob\mathfrak{H}$, we denote the intersection $\mathfrak{Supp}_{\mathfrak{H}}(x)\bigcap\mathfrak{Spec}^1(\mathfrak{H})$ by $\mathfrak{Supp}^1_{\mathfrak{H}}(x)$ and call it the *support of x in* $\mathfrak{Spec}^1(\mathfrak{H})$. Evidently, 2.3.1(b) is still true if $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is replaced by $\mathfrak{Supp}^1_{\mathfrak{H}}(x)$, as well as a half of 2.3.1(a): if $\mathfrak{H}(x,y)$ is not empty, then $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$.

2.4. The spectrum $\mathfrak{Spec}^0(\mathfrak{H})$. We denote by $\mathfrak{Spec}^0(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by $x \in Ob\mathfrak{H}$ such that $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is not empty and has a final object, $\widehat{x}$.

2.4.1. Proposition. *Let $\mathfrak{H}$ be local. Then initial objects of $\mathfrak{H}^1$ belong to $\mathfrak{Spec}^0(\mathfrak{H})$.*

Proof. Let $\mathfrak{H}_0$ be the full subcategory (groupoid) of $\mathfrak{H}$ generated by all initial objects of $\mathfrak{H}$. If x is an initial object of the category $\mathfrak{H}^1$, then $\mathfrak{Supp}_{\mathfrak{H}}(x)$ coincides with $\mathfrak{H}_0$.

In fact, suppose that there is an arrow, $x \xrightarrow{f} y$, for some $y \in Ob\mathfrak{H}_0$. Since y is an initial object of the category $\mathfrak{H}$, there exists a unique morphism $y \xrightarrow{g} x$. By the universal property of y, the composition $y \xrightarrow{fg} y$ is the identical morphism. Since x is an initial object of the category $\mathfrak{H}^1$, the composition $x \xrightarrow{gf} x$ is the identical morphism too. This means that the morphism $x \xrightarrow{f} y$ is an isomorphism which contradicts to the fact that x is not an initial object of the category $\mathfrak{H}$.

Thus, $\mathfrak{H}_0$ is a subcategory of $\mathfrak{Supp}_{\mathfrak{H}}(x)$. Since for every $z \in Ob\mathfrak{H}^1 = Ob\mathfrak{H} - Ob\mathfrak{H}_0$ there is a (unique) morphism $x \longrightarrow z$, the subcategory $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is contained in $\mathfrak{H}_0$; i.e. $\mathfrak{Supp}_{\mathfrak{H}}(x) = \mathfrak{H}_0$.

Since $\mathfrak{H}_0$ is a connected groupoid, every object of $\mathfrak{H}_0$ is final. ■

2.4.4. Lemma. *A choice for every $x \in Ob\mathfrak{Spec}^0(\mathfrak{H})$ of a final object, $\widehat{x}$, of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$ extends to a functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$.*

Proof. In fact, if $x, y \in Ob\mathfrak{Spec}^0(\mathfrak{H})$, and there is a morphism $x \longrightarrow y$, then $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$. Therefore there exists a unique morphism $\widehat{x} \longrightarrow \widehat{y}$. ■

2.4.5. Remark. Notice that the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ is faithful iff $\mathfrak{H}$ is a preorder, i.e. for any pair of objects, x, y, of $\mathfrak{H}$, there is at most one morphism $x \longrightarrow y$.

2.4.6. Proposition. *Suppose the category $\mathfrak{H}$ is a preorder with finite coproducts (i.e. supremums of pairs of objects). Then the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ takes values in $\mathfrak{Spec}^1(\mathfrak{H})$, i.e. it induces a functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\theta_{\mathfrak{H}}} \mathfrak{Spec}^1(\mathfrak{H})$.*

Proof. For any $x \in Ob\mathfrak{Spec}^0(\mathfrak{H})$, the final object, $\widehat{x}$, of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$ belongs to $\mathfrak{Spec}^1(\mathfrak{H})$. More expicitly, we claim that the canonical coprojection, $\widehat{x} \longrightarrow x \sqcup \widehat{x}$, is an initial object of the category $(\widehat{x}\backslash\mathfrak{H})^1$.

In fact, let $\widehat{x} \xrightarrow{g} y$ be a morphism. Then one of two things happens: either $y \in Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, or not. If $y \in Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, then, since $\widehat{x}$ is a final object of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$, there is a unique morphism $y \xrightarrow{h} \widehat{x}$. It follows from the universal property of $\widehat{x}$ that $h \circ g = id_{\widehat{x}}$. By hypothesis, $\mathfrak{H}$ is a preorder, in particular, h is a monomorphism. Therefore, h is an isomorphism inverse to g.

If $y \notin Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, then there exists an arrow $x \longrightarrow y$ which, together with $\widehat{x} \xrightarrow{g} y$, determines (and is determined by) a morphism $(x \sqcup \widehat{x}, \widehat{x} \to x \sqcup \widehat{x}) \longrightarrow (y, \widehat{x} \xrightarrow{g} y)$. Since $\mathfrak{H}$ is a preorder, this is all we need. ■

2.5. Relative spectra. Let $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ be a functor. We define the relative spectra,

$\mathfrak{Spec}^1(\mathfrak{G}, F)$ and $\mathfrak{Spec}^0(\mathfrak{G}, F)$, via cartesian squares

$$\begin{array}{ccccccccc} \mathfrak{Spec}^1(\mathfrak{G}, F) & \xrightarrow{\theta^1_F} & \mathfrak{G} & & & \mathfrak{Spec}^0(\mathfrak{G}, F) & \xrightarrow{\vartheta_F} & \mathfrak{G} & \\ {\scriptstyle \pi^F_1}\downarrow & & \downarrow{\scriptstyle F} & & \text{and} & {\scriptstyle \pi^F_0}\downarrow & & \downarrow{\scriptstyle F} & \quad (1) \\ \mathfrak{Spec}^1(\mathfrak{H}) & \xrightarrow{\theta^1_{\mathfrak{H}}} & \mathfrak{H} & & & \mathfrak{Spec}^0(\mathfrak{H}) & \xrightarrow{\vartheta_{\mathfrak{H}}} & \mathfrak{H} & \end{array}$$

(in the bicategorical sense, i.e. the squares quasi-commute), where $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ is the canonical functor of 2.4.4.

Explicitly, objects of the category $\mathfrak{Spec}^1(\mathfrak{G}, F)$ are triples $(z, x; \phi)$, where z is an object of $\mathfrak{Spec}^1(\mathfrak{H})$, $x \in Ob\mathfrak{G}$, and ϕ is an isomorphism $z \xrightarrow{\sim} F(x)$. Morphisms from $(z, x; \phi)$ to $(z', x'; \phi')$ are given by pairs of arrows, $z \xrightarrow{g} z'$ and $x \xrightarrow{h} x'$ such that the diagram

$$\begin{array}{ccc} z & \xrightarrow{g} & z' \\ {\scriptstyle \phi}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi'} \\ F(x) & \xrightarrow{F(h)} & F(x') \end{array}$$

commutes. The projections $\mathfrak{Spec}^1(\mathfrak{H}) \xleftarrow{\pi^F_1} \mathfrak{Spec}^1(\mathfrak{G}, F) \xrightarrow{\theta^1_F} \mathfrak{G}$ in the left diagram (1) are defined by $\pi^F_1(z, x; \phi) = z$ and $\theta^1_F(z, x; \phi) = x$.

Similarly, objects of the category $\mathfrak{Spec}^0(\mathfrak{G}, F)$ are triples $(z, x; \psi)$, where z is an object of $\mathfrak{Spec}^0(\mathfrak{H})$, $x \in Ob\mathfrak{G}$, and ψ is an isomorphism $\vartheta_{\mathfrak{H}}(z) \xrightarrow{\sim} F(x)$.

2.5.1. Proposition. *Let i be 0 or 1. The map $(\mathfrak{G}, F) \longmapsto \mathfrak{Spec}^i(\mathfrak{G}, F)$ extends to a pseudo-functor $\mathfrak{Spec}^i : Cat/\mathfrak{H} \longrightarrow Cat$.*

Proof. The assertion follows from the universal property of cartesian squares. ■

2.5.2. Proposition. *Suppose $\mathfrak{H}$ is a preorder with finite coproducts. Then for every functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$, there is a canonical functor*

$$\mathfrak{Spec}^0(\mathfrak{G}, F) \xrightarrow{\vartheta_{(\mathfrak{G},F)}} \mathfrak{Spec}^1(\mathfrak{G}, F). \qquad (2)$$

The family $\{\vartheta_{(\mathfrak{G},F)} \mid (\mathfrak{G}, F) \in ObCat/\mathfrak{H}\}$ is a morphism of pseudo-functors,

$$\mathfrak{Spec}^0 \xrightarrow{\vartheta} \mathfrak{Spec}^1. \qquad (3)$$

Proof. Since $\mathfrak{H}$ is a preorder with finite coproducts, the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ takes values in $\mathfrak{Spec}^1(\mathfrak{H})$, hence it factors through the embedding $\mathfrak{Spec}^1(\mathfrak{H}) \longrightarrow \mathfrak{H}$ (see 2.4.6). By the universal property of cartesian squares, there exists a unique functor (2) such that $\theta^1_F \circ \vartheta_{(\mathfrak{G},F)} = \vartheta_F$ and $\pi^F_1 \circ \vartheta_{(\mathfrak{G},F)} = \pi^F_0$ (see the diagram (2)).

It is useful to have an explicit description of the functor (2) in terms of the descripiton of $\mathfrak{Spec}^0(\mathfrak{G}, F)$ and $\mathfrak{Spec}^1(\mathfrak{G}, F)$ given above. The functor $\vartheta_{(\mathfrak{G},F)}$ maps an object $(z, x; \psi)$ of $\mathfrak{Spec}^0(\mathfrak{G}, F)$ to the object $(\vartheta_{\mathfrak{H}}(z), x; \psi)$ of $\mathfrak{Spec}^1(\mathfrak{G}, F)$.

It follows from this description that $\vartheta = \{\vartheta_{(\mathfrak{G},F)} \mid (\mathfrak{G}, F) \in ObCat/\mathfrak{H}\}$ is a morphism of pseudo-functors. ■

I. Spectra related to topologizing and thick subcategories.

Fix an abelian category C_X. Let $\mathfrak{Th}(X)$ (resp. $\mathfrak{T}(X)$) denote the preorder of all thick (resp. topologizing) subcategories of C_X. Applying the construction 2.5 to the inclusion functor $\mathfrak{Th}(X) \xrightarrow{\mathcal{J}_X^{\mathfrak{t}}} \mathfrak{T}(X)$, we obtain two spectra,

$$\mathbf{Spec}_{\mathfrak{t}}^i(X) = \mathfrak{Spec}^i(\mathfrak{Th}(X), \mathfrak{J}_X^{\mathfrak{t}}), \quad i = 0, 1.$$

and (by 2.5.2) a canonical monomorphism from one to another, $\mathbf{Spec}_{\mathfrak{t}}^0(X) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^1(X)$. The category $\mathbf{Spec}_{\mathfrak{t}}^1(X)$ is defined by the cartesian square

$$\begin{array}{ccc}
\mathbf{Spec}_{\mathfrak{t}}^1(X) & \longrightarrow & \mathfrak{Th}(X) \\
\big\downarrow & & \big\downarrow \\
\mathfrak{Spec}^1(\mathfrak{T}(X)) & \longrightarrow & \mathfrak{T}(X)
\end{array}$$

in which the right vertical arrow and the lower horizontal arrow are inclusions (see 2.5). It follows from this description (or from the explicit description of relative spectra in 2.5) that objects of $\mathbf{Spec}_{\mathfrak{t}}^1(X)$ are thick subcategories, $\mathcal{P}$ such that the intersection $\mathcal{P}^{\mathfrak{t}}$ of all topologizing subcategories containing $\mathcal{P}$ properly does not coincide with $\mathcal{P}$.

The spectrum $\mathbf{Spec}_{\mathfrak{t}}^0(X)$ is defined by the cartesian square

$$\begin{array}{ccc}
\mathbf{Spec}_{\mathfrak{t}}^0(X) & \longrightarrow & \mathfrak{Th}(X) \\
\big\downarrow & & \big\downarrow \\
\mathfrak{Spec}^0(\mathfrak{T}(X)) & \longrightarrow & \mathfrak{T}(X)
\end{array} \qquad (1)$$

(see 2.5). By definition, objects of $\mathfrak{Spec}^0(\mathfrak{T}(X))$ are topologizing subcategories, $\mathcal{P}$, such that $\mathfrak{Supp}_{\mathfrak{T}(X)}(\mathcal{P})$ has a final object. This means, precisely, that the union, $\widehat{\mathcal{P}}$, of all topologizing subcategories which do not contain $\mathcal{P}$ is a topologizing subcategory. The lower horizontal arrow of the diagram (1) maps an element $\mathcal{P}$ to $\widehat{\mathcal{P}}$. Thus, $\mathbf{Spec}_{\mathfrak{t}}^0(X)$ is naturally identified with the preorder of those topologizing subcategories $\mathcal{P}$ of C_X for which $\widehat{\mathcal{P}}$ is a thick subcategory.

Similarly, the construction 2.5 applied to the incluison functor $\mathfrak{Se}(X) \xrightarrow{\mathcal{J}_X^{\mathfrak{s}}} \mathfrak{T}(X)$ produces two spectra

$$\mathbf{Spec}_{\mathfrak{s},\mathfrak{t}}^i(X) = \mathfrak{Spec}^i(\mathfrak{Se}(X), \mathcal{J}_X^{\mathfrak{s}}), \quad i = 0, 1,$$

and a canonical injective map $\mathbf{Spec}_{\mathfrak{s},\mathfrak{t}}^0(X) \longrightarrow \mathbf{Spec}_{\mathfrak{s},\mathfrak{t}}^1(X)$.

Here $\mathbf{Spec}_{\mathfrak{s},\mathfrak{t}}^1(X)$ consists of all Serre subcategories $\mathcal{P}$ such that the intersection $\mathcal{P}^{\mathfrak{t}}$ of topogizing subcategories properly containing $\mathcal{P}$ contains $\mathcal{P}$ properly too. The spectrum $\mathbf{Spec}_{\mathfrak{s},\mathfrak{t}}^0(X)$ is identified with the preorder of all topogizing subcategories $\mathcal{Q}$ of C_X such that the union $\widehat{\mathcal{Q}}$ of all topologizing subcategories of C_X which do not contain $\mathcal{Q}$ is a Serre subcategory. It follows from these definitions that these four spectra are related by a commutative diagram

$$\begin{array}{ccc}
\mathbf{Spec}_{\mathfrak{t}}^0(X) & \longrightarrow & \mathbf{Spec}_{\mathfrak{t}}^1(X) \\
\big\uparrow & & \big\uparrow \\
\mathbf{Spec}_{\mathfrak{s},\mathfrak{t}}^0(X) & \longrightarrow & \mathbf{Spec}_{\mathfrak{s},\mathfrak{t}}^1(X)
\end{array} \qquad (2)$$

of injective morphisms.

In the next two sections, it is shown, among other facts, that $\mathbf{Spec}^0_{\mathfrak{s},\mathfrak{t}}(X)$ coincides with the spectrum $\mathbf{Spec}(X)$ and the canonical map $\mathbf{Spec}^0_{\mathfrak{s},\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{s},\mathfrak{t}}(X)$ is an isomorphism.

3. The decomposition of the spectrum $\mathbf{Spec}^1_{\mathfrak{t}}(X)$.

Fix an abelian category C_X. For any subcategory $\mathcal{S}$ of C_X, we denote by $\mathcal{S}^{\mathfrak{t}}$ the intersection of all topologizing subcategories of C_X properly containing $\mathcal{S}$. The spectrum $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ is formed by thick subcategories $\mathcal{P}$ such that $\mathcal{P} \neq \mathcal{P}^{\mathfrak{t}}$ (see above). We set

$$\mathbf{Spec}^{1,0}_{\mathfrak{t}}(X) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{t}}(X) \mid \mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp} = 0\}$$

and

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{t}}(X) \mid \mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp} \neq 0\},$$

where $\mathcal{P}^{\perp}$ is the right orthogonal to $\mathcal{P}$. Since the subcategory $\mathcal{P}$ is closed under taking quotients, $\mathcal{P}^{\perp}$ coincides with the full subcategory of C_X formed by all $\mathcal{P}$-torsion free objects (see 1.5.1(a)). Clearly $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ is the disjoint union of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ and $\mathbf{Spec}^{1,0}_{\mathfrak{t}}(X)$:

$$\mathbf{Spec}^1_{\mathfrak{t}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \coprod \mathbf{Spec}^{1,0}_{\mathfrak{t}}(X). \tag{1}$$

3.1. Remarks. (a) It follows from definitions that

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) = \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp} \neq 0\}.$$

(b) The notion of $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ is selfdual, i.e. $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ is naturally isomorphic to $\mathbf{Spec}^1_{\mathfrak{t}}(X^o)$, where X^o is the dual 'space': $C_{X^o} = C_X^{op}$. This is the consequence of the selfduality of the notions of a thick and topologizing subcategories.

The definition of $\mathcal{P}_0 = \mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp}$ is not selfdual (because of $\mathcal{P}^{\perp}$), in particular, the decomposition $\mathbf{Spec}^1_{\mathfrak{t}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \coprod \mathbf{Spec}^{1,0}_{\mathfrak{t}}(X)$ is not selfdual.

3.2. Proposition. *(i) A thick subcategory $\mathcal{P}$ from $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ iff it is a Serre subcategory. Thus,*

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) = \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P}^{-} = \mathcal{P} \subsetneqq \mathcal{P}^{\mathfrak{t}}\},$$

$$\mathbf{Spec}^{1,0}_{\mathfrak{t}}(X) = \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P} \subsetneqq \mathcal{P}^{\mathfrak{t}} \subseteq \mathcal{P}^{-}\}.$$

(ii) The map $[P] \longmapsto \langle P \rangle$ induces an isomorphism

$$\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X).$$

The inverse isomorphism assigns to every element $\mathcal{P}$ of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ the topologizing subcategory $[\mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}]$.

Proof. (i) Suppose that $\mathcal{P} = \mathcal{P}^{-} \subsetneqq \mathcal{P}^{\mathfrak{t}}$. Then $(\mathcal{P}^{\mathfrak{t}} - \mathcal{P}) \bigcap \mathcal{P}^{-} = 0$ which means that every nonzero object of $\mathcal{P}^{\mathfrak{t}} - \mathcal{P}$ has a nonzero subquotient which has no $\mathcal{P}$-torsion. Since

$\mathcal{P}^{\mathfrak{t}}$ is a topologizing subcategory, this subquotient belongs to $\mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$. In particular, this shows that $\mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp} \neq 0$, i.e. $\mathcal{P}$ is an element of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, which proves the inclusion $\mathbf{Spec}^{1}_{\mathfrak{t}}(X) \bigcap \mathfrak{Se}(X) \subseteq \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. Here $\mathfrak{Se}(X)$ denotes the preorder of all Serre subcategories of C_X.

Conversely, if $\mathcal{P}$ is a thick subcategory which is not a Serre subcategory, i.e. $\mathcal{P} \subsetneqq \mathcal{P}^{-}$, then $\mathcal{P}^{\mathfrak{t}} \subseteq \mathcal{P}^{-}$. Since $\mathcal{P}^{-} \bigcap \mathcal{P}^{\perp} = 0$, the inclusion $\mathcal{P}^{\mathfrak{t}} \subseteq \mathcal{P}^{-}$ implies that $\mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp} = 0$, i.e. $\mathcal{P} \notin \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. This shows that $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) = \mathbf{Spec}^{1}_{\mathfrak{t}}(X) \bigcap \mathfrak{Se}(X)$.

(ii) (a) The image of $\mathbf{Spec}(X)$ by the map $[P] \longmapsto \langle P \rangle$ is contained in $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ because $\langle P \rangle$ is a Serre (hence thick) subcategory and $P \in \langle P \rangle^{\mathfrak{t}} \bigcap \langle P \rangle^{\perp}$ for every P in $\mathit{Spec}(X)$.

(b) Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. If $\mathbb{T}$ is a topologizing subcategory of C_X such that $\mathcal{P}_0 = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp} \nsubseteq \mathbb{T}$, then $\mathbb{T} \subseteq \mathcal{P}$.

In fact, suppose $\mathbb{T} \nsubseteq \mathcal{P}$. Then the topologizing subcategory $[\mathbb{T}, \mathcal{P}]$ spanned by $\mathbb{T}$ and $\mathcal{P}$ contains properly $\mathcal{P}$, hence it contains $\mathcal{P}^{\mathfrak{t}}$. By 1.3.3(b), every $\mathcal{P}$-torsion free object of $[\mathbb{T}, \mathcal{P}]$ belongs to $\mathbb{T}$; in particular, $\mathcal{P}_0 = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp} \nsubseteq \mathbb{T}$.

(c) Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. Every nonzero object of $\mathcal{P}_0 = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$ belongs to $\mathit{Spec}(X)$.

Let L be a nonzero object of $\mathcal{P}_0$ and L_1 a nonzero subobject of L. Then $[L_1] \subseteq [L]$. If $[L_1] \nsubseteq [L]$, then it follows from (b) above that $[L_1] \subseteq \mathcal{P}$, or, equivalently, $L_1 \in \mathit{Ob}\mathcal{P}$. This contradicts to the assumption that the object L is $\mathcal{P}$-torsion free.

(d) Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. Then $\mathcal{P} = \langle L \rangle$ for any nonzero object of $\mathcal{P}_0 = \mathcal{P}^{\mathfrak{t}} \bigcap \mathcal{P}^{\perp}$.

Let L be a nonzero object of $\mathcal{P}_0$. Since L does not belong to the Serre subcategory $\langle L \rangle$, by (b), we have the inclusion $\langle L \rangle \subseteq \mathcal{P}$. On the other hand, if $\langle L \rangle \subsetneqq \mathcal{P}$, then $L \in \mathit{Ob}\mathcal{P}$ which is not the case. Therefore $\mathcal{P} = \langle L \rangle$.

(e) The topologizing subcategory $[\mathcal{P}_0]$ coincides with the subcategory $[L]$ for any nonzero object L of $\mathcal{P}_0$.

Clearly $[L] \subseteq [\mathcal{P}_0]$ for any $L \in \mathit{Ob}\mathcal{P}_0$. By (b), if $\mathcal{P}_0 \nsubseteq [L]$, then $[L] \subseteq \mathcal{P}$, hence $L = 0$.

Since, by (c), every nonzero object of $\mathcal{P}_0$ belongs to $\mathit{Spec}(X)$, this shows that $[\mathcal{P}_0]$ is an element of $\mathbf{Spec}(X)$.

(f) It follows from the argument above that the map

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X), \quad [P] \longmapsto \langle P \rangle,$$

is inverse to the map $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}(X)$ which assigns to every $\mathcal{P}$ the topologizing subcategory $[\mathcal{P}_0]$. ■

3.3. Note. Suppose that the category C_X has the property (sup). Then for any $\mathcal{P} \in \mathbf{Spec}^{1,0}_{\mathfrak{t}}(X)$, the subcategory $\mathcal{P}^{\mathfrak{t}} - \mathcal{P}$ does not have objects of finite type.

In fact, by 3.2(i), $\mathcal{P} \subseteq \mathcal{P}^{\mathfrak{t}} \subseteq \mathcal{P}^{-}$. By 2.4.7, objects of $\mathcal{P}^{-}$ are supremums of their subobjects from $\mathcal{P}$. Thus, every object of $\mathcal{P}^{-}$ of finite type belongs to $\mathcal{P}$.

4. The spectrum $\mathbf{Spec}^{0}_{\mathfrak{t}}(X)$.

For any subcategory (or a family of objects, or an object) $\mathcal{B}$ of C_X, we denote by $\widehat{\mathcal{B}}$ the union of all topologizing subcategories of C_X which do not contain $\mathcal{B}$.

4.1. Lemma. *Let $\mathcal{B}$ be a subcategory and $\mathbb{T}$ a topologizing subcategory of C_X. Then $\mathcal{B} \subseteq \mathbb{T}$ iff $\widehat{\mathcal{B}} \subseteq \widehat{\mathbb{T}}$.*

Proof. If $\mathcal{B} \subseteq \mathbb{T}$ and $\mathbb{S}$ is a topologizing subcategory of C_X such that $\mathcal{B} \nsubseteq \mathbb{S}$, then $\mathbb{T} \nsubseteq \mathbb{S}$. This shows that the inclusion $\mathcal{B} \subseteq \mathbb{T}$ implies that $\widehat{\mathcal{B}} \subseteq \widehat{\mathbb{T}}$.

Conversely, let $\widehat{\mathcal{B}} \subseteq \widehat{\mathbb{T}}$. If $\mathcal{B} \nsubseteq \mathbb{T}$, then $\mathbb{T}$, being a topologizing subcategory, is contained in $\widehat{\mathcal{B}}$. Together with the inclusion $\widehat{\mathcal{B}} \subseteq \widehat{\mathbb{T}}$, this implies that $\mathbb{T} \subseteq \widehat{\mathbb{T}}$ which is impossible by definition of $\widehat{\mathbb{T}}$. Therefore, $\mathcal{B} \subseteq \mathbb{T}$. ∎

By definition, the spectrum $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ is the preorder (with respect to $\subseteq$) of all topologizing subcategories $\mathcal{T}$ of C_X such that $\widehat{\mathcal{T}}$ is a thick subcategory.

4.2. Note. One can see that $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ is selfdual, i.e. $\mathbf{Spec}^0_{\mathfrak{t}}(X^o)$ is naturally isomorphic to $\mathbf{Spec}^0_{\mathfrak{t}}(X)$.

4.3. Proposition. *(a) The spectrum* $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ *consists of all topologizing subcategories* $\mathcal{P}$ *of* C_X *having the property:*

if $\mathcal{P} \subseteq \mathbb{T} \bullet \mathbb{S}$ *for some topologizing subcategories* $\mathbb{S}$ *and* $\mathbb{T}$*, then either* $\mathcal{P} \subseteq \mathbb{S}$*, or* $\mathcal{P} \subseteq \mathbb{T}$*.*

(b) The map $\mathcal{T} \longmapsto \widehat{\mathcal{T}}$ *induces a monomorphism*

$$\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{t}}(X). \tag{1}$$

(c) If $\mathcal{T} \in \mathbf{Spec}^0_{\mathfrak{t}}(X)$*, then* $\mathcal{T} = [L]$ *for every* $L \in Ob(\mathcal{T} - \widehat{\mathcal{T}})$*.*

Proof. (a) Suppose $\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{t}}(X)$, i.e. $\widehat{\mathcal{P}}$ is a thick subcategory. If $\mathbb{S}$ and $\mathbb{T}$ are topologizing subcategories which do not contain $\mathcal{P}$, then they both are contained in $\widehat{\mathcal{P}}$ and, since the latter is thick, $\mathbb{T} \bullet \mathbb{S} \subseteq \widehat{\mathcal{P}}$.

Conversely, if for any pair of topologizing subcategories $\mathbb{S}$, $\mathbb{T}$ of $\widehat{\mathcal{P}}$ their Gabriel product $\mathbb{T} \bullet \mathbb{S}$ is a subcategory of $\widehat{\mathcal{P}}$, then $\widehat{\mathcal{P}}$ is thick, i.e. $\mathcal{P}$ belongs to $\mathbf{Spec}^0_{\mathfrak{t}}(X)$.

(b) The assertion (b) is a special case of the general nonsense fact 2.4.6. It is useful to specialize the argument. Let $\mathcal{T} \in \mathbf{Spec}^0_{\mathfrak{t}}(X)$, and let $\mathbb{S}$ be a topologizing subcategory of C_X properly containing $\widehat{\mathcal{T}}$. Then it follows from the definition of $\widehat{\mathcal{T}}$ that $\mathcal{T} \subseteq \mathbb{S}$, or, equivalently, $[\mathcal{T}, \widehat{\mathcal{T}}] \subseteq \mathbb{S}$. This shows that $(\widehat{\mathcal{T}})^{\mathfrak{t}} = [\mathcal{T}, \widehat{\mathcal{T}}]$. Since $\mathcal{T} \nsubseteq \widehat{\mathcal{T}}$, the subcategory $\widehat{\mathcal{T}}$ is properly contained in $(\widehat{\mathcal{T}})^{\mathfrak{t}}$, i.e. $\widehat{\mathcal{T}}$ belongs to $\mathbf{Spec}^1_{\mathfrak{t}}(X)$. The injectivity of the map (1) follows from 4.1.

(c) If $\mathcal{B}$ is a subcategory of $\mathcal{T}$, then, by 4.1, $\widehat{\mathcal{B}} \subseteq \widehat{\mathcal{T}}$. If, in addition, $\mathcal{B} \nsubseteq \widehat{\mathcal{T}}$ and $\widehat{\mathcal{T}}$ is a topologizing subcategory, then $\widehat{\mathcal{T}} \subseteq \widehat{\mathcal{B}}$, hence $\widehat{\mathcal{T}} = \widehat{\mathcal{B}}$. If $\mathcal{B}$ is a topologizing subcategory, the latter equality means that $\mathcal{T} = \mathcal{B}$. Taking $\mathcal{B} = [L]$ for any object L from $\mathcal{T} - \widehat{\mathcal{T}}$, we obtain the equality $\mathcal{T} = [L]$. ∎

4.3.1. Note. Let $(\mathcal{G}, \leq)$ be a preordered monoid. We call a non-unit element x of $\mathcal{G}$ *prime* if the set $\{s \in \mathcal{G} \mid x \geq s\}$ is a submonoid of $\mathcal{G}$.

It follows from 4.3(a) that $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ is the set of all prime elements of the preordered monoid $((\mathfrak{T}(X), \bullet), \subseteq)$ of topologizing subcategories of C_X.

4.4. Proposition. *The decomposition* $\mathbf{Spec}^1_{\mathfrak{t}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \coprod \mathbf{Spec}^{1,0}_{\mathfrak{t}}(X)$ *induces, via the canonical map* $\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{t}}(X)$ *(cf. 4.3(b)), a decomposition*

$$\mathbf{Spec}^0_{\mathfrak{t}}(X) = \mathbf{Spec}(X) \coprod \mathbf{Spec}^{0,0}_{\mathfrak{t}}(X) \tag{2}$$

Proof. For every object L of $Spec(X)$, the category $\widehat{L} = \widehat{[L]}$ coincides with the subcategory $\langle L\rangle$, because if a topologizing subcategory $\mathbb{T}$ does not contain L, it is contained in $\langle L\rangle$. By 3.2(ii), the map $[L] \longmapsto \langle L\rangle$ is an isomorphism $\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Therefore, the preimage of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ in $\mathbf{Spec}_{\mathfrak{t}}^{0}(X)$ coincides with $\mathbf{Spec}(X)$.

Thus, the map (1) in 4.3 is the coproduct of the isomorphism $\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ and the map

$$\mathbf{Spec}_{\mathfrak{t}}^{0,0}(X) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^{1,0}(X), \quad \mathcal{T} \longmapsto \widehat{\mathcal{T}}. \tag{3}$$

It follows that $\mathbf{Spec}_{\mathfrak{t}}^{0,0}(X)$ consists of all topologizing subcategories $\mathcal{T}$ such that $\widehat{\mathcal{T}}$ belongs to $\mathfrak{Th}(X) - \mathfrak{Se}(X)$. ■

4.4.1. Note. Proposition 4.4 gives the following description of $\mathbf{Spec}(X)$: it consists of all topologizing subcategories $\mathcal{P}$ of C_X such that the union $\widehat{\mathcal{P}}$ of all topologizing subcategories which do not contain $\mathcal{P}$ is a Serre subcategory.

4.4.2. Corollary. *Suppose that C_X is a noetherian category (i.e. every object of C_X is noetherian). Then $\mathbf{Spec}_{\mathfrak{t}}^{1}(X) = \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ and $\mathbf{Spec}_{\mathfrak{t}}^{0}(X) = \mathbf{Spec}(X)$.*

Proof. Let C_X be an arbitrary abelian category, and let $\mathcal{T}$ be a full subcategory of C_X closed under finite coproducts and quotients. Then every noetherian object of C_X has the largest subobject which belongs to $\mathcal{T}$.

In fact, if M is a noetherian object, then the set $S_{\mathcal{T}}(M)$ of subobjects of M which belong to $\mathcal{T}$ has a maximal element, $M_{\mathcal{T}}$. If N is any other element of $S_{\mathcal{T}}(M)$, then the image of the natural morphism $M_{\mathcal{T}} \oplus N \longrightarrow M$ belongs to $\mathcal{T}$ (thanks to the assumptions on $\mathcal{T}$) and contains $M_{\mathcal{T}}$. Due to the maximality of $M_{\mathcal{T}}$, the image of $M_{\mathcal{T}} \oplus N \longrightarrow M$ coincides with $M_{\mathcal{T}}$, hence N is a subobject of $M_{\mathcal{T}}$.

This shows that if all objects of the category C_X are noetherian, than the subcategory $\mathcal{T}$ is coreflective. In particular, every topologizing and every thick subcategory of C_X is coreflective. By 2.4.5, thick coreflective subcategories are Serre subcategories. The assertion follows now from 3.2 and 4.4. ■

4.4.3. Examples. The conditions of 4.4.2 are fulfilled if C_X is the category of finitely generated modules over a left noetherian ring, or the category of quasi-coherent sheaves on a noetherian scheme, or the category of coherent D-modules on a noetherian scheme.

4.5. Representatives of $\mathbf{Spec}_{\mathfrak{t}}^{0}(X)$ and representatives of $\mathbf{Spec}(X)$. Let $Spec_{\mathfrak{t}}^{0}(X)$ denote the family of all objects M of C_X such that $[M] \in \mathbf{Spec}_{\mathfrak{t}}^{0}(X)$. In other words, an object M belongs to $Spec_{\mathfrak{t}}^{0}(X)$ iff the union $\widehat{[M]}$ of all topologizing subcategories of C_X which do not contain M is a thick subcategory. We consider $Spec_{\mathfrak{t}}^{0}(X)$ together with the specialization preorder $\succ$, which is precisely the preorder induced by the specialization preorder $\supseteq$ on $\mathbf{Spec}_{\mathfrak{t}}^{0}(X)$.

The decomposition

$$\mathbf{Spec}_{\mathfrak{t}}^{0}(X) = \mathbf{Spec}(X) \coprod \mathbf{Spec}_{\mathfrak{t}}^{0,0}(X)$$

induces the decomposition

$$Spec_{\mathfrak{t}}^{0}(X) = Spec(X) \coprod Spec_{\mathfrak{t}}^{0,0}(X).$$

4.5.1. Proposition. *Let C_X have the property (sup). Then an object M of finite type belongs to $Spec^0_{\mathfrak{t}}(X)$ iff it belongs to $Spec(X)$.*

Proof. Let M be an object of $Spec^0_{\mathfrak{t}}(X)$. Suppose that $M \in \widehat{[M]}^{-}$. Then, by 2.4.7, M is the supremum of its subobjects from $\widehat{[M]}$. Therefore, if M is of finite type, it belongs to $\widehat{[M]}$ which is impossible by the definition of $\widehat{[M]}$. This shows that $\widehat{[M]} = \widehat{[M]}^{-}$, i.e. $\widehat{[M]}$ is a Serre subcategory. Therefore, by 3.2(i), $[M] \in \mathbf{Spec}(X)$. ■

4.5.2. Corollary. *Let C_X have the property (sup) and enough objects of finite type (i.e. every nonzero object of C_X has a nonzero subobject of finite type). Then $\mathbf{Spec}^{0,0}_{\mathfrak{t}}(X)$ is a subset of all $[M] \in \mathbf{Spec}^0_{\mathfrak{t}}(X)$ such that the subcategory $[M] - \widehat{[M]}$ does not contain objects of finite type. Or, equivalently, $\mathbf{Spec}(X) = \{[M] \in \mathbf{Spec}^0_{\mathfrak{t}}(X) \mid M$ is of finite type$\}$.*

4.5.3. Corollary. *Let $C_X = R - mod$ for an associative ring R. The set of all left ideals m such that R/m belongs to $Spec^0_{\mathfrak{t}}(X)$ coincides with the left spectrum, $Spec_\ell(R)$ of the ring R.*

Proof. By 4.5.2, if the module R/m belongs to $Spec^0_{\mathfrak{t}}(X)$, then it belongs to $Spec(X)$. By [R, III.4.2], the left spectrum $Spec_\ell(R)$ consists of all left ideals $\mathfrak{m}$ such that the quotient module $R/\mathfrak{m}$ belongs to $Spec(X)$. ■

The reader is encouraged to look into Appendix 2 for more details about the spectra $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ and $\mathbf{Spec}(X)$.

5. The complete spectrum and the general construction.

5.1. Invariant topologizing subcategories and localizations. Fix an abelian category C_X. Let $\mathbb{T}$ and $\mathbb{S}$ be topologizing subcategories of C_X. We say that the subcategory $\mathbb{T}$ is $\mathbb{S}$-*invariant* if $\mathbb{T} = \mathbb{S} \bullet \mathbb{T} \bullet \mathbb{S}$.

It follows from this definition that for any topologizing subcategory $\mathbb{T}$, the smallest $\mathbb{T}$-invariant topologizing subcategory is the *thick envelope* $\mathbb{T}^\infty = \bigcup\limits_{n \geq 1} \mathbb{T}^{\bullet n}$ of $\mathbb{T}$, which is the smallest thick subcategory containing $\mathbb{T}$. In particular, $\mathbb{T}$ is $\mathbb{T}$-invariant iff it is thick.

On the other hand, $\mathbb{T}$ is $\mathbb{S}$-invariant iff $\mathbb{T}$ is $\mathbb{S}^\infty$-invariant. In particular, if $\mathbb{T}$ is $\mathbb{S}$-invariant, then $\mathbb{S}^\infty \subseteq \mathbb{T}$.

If $\mathbb{T}_1$ is $\mathbb{T}_0$-invariant and $\mathbb{T}_2$ is $\mathbb{T}_1$-invariant, then $\mathbb{T}_2$ is $\mathbb{T}_0$-invariant, because in this case we have: $\mathbb{T}_2 \subseteq \mathbb{T}_0 \bullet \mathbb{T}_2 \bullet \mathbb{T}_0 \subseteq \mathbb{T}_1 \bullet \mathbb{T}_2 \bullet \mathbb{T}_1 = \mathbb{T}_2$.

Given a thick subcategory $\mathbb{S}$ of the category C_X, let $\mathcal{T}(X, \mathbb{S})$ denote the preorder (with respect to $\subseteq$) of $\mathbb{S}$-invariant topologizing subcategories. Note that the inclusion $\mathcal{T}(X, \mathbb{S}) \hookrightarrow \mathcal{T}(X)$ has a left inverse (a left adjoint), $\mathcal{T}(X) \longrightarrow \mathcal{T}(X, \mathbb{S})$ which maps every topologizing subcategory $\mathbb{T}$ to the $\mathbb{S}$-invariant topologizing subcategory $\mathbb{S} \bullet \mathbb{T} \bullet \mathbb{S}$.

5.1.1. Lemma. *Let $\mathbb{S}$ be a thick subcategory of the category C_X and $C_X \xrightarrow{q^*} C_X/\mathbb{S}$ the localization functor. Then for any topologizing subcategory $\mathbb{T}$ of C_X, the topologizing subcategory $[q^*(\mathbb{T})]$ of $C_X/\mathbb{S}$ spanned by $q^*(T)$ is generated by all objects of $C_X/\mathbb{S}$ isomorphic to objects of $q^*(T)$, and its preimage in C_X, $q^{*^{-1}}([q^*(\mathbb{T})])$, coincides with $\mathbb{S} \bullet \mathbb{T} \bullet \mathbb{S}$. In other words, $q^{*^{-1}}([q^*(\mathbb{T})])$ is the smallest $\mathbb{S}$-invariant subcategory of C_X containing $\mathbb{T}$.*

Proof. It follows that $[q^*(\mathbb{T})] = \bigcup_{M\in Ob\mathbb{T}} [q^*(M)]$. Let $q^*(M) \succ q^*(L)$, i.e. there exists a diagram $q^*(M)^{\oplus n} \longleftarrow q^*(N) \longrightarrow q^*(L)$ in which the left arrow is a monomorphism and the right arrow is an epimorphism. To this diagram, there corresponds a diagram $M^{\oplus n} \longleftarrow N' \longrightarrow N \longleftarrow N'' \longrightarrow L$ such that q^* maps the arrows $N' \longrightarrow N \longleftarrow N''$ to isomorphisms. Completing $N' \longrightarrow N \longleftarrow N''$ to a cartesian square

$$\begin{array}{ccc} N' & \longrightarrow & N \\ \uparrow & & \uparrow \\ K' & \longrightarrow & N'' \end{array}$$

and taking the compositions of the arrows $M^{\oplus n} \longleftarrow N' \longleftarrow K'$ and $K' \longrightarrow N'' \longrightarrow L$, we obtain a diagram, $M^{\oplus n} \xleftarrow{\mathfrak{j}'} K' \xrightarrow{\mathfrak{e}'} L$ such that $q^*(\mathfrak{j}')$ is a monomorphism and $q^*(\mathfrak{e}')$ is an epimorphism. Representing arrows $\mathfrak{j}'$ and $\mathfrak{e}'$ as compositions of an epimorphism and a monomorphism, we obtain the diagram

$$M^{\oplus n} \xleftarrow{\mathfrak{j}} K \xleftarrow{s'} K' \xrightarrow{e'} L' \xrightarrow{t'} L$$

in which $\mathfrak{j}$ is a monomorphism, e' is an epimorphism, and $q^*(s')$, $q^*(t')$ are isomorphisms. Taking a push-forward of the pair of arrows $K \xleftarrow{s'} K' \xrightarrow{e'} L'$, we obtain a commutative diagram

$$\begin{array}{ccccccc} M^{\oplus n} & \xleftarrow{\mathfrak{j}} & K & \xrightarrow{\mathfrak{e}} & L_1 & & \\ & & {\scriptstyle s'}\uparrow & & \uparrow{\scriptstyle t} & & \\ & & K' & \xrightarrow{e'} & L' & \xrightarrow{t'} & L \end{array}$$

in which $\mathfrak{j}$ is a monomorphism, $\mathfrak{e}$ is an epimorphism, and $q^*(t)$, $q^*(t')$ are isomorphisms. Therefore $q^*(L_1) \simeq q^*(L)$ and $M \succ L_1$. The latter implies that if $M \in Ob\mathbb{T}$, then $L_1 \in Ob\mathbb{T}$. ∎

5.1.2. Corollary. *Let $\mathbb{S}$ be a thick subcategory of C_X and $C_X \xrightarrow{q^*} C_{X/\mathbb{S}} = C_X/\mathbb{S}$ the localization functor. The map $\mathbb{T} \longmapsto q^{*^{-1}}(\mathbb{T})$ induces an isomorphism*

$$\mathcal{T}(X/\mathbb{S}) \xrightarrow{\sim} \mathcal{T}(X,\mathbb{S})$$

between the preorder of topologizing subcategories of the quotient category $C_{X/\mathbb{S}} = C_X/\mathbb{S}$ the preorder of $\mathbb{S}$-invariant topologizing subcategories of C_X.

5.2. The complete spectrum. Fix an abelian category C_X. Recall that the *complete spectrum*, $\mathbf{Spec}^1(X)$, of X is formed by thick subcategories $\mathcal{P}$ of C_X such that the quotient category $C_{X/\mathcal{P}} = C_X/\mathcal{P}$ is *local*. The latter means that the category $C_{X/\mathcal{P}}$ has the smallest nonzero topologizing subcategory.

For every thick subcategory $\mathbb{S}$ of C_X, let $\mathbb{S}^{\circledast}$ denote the intersection of all $\mathbb{S}$-invariant topologizing subcategories of C_X properly containing $\mathbb{S}$.

5.2.1. Proposition. *A thick subcategory $\mathcal{P}$ of the category C_X belongs to $\mathbf{Spec}^1(X)$ iff there exists the smallest $\mathcal{P}$-invariant topologizing subcategory properly containing $\mathcal{P}$. In other words,* $\mathbf{Spec}^1(X) = \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P} \neq \mathcal{P}^{\circledast}\}$.

Proof. The assertion follows from 5.1.2. ∎

5.2.2. Corollary. $\mathbf{Spec}^1_{\mathfrak{t}}(X) \subseteq \mathbf{Spec}^1(X)$.

5.2.3. The preorder $\widetilde{\mathfrak{T}}(X)$ and the complete spectrum. Let $\widetilde{\mathfrak{T}}(X)$ denote the preorder whose objects are topologizing subcategories of C_X and morphisms are identical morphisms and inclusions $\mathbb{S} \hookrightarrow \mathbb{T}$ such that $\mathbb{T}$ is $\mathbb{S}$-invariant. It follows from the discussion at the beginning of 5.1 that the composition of morphisms is a morphism.

Notice that $\mathfrak{Th}(X)$ is a subpreorder of $\widetilde{\mathfrak{T}}(X)$, because if $\mathbb{S} \hookrightarrow \mathbb{T}$ and $\mathbb{T}$ is thick, then $\mathbb{T} \subseteq \mathbb{S} \bullet \mathbb{T} \bullet \mathbb{S} \subseteq \mathbb{T} \bullet \mathbb{T} \bullet \mathbb{T} = \mathbb{T}$, that is $\mathbb{T}$ is $\mathbb{S}$-invariant. In particular, the preorder $\mathfrak{Se}(X)$ of Serre subcategories is a subpreorder of $\widetilde{\mathfrak{T}}(X)$.

5.2.3.1. Lemma. *For every topologizing subcategory $\mathbb{S}$ of C_X, there is a natural isomorphism of preorders $\mathbb{S}\backslash\widetilde{\mathfrak{T}}(X) \xrightarrow{\sim} \widetilde{\mathfrak{T}}(X/\mathbb{S}^{\infty})$, where $\mathbb{S}^{\infty}$ is the smallest thick subcategory containing $\mathbb{S}$.*

Proof. The assertion follows from 5.1.2. ∎

Applying the general construction to the inclusion functor $\mathfrak{Th}(X) \xrightarrow{\widetilde{\mathcal{J}}_X} \widetilde{\mathfrak{T}}(X)$, we obtain the spectra $\mathfrak{Spec}^i(\mathfrak{Th}(X), \mathfrak{J}_X)$, $i = 0, 1$. and a canonical monomorphism

$$\mathfrak{Spec}^0(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X).$$

5.2.3.2. Proposition. $\mathfrak{Spec}^1(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X) = \mathbf{Spec}^1(X)$.

Proof. The preorder $\mathfrak{Spec}^1(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X)$ is defined by the cartesian square

$$\begin{array}{ccc}
\mathfrak{Spec}^1(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X) & \longrightarrow & \mathfrak{Th}(X) \\
\downarrow & & \downarrow \\
\mathfrak{Spec}^1(\widetilde{\mathfrak{T}}(X)) & \longrightarrow & \widetilde{\mathfrak{T}}(X)
\end{array}$$

In other words, the objects of $\mathfrak{Spec}^1(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X)$ are thick subcategories $\mathcal{P}$ of C_X such that there exists the smallest $\mathcal{P}$-invariant topologizing subcategory of C_X properly containing $\mathcal{P}$. Comparing this description with 5.2.1, one can see that $\mathfrak{Spec}^1(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X)$ coincides with $\mathbf{Spec}^1(X)$. ∎

5.2.4. Note. The 'abslolute' spectrum, $\mathfrak{Spec}^1(\widetilde{\mathfrak{T}}(X))$, of the preorder $\widetilde{\mathfrak{T}}(X)$ is the (obviously, disjoint) union of $\mathfrak{Spec}^1(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X) = \mathbf{Spec}^1(X)$ and $\widetilde{\mathfrak{T}}(X) - \mathfrak{Th}(X)$.

In fact, if $\mathcal{P} \in \mathbf{Spec}^1(X)$, then there exists the smallest $\mathcal{P}$-invariant topologizing subcategory $\mathcal{P}^{\circledast}$ of C_X properly containing $\mathcal{P}$. The pair $(\mathcal{P}, \mathcal{P} \hookrightarrow \mathcal{P}^{\circledast})$ is, evidently, the initial object of the category $\mathcal{P}\backslash\widetilde{\mathfrak{T}}(X)$. If $\mathbb{T}$ is a topologizing, but not thick, subcategory of C_X, then $(\mathbb{T}, \mathbb{T} \hookrightarrow \mathbb{T}^{\infty})$ is the initial object of the category $\mathbb{T}\backslash\widetilde{\mathfrak{T}}(X)$; hence $\mathbb{T}$ is an object of $\mathfrak{Spec}^1(\widetilde{\mathfrak{T}}(X))$.

5.2.5. The spectrum $\mathbf{Spec}^0(X)$. We denote $\mathfrak{Spec}^0(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X)$ by $\mathbf{Spec}^0(X)$. It follows that objects of $\mathbf{Spec}^0(X)$ are topologizing subcategories $\mathcal{T}$ such that the union $\widehat{\mathcal{T}}$ of all topologizing subcategories which are not $\mathcal{T}$-invariant belongs to $\mathbf{Spec}^1(X)$. In particular, $\widehat{\mathcal{T}}$ is thick, hence it coincides with the union of all thick subcategories of C_X which do not contain $\mathcal{T}$. This discussion together with the inclusion $\mathbf{Spec}^1_{\mathfrak{t}}(X) \subseteq \mathbf{Spec}^1(X)$ of 5.2.2 shows that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^0(X) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Spec}^1_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^1(X) \end{array} \qquad (1)$$

whose horizontal arrows are inclusions and vertical arrows are injective.

5.3. A canonical decomposition of the complete spectrum and the S-spectrum. For every element $\mathcal{P}$ of $\mathbf{Spec}^1(X)$, we set $\mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \bigcap \mathcal{P}^{\perp}$; i.e. $\mathcal{P}_{\circledast}$ is the full subcategory of $\mathcal{P}^{\circledast}$ generated by all $\mathcal{P}$-torsion free objects of $\mathcal{P}^{\circledast}$. Similarly to Section 3, we have a decomposition

$$\mathbf{Spec}^1(X) = \mathbf{Spec}^{1,1}(X) \coprod \mathbf{Spec}^{1,0}(X), \qquad (2)$$

where

$$\mathbf{Spec}^{1,0}(X) = \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P}_{\circledast} = 0\} \quad \text{and} \quad \mathbf{Spec}^{1,1}(X) = \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P}_{\circledast} \neq 0\}.$$

Recall that the *S-spectrum*, $\mathbf{Spec}^-(X)$, of the 'space' X is formed by all Serre subcategories $\mathcal{P}$ of C_X such that the quotient 'space' $X/\mathcal{P}$ is local. In other words, $\mathbf{Spec}^-(X) = \mathbf{Spec}^1(X) \bigcap \mathfrak{Se}(X)$, where $\mathfrak{Se}(X)$ is the preorder of all Serre subcategories of C_X.

We have the following analogue of 3.2(i):

5.3.1. Proposition. *A thick subcategory $\mathcal{P}$ from $\mathbf{Spec}^1(X)$ belongs to $\mathbf{Spec}^{1,1}(X)$ iff it is a Serre subcategory. Thus,*

$$\begin{aligned} \mathbf{Spec}^{1,1}(X) =& \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P}^- = \mathcal{P} \subsetneqq \mathcal{P}^{\circledast}\} = \mathbf{Spec}^-(X) \\ \mathbf{Spec}^{1,0}(X) =& \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P} \subsetneqq \mathcal{P}^{\circledast} \subseteq \mathcal{P}^-\}. \end{aligned}$$

Proof. Suppose $\mathcal{P} = \mathcal{P}^- \subsetneqq \mathcal{P}^{\circledast}$. Then every nonzero object of $\mathcal{P}^{\circledast} - \mathcal{P}$ has a nonzero subquotient which has no $\mathcal{P}$-torsion. Since $\mathcal{P}^{\circledast}$ is a topologizing subcategory, this subquotient belongs to $\mathcal{P}^{\circledast} \bigcap \mathcal{P}^{\perp}$. In particular, this shows that $\mathcal{P}^{\circledast} \bigcap \mathcal{P}^{\perp} \neq 0$, that is $\mathcal{P}$ is an element of $\mathbf{Spec}^{1,1}(X)$, which proves the inclusion $\mathbf{Spec}^1(X) \bigcap \mathfrak{Se}(X) \subseteq \mathbf{Spec}^{1,1}(X)$. Here $\mathfrak{Se}(X)$ denotes the preorder of all Serre subcategories of C_X.

Conversely, if $\mathcal{P}$ is a thick subcategory which is not a Serre subcategory, i.e. $\mathcal{P} \subsetneqq \mathcal{P}^-$, then $\mathcal{P}^{\circledast} \subseteq \mathcal{P}^-$, because $\mathcal{P}^-$ (as any thick subcategory containing $\mathcal{P}$) is $\mathcal{P}$-invariant. Since $\mathcal{P}^- \bigcap \mathcal{P}^{\perp} = 0$, the inclusion $\mathcal{P}^{\circledast} \subseteq \mathcal{P}^-$ implies that $\mathcal{P}^{\circledast} \bigcap \mathcal{P}^{\perp} = 0$, i.e. $\mathcal{P} \notin \mathbf{Spec}^{1,1}(X)$. This shows that $\mathbf{Spec}^{1,1}(X) = \mathbf{Spec}^1(X) \bigcap \mathfrak{Se}(X) = \mathbf{Spec}^-(X)$. ∎

5.4. The preorder $\widetilde{\mathfrak{T}}(X)$ and the S-spectrum. General pattern associate with the inclusion functor $\mathfrak{Se}(X) \xrightarrow{\widetilde{\mathcal{J}}^{\mathfrak{s}}_X} \widetilde{\mathfrak{T}}(X)$ the spectra $\mathfrak{Spec}^i(\mathfrak{Se}(X), \mathcal{J}^{\mathfrak{s}}_X)$, $i = 0, 1$, and a canonical monomorphism

$$\mathfrak{Spec}^0(\mathfrak{Se}(X), \widetilde{\mathcal{J}}^{\mathfrak{s}}_X) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X), \widetilde{\mathcal{J}}^{\mathfrak{s}}_X).$$

5.4.1. Proposition. $\mathfrak{Spec}^1(\mathfrak{Se}(X), \widetilde{\mathcal{J}}^{\mathfrak{s}}_X) = \mathbf{Spec}^-(X)$.

Proof. By 5.2.3.2, $\mathfrak{Spec}^1(\mathfrak{Th}(X), \widetilde{\mathcal{J}}_X)$ coincides with the complete spectrum $\mathbf{Spec}^1(X)$ This implies that $\mathfrak{Spec}^1(\mathfrak{Se}(X), \widetilde{\mathcal{J}}^{\mathfrak{s}}_X)$ is the S-spectrum $\mathbf{Spec}^-(X)$ of the 'space' X. ■

6. The spectrum $\mathbf{Spec}_{\rtimes}(X)$.

For every subcategory (or a family of objects) $\mathcal{B}$ of the category C_X, let $\mathcal{B}_{\rtimes}$ denote the union of all thick subcategories of C_X which are left orthogonal to $\mathcal{B}$.

Clearly, $\mathcal{B}_{\rtimes} \subseteq \mathcal{D}_{\rtimes}$ if $\mathcal{D} \subseteq \mathcal{B}$, and $(\mathcal{B}_{\rtimes})^{\perp}$ is the largest subcategory among the subcategories $\mathcal{B}'$ of C_X such that $\mathcal{B}'_{\rtimes} = \mathcal{B}_{\rtimes}$.

6.1. Lemma. *If the subcategory $\mathcal{B}$ is closed under taking subobjects, then $\mathcal{B}_{\rtimes}$ is a Serre subcategory.*

Proof. In fact, suppose that $\mathcal{B}^-_{\rtimes} \neq \mathcal{B}_{\rtimes}$, and let M be an object of $\mathcal{B}^-_{\rtimes} - \mathcal{B}_{\rtimes}$.

The condition $M \notin Ob\mathcal{B}_{\rtimes}$ means that for some object M' of $[M]_{\bullet}$, there is a nonzero morphism $N \xrightarrow{g} M'$ with $N \in Ob\mathcal{B}$. The image M'' of the morphism g is a nonzero object of $\mathcal{B}^-_{\rtimes}$. In particular, it has a nonzero subobject L from $\mathcal{B}_{\rtimes}$. Thus, we have a cartesian square

$$\begin{array}{ccc}
N \times_{M''} L & \longrightarrow & L \\
\big\downarrow & & \big\downarrow \\
N & \longrightarrow & M''
\end{array}$$

whose vertical arrows are monomorphisms and horizontal arrows are epimorphisms. Since $\mathcal{B}$ is closed under taking subobjects, in particular $N \times_{M''} L$ is an object of $\mathcal{B}$, we run into a contradiction. ■

6.2. The functor $\mathfrak{L}_{\rtimes}$ and the related spectrum. Let $\mathfrak{T}_{\le}(X)$ denote the preorder (with respect to $\subseteq$) of full subcategories of C_X closed under taking subobjects and finite coproducts. Let $\mathfrak{L}_{\rtimes}$ denote the morphism of preorders $\mathfrak{T}_{\le}(X) \longrightarrow \mathfrak{Se}(X)$ which maps every object $\mathcal{B}$ of $\mathfrak{T}_{\le}(X)$ to the Serre subcategory $\mathcal{B}_{\rtimes}$ (defined in 6.1). And let $\mathfrak{L}'_{\rtimes}$ be the composition of $\mathfrak{L}_{\rtimes}$ and the inclusion $\mathfrak{Se}(X) \hookrightarrow \widetilde{\mathfrak{T}}(X)$. The morphism $\mathfrak{L}'_{\rtimes}$ gives rise to the spectrum $\mathfrak{Spec}^1(\mathfrak{T}_{\le}(X), \mathfrak{L}_{\rtimes})$ defined via the cartesian square

$$\begin{array}{ccc}
\mathfrak{Spec}^1(\mathfrak{T}_{\le}(X), \mathfrak{L}'_{\rtimes}) & \longrightarrow & \mathfrak{T}_{\le}(X) \\
\big\downarrow & & \big\downarrow {\scriptstyle \mathfrak{L}'_{\rtimes}} \\
\mathfrak{Spec}^1(\widetilde{\mathfrak{T}}(X)) & \longrightarrow & \widetilde{\mathfrak{T}}(X)
\end{array} \qquad (1)$$

Since the morphism $\mathfrak{L}'_{\rtimes}$ takes values in the preorder $\mathfrak{Se}(X)$ of Serre subcategories, the spectrum $\mathfrak{Spec}^1(\mathfrak{T}_{\leq}(X), \mathfrak{L}'_{\rtimes})$ coincides with the spectrum $\mathfrak{Spec}^1(\mathfrak{T}_{\leq}(X), \mathfrak{L}_{\rtimes})$ defined by the cartesian square

$$\begin{array}{ccc} \mathfrak{Spec}^1(\mathfrak{T}_{\leq}(X), \mathfrak{L}_{\rtimes}) & \longrightarrow & \mathfrak{T}_{\leq}(X) \\ \big\downarrow & & \big\downarrow \mathfrak{L}_{\rtimes} \\ \mathbf{Spec}^-(X) & \longrightarrow & \mathfrak{Se}(X) \end{array} \qquad (2)$$

In other words, objects of $\mathfrak{Spec}^1(\mathfrak{T}_{\leq}(X), \mathfrak{L}_{\rtimes})$ are naturally identified with objects $\mathcal{B}$ of $\mathfrak{T}_{\leq}(X)$ such that the Serre subcategory $\mathcal{B}_{\rtimes}$ belongs to the S-spectrum $\mathbf{Spec}^-(X)$.

6.3. The spectrum $\mathbf{Spec}_{\rtimes}(X)$. We denote by $\mathbf{Spec}_{\rtimes}(X)$ the preorder (with respect to the inclusion) of full nonzero subcategories $\mathcal{B}$ of the category C_X such that $\mathcal{B} = (\mathcal{B}_{\rtimes})^{\perp} \bigcap (\mathcal{B}_{\rtimes})^{\circledast}$ and $\mathcal{B}_{\rtimes} \in \mathbf{Spec}^-(X)$.

It follows from the discussion above that $\mathbf{Spec}_{\rtimes}(X) \subseteq \mathfrak{Spec}^1(\mathfrak{T}_{\leq}(X), \mathfrak{L}_{\rtimes})$.

6.4. Proposition. *The map $\mathcal{B} \longmapsto \mathcal{B}_{\rtimes}$ induces an isomorphism*

$$\mathbf{Spec}_{\rtimes}(X) \longrightarrow \mathbf{Spec}^-(X)$$

with the inverse map $\mathcal{P} \longmapsto \mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$.

Proof. Let $\mathcal{P} \in \mathbf{Spec}^-(X)$. The subcategory $\mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$ is closed under taking subobjects, hence, by 6.1, $(\mathcal{P}_{\star})_{\rtimes}$ is a Serre subcategory. This Serre subcategory contains $\mathcal{P}$, because all objects of $\mathcal{P}_{\circledast}$ are right orthogonal to $\mathcal{P}$. If $(\mathcal{P}_{\circledast})_{\rtimes}$ would contain $\mathcal{P}$ properly, then it would contain $\mathcal{P}_{\circledast}$, which is impossible. Therefore $(\mathcal{P}_{\circledast})_{\rtimes} = \mathcal{P}$. This shows that $\mathcal{P} \longmapsto \mathcal{P}_{\circledast}$ maps $\mathbf{Spec}^-(X)$ to $\mathbf{Spec}_{\rtimes}(X)$.

On the other hand, if $\mathcal{B}$ belongs to $\mathbf{Spec}_{\rtimes}(X)$, i.e. $0 \neq \mathcal{B} = (\mathcal{B}_{\rtimes})^{\perp} \bigcap (\mathcal{B}_{\rtimes})^{\circledast}$, then $\mathcal{B}_{\rtimes} \in \mathbf{Spec}^-(X)$. ∎

6.5. Representatives of points of $\mathbf{Spec}_{\rtimes}(X)$. For every subcategory $\mathcal{A}$ of C_X, we denote by $[\mathcal{A})$ the smallest full subcategory of C_X which contains $\mathcal{A}$ and is closed under taking subobjects and finite coproducts.

For every $M \in C_X$, we write $[M)$ instead of $[M, id_M)$. We denote by $Spec_{\rtimes}(X)$ the family of all objects M of C_X such that $[M)_{\rtimes} \in \mathbf{Spec}^-(X)$. Objects of $Spec_{\rtimes}(X)$ are regarded as representatives of the corresponding points of $\mathbf{Spec}_{\rtimes}(X)$.

It follows that $Spec(X) \subseteq Spec_{\rtimes}(X)$ and the map

$$Spec_{\rtimes}(X) \longrightarrow \mathbf{Spec}^-(X), \quad M \longmapsto [M)_{\rtimes}, \qquad (3)$$

is an extension of the map $Spec(X) \longrightarrow \mathbf{Spec}^-(X)$, $P \longmapsto \langle P \rangle$.

We define a preorder $\succcurlyeq$ on $Spec_{\rtimes}(X)$ by $M \succcurlyeq N \Leftrightarrow [N)_{\rtimes} \subseteq [M)_{\rtimes}$. If M and N are objects of $Spec(X)$, then $[N)_{\rtimes} = \langle N \rangle$ and $[M)_{\rtimes} = \langle M \rangle$. Therefore in this case $M \succcurlyeq N \Leftrightarrow \langle N \rangle \subseteq \langle M \rangle \Leftrightarrow [N] \subseteq [M] \Leftrightarrow M \succ N$; i.e. the restriction of the preorder $\succcurlyeq$ to $Spec(X)$ coincides with the specialization preorder $\succ$ on $Spec(X)$.

6.5.1. Proposition. *Suppose that $\mathcal{P} \in \mathbf{Spec}^1(X)$ is such that the localization functor $C_X \xrightarrow{q^*_{\mathcal{P}}} C_X/\mathcal{P}$ has a right adjoint, $q_{\mathcal{P}*}$ (in particular, $\mathcal{P} \in \mathbf{Spec}^-(X)$) and the quotient category $C_X/\mathcal{P}$ has a simple object, L. Then $[q_{\mathcal{P}*}(L)) = \mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$.*

Proof. In fact, for every $L \in ObC_X/\mathcal{P}$, the object $q_{\mathcal{P}*}(L)$ is $\mathcal{P}$-torsion free, i.e. it belongs to $\mathcal{P}^{\perp}$, and $q^*_{\mathcal{P}}q_{\mathcal{P}*}(L) \simeq L$. Therefore, if L is a quasi-final object, then $q_{\mathcal{P}*}(L)$ belongs to $\mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$. In particular, $[q_{\mathcal{P}*}(L)) \subseteq \mathcal{P}_{\circledast}$. In order to prove the inverse inclusion, notice that any nonzero object M of $\mathcal{P}_{\circledast}$ is a subobject of $q_{\mathcal{P}*}(q^*_{\mathcal{P}}(M))$; in particular, M is an object of the subcategory $[q_{\mathcal{P}*}(q^*_{\mathcal{P}}(M)))$. Since M is a nonzero object of $\mathcal{P}_{\circledast}$, its image $q^*_{\mathcal{P}}(M)$ is a quasi-final object of $C_X/\mathcal{P}$. Therefore $q^*_{\mathcal{P}}(M)$ is isomorphic to the coproduct of a finite set of copies of the simple object L, which implies that $q_{\mathcal{P}*}q^*_{\mathcal{P}}(M)$ is isomorphic to the coproduct of a finite number of copies of $q_{\mathcal{P}*}(L)$. It follows that the subcategories $[q_{\mathcal{P}*}(q^*_{\mathcal{P}}(M)))$ and $[q_{\mathcal{P}*}(L))$ coincide. In particular, M is an object of $[q_{\mathcal{P}*}(L))$. This proves the inverse inclusion $\mathcal{P}_{\circledast} \subseteq [q_{\mathcal{P}*}(L))$. ■

6.5.2. The case of Grothendieck categories. Suppose that C_X is a Grothendieck category. Then the localization functor at any Serre subcategory has a right adjoint. If C_X has Gabriel-Krull dimension (cf. A3.7), then for every $\mathcal{P} \in \mathbf{Spec}^-(X)$, the quotient category $C_X/\mathcal{P}$ has a unique (up to isomorphism) simple object, $L_{\mathcal{P}}$.

The map $\mathcal{P} \longmapsto q_{\mathcal{P}*}(L_{\mathcal{P}})$ takes values in $Spec_{\rtimes}(X)$ and is a section of the map

$$Spec_{\rtimes}(X) \longrightarrow \mathbf{Spec}^-(X), \quad M \longmapsto [M)_{\rtimes}.$$

Thus, in the case when C_X has a Gabriel-Krull dimension, the map $\mathcal{P} \longmapsto q_{\mathcal{P}*}(L_{\mathcal{P}})$ is a canonical (defined uniquely up to isomorphism) choice of representatives of points of the spectrum $\mathbf{Spec}_{\rtimes}(X)$. For a general Grothendieck category C_X, this gives a canonical choice of a representative for every point $\mathcal{Q}$ of $\mathbf{Spec}_{\rtimes}(X)$ such that the category $C_X/\mathcal{Q}_{\rtimes}$ has simple objects.

7. Functorialities.

7.1. Lemma. *Let $\mathbb{T}$ be a topologizing subcategory of an abelian category C_X and $|\mathbb{T}|$ a 'space' defined by $C_{|\mathbb{T}|} = \mathbb{T}$.*

(a) For every $\mathcal{S} \in \mathfrak{Th}(|\mathbb{T}|)$, there exists the biggest and the smallest thick subcategory of C_X, resp. $\mathcal{S}_-$ and $\mathcal{S}_+$, such that $\mathcal{S}_- \cap \mathbb{T} = \mathcal{S} = \mathcal{S}_+ \cap \mathbb{T}$.

(b) If $\mathcal{S}$ is a Serre subcategory of $\mathbb{T}$, then $\mathcal{S}_+$ is a Serre subcategory of C_X, and $\mathcal{S}^- \cap \mathbb{T} = \mathcal{S}$.

Proof. (a) Let $\mathfrak{T}(X;\mathbb{T},\mathcal{S})$ denote the preorder of all topologizing subcategories $\mathcal{B}$ of C_X such that $\mathcal{B} \cap \mathbb{T} \subseteq \mathcal{S}$. Notice that $\mathfrak{T}(X;\mathbb{T},\mathcal{S})$ is closed under the Gabriel multiplication: if $\mathcal{B}'$, $\mathcal{B}''$ are elements of $\mathfrak{T}(X;\mathbb{T},\mathcal{S})$ and $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence such that $M' \in Ob\mathcal{B}'$, $M'' \in Ob\mathcal{B}''$ and $M \in Ob\mathbb{T}$, then M' and M'' are objects of $\mathcal{S}$, hence M is an object of $\mathcal{S}$. Therefore, the union $\mathcal{S}_+$ of all $\mathcal{B}$ from $\mathfrak{T}(X;\mathbb{T},\mathcal{S})$ is a thick subcategory of C_X such that $\mathcal{S}_+ \cap \mathbb{T} \subseteq \mathcal{S}$. Since $\mathcal{S}$ is a subcategory of $\mathcal{S}_+$, the intersection $\mathcal{S}_+ \cap \mathbb{T}$ coincides with $\mathcal{S}$. This also implies that the intersection $\mathcal{S}_- \cap \mathbb{T} \subseteq \mathcal{S}$ of the smallest thick subcategory of C_X containing $\mathcal{S}$ with $\mathbb{T}$ coincides with $\mathcal{S}$.

(b) Suppose now that $\mathcal{S}$ is a Serre subcategory of $\mathbb{T}$. Then $\mathcal{S}^-_+ \cap \mathbb{T} = \mathcal{S}$.

In fact, let M be an object of $\mathcal{S}_+^- \cap \mathbb{T}$; and let L be a nonzero subquotient of M. Since $\mathbb{T}$ is closed with respect to taking subquotients, there exists a diagram in $M \longleftarrow K \longrightarrow L$ in $\mathbb{T}$ whose left arrow is a monomorphism and the right arrow is an epimorphism. By definition of $\mathcal{S}_+^-$, the object L has a nonzero subobject which belongs to $\mathcal{S}_+$, hence it belongs to $\mathcal{S}_+ \cap \mathbb{T} = \mathcal{S}$. This shows that M is an object of $\mathcal{S}^-$. Therefore $\mathcal{S}_+^- \cap \mathbb{T} = \mathcal{S}_{\mathbb{T}}^-$. Here $\mathcal{S}_{\mathbb{T}}^-$ denotes the Serre envelope of $\mathcal{S}$ in $\mathbb{T}$. In particular, if $\mathcal{S}$ is a Serre subcategory of $\mathbb{T}$ (i.e. $\mathcal{S} = \mathcal{S}_{\mathbb{T}}^-$), then $\mathcal{S}_+^- \cap \mathbb{T} = \mathcal{S} = \mathcal{S}^- \cap \mathbb{T}$. ∎

For a subcategory $\mathbb{T}$ of C_X, we set

$$\begin{aligned} &U^1(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1(X) \mid \mathbb{T} \subseteq \mathcal{P}\} \\ &V^1(\mathbb{T}) = \mathbf{Spec}^1(X) - U^1(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1(X) \mid \mathbb{T} \not\subseteq \mathcal{P}\}. \\ &U^-(\mathbb{T}) = U^1(\mathbb{T}) \cap \mathfrak{Se}(X) \quad \text{and} \quad V^-(\mathbb{T}) = V^1(\mathbb{T}) \cap \mathfrak{Se}(X) \end{aligned} \tag{1}$$

The following assertion can be extracted from [R, Ch.6]. For the reader's convenience, we give here a (mostly different) proof.

7.2. Proposition. *Let $\mathbb{T}$ be a topologizing subcategory of C_X and $\mathbb{T}^\infty$ the smallest thick subcategory of C_X containing $\mathbb{T}$.*

(a) $V^1(\mathbb{T}) = V^1(\mathbb{T}^\infty)$ and $U^1(\mathbb{T}) = U^1(\mathbb{T}^\infty)$. The map

$$U^1(\mathbb{T}) \longrightarrow \mathbf{Spec}^1(X/\mathbb{T}^\infty), \quad \mathcal{P} \longmapsto \mathcal{P}/\mathbb{T}^\infty, \tag{2}$$

is an isomorphism.

(b) $V^-(\mathbb{T}) = V^-(\mathbb{T}^-)$ and $U^-(\mathbb{T}) = U^1(\mathbb{T}^-)$. The map

$$U^-(\mathbb{T}) \longrightarrow \mathbf{Spec}^-(X/\mathbb{T}^-), \quad \mathcal{P} \longmapsto \mathcal{P}/\mathbb{T}^-, \tag{3}$$

is an isomorphism.

(c) For every $\mathcal{P} \in V^1(\mathbb{T})$, the intersection $\mathcal{P} \cap \mathbb{T}$ is an element of $\mathbf{Spec}^1(|\mathbb{T}|)$, and the map

$$V^1(\mathbb{T}) \longrightarrow \mathbf{Spec}^1(|\mathbb{T}|), \quad \mathcal{P} \longmapsto \mathcal{P} \cap \mathbb{T}, \tag{4}$$

is an isomorphism. The inverse map is given by $\widetilde{\mathcal{P}} \longmapsto \widetilde{\mathcal{P}}_+$ (see 7.1). Here $|\mathbb{T}|$ is defined by $C_{|\mathbb{T}|} = \mathbb{T}$.

(d) For every $\mathcal{P} \in V^-(\mathbb{T})$, the intersection $\mathcal{P} \cap \mathbb{T}$ is an element of $\mathbf{Spec}^1(|\mathbb{T}|)$, and the map (2) induces an isomorphism $V^-(\mathbb{T}) \longrightarrow \mathbf{Spec}^-(|\mathbb{T}|)$.

Proof. (a) Since elements of $\mathbf{Spec}^1(X)$ are thick subcategories, $U^1(\mathbb{T}) = U^1(\mathbb{T}^\infty)$, hence $V^1(\mathbb{T}) = V^1(\mathbb{T}^\infty)$. Clearly $\mathcal{P} \longmapsto \mathcal{P}/\mathbb{T}^\infty$ maps $U^1(\mathbb{T}^\infty)$ to $\mathbf{Spec}^1(X/\mathbb{T}^\infty)$. The inverse map assigns to every element $\widetilde{\mathcal{P}}$ of $\mathbf{Spec}^1(X/\mathbb{T}^\infty)$ the thick subcategory ${q^*}^{-1}(\widetilde{\mathcal{P}})$, where q^* is the localization functor $C_X \longrightarrow C_{X/\mathbb{T}^\infty}$.

(b) The argument is similar to that of (a).

(c) Let $\mathcal{P} \in V^1(\mathbb{T})$, that is $\mathcal{P}$ is a thick subcategory of C_X such that $X/\mathcal{P}$ is local and $\mathbb{T} \not\subseteq \mathcal{P}$. The composition of the inclusion functor $\mathbb{T} \hookrightarrow C_X$ and the localization

$C_X \xrightarrow{q^*_{\mathcal{P}}} C_{X/\mathcal{P}}$ is an exact functor with the kernel $\mathcal{P} \cap \mathbb{T}$, hence it factors uniquely through the localization $\mathbb{T} \longrightarrow \mathbb{T}/(\mathbb{T} \cap \mathcal{P})$. Thus, we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{T} \cap \mathcal{P} & \longrightarrow & \mathbb{T} & \xrightarrow{q^*} & \mathbb{T}/(\mathbb{T} \cap \mathcal{P}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \mathfrak{j}^* \\ \mathcal{P} & \longrightarrow & C_X & \xrightarrow{q^*_{\mathcal{P}}} & C_{X/\mathcal{P}} \end{array} \tag{5}$$

in which the two left vertical arrows are inclusion functors. It follows that the right arrow in (5) is a fully faithful functor which induces an equivalence between $\mathbb{T}/(\mathbb{T} \cap \mathcal{P})$ and a nonzero topologizing subcategory of $C_{X/\mathcal{P}}$. Every nonzero topologizing subcategory of a local category is local. Therefore $\mathbb{T}/(\mathbb{T} \cap \mathcal{P})$ is a local category. The latter means that $\mathbb{T} \cap \mathcal{P} \in \mathbf{Spec}^1(|\mathbb{T}|)$.

(c1) Let now $\widetilde{\mathcal{P}}$ be an element of $\mathbf{Spec}^1(|\mathbb{T}|)$. By 7.1, there is the biggest topologizing subcategory $\widetilde{\mathcal{P}}_+$ of C_X (which happens to be thick) such that $\widetilde{\mathcal{P}} = \mathbb{T} \cap \widetilde{\mathcal{P}}_+$. We claim that $\widetilde{\mathcal{P}}_+$ is an element of $\mathbf{Spec}^1(X)$.

In fact, let $\mathcal{B}$ be a topologizing $\widetilde{\mathcal{P}}_+$-invariant subcategory of C_X properly containing $\widetilde{\mathcal{P}}_+$. Then $\mathcal{B} \cap \mathbb{T}$ contains $\widetilde{\mathcal{P}}$ properly; because if $\mathcal{B} \cap \mathbb{T} = \widetilde{\mathcal{P}}$, then $\mathcal{B} \subseteq \widetilde{\mathcal{P}}_+$. Clearly, the subcategory $\mathcal{B} \cap \mathbb{T}$ is $\widetilde{\mathcal{P}}$-invariant. Therefore it contains the subcategory $\widetilde{\mathcal{P}}^{\circledast}$. This shows that $(\widetilde{\mathcal{P}}_+)^{\circledast} = \widetilde{\mathcal{P}}_+ \bullet \widetilde{\mathcal{P}}^{\circledast} \bullet \widetilde{\mathcal{P}}_+$ is the smallest $\widetilde{\mathcal{P}}_+$-invariant topologizing subcategory of C_X properly containing $\widetilde{\mathcal{P}}_+$; so that $\widetilde{\mathcal{P}}_+$ belongs to $\mathbf{Spec}^1(X)$. Thus, the map

$$\mathbf{Spec}^1(|\mathbb{T}|) \longrightarrow V^1(\mathbb{T}), \quad \widetilde{\mathcal{P}} \longmapsto \widetilde{\mathcal{P}}_+,$$

is a right inverse to the map (4). It remains to show that the map (4) is injective.

(c2) Let $\widetilde{\mathcal{P}} \in \mathbf{Spec}^1(|\mathbb{T}|)$; and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be elements of $V^1(\mathbb{T})$ such that $\mathcal{P}_1 \cap \mathbb{T} = \widetilde{\mathcal{P}} = \mathcal{P}_2 \cap \mathbb{T}$. Replacing X by $X/(\mathcal{P}_1 \cap \mathcal{P}_2)$, we can (and will) assume that $\widetilde{\mathcal{P}} = 0$. In particular, $|\mathbb{T}|$ is local. Let M be a quasi-final object of $\mathbb{T}$. Notice that the composition of the embedding $\mathbb{T} \hookrightarrow C_X$ with the localization functor $C_X \xrightarrow{q^*_{\mathcal{P}_1}} C_{X/\mathcal{P}_1}$ is a fully faithful functor $\mathbb{T} \longrightarrow C_{X/\mathcal{P}_1}$ which induces an equivalence between $\mathbb{T}$ and a nonzero topologizing subcategory of the category $C_{X/\mathcal{P}_1}$ (see the diagram (5)). Therefore the image $q^*_{\mathcal{P}_1}(M)$ of the object M is a quasi-final object of $C_{X/\mathcal{P}_1}$. This implies that $\mathcal{P}_1 = \langle M \rangle$.

In fact, $\mathcal{P}_1 \subseteq \langle M \rangle$, because $M \notin Ob\mathcal{P}_1$. On the other hand, if $\mathcal{P}_1 \neq \langle M \rangle$, then $\langle M \rangle$ (being a thick subcategory containing $\mathcal{P}_1$, hence $\mathcal{P}_1$-invariant) contains the object M which cannot happen, because $M \in Spec(|\mathbb{T}|) \subseteq Spec(X)$.

By the same reason, $\mathcal{P}_2 = \langle M \rangle$, hence $\mathcal{P}_2 = \mathcal{P}_1$.

(d) The assertion follows from (c) and 7.1(b). Details are left to the reader. ∎

7.3. Corollary. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be elements of* $\mathbf{Spec}^1(X)$ *such that $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_2 \subseteq \mathcal{P}_1$. Then $\mathcal{P}_2 \subseteq \mathcal{P}_1$.*

Proof. The assertion follows from 7.2(c) applied to $\mathbb{T} = \mathcal{P}^{\circledast}$. ∎

For a subcategory $\mathbb{T}$ of C_X, we set

$$\begin{aligned} U_{\rtimes}(\mathbb{T}) &= \{\mathcal{P}_{\circledast} \in \mathbf{Spec}_{\rtimes}(X) \mid \mathbb{T} \cap \mathcal{P}_{\circledast} = 0\} \\ V_{\rtimes}(\mathbb{T}) &= \mathbf{Spec}_{\rtimes}(X) - U_{\rtimes}(\mathbb{T}) = \{\mathcal{P}_{\circledast} \in \mathbf{Spec}_{\rtimes}(X) \mid \mathbb{T} \cap \mathcal{P} \neq 0\}. \end{aligned} \tag{6}$$

7.4. Proposition. *Let $\mathbb{T}$ be a topologizing subcategory of C_X.*

(a) $V_{\rtimes}(\mathbb{T}) = V_{\rtimes}(\mathbb{T}^-)$ and $U_{\rtimes}(\mathbb{T}) = U_{\rtimes}(\mathbb{T}^-)$. The map

$$U_{\rtimes}(\mathbb{T}) \longrightarrow \mathbf{Spec}_{\rtimes}(X/\mathbb{T}^-) \tag{7}$$

which assigns to an element $\mathcal{P}$ of $U_{\rtimes}(\mathbb{T})$ the strictly full subcategory of $C_{X/\mathbb{T}^-}$ generated by the image of $\mathcal{P}$, is an isomorphism.

(b) For every $\mathcal{P} \in V_{\rtimes}(\mathbb{T})$, the intersection $\mathcal{P} \cap \mathbb{T}$ is an element of $\mathbf{Spec}_{\rtimes}(|\mathbb{T}|)$, and the map

$$V_{\rtimes}(\mathbb{T}) \longrightarrow \mathbf{Spec}_{\rtimes}(|\mathbb{T}|), \quad \mathcal{P} \longmapsto \mathcal{P} \cap \mathbb{T}, \tag{8}$$

is an isomorphism.

Proof. The assertion follows from 7.2 and 6.4. Details are left to the reader. ■

7.5. Proposition. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be elements of $\mathbf{Spec}^-(X)$. Then either $\mathcal{P}_2 \subseteq \mathcal{P}_1$, or $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp} \cap \mathcal{P}_2 \neq 0$.*

Proof. Replacing X by $X/(\mathcal{P}_1 \cap \mathcal{P}_2)$, we assume that $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$. If $\mathcal{P}_1$ is nonzero, then (by 7.2(c)) $|\mathcal{P}_1|$ is local. Let M_1 be a quasi-final object of $\mathcal{P}_1$. The argument of 7.2(c) shows that $\mathcal{P}_2 = \langle M_1 \rangle$. If $\mathcal{P}_2$ is nonzero, then, by the same reason, $|\mathcal{P}_2|$ is local and $\mathcal{P}_1 = \langle M_2 \rangle$ for a quasi-final object M_2 of $\mathcal{P}_2$. Therefore, $\mathcal{P}_1^{\mathfrak{t}} = [[M_2], \langle M_2 \rangle] = [[M_2], \mathcal{P}_1]$ is the smallest topologizing subcategory $\mathcal{P}^{\mathfrak{t}}$ properly containing $\mathcal{P}_1$. By 1.3.3, $\mathcal{P}_1^{\mathfrak{t}} \cap \mathcal{P}_1^{\perp}$ is contained in $\mathcal{P}_2$. ■

8. Spectra related with localizations and their canonical decompositions.

Spectra discussed in this section were first introduced in [R5]. Here we follow the general pattern dictated by [R6] (Section 2) and similar to those of the previous sections.

8.1. The thick spectra. Fix an abelian category C_X. The *thick spectra*, $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$, of the 'space' X are the spectra of the preorder $(\mathfrak{Th}(X), \subseteq)$, i.e. they are defined by

$$\mathbf{Spec}^i_{\mathfrak{Th}}(X) = \mathfrak{Spec}^i(\mathfrak{Th}(X)), \quad i = 0, 1,$$

and there is a canonical morphism

$$\mathbf{Spec}^0_{\mathfrak{Th}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{Th}}(X). \tag{1}$$

For any subcategory $\mathcal{B}$ of C_X, let $\mathcal{B}^{\star}$ denote the intersection of all thick subcategories of C_X containing properly $\mathcal{B}$, and let $\widehat{\mathcal{B}}$ denote the union of all thick subcategories which do not contain $\mathcal{B}$. The spectra $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$ and the morphism (1) can be described as follows.

Objects of $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ are thick subcategories $\mathbb{T}$ of C_X such that $\mathbb{T}^{\star} \neq \mathbb{T}$.

The spectrum $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$ is formed by all thick subcategories $\mathcal{Q}$ such that the union $\widehat{\mathcal{Q}}$ of all thick subcategories of C_X which do not contain $\mathcal{Q}$ is a thick subcategory.

The canonical morphism $\mathbf{Spec}^0_{\mathfrak{Th}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{Th}}(X)$ maps every object $\mathcal{Q}$ of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$ to the corresponding thick subcategory $\widehat{\mathcal{Q}}$.

8.2. Proposition. *(a) If $\mathcal{T}_1,\ \mathcal{T}_2$ are thick subcategories, then $\mathcal{T}_1 \subseteq \mathcal{T}_2$ iff $\widehat{\mathcal{T}_1} \subseteq \widehat{\mathcal{T}_2}$*
(b) The canonical map

$$\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X), \quad \mathcal{Q} \longmapsto \widehat{\mathcal{Q}}, \tag{4}$$

is injective. In particular, the induced maps

$$\mathbf{Spec}^{0,1}_{\mathfrak{T}\mathfrak{h}}(X) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}}(X) \quad \textit{and} \quad \mathbf{Spec}^{0,0}_{\mathfrak{T}\mathfrak{h}}(X) \longrightarrow \mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}}(X). \tag{5}$$

are injective.

Proof. (a) The implication $\mathcal{T}_1 \subseteq \mathcal{T}_2 \Rightarrow \widehat{\mathcal{T}_1} \subseteq \widehat{\mathcal{T}_2}$ holds for any pair subcategories $\mathcal{T}_1,\ \mathcal{T}_2$.

If the category $\mathcal{T}_2$ is thick and $\mathcal{T}_1 \nsubseteq \mathcal{T}_2$, then $\mathcal{T}_2 \subseteq \widehat{\mathcal{T}_1}$. In particular, $\widehat{\mathcal{T}_1} \nsubseteq \widehat{\mathcal{T}_2}$, whence the assertion (a).

(b) It follows from (a) that the map $\mathcal{T} \longmapsto \widehat{\mathcal{T}}$ induces a monomorphism of preorders $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$. ∎

8.3. The canonical decompositions of the thick spectra. We represent the thick spectrum $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$ as the disjoint union of $\mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}}(X)$ and $\mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}}(X)$, where

$$\begin{aligned} \mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}}(X) =& \{\mathcal{T} \in \mathfrak{T}\mathfrak{h}(X) \mid \mathcal{T}^\star \cap \mathcal{T}^\perp \neq 0\} \\ \mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}}(X) =& \{\mathcal{T} \in \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X) \mid \mathcal{T}^\star \cap \mathcal{T}^\perp = 0\}. \end{aligned} \tag{1}$$

8.3.1. Proposition. *A thick subcategory $\mathcal{T}$ from $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}}(X)$ iff it is a Serre subcategory. Thus,*

$$\begin{aligned} \mathbf{Spec}^{1,1}_{\mathfrak{T}\mathfrak{h}}(X) =& \{\mathcal{P} \in \mathfrak{T}\mathfrak{h}(X) \mid \mathcal{P}^- = \mathcal{P} \subsetneqq \mathcal{P}^\star\} \\ \mathbf{Spec}^{1,0}_{\mathfrak{T}\mathfrak{h}}(X) =& \{\mathcal{P} \in \mathfrak{T}\mathfrak{h}(X) \mid \mathcal{P} \subsetneqq \mathcal{P}^\star \subseteq \mathcal{P}^-\}. \end{aligned} \tag{2}$$

Proof. The argument is similar to that of 3.2(i). ∎

8.3.2. The decomposition of $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X)$. The decomposition (1) induces, via the map $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$, a decomposition

$$\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X) = \mathbf{Spec}^{0,1}_{\mathfrak{T}\mathfrak{h}}(X) \coprod \mathbf{Spec}^{0,0}_{\mathfrak{T}\mathfrak{h}}(X). \tag{3}$$

8.4. The spectra $\mathbf{Spec}^0_{\mathfrak{s}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{s}}(X)$. Applying the general construction to the inclusion functor $\mathfrak{Se}(X) \xrightarrow{\mathcal{J}^{\mathfrak{s}}_X} \mathfrak{T}\mathfrak{h}(X)$, we obtain the spectra $\mathfrak{Spec}^i(\mathfrak{Se}(X), \mathcal{J}^{\mathfrak{s}}_X)$, $i = 0, 1$, together with the canonical monomorphism

$$\mathfrak{Spec}^0(\mathfrak{Se}(X), \mathcal{J}^{\mathfrak{s}}_X) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X), \mathcal{J}^{\mathfrak{s}}_X).$$

We shall write $\mathbf{Spec}^i_{\mathfrak{s}}(X)$ instead of $\mathfrak{Spec}^i(\mathfrak{Se}(X), \mathcal{J}^{\mathfrak{s}}_X)$. Consider the canonical commutative diagram

$$\begin{array}{ccc}
\mathbf{Spec}^0_{\mathfrak{s}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{s}}(X) \\
\big\downarrow & & \big\downarrow \\
\mathbf{Spec}^0_{\mathfrak{Th}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{Th}}(X)
\end{array} \qquad (6)$$

8.4.1. Proposition. *The diagram (6) induces the commutative diagram of isomorphisms*

$$\begin{array}{ccc}
\mathbf{Spec}^0_{\mathfrak{s}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{s}}(X) \\
\wr\big\downarrow & & \big\downarrow\wr \\
\mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X) & \longrightarrow & \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X)
\end{array} \qquad (7)$$

whose vertical arrows are isomorphisms.

Proof. It follows from the general definitions that $\mathbf{Spec}^1_{\mathfrak{s}}(X)$ is naturally identified with the preorder of all Serre subcategories $\mathcal{P}$ of C_X such that $\mathcal{P} \neq \mathcal{P}^\star$. By 8.3.1, this preorder coincides with $\mathbf{Spec}^{1,1}_{\mathfrak{Th}}$. The rest is a consequence of the definition of $\mathbf{Spec}^0_{\mathfrak{s}}(X)$. Details are left to the reader. ∎

Our next objective is to describe the component $\mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X) = \mathbf{Spec}^0_{\mathfrak{s}}(X)$ of the spectrum $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$.

8.4.2. $Spec_{\mathfrak{s}}(X)$**.** We denote by $Spec_{\mathfrak{s}}(X)$ the class of all nonzero objects L of C_X such that $[L_1]_\bullet = [L]_\bullet$ for any nonzero subobject L_1 of L. Equivalently, $Spec_{\mathfrak{s}}(X)$ consists of objects L such that if there exists a nonzero morphism $N \longrightarrow L$, then $L \in Ob[M]_\bullet$. We consider $Spec_{\mathfrak{s}}(X)$ together with the preorder $\succcurlyeq$ defined by $L \succcurlyeq M$ iff $[M]_\bullet \subseteq [L]_\bullet$.

8.4.3. Proposition. *The spectrum* $\mathbf{Spec}^0_{\mathfrak{s}}(X)$ *coincides with* $\{[L]_\bullet \mid L \in Spec_{\mathfrak{s}}(X)\}$.

Proof. (a) If a nonzero object L is such that $[L_1]_\bullet = [L]_\bullet$ for any nonzero subobject L_1 of L, then $[L]_\bullet \in \mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X)$, i.e. $\widehat{L} \in \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X)$.

(a1) In fact, if $\mathbb{T}$ is a thick subcategory, then $L \notin Ob\mathbb{T}$ iff L belongs to $\mathbb{T}^\perp$. Because if there is a nonzero arrow $N \xrightarrow{f} L$ with $N \in Ob\mathbb{T}$, then $\mathbb{T} \supseteq [N]_\bullet \supseteq [im(f)]_\bullet = [L]_\bullet$. This shows that L is right orthogonal to $\widehat{L}$.

(a2) If $L \in Ob\widehat{L}^-$, then, since $L \neq 0$, it has a nonzero subobject, M, which belongs to $\widehat{L}$. But, this is impossible, because, according to (b1), L is right orthogonal to $\widehat{L}$. Thus, $L \notin Ob\widehat{L}^-$ which means that $\widehat{L} = \widehat{L}^-$, i.e. $\widehat{L}$ is a Serre subcategory.

(b) Notice that $\mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X)$ consists precisely of all $\mathcal{T} \in \mathbf{Spec}^0_{\mathfrak{Th}}(X)$ such that the intersection $\mathcal{T} \cap \widehat{\mathcal{T}}^\perp$ is nonzero.

In fact, if $\widehat{\mathcal{T}}$ is a Serre subcategory, then every nonzero object of $\mathcal{T} - \widehat{\mathcal{T}}$ has a nonzero $\widehat{\mathcal{T}}$-torsion free subquotient. The latter is an object of $\mathcal{T} \cap \widehat{\mathcal{T}}^\perp$.

(c) If $\mathcal{T} \in \mathbf{Spec}^0_{\mathfrak{Th}}(X)$, then $\mathcal{T} = [L]_\bullet$ for every object L of $\mathcal{T} - \widehat{\mathcal{T}}$.

Since $L \notin Ob\widehat{\mathcal{T}}$ and $\widehat{\mathcal{T}}$ is a thick subcategory, we have the inclusion $\widehat{\mathcal{T}} \subseteq \widehat{[L]_\bullet} = \widehat{L}$. The inverse inclusion follows from the inclusion $[L]_\bullet \subseteq \mathcal{T}$ and 8.2(a).

(d) If L is a nonzero object of $\mathcal{T} \cap \widehat{\mathcal{T}}^{\perp}$, then every nonzero object, L_1, of L belongs to $\mathcal{T} \cap \widehat{\mathcal{T}}^{\perp}$. By (c), $[L]_{\bullet} = \mathcal{T} = [L_1]_{\bullet}$. This shows that L belongs to $Spec_{\mathfrak{s}}(X)$, hence, by (a), $\mathcal{T} = [L]_{\bullet}$ is an element of $\mathbf{Spec}^0_{\mathfrak{s}}(X)$. ∎

8.5. The spectrum $\mathbf{Spec}^{\mathfrak{s}}_{\rtimes}(X)$. Starting with the functor

$$\mathfrak{T}_{\leq}(X) \xrightarrow{\mathfrak{L}_{\rtimes}} \mathfrak{Se}(X), \quad \mathcal{B} \longmapsto \mathcal{B}_{\rtimes},$$

(see 6.2) and the embedding $\mathbf{Spec}^1_{\mathfrak{s}}(X) \longrightarrow \mathfrak{Se}(X)$, we come, mimiking 6.2 and 6.3, to the preorder $\mathbf{Spec}^{\mathfrak{s}}_{\rtimes}(X)$ of all full nonzero subcategories $\mathcal{B}$ of the category C_X such that $\mathcal{B} = (\mathcal{B}_{\rtimes})^{\perp} \bigcap (\mathcal{B}_{\rtimes})^{\star}$ and $\mathcal{B}_{\rtimes} \in \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X)$.

8.5.1. Proposition. *The map $\mathcal{B} \longmapsto \mathcal{B}_{\rtimes}$ induces an isomorphism*

$$\mathbf{Spec}^{\mathfrak{s}}_{\rtimes}(X) \xrightarrow{\ \sim\ } \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X) \tag{8}$$

with the inverse map $\mathcal{P} \longmapsto \mathcal{P}_{\star} = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp}$.

Proof. The argument is similar to the proof of 6.4. Details are left to the reader. ∎

8.6. Connections with the other spectra. The inclusion maps

$$\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \hookrightarrow \mathbf{Spec}^{-}(X) \hookrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X)$$

are a part of the commutative diagram

$$\begin{tikzcd}
\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \arrow[r] & \mathbf{Spec}^{-}(X) \arrow[r] & \mathbf{Spec}^{1,1}_{\mathfrak{Th}}(X) & \mathbf{Spec}^{1}_{\mathfrak{s}}(X) \arrow[l, "\sim"'] \\
\mathbf{Spec}(X) \arrow[u, "\wr"] \arrow[r, "\alpha"] & \mathbf{Spec}_{\rtimes}(X) \arrow[u, "\wr"] \arrow[r, "\beta"] & \mathbf{Spec}^{\mathfrak{s}}_{\rtimes}(X) \arrow[u, "\wr"'] & \mathbf{Spec}^{0}_{\mathfrak{s}}(X) \arrow[l] \arrow[u]
\end{tikzcd} \tag{9}$$

in which four arrows are isomorphisms and the remaining arrows are injective. Here the map $\mathbf{Spec}(X) \xrightarrow{\alpha} \mathbf{Spec}_{\rtimes}(X)$ assigns to each element $\mathcal{P}$ of $\mathbf{Spec}(X)$ the intersection $(\langle\mathcal{P}\rangle \bullet \mathcal{P} \bullet \langle\mathcal{P}\rangle) \cap \langle\mathcal{P}\rangle^{\perp}$, and $\mathbf{Spec}_{\rtimes}(X) \xrightarrow{\beta} \mathbf{Spec}^{0,1}_{\mathfrak{Th}}(X)$ transforms any element $\mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$ of $\mathbf{Spec}_{\rtimes}(X)$ (– the image of $\mathcal{P} \in \mathbf{Spec}^{-}(X)$) into the subcategory $\mathcal{P}^{\star} \cap \mathcal{P}^{\perp}$. The vertical isomorphisms are defined by resp.

$$\mathcal{P} \longmapsto [\mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp}], \quad \mathcal{P} \longmapsto \mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp} \quad \text{and} \quad \mathcal{P} \longmapsto \mathcal{P}_{\star} = \mathcal{P}^{\star} \cap \mathcal{P}^{\perp}.$$

8.7. Spectra defined by the preorder of Serre subcategories.

Let $\mathfrak{H}$ be the preorder $\mathfrak{Se}(X)$ of all Serre subcategories of the category C_X. Thus, we have two spectra and an embedding:

$$\mathfrak{Spec}^0(\mathfrak{Se}(X)) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X)).$$

We change the notations setting $\mathbf{Spec}^i_{\mathfrak{Se}}(X) = \mathfrak{Spec}^i(\mathfrak{Se}(X)),\ i = 0,1.$

It follows from the definitions that objects of $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$ are all Serre subcategories $\mathcal{P}$ such that the intersection $\mathcal{P}^{\mathfrak{s}}$ of all Serre subcategories of C_X properly containing $\mathcal{P}$ does not coincide with $\mathcal{P}$. The spectrum $\mathbf{Spec}^0_{\mathfrak{Se}}(X)$ is formed by all Serre subcategories $\mathcal{Q}$ such that the union $\langle\mathcal{Q}\rangle_{\mathfrak{s}}$ of all Serre subcategories of C_X which do not contain $\mathcal{Q}$ is a Serre subcategory: $\langle\mathcal{Q}\rangle_{\mathfrak{s}} = \langle\mathcal{Q}\rangle^-_{\mathfrak{s}}$.

8.7.1. Proposition. *There are natural injective morphisms of preorders*

$$\mathbf{Spec}^i_{\mathfrak{s}}(X) \longrightarrow \mathbf{Spec}^i_{\mathfrak{Se}}(X), \quad i = 0,1,$$

such that the diagram

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{s}}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{Se}}(X) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Spec}^1_{\mathfrak{s}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{Se}}(X) \end{array} \tag{10}$$

commutes.

Proof. The morphism $\mathbf{Spec}^1_{\mathfrak{s}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{Se}}(X)$ is the inclusion. The morphism $\mathbf{Spec}^0_{\mathfrak{s}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{Se}}(X)$ assigns to each object $\mathcal{Q}$ of $\mathbf{Spec}^0_{\mathfrak{s}}(X)$ the associated Serre subcategory $\mathcal{Q}^-$. ∎

The diagram (10) can be combined with the diagram (9) above. In particular, we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{s}}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{Se}}(X) \\ \wr\big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{s}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{Se}}(X) \end{array} \tag{11}$$

whose lower horizontal arrows are embeddings. The diagram (11) can be, in turn, extended to a commutative diagram including other spectra of this work. This exercise is left to the reader. If C_X is a category with Gabriel-Krull dimension, then it follows from (the argument of) [R5, 8.7.1] that $\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{s}}(X) = \mathbf{Spec}^1_{\mathfrak{Se}}(X)$.

II. Local properties of spectra.

9. Spectra and (co)covers. Local properties of the spectra with respect to finite (co)covers.

Fix an abelian category C_X. We call a set $\{\mathcal{T}_i \mid i \in J\}$ of thick subcategories of C_X a *cocover of* X, or a *thick cocover of* X if $\bigcap\limits_{i\in J} \mathcal{T}_i = 0$. The latter condition means that the corresponding family of exact localizations $\{C_X \xrightarrow{u_i^*} C_X/\mathcal{T}_i \mid i \in J\}$ is conservative (i.e. it reflects isomorphisms), i.e. the morphisms $\{X/\mathcal{T}_i \xrightarrow{u_i} X \mid i \in J\}$ form a cover of X.

9.1. Proposition. *Let a set $\{\mathcal{T}_i \mid i \in J\}$ of thick subcategories of an abelian category C_X be a cocover, i.e. $\bigcap\limits_{i\in J} \mathcal{T}_i = 0$. Then for every $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, there exists an $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.*

Proof. Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, and let $\mathcal{Q}$ be the image of $\mathcal{P}$ in $\mathbf{Spec}(X)$; that is $\mathcal{P} = \widehat{\mathcal{Q}}$. It follows that $\mathcal{T}_i \nsubseteq \mathcal{P} = \widehat{\mathcal{Q}}$ iff $\mathcal{Q} \subseteq \mathcal{T}_i$. Since $\bigcap_{i \in J} \mathcal{T}_i = 0$, there are $i \in J$ such that $\mathcal{Q} \nsubseteq \mathcal{T}_i$, or, equivalently, $\mathcal{T}_i \subseteq \widehat{\mathcal{Q}} = \mathcal{P}$. ■

9.2. Corollary. *For any thick cocover $\{\mathcal{T}_i \mid i \in J\}$ of X, there is an embedding $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \hookrightarrow \bigcup_{i \in J} \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$. Here $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ and $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ are identified with their images in $\mathbf{Spec}^{1}_{\mathfrak{Th}}(X)$.*

Proof. It follows from 9.1 that $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) = \bigcup_{i \in J} U^{1,1}_{\mathfrak{t}}(\mathcal{T}_i)$, where $U^{1,1}_{\mathfrak{t}}(\mathcal{T}_i) = \{\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \mid \mathcal{T}_i \subseteq \mathcal{P}\}$. By A2.2.1.1, for each $i \in J$, there is a natural embedding $U^{1,1}_{\mathfrak{t}}(\mathcal{T}_i) \longrightarrow \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$, hence the assertion. ■

9.3. Proposition. *Let a set $\{\mathcal{T}_i \mid i \in J\}$ of thick subcategories of an abelian category C_X be a cocover. Then for every $\mathcal{Q} \in \mathbf{Spec}^{0}_{\mathfrak{Th}}(X)$, there exists $i \in J$ such that $\mathcal{Q} \nsubseteq \mathcal{T}_i$.*

If J is finite, then for every $\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{Th}}(X)$, there exists $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

Proof. The first assertion is true by a trivial reason: elements of $\mathbf{Spec}^{0}_{\mathfrak{Th}}(X)$ are nonzero subcategories and $\bigcap_{i \in J} \mathcal{T}_i = 0$.

Let $\mathcal{P} \in \mathbf{Spec}^{1}_{\mathfrak{Th}}(X)$. Notice that $\mathcal{T}_i \nsubseteq \mathcal{P}$ iff the thick subcategory $\mathcal{T}_i \sqcup \mathcal{P}$ spanned by $\mathcal{T}_i$ and $\mathcal{P}$ contains $\mathcal{P}$ properly, hence it contains $\mathcal{P}^\star$. Therefore, if $\mathcal{T}_i \nsubseteq \mathcal{P}$ for all $i \in J$, then $\mathcal{P}^\star \subseteq \bigcap_{i \in J} (\mathcal{T}_i \sqcup \mathcal{P})$. Since J is finite, it follows from A1.4.1 that

$$\mathcal{P}^\star \subseteq \bigcap_{i \in J} (\mathcal{T}_i \sqcup \mathcal{P}) = \Big(\bigcap_{i \in J} \mathcal{T}_i\Big) \sqcup \mathcal{P} = 0 \sqcup \mathcal{P} = \mathcal{P},$$

which is impossible because $\mathcal{P}^\star \neq \mathcal{P}$. ■

9.4. Corollary. *If $\{\mathcal{T}_i \mid i \in J\}$ is a finite thick cocover of X, then*

$$\mathbf{Spec}^{1}_{\mathfrak{Th}}(X) = \bigcup_{i \in J} \mathbf{Spec}^{1}_{\mathfrak{Th}}(X/\mathcal{T}_i). \qquad (1)$$

Here $\mathbf{Spec}^{1}_{\mathfrak{Th}}(X/\mathcal{T}_i)$ is identified with its image in $\mathbf{Spec}^{1}_{\mathfrak{Th}}(X)$.

Proof. The fact follows from 9.3. ■

9.5. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite thick cocover of X. Then*

$$\mathbf{Spec}^{1}(X) = \bigcup_{i \in J} \mathbf{Spec}^{1}(X/\mathcal{T}_i). \qquad (2)$$

If C_X has the property (sup), then

$$\begin{aligned} \mathbf{Spec}^-(X) &= \bigcup_{i\in J} \mathbf{Spec}^-(X/\mathcal{T}_i^-), \\ \mathbf{Spec}^1_{\mathfrak{Se}}(X) &= \bigcup_{i\in J} \mathbf{Spec}^1_{\mathfrak{Se}}(X/\mathcal{T}_i^-). \end{aligned} \tag{3}$$

Proof. (a) The equality (2) follows from 9.4, because $\mathbf{Spec}^1(X) \subseteq \mathbf{Spec}^1_{\mathfrak{Th}}(X)$.

(b) By A1.4.1, the equality $\bigcap_{i\in J} \mathcal{T}_i = 0$ implies that $\bigcap_{i\in J} \mathcal{T}_i^- = 0$, i.e. the set $\{\mathcal{T}_i^- \mid i \in J\}$ of Serre subcategories form a cocover. Therefore, the inclusions

$$\mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \bigcup_{i\in J} \mathbf{Spec}^1_{\mathfrak{Se}}(X/\mathcal{T}_i^-) \quad \textit{and} \quad \mathbf{Spec}^1(X) \subseteq \bigcup_{i\in J} \mathbf{Spec}^1(X/\mathcal{T}_i)$$

follows from 9.3 and the inclusions $\mathbf{Spec}^1(X) \subseteq \mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{Th}}(X)$.

(b1) If $\mathcal{T}$ is a coreflective thick subcategory of C_X, then a thick subcategory $\mathcal{S}$ containing $\mathcal{T}$ is a Serre subcategory iff $\mathcal{S}/\mathcal{T}$ is a Serre subcategory.

Let u^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}$. Let M be an object of $\mathcal{S}^- - \mathcal{T}$, and $\widetilde{L}$ a nonzero subquotient of the image $u*(M)$ of M in $C_X/\mathcal{T}^-$. By a standard argument, $\widetilde{L}$ is isomorphic to $u*(L)$, where L is a subquotient of M. Since $\mathcal{T}^-$ is a coreflective subcategory (thanks to the property (sup)), we can choose L to be $\mathcal{T}^-$-torsion free. Since $M \in Ob\mathcal{S}^-$, the object L, being a nonzero subquotient of M, has a nonzero subobject L_1 from $\mathcal{S}$ which is $\mathcal{T}^-$-torsion free. Therefore, $u^*(L_1)$ is a nonzero subobject of $\widetilde{L}$ which belongs to $\mathcal{S}/\mathcal{T}^-$. This shows that $u_i^*(M)$ is an object of $(\mathcal{S}/\mathcal{T}^-)^-$. But, $(\mathcal{S}/\mathcal{T}^-)^- = \mathcal{S}/\mathcal{T}^-$ by hypothesis, hence M belongs to $\mathcal{S}$.

(b2) Since C_X has the property (sup), every Serre subcategory of C_X is coreflective, in particular, $\mathcal{T}_i^-$, $i \in J$, are coreflective subcategories. The assertion follows now from (b1) above: $\mathbf{Spec}^1_{\mathfrak{Se}}(X/\mathcal{T}_i^-)$ realized as a subset of $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ consists of Serre subcategories, hence is contained in $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$. Similarly with $\mathbf{Spec}^-(X/\mathcal{T}_i^-)$ realized as a subset of $\mathbf{Spec}^1(X)$. ∎

9.6. Local property of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.

9.6.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of thick subcategories of the category C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$. The following conditions on a thick subcategory $\mathcal{P}$ of C_X are equivalent:*

(a) $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$,

(b) $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{Se}}(X)$ and $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

If the category C_X has the property (sup), then the conditions (a) and (b) are equivalent to the condition

(c) $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{Th}}(X)$ and $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

Proof. $(a) \Rightarrow (b)$. Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, and let $\mathcal{T}$ be a thick subcategory of C_X contained in $\mathcal{P}$. Then $(\mathcal{T} \bullet \mathcal{P}^{\mathfrak{t}} \bullet \mathcal{T})/\mathcal{T}$ is the smallest topologizing subcategory of $C_X/\mathcal{T}$ properly containing $\mathcal{P}/\mathcal{T}$, and the localization functor $C_X \xrightarrow{u^*} C_X/\mathcal{T}$ maps nonzero objects of $\mathcal{P}_{\mathfrak{t}} = \mathcal{P}^{\mathfrak{t}} \cap \mathcal{P}^{\perp}$ to nonzero objects of $(\mathcal{P}/\mathcal{T})_{\mathfrak{t}} = (\mathcal{P}/\mathcal{T})^{\mathfrak{t}} \bigcap (\mathcal{P}/\mathcal{T})^{\perp}$.

$(b) \Rightarrow (a)$. Let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. Set $J_{\mathcal{P}} = \{j \in J \mid \mathcal{T}_j \subseteq \mathcal{P}\}$. For every $i \in J_{\mathcal{P}}$, we denote by $\widetilde{\mathcal{Q}}_i$ the intersection $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp}$ and by $\mathcal{Q}_i$ the topologizing subcategory $[\widetilde{\mathcal{Q}}_i]$ spanned by $\widetilde{\mathcal{Q}}_i$. By assumption, $\widetilde{\mathcal{Q}}_i \neq 0$ for each $i \in J_{\mathcal{P}}$, hence $\mathcal{Q}_i \not\subseteq \mathcal{P}$. The latter implies that, for every $j \in J_{\mathcal{P}}$, the topologizing subcategory $[u_j^*(\mathcal{Q}_i \bullet \mathcal{P})]$ contains $(\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}$, or, equivalently, $u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i \bullet \mathcal{P}$. Therefore,

$$\widetilde{\mathcal{Q}}_j = u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp} \subseteq (\mathcal{T}_j \bullet \mathcal{Q}_i \bullet \mathcal{P}) \bigcap \mathcal{P}^{\perp} = (\mathcal{T}_j \bullet \mathcal{Q}_i) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i$$

which implies the inclusion $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i$ for every $(i,j) \in J_{\mathcal{P}} \times J_{\mathcal{P}}$, hence

$$\mathcal{Q}_j \subseteq \bigcap_{i \in J_{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{Q}_i) = \mathcal{T}_j \bullet \big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{Q}_i \big).$$

Here the equality is due to the finiteness of $J_{\mathcal{P}}$.

It follows from the inclusion $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet \big(\bigcap\limits_{i \in J_{\mathcal{P}}} \mathcal{Q}_i \big)$ that $\bigcap\limits_{i \in J_{\mathcal{P}}} \mathcal{Q}_i \neq 0$, because otherwise $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet 0 = \mathcal{T}_i$, which is impossible, since $\mathcal{T}_j \subseteq \mathcal{P}$ and $\mathcal{Q}_j \not\subseteq \mathcal{P}$.

There are two cases: $J = J_{\mathcal{P}}$ and $J \neq J_{\mathcal{P}}$. Consider each of them.

(i) Let $J_{\mathcal{P}} = J$. We set $\mathcal{Q} = \bigcap\limits_{i \in J_{\mathcal{P}}} \mathcal{Q}_i$ and claim that $\mathcal{Q}$ is an element of $\mathbf{Spec}(X)$ corresponding to $\mathcal{P}$, that is $\mathcal{P} = \langle \mathcal{Q} \rangle$.

In fact, let $\mathcal{S}$ is a topologizing subcategory of C_X which is not contained in $\mathcal{P}$. Then $\mathcal{P}$ is properly contained in $\mathcal{S} \bullet \mathcal{P}$ and, therefore, $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \subseteq \mathcal{T}_i \bullet \mathcal{S} \bullet \mathcal{P}$ for each $i \in J$. This implies that $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet \mathcal{S} \bullet \mathcal{P} \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet \mathcal{S}$. Therefore,

$$\widetilde{\mathcal{Q}} = \bigcap_{i \in J} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp} \subseteq \bigcap_{i \in J} (\mathcal{T}_i \bullet \mathcal{S}) = \big(\bigcap_{i \in J} \mathcal{T}_i \big) \bullet \mathcal{S} = 0 \bullet \mathcal{S} = \mathcal{S},$$

which implies that $\mathcal{Q} = [\widetilde{\mathcal{Q}}] \subseteq \mathcal{S}$.

(ii) Consider now the second case: $J_{\mathcal{P}} \neq J$, i.e. $J^{\mathcal{P}} = J - J_{\mathcal{P}}$ is non-empty. This case can be reduced to the first case as follows.

1) Set $C_{\mathcal{V}_{\mathcal{P}}} = \bigcap\limits_{j \in J^{\mathcal{P}}} \mathcal{T}_j$. Notice that $C_{\mathcal{V}_{\mathcal{P}}} \not\subseteq \mathcal{P}$.

In fact, if $i \in J^{\mathcal{P}} = J - J_{\mathcal{P}}$, then $\mathcal{T}_i \not\subseteq \mathcal{P}$. Therefore, for every $j \in J_{\mathcal{P}}$, the topologizing subcategory $[u_j^*(\mathcal{T}_i \bullet \mathcal{P})]$ contains $(\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}$, or, equivalently, $u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}) \subseteq \mathcal{T}_j \bullet \mathcal{T}_i \bullet \mathcal{P}$, which implies that $\widetilde{\mathcal{Q}}_j = u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{t}}) \bigcap \mathcal{P}^{\perp} \subseteq (\mathcal{T}_j \bullet \mathcal{T}_i) \bigcap \mathcal{P}^{\perp}$. Thanks to the finiteness of $J^{\mathcal{P}}$, we obtain:

$$\widetilde{\mathcal{Q}}_j \subseteq \big(\bigcap_{i \in J^{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{T}_i) \big) \bigcap \mathcal{P}^{\perp} = \big(\mathcal{T}_j \bullet \big(\bigcap_{i \in J^{\mathcal{P}}} \mathcal{T}_i \big) \big) \bigcap \mathcal{P}^{\perp}. \tag{4}$$

The inclusion $\bigcap_{i\in J^{\mathcal{P}}} \mathcal{T}_i \subseteq \mathcal{P}$ implies (together with (4)) that $\widetilde{\mathcal{Q}}_j \subseteq \mathcal{T}_j \subseteq \mathcal{P}$, which is impossible. So that $\bigcap_{i\in J^{\mathcal{P}}} \mathcal{T}_i \nsubseteq \mathcal{P}$.

2) Since $C_{\mathcal{V}_{\mathcal{P}}} \nsubseteq \mathcal{P}$, the intersection $\mathcal{P}_0 = C_{\mathcal{V}_{\mathcal{P}}} \bigcap \mathcal{P}$ is an element of $\mathbf{Spec}^1_{\mathfrak{Se}}(\mathcal{V}_{\mathcal{P}})$. Notice that $\{\mathcal{T}_i \cap C_{\mathcal{V}_{\mathcal{P}}} = \widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$ is a cocover of $\mathcal{V}_{\mathcal{P}}$, i.e. $\bigcap_{j\in J_{\mathcal{P}}} \widetilde{\mathcal{T}}_j = 0$. It remains to notice that $\mathcal{P}_0/\widetilde{\mathcal{T}}_j \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathcal{V}_{\mathcal{P}}/\widetilde{\mathcal{T}}_j)$ for each $j \in J_{\mathcal{P}}$.

In fact, the localization functor $C_X \longrightarrow C_X/\mathcal{T}_j$ induces an equivalence of $C_{\mathcal{V}_{\mathcal{P}}}/\widetilde{\mathcal{T}}_j$ and the topologizing subcategory $(\mathcal{T}_j \bullet C_{\mathcal{V}_{\mathcal{P}}} \bullet \mathcal{T}_j)/\mathcal{T}_j$ of $C_X/\mathcal{T}_j$. The subcategory $\mathcal{P}_0/\widetilde{\mathcal{T}}_j$ of $C_{\mathcal{V}_{\mathcal{P}}}/\widetilde{\mathcal{T}}_j$ is the preimage of the intersection of the $\mathcal{P}/\mathcal{T}_j \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_j)$ with the topologizing subcategory $(\mathcal{T}_j \bullet C_{\mathcal{V}_{\mathcal{P}}} \bullet \mathcal{T}_j)/\mathcal{T}_j$, hence it belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathcal{V}_{\mathcal{P}}/\widetilde{\mathcal{T}}_j)$.

3) Thus, the 'space' $\mathcal{V}_{\mathcal{P}}$, the cocover $\{\widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$, and the point $\mathcal{P}_0 = \mathcal{P} \bigcap C_{\mathcal{V}_{\mathcal{P}}}$ of the spectrum $\mathbf{Spec}^1_{\mathfrak{Se}}(\mathcal{V}_{\mathcal{P}})$ satisfy the conditions (b) with all $\widetilde{\mathcal{T}}_i$ being subcategories of $\mathcal{P}_0$. By 2) above, $\mathcal{P}_0$ belongs to the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(\mathcal{V}_{\mathcal{P}})$, and $\mathcal{P}_0 = \langle \widetilde{\mathcal{Q}}_0 \rangle_{\mathcal{V}_{\mathcal{P}}} = \langle \mathcal{Q}_0 \rangle_{\mathcal{V}_{\mathcal{P}}}$, where $\mathcal{Q}_0$ is the smallest topologizing subcategory of ($C_{\mathcal{V}_{\mathcal{P}}}$, hence) C_X containing $\widetilde{\mathcal{Q}}_0$. Therefore, $\mathcal{Q}_0$ is a point of the spectrum $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ and $\langle \mathcal{Q}_0 \rangle_X = \mathcal{P}$.

Obviously, $(b) \Rightarrow (c)$ without additional conditions on the category C_X. Suppose now that C_X has the property (sup).

$(c) \Rightarrow (b)$. It follows from (c) that $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{s}}(X)$, i.e. $\mathcal{P}$ is a Serre subcategory.

In fact, by A1.4.1, the equality $\bigcap_{i\in J} \mathcal{T}_i = 0$ implies that $\bigcap_{i\in J} \mathcal{T}_i^- = 0$. In other words, $\{\mathcal{T}_i^- \mid i \in J\}$ is a finite cocover, which implies, by the local property of $\mathbf{Spec}^1_{\mathfrak{Th}}(X)$ that $\mathcal{T}_i^- \subseteq \mathcal{P}$ for some $i \in J$. Notice that if $\mathcal{T}_i^- \subseteq \mathcal{P}$, then $\mathcal{P}/\mathcal{T}_i^-$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i^-)$. This follows from the fact that $\mathcal{P}/\mathcal{T}_i$ is, by the condition (c), an element of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$, and the spectrum $\mathbf{Spec}^{1,1}_{\mathfrak{t}}$ is functorial with respect to localizations (see the proof of $(a) \Rightarrow (b)$ above), in particular, with respect to $C_X/\mathcal{T}_i \longrightarrow C_X/\mathcal{T}_i^-$.

Therefore, $\mathcal{P}/\mathcal{T}_i^-$ is a Serre subcategory of the quotient category $C_X/\mathcal{T}_i^-$. Thanks to the property (sup), the Serre subcategory $\mathcal{T}_i^-$ is coreflective. By the argument 9.5(b1), this implies that $\mathcal{P}$ is a Serre subcategory of C_X. ■

9.6.2. Note. Proposition 9.6.1 is a stronger statement than [R4, 6.3] in all respects. The equivalence $(a) \Leftrightarrow (b)$ is essentially the assertion of [R4, 6.3], but the argument presented here is valid for arbitrary abelian categories, while the proof of [R4, 6.3] used the property (sup). The equivalence (a) and (b) to (c) (when C_X has the property (sup)) is a new observation (which could of be made in [R4]).

9.7. $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ and $\mathbf{Spec}^-(X)$. Let $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a set of morphisms such that their inverse image functors $C_X \xrightarrow{u_i^*} C_{U_i}$ are exact localizations. We call $\mathfrak{U}$ a *cover* if the family $\{u_i^* \mid i \in J\}$ of inverse image functors is conservative, i.e. it reflects isomorphisms. We set

$$\mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U}) = \{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \text{if } Ker(u_i^*) \subseteq \mathcal{P}, \text{ then } \mathcal{P}/Ker(u_i^*) \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i)\}.$$

There is a canonical embedding $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^{1,1}_{\wp}(\mathfrak{U})$. By 9.6.1, if the cover $\mathfrak{U}$ is finite then this embedding is an isomorphism.

Recall that the *associated points of the object M in* $\mathbf{Spec}(X)$ are elements $\mathcal{Q}$ of $\mathbf{Spec}(X)$ such that M has a nonzero subobject from $\mathcal{Q} \cap \widehat{\mathcal{Q}}^{\perp}$.

9.7.1. Proposition. *Let C_X be an abelian category with property (sup). Let $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a finite cover of the 'space' X such that all morphisms $U_{ij} = U_i \cap U_j \xrightarrow{u_{ij}} U_i$ are continuous. Let $\mathcal{P}_i \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_i)$, and let L_i be an object of $Spec(U_i)$ such that $\mathcal{P}_i = \langle L_i \rangle$.*

The following conditions are equivalent:

(a) $\mathcal{P} = u_i^{*^{-1}}(\mathcal{P}_i)$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$; i.e. $\mathcal{P} = \langle M \rangle$ for some object M of $Spec(X)$;

(b) for any $j \in J$ such that $u_{ij}^*(L_i) \neq 0$, the object $u_{ji*}u_{ij}^*(L_i)$ of C_{U_j} has an associated point; i.e. it has a subobject L_j which belongs to $Spec(U_j)$;

(c) $\mathcal{P}/Ker(u_j^*) = \mathcal{P}_j$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(U_j)$ for all j such that $Ker(u_j^*) \subseteq \mathcal{P}$.

Proof. (a)$\Rightarrow$(c) follows from 3.2(ii) and the functoriality of $\mathbf{Spec}$ (hence $\mathbf{Spec}^{1,1}_{\mathfrak{t}}$) with respect to localizations.

(c)$\Rightarrow$(a) follows from 9.6.1.

(b)$\Rightarrow$(c). Suppose that $Ker(u_j^*) \subseteq \mathcal{P}$, or, equivalently, $u_{ij}^*(L_i) \neq 0$. Then $\mathcal{P}_j = \mathcal{P}/Ker(u_j^*)$ is a point of $\mathbf{Spec}^1(U_j)$. Let $U_i \xleftarrow{u_{ij}} U_i \cap U_j = U_{ij} \xrightarrow{u_{ji}} U_j$ be the canonical embeddings. Since $L_i \in Spec(U_i)$ and $u_{ij}^*(L_i) \neq 0$, it follows that $u_{ij}^*(L_i) \in Spec(U_{ij})$.

Let L_j be a nonzero subobject of $u_{ji*}u_{ij}^*(L_i)$, and $L_j \in Spec(U_j)$. Then $u_{ji}^*(L_j)$ is a nonzero subobject of $u_{ij}^*(L_i)$. Therefore, since $u_{ij}^*(L_i)$ belongs to $Spec(U_{ij})$, the objects $u_{ji}^*(L_j)$ and $u_{ij}^*(L_i)$ are equivalent. Notice that, it follows from $\mathcal{P}_i = \langle L_i \rangle$ that $\mathcal{P}_i/Ker(u_{ij}^*) = \langle u_{ij}^*(L_i) \rangle$. But, $\mathcal{P}_i/Ker(u_{ij}^*) = \mathcal{P}_j/Ker(u_{ji}^*) = \mathcal{P}/Ker(u_{ji}^*u_j^*)$ and, by the argument above, $\langle u_{ij}^*(L_i) \rangle = \langle u_{ji}^*(L_j) \rangle$. Together with the fact that L_j is an object of $Spec(U_j)$, this shows that $\mathcal{P}_j = \langle L_j \rangle$.

(a)$\Rightarrow$(b). Suppose that $\mathcal{P} = u_i^{*^{-1}}(\mathcal{P}_i)$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$; i.e. $\mathcal{P} = \langle M \rangle$ for some object M of $Spec(X)$. Let $\widetilde{L}_i$ be a $\mathcal{P}$-torsion free object of C_X such that $u_i^*(\widetilde{L}_i) \simeq L_i$. The relation $u_i^*(M) \succ L_i$ means that there exists a diagram $M^{\oplus n} \xleftarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}} L_1 \xrightarrow{g} \widetilde{L}_i$ in which $\mathfrak{e}$ is an epimorphism, the arrows $\mathfrak{j}$ and g are nonzero monomorphisms; in particular, $M \succ L_1$. Notice that $L_1 \succ M$, i.e. M and L_1 are equivalent. In fact, $u_i^*(L_1)$ is a nonzero subobject of L_i. Since the latter belongs to $Spec(U_i)$, they are equivalent. Therefore, $u_i^*(L_1)$ is equivalent to $u_i^*(M)$. The relation $u_i^*(L_1) \succ u_i^*(M)$ is expressed by a diagram $L_1^{\oplus m} \xleftarrow{\mathfrak{j}'} \widetilde{K} \xrightarrow{\mathfrak{e}'} M_1 \xrightarrow{h} M$ in which $\mathfrak{e}'$ is an epimorphism and $\mathfrak{j}'$ and h are nonzero monomorphisms. Since $M \in Spec(X)$, M_1 is equivalent to M, hence the relation $L_1 \succ M_1$ which is explicit in the diagram above, implies that $L_1 \succ M$. Thus $L_1 \in ObSpec(X)$.

By the functoriality of $Spec$ with respect to exact localizations, $u_j^*(L_1) = L_j$ belongs to $Spec(U_j)$. Since L_1 is $\mathcal{P}$-torsion free, the adjunction arrow $L_j = u_j^*(L_1) \longrightarrow u_{ji*}u_{ji}^*u_j^*(L_1)$ is a monomorphism. On the other hand,

$$u_{ji*}u_{ji}^*u_j^*(L_1) \simeq u_{ji*}u_{ij}^*u_i^*(L_1) \longrightarrow u_{ji*}u_{ij}^*u_i^*(\widetilde{L}_i) \simeq u_{ji*}u_{ij}^*(L_i), \tag{5}$$

where the arrow in the middle is the image of the monomorphism $L_1 \longrightarrow \widetilde{L}_i$. Since all functors in the diagram (5) are left exact, this arrow is a monomorphism. Altogether gives the desired monomorphism $L_j \longrightarrow u_{ji*}u_{ij}^*(L_i)$. ∎

9.7.2. Example. Let C_X be the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Let $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ be an affine cover and C_{U_i} the category of quasi-coherent sheaves on $(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$. Then all morphisms $U_i \cap U_j \longrightarrow U_i$ are continuous and the equivalent conditions (a), (b), (c) hold for every point $\mathcal{P}_i \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i)$. This reflects the fact that $\mathbf{Spec}(U_i)$ is naturally identified with $\mathcal{U}_i$ and is an open subset of the spectrum $\mathbf{Spec}(X) \simeq \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. It follows from 9.6.1 that $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \bigcup_{i\in J} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i)$. So, Proposition 9.7.1 becomes trivial in the case of commutative schemes. It is non-trivial and meaningful in the case of noncommutative schemes, even in the case of D-schemes.

9.7.3. Example: simple holonomic D-modules. Let C_X be the category of holonomic D-modules on a smooth quasi-compact scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Let $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ be an affine cover of $\mathbf{X}$, and let C_{U_i} be the category of holonomic D-modules on the affine subscheme $(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$. Then all morphisms $U_i \cap U_j \longrightarrow U_i$ are continuous and the equivalent conditions (a), (b), (c) hold for every simple object L_i of C_{U_i}. The latter is due to the fact that direct and inverse image functors of open immersions preserve holonomicity. Thanks to the fact that all holonomic D-modules are of finite length, the 'space' X (i.e. the category C_X) has the Gabriel-Krull dimension zero, hence elements of $\mathbf{Spec}(X)$ are in a bijective correspondence with isomorphism classes of holonomic simple objects. Therefore, it follows from 9.7.1 and 9.6.1 that $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \bigcup_{i\in J} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i)$. Thus, the problem of the description of simple holonomic modules on a smooth quasi-compact scheme is local: it can be reduced to the affine case.

Consider, for instance, the cover of the flag variety G/B of a reductive algebraic connected group G over $\mathbb{C}$ (or any other algebraicly closed field of zero characteristic) by translations U_w, $w \in W$, of the big Schubert cell (here, as usual, W denotes the Weyl group of G. Then for any $w \in W$, the category C_{U_w} is equivalent to the category $A_n - mod$ of left modules over the Weyl algebra A_n. So the problem of a classification of holonomic D-modules on G/B is reduced to the problem of classification of holonomic modules on the affine n-dimensional space $\mathbb{A}^n$, that is holonomic A_n-modules.

9.7.4. Proposition. *Let C_X have the property (sup), and let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of the category C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$. Suppose that $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i) = \mathbf{Spec}^-(X/\mathcal{T}_i)$ for all $i \in J$. Then $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) = \mathbf{Spec}^-(X)$; i.e. the map*

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^-(X), \quad \mathcal{P} \longmapsto \langle \mathcal{P} \rangle,$$

is an isomorphism.

Proof. By 9.6.1, $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ coincides with

$$\{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i) \text{ if } \mathcal{T}_i \subseteq \mathcal{P}\}.$$

By 9.5, $\mathbf{Spec}^-(X) = \bigcup_{i\in J} \mathbf{Spec}^-(X/\mathcal{T}_i)$. In particular,

$$\mathbf{Spec}^-(X) = \{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^-(X/\mathcal{T}_i) \text{ if } \mathcal{T}_i \subseteq \mathcal{P}\}.$$

Hence the assertion. ∎

9.7.5. Example. Let C_X be the category of quasi-coherent sheaves on a smooth quasi-compact scheme $\mathcal{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of dimension n. Let $C_{\mathfrak{A}}$ be the category of D-modules on $\mathcal{X}$ and $C_{\mathfrak{A}} \xrightarrow{u_*} C_X$ the pull-back functor corresponding to the embedding of the structure sheaf $\mathcal{O}_{\mathcal{X}}$ into the sheaf $\mathcal{D}_{\mathcal{X}}$ of differential operators on $\mathcal{X}$. Let $\mathfrak{U} = \{U_i \xrightarrow{u_i} \mathcal{X} \mid i \in J\}$ be an affine finite cover of $\mathcal{X}$ such that each U_i is isomorphic to the affine space $\mathbb{A}^n$. Then $\mathbf{Spec}^-(\mathfrak{A}) = \bigcup_{i\in J} \mathbf{Spec}^-(|A_n - mod|)$, where A_n is the n-th Weyl algebra.

9.7.5.1. The case of a curve. Suppose $n = 1$, i.e. $\mathcal{X}$ is a curve. Then $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ and $\mathbf{Spec}^-(X)$ coincide.

In fact, the equality holds when C_X is the category of left modules over the first Weyl algebra A_1. This follows from the fact that A_1 has Gabriel-Krull dimension one, hence $\mathbf{Spec}^-(X)$ consists of closed points and one generic point.

In the general case, the equality follows from this and 9.7.4.

9.7.5.2. Corollary. *Let $C_{\mathfrak{A}}$ be the category $U(sl_2) - mod_0$ of $U(sl_2)$-modules with the trivial central character. Then* $\mathbf{Spec}^-(\mathfrak{A}) = \mathbf{Spec}(\mathfrak{A})$.

Proof. The fact is true if the base field is of positive characteristic, because then $U(sl_2)$ is finite-dimensional over its center.

Suppose that the base field is of characteristic zero. The category $C_{\mathfrak{A}} = U(sl_2) - mod_0$ is equivalent to the category $D(\mathbb{P}^1)$ of D-modules on the one-dimensional projective space. The assertion follows from 9.7.4. ∎

9.8. Reconstruction of quasi-compact schemes.

9.8.1. Geometric center of a 'space'. Let C_X be an abelian category. Fix a topology τ on $\mathbf{Spec}(X)$. The map $\widetilde{\mathcal{O}}_{X,\tau}$ which assigns to every open subset W of $\mathbf{Spec}(X)$ the center of the quotient category $C_X/\mathcal{S}_W$, where $\mathcal{S}_W = \bigcap_{\mathcal{Q}\in W} \widehat{\mathcal{Q}}$, is a presheaf on $(\mathbf{Spec}(X), \tau)$. Recall that the *center* of the category C_Y is the (commutative) ring of endomorphisms of its identical functor. If C_Y is a category of left modules over a ring R, then the center of C_Y is naturally isomorphic to the center of R.

We denote by $\mathcal{O}_{X,\tau}$ the associated sheaf. The ringed space $((\mathbf{Spec}(X), \tau), \mathcal{O}_{X,\tau})$ is called the *geometric center* of the 'space' X. If τ is the Zariski topology, then we write simply $(\mathbf{Spec}(X), \mathcal{O}_X)$ and call this ringed space the *Zariski geometric center* of X. Recall that open sets in Zariski topology are sets of the form $U(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}(X) \mid \mathcal{Q} \not\subseteq \mathbb{T}\}$, where $\mathbb{T}$ is an arbitrary bireflective topologizing subcategory of C_X. Recall that 'bireflective' means that the inclusion functor $\mathbb{T} \hookrightarrow C_X$ has right and left adjoints.

Let $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed topological space and $\mathbf{U} = (\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \xrightarrow{\mathfrak{j}} (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ an open immersion. Then the morphism $\mathfrak{j}$ has an exact inverse image functor $\mathfrak{j}^*$ and a fully faithful direct image functor j_*. This implies that $Ker(\mathrm{j}^*)$ is a Serre subcategory of the category $\mathcal{O}_X - \mathcal{M}od$ of sheaves of $\mathcal{O}_X$-modules and the unique functor

$$\mathcal{O}_X - \mathcal{M}od/Ker(\mathrm{j}^*) \longrightarrow \mathcal{O}_{\mathcal{U}} - \mathcal{M}od$$

induced by j^* is an equivalence of categories [Gab, III.5].

Suppose now that $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a scheme and $Qcoh_{\mathbf{X}}$ the category of quasi-coherent sheaves on $\mathbf{X}$. The inverse image functor j^* of the immersion j maps quasi-coherent sheaves to quasi-coherent sheaves. Let u^* denote the functor $Qcoh_{\mathbf{X}} \longrightarrow Qcoh_{\mathbf{U}}$ induced by j^*. The funtor u^*, being the composition of the exact full embedding of $Qcoh_{\mathbf{X}}$ into $\mathcal{O}_X - \mathcal{M}od$ and the exact functor j^*, is exact; hence it is represented as the composition of an exact localization $Qcoh_{\mathbf{X}} \longrightarrow Qcoh_{\mathbf{X}}/Ker(u^*)$ and a uniquely defined exact functor $Qcoh_{\mathbf{X}}/Ker(u^*) \longrightarrow Qcoh_{\mathbf{U}}$. If the direct image functor j_* of the immersion j maps quasi-coherent sheaves to quasi-coherent sheaves, then it induces a fully faithful functor $Qcoh_{\mathbf{U}} \xrightarrow{u_*} Qcoh_{\mathbf{X}}$ which is right adjoint to u^*. In particular, the canonical functor $Qcoh_{\mathbf{X}}/Ker(u^*) \longrightarrow Qcoh_{\mathbf{U}}$ is an equivalence of categories.

The reconstruction of a scheme $\mathbf{X}$ from the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on $\mathbf{X}$ is based on the existence of an affine cover $\{\mathbf{U}_i \xrightarrow{u_i} \mathbf{X} \mid i \in J\}$ such that the canonical functors $Qcoh_{\mathbf{X}}/Ker(u_i^*) \longrightarrow Qcoh_{\mathbf{U}_i}$, $i \in J$, are category equivalences. It follows from the discussion above (or from [GZ, I.2.5.2]) that this is garanteed if the inverse image functor $Qcoh_{\mathbf{X}} \xrightarrow{u_i} Qcoh_{\mathbf{U}_i}$ has a fully faithful right adjoint.

9.8.2. Proposition. *Let $\mathbf{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-compact scheme such that there exists an affine cover $\{\mathbf{U}_i \xrightarrow{u_i} \mathbf{X} \mid i \in J\}$ such that the canonical functors*

$$Qcoh_{\mathbf{X}}/Ker(u_i^*) \longrightarrow Qcoh_{\mathbf{U}_i}, \quad i \in J,$$

are category equivalences. Then

(a) The scheme $\mathbf{X}$ is isomorphic to the Zariski geometric center $((\mathbf{Spec}(X), \tau_{\mathfrak{z}}), \mathcal{O}_X)$ of the 'space' X, where $C_X = Qcoh_{\mathbf{X}}$, $\tau_{\mathfrak{z}}$ is the Zariski topology on X and $\mathcal{O}_X$ is the sheaf of commutative rings defined in 9.6.4.

(b) For every open immersion $\mathbf{U} \xrightarrow{u} \mathbf{X}$ such that $Qcoh_{\mathbf{X}}/Ker(u^*) \longrightarrow Qcoh_{\mathbf{U}}$ is a category equivalence, $Ker(u^*)$ is a Serre subcategory of $Qcoh_{\mathbf{X}}$. In particular, $Ker(u_i^*)$ is a Serre subcategory for all $i \in J$.

Proof. (a) Set $C_{U_i} = Qcoh_{\mathbf{U}_i}$ and $\mathcal{T}_i = Ker(u_i^*)$. Since $\mathbf{X}$ is quasi-compact, we can and will assume that J is finite. The condition that $\{\mathbf{U}_i \xrightarrow{u_i} \mathbf{X} \mid i \in J\}$ is a cover means precisely that $\bigcap_{i\in J} \mathcal{T}_i = 0$.

(a1) Let x be a point of the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$. Let $\mathfrak{I}_{\bar{x}}$ be the defining ideal of the closure $\bar{x}$ of the point x and $\mathcal{M}_{\bar{x}}$ the quotient sheaf $\mathcal{O}/\mathfrak{I}_{\bar{x}}$. Set $J_x = \{i \in J \mid \mathcal{M}_{\bar{x}} \notin Ob\mathcal{T}_i\}$. We claim that $\mathcal{Q}_x = [\mathcal{M}_{\bar{x}}]$ is an element of $\mathbf{Spec}(X)$.

For every $i \in J_x$, the object $u_i^*(\mathcal{M}_{\bar{x}})$ of the category C_{U_i} belongs to $Spec(U_i)$ and $Spec(U_i)$, because C_{U_i} is (equivalent to) the category of modules over a ring. Therefore, $[u_i^*(\mathcal{Q}_x)] = [u_i^*(\mathcal{M}_{\bar{x}})]$ is an element of $\mathbf{Spec}(U_i)$. By 9.6.1, $\mathcal{Q}_x \in \mathbf{Spec}(X)$.

(a2) Conversely, let $\mathcal{Q}$ be an element of $\mathbf{Spec}(X)$. Let $\mathcal{Q} \nsubseteq \mathcal{T}_i$, or, equivalently, $\mathcal{T}_i \subseteq \widehat{\mathcal{Q}}$. By the functoriality of $\mathbf{Spec}(X)$ under exact localizations, $[u_i^*(\mathcal{Q})]$ is an element of $\mathbf{Spec}(U_i)$. Since U_i is affine, $\mathbf{Spec}(U_i)$ is in bijective correspondence with the underlying space $\mathcal{U}_i$ of the subscheme $\mathbf{U}_i = (\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$; in particular, to the element $[u_i^*(\mathcal{Q})]$ there corresponds a point x of $\mathcal{U}_i$ which we identify with its image in $\mathcal{X}$. Notice that the point x does not depend on the choice of $i \in J_{\widehat{\mathcal{Q}}} = \{j \in J \mid \mathcal{T}_j \subseteq \widehat{\mathcal{Q}}\}$. This gives a map $\mathbf{Spec}(X) \longrightarrow \mathcal{X}$ which is inverse to the map $\mathcal{X} \longrightarrow \mathbf{Spec}(X)$ constructed in (a1) above. These maps are homeomorphisms in the case if the cover consists of one element, i.e. the scheme is affine. The general case follows from the commutative diagrams

$$\begin{array}{ccc} \mathbf{Spec}(U_i) & \xrightarrow{\sim} & \mathcal{U}_i \\ \downarrow & & \downarrow \\ \mathbf{Spec}(X) & \longrightarrow & \mathcal{X}, \qquad i \in J. \end{array} \tag{5}$$

in which vertical arrows are open immersions and the upper horizontal arrow is a homeomorphism; hence the lower horizontal arrow is a homeomorphism.

(a3) The diagrams (5) extend to the commutative diagrams of ringed spaces

$$\begin{array}{ccc} (\mathbf{Spec}(U_i), \mathcal{O}_{U_i}) & \xrightarrow{\sim} & (\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i}) \\ \downarrow & & \downarrow \\ (\mathbf{Spec}(X), \mathcal{O}_X) & \longrightarrow & (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \qquad i \in J. \end{array} \tag{6}$$

in which $\mathbf{Spec}^0_{\mathfrak{c}}(U_i), \mathcal{O}_{U_i})$ and $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \mathcal{O}_X)$ are Zariski geometric centra of resp. U_i and X, vertical arrows are open immersions and upper horizontal arrow is an isomorphism. Therefore the lower horizontal arrow is an isomorphism.

(b) Let x be a point of $\mathcal{U}_i$, which we identify with its image in the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$. Let $\mathcal{Q}_x$ denote the corresponding element of $\mathbf{Spec}(X)$. It follows that $\mathcal{Q}_x \nsubseteq \mathcal{T}_i$, or, equivalently, $\mathcal{T}_i \subseteq \widehat{\mathcal{Q}}_x$. Thus, $\mathcal{T}_i \subseteq \bigcap\limits_{x \in \mathcal{U}_i} \widehat{\mathcal{Q}}_x$. We claim that $\mathcal{T}_i = \bigcap\limits_{x \in \mathcal{U}_i} \widehat{\mathcal{Q}}_x$. In fact, if M is an object of $C_X - \mathcal{T}_i$, then $u_i^*(M) \neq 0$. Since the category C_{U_i} is equivalent to the category of modules over a ring, every nonzero object of C_{U_i} has a non-empty support. In particular, there is a point $x \in \mathcal{U}_i$ which belongs to the support of $u_i^*(M)$. The latter means that $u_i^*(M) \notin Ob(\widehat{\mathcal{Q}}_x/\mathcal{T}_i)$, or, what is the same, $M \notin Ob\widehat{\mathcal{Q}}_x$.

Since each $\widehat{\mathcal{Q}}_x$ is a Serre subcategory and the intersection of any family of Serre subcategories is a Serre subcategory, $\mathcal{T}_i$ is a Serre subcategory. ■

9.8.3. Remarks. (i) A comment to the assertion 9.8.2(b): if C_Y is a Grothendieck category and $\mathcal{T}$ is a Serre subcategory than the localization functor $C_Y \longrightarrow C_Y/\mathcal{T}$ has a right adjoint.

(ii) The quasi-compactness of the scheme in 9.8.2 is an essential requirement. If the scheme is not quasi-compact, the spectrum $\mathbf{Spec}(X)$ might be not sufficiently big to reconstruct the underlying space. This was observed by O. Gabber who produced an example of a scheme which is not isomorphic to the ringed space $(\mathbf{Spec}(X), \mathcal{O}_X)$ associated with its category of quasi-coherent sheaves. He also mentioned that his example does not work for slightly different version of a spectrum. This version of a spectrum is nothing else, but the spectrum related to coreflective topologizing subcategories studied in the next section.

10. The spectra related to coreflective topologizing subcategories.

10.1. The spectra $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{c}}(X)$. These are spectra obtained via an application of the construction of the relative spectra (see 2.5) to the inclusion functor $\mathfrak{Th}_{\mathfrak{c}}(X) \hookrightarrow \mathfrak{T}_{\mathfrak{c}}(X)$ from the preorder $\mathfrak{Th}_{\mathfrak{c}}(X)$ of coreflective thick subcategories to the preorder $\mathfrak{T}_{\mathfrak{c}}(X)$ of coreflective topologizing subcategories of the category C_X. Objects of the spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ are coreflective thick subcategories $\mathcal{P}$ such that the intersection $\mathcal{P}^{\mathfrak{c}}$ of all coreflective topologizing subcategories properly containing $\mathcal{P}$ contains $\mathcal{P}$ properly too. The spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ is formed by coreflective topologizing subcategories $\mathcal{Q}$ of C_X such that the union $\langle\mathcal{Q}\rangle$ of all coreflective subcategories of C_X which do not contain $\mathcal{Q}$ is a coreflective thick subcategory. The canonical injective morphism $\mathbf{Spec}^0_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)$ maps $\mathcal{Q}$ to $\langle\mathcal{Q}\rangle$. It follows from the definition of $\langle\mathcal{Q}\rangle$ that every coreflective topologizing subcategory properly containing $\langle\mathcal{Q}\rangle$ contains $\mathcal{Q}$, hence the smallest coreflective subcategory $[\langle\mathcal{Q}\rangle, \mathcal{Q}]_{\mathfrak{c}}$ containing $\mathcal{Q}$ and $\langle\mathcal{Q}\rangle$ coincides with $\langle\mathcal{Q}\rangle^{\mathfrak{c}}$. The injectivity of the map $\mathcal{Q} \longmapsto \langle\mathcal{Q}\rangle$ is a consequence of the following fact which is going to be used more than once.

10.1.1. Lemma. *If $\mathcal{Q}_1$, $\mathcal{Q}_2$ are elements of $\mathcal{T}_{\mathfrak{c}}(X)$, then $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ iff $\widehat{\mathcal{Q}}_1 \subseteq \widehat{\mathcal{Q}}_2$.*

Proof. The argument is the same as in 4.1. ∎

10.1.2. Proposition. *Let C_X be an abelian category with the property (sup).*
(a) The canonical morphism

$$\mathbf{Spec}^0_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X), \quad \mathcal{Q} \longmapsto \langle\mathcal{Q}\rangle, \tag{1}$$

is an isomorphism.
(b) There are natural injective morphisms

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X) \quad and \quad \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X) \tag{2}$$

such that the diagram

$$\begin{array}{ccc} \mathbf{Spec}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(X) \\ \wr\downarrow & & \downarrow\wr \\ \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(X) \end{array}$$

commutes.
(c) If C_X has enough objects of finite type, then the morphisms (2) are isomorphisms.

Proof. (a) For every $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$, the intersection $\mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}$ is nonzero, because $\mathcal{P}$ is a Serre subcategory. The claim is that the coreflective topologizing subcategory $[\mathcal{P}_*]_{\mathfrak{c}}$ spanned by the subcategory $\mathcal{P}_* = \mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}$ belongs to the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\langle[\mathcal{P}_*]_{\mathfrak{c}}\rangle = \mathcal{P}$. The map

$$\mathbf{Spec}^1_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X), \quad \mathcal{P} \longmapsto \langle[\mathcal{P}_*]_{\mathfrak{c}}\rangle, \tag{3}$$

is inverse to the map (1) above.

(a1) Notice that $\langle [\mathcal{P}_*]_\mathfrak{c} \rangle = \langle \mathcal{P}_* \rangle$ because a coreflective topologizing subcategory does not contain $[\mathcal{P}_*]_\mathfrak{c}$ iff it does not contain $\mathcal{P}_*$. Therefore, our claim is that $\langle \mathcal{P}_* \rangle = \mathcal{P}$.

(a2) If $\mathcal{T}$ is a coreflective topologizing subcategory of C_X which is not contained in $\mathcal{P}$, then $\mathcal{P}_* = \mathcal{P}^\mathfrak{c} \cap \mathcal{P}^\perp \subseteq \mathcal{T}$.

In fact, if $\mathcal{T} \nsubseteq \mathcal{P}$, then the coreflective topologizing subcategory $\mathcal{T} \bullet \mathcal{P}$ contains $\mathcal{P}$ properly, hence it contains $\mathcal{P}^\mathfrak{c}$. Notice that every $\mathcal{P}$-torsion free object of $\mathcal{T} \bullet \mathcal{P}$ belongs to $\mathcal{T}$. In particular, $\mathcal{P}_* \subseteq \mathcal{T}$.

(a3) It follows from (a1) that if $\mathcal{T} \in \mathfrak{T}_\mathfrak{c}(X)$ is such that $\mathcal{P}_* \nsubseteq \mathcal{T}$, then $\mathcal{T} \subseteq \mathcal{P}$. This means that $\langle \mathcal{P}_* \rangle \subseteq \mathcal{P}$. On the other hand, $\mathcal{P}_* \nsubseteq \mathcal{P}$ and $\mathcal{P}$ is a Serre subcategory; in particular, it is coreflective and topologizing; hence the inverse inclusion, $\mathcal{P} \subseteq \langle \mathcal{P}_* \rangle$.

(a4) Since the map (1) is injective and has a right inverse, $\mathcal{P} \longmapsto [\mathcal{P}_*]_\mathfrak{c}$, it is bijective.

(b) Thanks to the property (sup), a thick subcategory of the category C_X is coreflective iff it is a Serre subcategory. In particular, since elements of $\mathbf{Spec}^{1,1}_\mathfrak{t}(X)$ are Serre subcategories, $\mathbf{Spec}^{1,1}_\mathfrak{t}(X) \subseteq \mathfrak{Th}_\mathfrak{c}(X)$. A Serre subcategory $\mathcal{P}$ belongs to $\mathbf{Spec}^{1,1}_\mathfrak{t}(X)$ iff the intersection $\mathcal{P}^\mathfrak{t}$ of topologizing subcategories properly containing $\mathcal{P}$ contains $\mathcal{P}$ properly. Therefore, $\mathcal{P}^\mathfrak{c}$ contains $\mathcal{P}$ properly. The map $\mathbf{Spec}^{1,1}_\mathfrak{t}(X) \longrightarrow \mathbf{Spec}^1_\mathfrak{c}(X)$ is the inclusion.

Let $\mathcal{Q} \in \mathbf{Spec}(X)$, and let $[\mathcal{Q}]_\mathfrak{c}$ be the smallest coreflective topologizing subcategory of C_X containing $\mathcal{Q}$. Clearly $[\mathcal{Q}]_\mathfrak{c} \nsubseteq \widehat{\mathcal{Q}}$, hence $\widehat{\mathcal{Q}} \subseteq \langle [\mathcal{Q}]_\mathfrak{c} \rangle$. On the other hand, if $\mathcal{T}$ is a coreflective topologizing subcategory of C_X such that $[\mathcal{Q}]_\mathfrak{c} \nsubseteq \mathcal{T}$, then $\mathcal{Q} \nsubseteq \mathcal{T}$, or, equivalently, $\mathcal{T} \subseteq \widehat{\mathcal{Q}}$. This shows the inverse inclusion, $\langle [\mathcal{Q}]_\mathfrak{c} \rangle \subseteq \widehat{\mathcal{Q}}$. The equality $\langle [\mathcal{Q}]_\mathfrak{c} \rangle = \widehat{\mathcal{Q}}$, together with the fact that $\widehat{\mathcal{Q}}$ is a Serre subcategory, shows that $[\mathcal{Q}]_\mathfrak{c} \in \mathbf{Spec}^0_\mathfrak{c}(X)$ for every $\mathcal{Q} \in \mathbf{Spec}(X)$. The map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_\mathfrak{c}(X)$ assigns to every element $\mathcal{Q}$ of $\mathbf{Spec}(X)$ the coreflective topologizing subcategory $[\mathcal{Q}]_\mathfrak{c}$ spanned by $\mathcal{Q}$.

(c) Let $\mathcal{Q}$ be an object of $\mathbf{Spec}^0_\mathfrak{c}(X)$. One can see that $\langle \mathcal{Q} \rangle = \langle M \rangle$ for every object M of $\mathcal{Q} - \langle \mathcal{Q} \rangle$: the inclusion $\langle M \rangle \subseteq \langle \mathcal{Q} \rangle$ is due to the fact that $M \in Ob\mathcal{Q}$ and the inverse inclusion holds because $M \notin Ob\langle \mathcal{Q} \rangle$. This implies that $\mathcal{Q} = [M]_\mathfrak{c}$ for any object M of $\mathcal{Q} - \langle \mathcal{Q} \rangle$. In particular, $\mathcal{Q} = [M]_\mathfrak{c}$ for any nonzero object M of $\mathcal{Q} \cap \langle \mathcal{Q} \rangle^\perp$.

Suppose that the category C_X has enough objects of finite type, i.e. every nonzero object of C_X has a nonzero subobject of finite type. In particular, any nonzero object of the subcategory $\mathcal{Q} \cap \langle \mathcal{Q} \rangle^\perp$ has a nonzero subobject L. Since L belongs to $\mathcal{Q} \cap \langle \mathcal{Q} \rangle^\perp$ and is nonzero, $[L]_\mathfrak{c} = \mathcal{Q}$. We claim that L is an object of $Spec(X)$, which implies that the topologizing subcategory $[L]$ generated by L belongs to $\mathbf{Spec}(X)$.

In fact, let N be a nonzero subobject of L. Then $[N]_\mathfrak{c} = \mathcal{Q} = [L]_\mathfrak{c}$. In particular, L is an object of the coreflective topologizing subcategory of C_X spanned by N. Objects of the subcategory $[N]_\mathfrak{c}$ are precisely objects of the category C_X which are supremums of their subobjects from $[N]$. In particular, L is a supremum of its subobjects from $[N]$. Since subobjects of L which belong to the topologizing subcategory $[N]$ form a filtered system and L is of finite type, it follows that L is isomorphic to one of its subobjects from $[N]$, i.e. $L \in Ob[N]$. This proves that L belongs to $Spec(X)$. ■

10.2. The spectra $\mathbf{Spec}^i_\mathfrak{c}(X)$ and $\mathbf{Spec}^i_{\mathfrak{Se}}(X)$. Recall that $\mathbf{Spec}^i_{\mathfrak{Se}}(X)$, $i = 0, 1$, are the spectra of the preorder $\mathfrak{Se}(X)$ of Serre subcategories of C_X (see 8.7): points of $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$ are Serre subcategories $\mathcal{P}$ of C_X such that the intersection $\mathcal{P}^\mathfrak{s}$ of all Serre subcategories of C_X properly containing $\mathcal{P}$ does not coincide with $\mathcal{P}$; and $\mathbf{Spec}^0_{\mathfrak{Se}}(X)$

is formed by Serre subcategories $\mathcal{Q}$ such that the union $\langle \mathcal{Q} \rangle_{\mathfrak{s}}$ of Serre subcategories not containing $\mathcal{Q}$ is a Serre subcategory.

10.2.1. Proposition. *There are natural injective morphisms*

$$\mathbf{Spec}^i_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^i_{\mathfrak{Se}}(X) \quad i = 0, 1,$$

such that the diagram

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{c}}(X) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{Se}}(X) \\ \wr\downarrow & & \downarrow \\ \mathbf{Spec}^1_{\mathfrak{c}}(X) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{Se}}(X) \end{array} \qquad (1)$$

commutes.

Proof. The spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ is contained in the spectrum $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$, because if $\mathcal{P}^{\mathfrak{c}} \neq \mathcal{P}$, then $(\mathcal{P}^{\mathfrak{c}})^-$ is the smallest Serre subcategory properly containing $\mathcal{P}$, hence $\mathcal{P}$ belongs to $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$. The map $\mathbf{Spec}^0_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{Se}}(X)$ assigns to every $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ the Serre subcategory $\mathcal{Q}^-$ spanned by $\mathcal{Q}$ (cf. 1.5). ■

10.2.2. Extended spectra. Extended spectra are obtained via adjoining to the original spectra a *marked point*. In the case of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{Se}}(X)$, this marked point might be realized the zero subcategory. In the case of the spectra $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^1_{\mathfrak{Se}}(X)$, the marked point is realized as the empty subcategory which reflects the equalities $\langle 0 \rangle = \emptyset = \langle 0 \rangle_{\mathfrak{s}}$.

Morphisms between the original spectra determine morphisms between the corresponding extended spectra mapping marked points to marked points. In particular, the commutative diagram (1) extends to the commutative diagram

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{c}}(X)_\star & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{Se}}(X)_\star \\ \wr\downarrow & & \downarrow \\ \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{Se}}(X)_\star \end{array} \qquad (1_\star)$$

10.2.2.1 Proposition. *There are natural maps*

$$\mathbf{Spec}^i_{\mathfrak{Se}}(X)_\star \longrightarrow \mathbf{Spec}^i_{\mathfrak{c}}(X)_\star, \quad i = 0, 1,$$

such that the diagram

$$\begin{array}{ccccc} \mathbf{Spec}^0_{\mathfrak{c}}(X)_\star & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{Se}}(X)_\star & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(X)_\star \\ \wr\downarrow & & \downarrow & & \downarrow\wr \\ \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{Se}}(X)_\star & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star \end{array} \qquad (2)$$

commutes and the compositions of its horizontal arrows are identical morphisms.

Proof. The map $\mathbf{Spec}^1_{\mathfrak{Se}}(X)_\star \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star$ assings to each $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{Se}}(X)$ the Serre subcategory $\langle \mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp} \rangle$ if $\mathcal{P}^{\mathfrak{c}} \neq \mathcal{P}$ (i.e. if $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$) and the marked point,

$\langle 0\rangle = \emptyset$, if $\mathcal{P}^{\mathfrak{c}} = \mathcal{P}$. The map $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{c}}(X)_\star \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)_\star$ is uniquely defined by the commutativity of the right square in the diagram (2). It follows from the (argument of) 10.1.2 that the composition of the lower horizontal arrows in (2) is the identical map. The similar fact for the upper horizontal arrows is a consequence of this and the commutativity of the diagram (2). ∎

10.3. Functorial properties of $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ and $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. For any topologizing subcategory $\mathbb{T}$ of the category C_X, we set

$$\begin{aligned} &U^1_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X) \mid \mathbb{T} \subseteq \mathcal{P}\} \\ &V^1_{\mathfrak{c}}(\mathbb{T}) = \mathbf{Spec}^1_{\mathfrak{c}}(X) - U^1_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X) \mid \mathbb{T} \nsubseteq \mathcal{P}\}. \\ &U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \nsubseteq [\mathbb{T}]_{\mathfrak{c}}\} \quad \text{and} \\ &V^0_{\mathfrak{c}}(\mathbb{T}) = \mathbf{Spec}^0_{\mathfrak{c}}(X) - U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \subseteq [\mathbb{T}]_{\mathfrak{c}}\} \end{aligned} \tag{1}$$

10.3.1. Proposition. *Let $\mathbb{T}$ be a topologizing subcategory of the category C_X.*

(a) The isomorphism

$$\mathbf{Spec}^0_{\mathfrak{c}}(X) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{c}}(X), \quad \mathcal{Q} \longmapsto \langle \mathcal{Q}\rangle,$$

(cf. 10.1.2) induces isomorphisms

$$U^0_{\mathfrak{c}}(\mathbb{T}) \xrightarrow{\sim} U^1_{\mathfrak{c}}(\mathbb{T}) \quad \text{and} \quad V^0_{\mathfrak{c}}(\mathbb{T}) \xrightarrow{\sim} V^1_{\mathfrak{c}}(\mathbb{T}). \tag{2}$$

(b) There are equalities $V^i_{\mathfrak{c}}(\mathbb{T}) = V^i_{\mathfrak{c}}(\mathbb{T}^-)$ *and* $U^i_{\mathfrak{c}}(\mathbb{T}) = U^i_{\mathfrak{c}}(\mathbb{T}^-)$, $i = 0, 1$.

(c) For every $\mathcal{P} \in V^1_{\mathfrak{c}}(\mathbb{T})$, the intersection $\mathcal{P} \cap \mathbb{T}$ is an element of $\mathbf{Spec}^1(|\mathbb{T}|)$, where $C_{|\mathbb{T}|} = \mathbb{T}$, and the map

$$V^1_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(|\mathbb{T}|), \quad \mathcal{P} \longmapsto \mathcal{P} \cap \mathbb{T}, \tag{3}$$

is an isomorphism. The inverse map is given by $\widetilde{\mathcal{P}} \longmapsto \widetilde{\mathcal{P}}_+$ (see 7.1).

Similarly, the map $\mathcal{Q} \longmapsto \mathcal{Q} \cap \mathbb{T}$ induces an isomorphism $V^0_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|)$.

(c^{bis}) If $\mathbb{T}$ is coreflective, then the inverse isomorphism, $\mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|) \longrightarrow V^0_{\mathfrak{c}}(\mathbb{T})$, is given by the identical map.

(d) The maps $\mathcal{P} \longmapsto \mathcal{P}/\mathbb{T}^-$ and $\mathcal{Q} \longmapsto (\mathbb{T}^- \bullet \mathcal{Q} \bullet \mathbb{T}^-)/\mathbb{T}^-$ define injective morphisms resp.

$$U^1_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X/\mathbb{T}^-) \quad \text{and} \quad U^0_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathbb{T}^-) \tag{4}$$

such that the diagram

$$\begin{array}{ccc} U^0_{\mathfrak{c}}(\mathbb{T}) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathbb{T}^-) \\ \wr\big\downarrow & & \big\downarrow\wr \\ U^1_{\mathfrak{c}}(\mathbb{T}) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(X/\mathbb{T}^-) \end{array} \tag{5}$$

commutes.

Proof. (a) Let $\mathcal{Q} \in U^0_{\mathfrak{c}}(\mathbb{T})$, i.e. $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathcal{Q} \nsubseteq [\mathcal{T}]_{\mathfrak{c}}$. This means precisely that $\langle \mathcal{Q}\rangle \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\mathcal{T} \subseteq \langle \mathcal{Q}\rangle$, i.e. $\mathcal{Q}$ is an element of $U^0_{\mathfrak{c}}(\mathbb{T})$ iff $\langle \mathcal{Q}\rangle$ is an element of $U^1_{\mathfrak{c}}(\mathbb{T})$. The isomorphism $V^0_{\mathfrak{c}}(\mathbb{T}) \xrightarrow{\sim} V^1_{\mathfrak{c}}(\mathbb{T})$ follows from this and the isomorphism $\mathbf{Spec}^0_{\mathfrak{c}}(X) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{c}}(X)$.

(b) The equalities $V^1_{\mathfrak{c}}(\mathbb{T}) = V^1_{\mathfrak{c}}(\mathbb{T}^-)$ and $U^1_{\mathfrak{c}}(\mathbb{T}) = U^1_{\mathfrak{c}}(\mathbb{T}^-)$ follow from an observation that elements of $\mathbf{Spec}^1_{\mathfrak{c}}(X)$ are Serre subcategories, and if $\mathcal{P}$ is a Serre subcategory, then $\mathbb{T} \subseteq \mathcal{P}$ iff $\mathbb{T}^- \subseteq \mathcal{P}$. The other two equalities follow from these isomorphisms (2) above.

(c) Let $\mathcal{P} \in V^1_{\mathfrak{c}}(\mathbb{T})$, i.e. $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$ and $\mathbb{T} \not\subseteq \mathcal{P}$. The latter implies that $[\mathbb{T}]_{\mathfrak{c}} \bullet \mathcal{P}$ is a coreflective topologizing subctegory of C_X properly containing $\mathcal{P}$. Therefore, it contains $\mathcal{P}^{\mathfrak{c}}$, and we have:

$$\mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp} \subseteq ([\mathbb{T}]_{\mathfrak{c}} \bullet \mathcal{P}) \cap \mathcal{P}^{\perp} = [\mathbb{T}]_{\mathfrak{c}} \cap \mathcal{P}^{\perp} \subseteq [\mathbb{T}]_{\mathfrak{c}}.$$

In particular, the intersection $\widetilde{\mathcal{Q}}_{\mathbb{T}} = \mathbb{T} \cap \mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}$ is nonzero. This implies that $\langle \widetilde{\mathcal{Q}}_{\mathbb{T}} \rangle = \mathcal{P}$ (see the argument 10.1.2(c)). Notice that if $\mathcal{S}$ is a coreflective topologizing subcategory of $\mathbb{T}$, then $\mathcal{S}$ coincides with the intersection of $\mathbb{T}$ with the smallest coreflective topologizing subcategory of C_X containing $\mathcal{S}$. Therefore, the union $\langle \widetilde{\mathcal{Q}}_{\mathbb{T}} \rangle_{\mathbb{T}}$ of coreflective topologizing subctegories of $\mathbb{T}$ which do not contain $\widetilde{\mathcal{Q}}_{\mathbb{T}}$ coincides with the intersection $\langle \widetilde{\mathcal{Q}}_{\mathbb{T}} \rangle \cap \mathbb{T}$. Thus, $\langle \widetilde{\mathcal{Q}}_{\mathbb{T}} \rangle_{\mathbb{T}} = \mathcal{P} \cap \mathbb{T}$; in particular, $\mathcal{P} \cap \mathbb{T} \in \mathbf{Spec}^1_{\mathfrak{c}}(|\mathbb{T}|)$ and the corresponding element of $\mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|)$ is $[\mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}]_{\mathfrak{c}} \cap \mathbb{T}$. In other words, it is obtained from $\mathcal{P}$ by applying the composition of the isomorphism $\mathbf{Spec}^1_{\mathfrak{c}}(X) \xrightarrow{\sim} \mathbf{Spec}^0_{\mathfrak{c}}(X)$ and the intersection with $\mathbb{T}$, i.e. the diagram

$$\begin{array}{ccc} V^0_{\mathfrak{c}}(\mathbb{T}) & \longrightarrow & \mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|) \\ \wr\downarrow & & \downarrow\wr \\ V^1_{\mathfrak{c}}(\mathbb{T}) & \longrightarrow & \mathbf{Spec}^1_{\mathfrak{c}}(|\mathbb{T}|) \end{array} \qquad (6)$$

whose horizontal arrows are given by $\mathcal{P} \longmapsto \mathcal{P} \cap \mathbb{T}$, commutes. It follows from the argument above that the map $V^0_{\mathfrak{c}}(\mathbb{T}) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|)$, $\mathcal{Q} \longmapsto \mathcal{Q} \cap \mathbb{T}$, is an isomorphism with the inverse map which assigns to every element $\mathcal{Q}'$ of $\mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|)$ the coreflective topologizing subcategory $[\mathcal{Q}']_{\mathfrak{c}}$ in C_X spanned by $\mathcal{Q}'$. Therefore the lower horizontal arrow in (6) is an isomorphism too.

(c^{bis}) If $\mathbb{T}$ is coreflective, then every coreflective topologizing subcategory of $\mathbb{T}$ is a coreflective topologizing subcategory of C_X, hence the isomorphism

$$\mathbf{Spec}^0_{\mathfrak{c}}(|\mathbb{T}|) \xrightarrow{\sim} V^0_{\mathfrak{c}}(\mathbb{T}), \quad \mathcal{Q}' \longmapsto [\mathcal{Q}']_{\mathfrak{c}},$$

discussed above becomes an identical map.

(d) Let q^* denote the localization functor $C_X \longrightarrow C_X/\mathbb{T}^-$.

If $\mathcal{P}$ belongs to $U^1_{\mathfrak{c}}(\mathbb{T})$, i.e. $\mathbb{T} \subseteq \mathcal{P} \subsetneq \mathcal{P}^{\mathfrak{c}}$, then

$$(\mathcal{P}/\mathbb{T}^-)^{\mathfrak{c}} = [q^*(\mathcal{P}^{\mathfrak{c}})]_{\mathfrak{c}} = (\mathbb{T}^- \bullet \mathcal{P}^{\mathfrak{c}} \bullet \mathbb{T}^-)/\mathbb{T}^- \supsetneq \mathcal{P}/\mathbb{T}^-.$$

This shows that $\mathcal{P}/\mathbb{T}^-$ is an element of $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathbb{T}^-)$.

If $\mathcal{Q}$ is the image of the element $\mathcal{P}$ in $U^0_{\mathfrak{c}}(\mathbb{T})$ (see the assertion (a) above), then the coreflective subcategory $[q^*(\mathcal{Q})]_{\mathfrak{c}} = (\mathbb{T}^- \bullet \mathcal{Q} \bullet \mathbb{T}^-)/\mathbb{T}^-$ is the image of $\mathcal{P}/\mathbb{T}^-$ in $\mathbf{Spec}^0_{\mathfrak{c}}(X/\mathbb{T}^-)$. The commutativity of the diagram (5) follows from the definition of its arrows. ■

10.4. The local property of the spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(X)$.

10.4.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of Serre subcategories of the category C_X such that $\bigcap_{i\in J} \mathcal{T}_i = 0$; and let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$.*

1) The following conditions on $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(X)$ are equivalent:

(a) $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$,

(b) $\bigcap_{i\in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \nsubseteq \mathcal{P}$, where $J_{\mathcal{P}} = \{j \in J \mid \mathcal{T}_j \subseteq \mathcal{P}\}$, and if $J^{\mathcal{P}} = J - J_{\mathcal{P}} \neq \emptyset$, then $\bigcap_{j\in J^{\mathcal{P}}} \mathcal{T}_j \nsubseteq \mathcal{P}$.

2) The conditions (a) and (b) imply the condition

(c) $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i)$ for each $i \in J_{\mathcal{P}}$.

If J is finite, then the conditions (a) and (b) are equivalent to the condition (c).

Proof. 1) Since $\mathcal{P}$ is a Serre subcategory, the condition $\bigcap_{i\in J} \mathcal{T}_i^- = 0$ implies that $J_{\mathcal{P}} = \{i \in J \mid \mathcal{T}_i \subseteq \mathcal{P}\}$ is not empty.

In fact, if $\mathcal{T}_i \nsubseteq \mathcal{P}$ for all $i \in J$, then $\mathcal{T}_i \bullet \mathcal{P} \supseteq \mathcal{P}^{\mathfrak{c}}$, hence $\mathcal{P}^{\mathfrak{c}} \subseteq \bigcap_{i\in J}(\mathcal{T}_i \bullet \mathcal{P})$. But, by A1.2.1, $\bigcap_{i\in J}(\mathcal{T}_i \bullet \mathcal{P}) = \big(\bigcap_{i\in J}\mathcal{T}_i\big) \bullet \mathcal{P} = 0 \bullet \mathcal{P} = \mathcal{P}$, hence $\mathcal{P}^{\mathfrak{c}} = \mathcal{P}$, which contradicts to the assumption that $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$, i.e. $\mathcal{P} \subsetneq \mathcal{P}^{\mathfrak{c}}$.

2) Notice that if $\mathcal{S}$ is a Serre subcategory, and $\mathbb{T}$ a subcategory of C_X closed under taking subquotients, then $\mathbb{T} \nsubseteq \mathcal{S}$ iff $\mathbb{T} \cap \mathcal{S}^{\perp} \neq 0$, because an object M of C_X does not belong to $\mathcal{S}$ iff it has a nonzero $\mathcal{S}$-torsion free subquotient.

In particular, the condition (b) above can be written as follows:

(b) $\bigcap_{i\in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \neq 0$, and $\big(\bigcap_{j\in J^{\mathcal{P}}} \mathcal{T}_j\big) \bigcap \mathcal{P}^{\perp} \neq 0$, if $J^{\mathcal{P}} = J - J_{\mathcal{P}} \neq \emptyset$.

$(a) \Rightarrow (b)$. Let $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$, i.e. $\mathcal{P} \neq \mathcal{P}^{\mathfrak{c}}$.

If $i \in J_{\mathcal{P}}$, that is $\mathcal{T}_i \subseteq \mathcal{P}$, then $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) = \mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}} \bullet \mathcal{T}_i$.

Since $\mathcal{T}_i \subseteq \mathcal{P}$, the intersection $(\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}} \bullet \mathcal{T}_i) \bigcap \mathcal{P}^{\perp}$ coincides with $(\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}$. Therefore

$$\begin{gathered}\bigcap_{i\in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} = \bigcap_{i\in J_{\mathcal{P}}} ((\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}) = \\ \big(\bigcap_{i\in J_{\mathcal{P}}} (\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}})\big) \bigcap \mathcal{P}^{\perp} = \big(\big(\bigcap_{i\in J_{\mathcal{P}}} \mathcal{T}_i\big) \bullet \mathcal{P}^{\mathfrak{c}}\big) \bigcap \mathcal{P}^{\perp} \supseteq \mathcal{P}^{\mathfrak{c}} \bigcap \mathcal{P}^{\perp} \neq 0.\end{gathered} \tag{1}$$

Here we used the equality $\bigcap_{i\in J_{\mathcal{P}}} (\mathcal{T}_i \bullet \mathcal{P}^{\mathfrak{c}}) = \big(\bigcap_{i\in J_{\mathcal{P}}} \mathcal{T}_i\big) \bullet \mathcal{P}^{\mathfrak{c}}$ which is due, by A1.2.1, to the fact that the subcategory $\mathcal{P}^{\mathfrak{c}}$ is coreflective.

Notice that if $J_{\mathcal{P}} = J$, that is $\mathcal{T}_i \subseteq \mathcal{P}$ for all $j \in J$, then $\bigcap_{i\in J_{\mathcal{P}}} \mathcal{T}_i = 0$, hence the last inclusion in (1) can be replaced by the equality, i.e. the intersection $\bigcap_{i\in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}$

coincides with $\mathcal{P}^{\mathfrak{c}} \cap \mathcal{P}^{\perp}$.

Suppose that $J^{\mathcal{P}} = J - J_{\mathcal{P}}$ is non-empty. If $j \in J^{\mathcal{P}}$, that is $\mathcal{T}_j$ is not contained in $\mathcal{P}$, then $\mathcal{T}_j \bullet \mathcal{P}$ is a coreflective topologizing subcategory properly containing both $\mathcal{T}_j$ and $\mathcal{P}$, hence properly containing $\mathcal{P}$. Therefore $\mathcal{P}^{\mathfrak{c}} \subseteq \mathcal{T}_j \bullet \mathcal{P}$ for all $j \in J^{\mathcal{P}}$, or $\mathcal{P}^{\mathfrak{c}} \subseteq \bigcap\limits_{j \in J^{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{P})$.

Since $\mathcal{P}$ is a coreflective subcategory, $\bigcap\limits_{j \in J^{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{P}) = \big(\bigcap\limits_{j \in J^{\mathcal{P}}} \mathcal{T}_j \big) \bullet \mathcal{P}$ (see A1.2.1). Thus, $\mathcal{P}^{\mathfrak{c}} \subseteq \big(\bigcap\limits_{j \in J^{\mathcal{P}}} \mathcal{T}_j \big) \bullet \mathcal{P}$, which implies (actually, is equivalent to) that $\bigcap\limits_{j \in J^{\mathcal{P}}} \mathcal{T}_j \nsubseteq \mathcal{P}$.

$(b) \Rightarrow (a)$. There are two cases: $J_{\mathcal{P}} = J$ and $J_{\mathcal{P}} \neq J$.

(i) We start with the first case; i.e. we assume that $\mathcal{T}_i \subseteq \mathcal{P}$ for all $i \in J$. Set $\widetilde{\mathcal{Q}} = \bigcap\limits_{i \in J} {u_i^*}^{-1}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}$ and $\mathcal{Q} = [\widetilde{\mathcal{Q}}]_{\mathfrak{c}}$ – the smallest coreflective topologizing subcategory of C_X containing $\widetilde{\mathcal{Q}}$. We claim that $\mathcal{Q}$ belongs to $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\langle \mathcal{Q} \rangle = \mathcal{P}$. Since, by condition (b), $\mathcal{Q} \nsubseteq \mathcal{P}$, it suffices to show that $\langle \mathcal{Q} \rangle = \mathcal{P}$.

Let $\mathcal{S}$ be a coreflective topologizing subcategory of C_X which is not contained in $\mathcal{P}$. Then $\mathcal{P}$ is properly contained in $\mathcal{S} \bullet \mathcal{P}$ and, therefore, ${u_i^*}^{-1}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \subseteq \mathcal{T}_i \bullet \mathcal{S} \bullet \mathcal{P}$ for each $i \in J$. This implies that ${u_i^*}^{-1}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet \mathcal{S} \bullet \mathcal{P} \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet \mathcal{S}$. Therefore,

$$\widetilde{\mathcal{Q}} = \bigcap_{i \in J} {u_i^*}^{-1}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \subseteq \bigcap_{i \in J} (\mathcal{T}_i \bullet \mathcal{S}) = \big(\bigcap_{i \in J} \mathcal{T}_i \big) \bullet \mathcal{S} = 0 \bullet \mathcal{S} = \mathcal{S},$$

so that $\mathcal{Q} = [\widetilde{\mathcal{Q}}]_{\mathfrak{c}} \subseteq \mathcal{S}$.

(ii) Consider now the second case: $J_{\mathcal{P}} \neq J$, i.e. $J^{\mathcal{P}} = J - J_{\mathcal{P}}$ is non-empty. This case can be reduced to the first case as follows.

Set $C_{\mathcal{V}_{\mathcal{P}}} = \bigcap\limits_{j \in J^{\mathcal{P}}} \mathcal{T}_j$. Since $\bigcap\limits_{j \in J^{\mathcal{P}}} \mathcal{T}_j \nsubseteq \mathcal{P}$, the intersection $\mathcal{P}_0 = C_{\mathcal{V}_{\mathcal{P}}} \bigcap \mathcal{P}$ is an element of $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(\mathcal{V}_{\mathcal{P}})$. Notice that $\{\mathcal{T}_i \cap C_{\mathcal{V}_{\mathcal{P}}} = \widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$ is a cocover of $\mathcal{V}_{\mathcal{P}}$, i.e. $\bigcap\limits_{j \in J_{\mathcal{P}}} \widetilde{\mathcal{T}}_j = 0$. It remains to show that the condition $\widetilde{\mathcal{Q}} = \bigcap\limits_{i \in J} {u_i^*}^{-1}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \neq 0$ implies the analogous condition for the object $\mathcal{P}_0 = C_{\mathcal{V}_{\mathcal{P}}} \bigcap \mathcal{P}$ of $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{e}}(\mathcal{V}_{\mathcal{P}})$ and the cover $\{\widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$; that is

$$\widetilde{\mathcal{Q}}_0 = \bigcap_{i \in J} {\widetilde{u}_i^*}^{-1}((\mathcal{P}_0/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}_0^{\perp} \neq 0.$$

In fact, let $\widetilde{u}_i^*$ denote the localization functor $C_{\mathcal{V}_{\mathcal{P}}} \longrightarrow C_{\mathcal{V}_{\mathcal{P}}}/\widetilde{\mathcal{T}}_i$. Then

$$\begin{aligned} &{\widetilde{u}_i^*}^{-1}((\mathcal{P}_0/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) = {u_i^*}^{-1}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap C_{\mathcal{V}_{\mathcal{P}}}, \quad \text{and} \\ &{\widetilde{u}_i^*}^{-1}((\mathcal{P}_0/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}_0^{\perp} = {u_i^*}^{-1}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap C_{\mathcal{V}_{\mathcal{P}}} \bigcap \mathcal{P}^{\perp}. \end{aligned}$$

Therefore,

$$\bigcap_{i \in J} {\widetilde{u}_i^*}^{-1}((\mathcal{P}_0/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}_0^{\perp} = \bigcap_{i \in J} {u_i^*}^{-1}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \bigcap C_{\mathcal{V}_{\mathcal{P}}} = \widetilde{\mathcal{Q}} \bigcap C_{\mathcal{V}_{\mathcal{P}}}. \tag{2}$$

On the other hand, for every $i \in J_{\mathcal{P}}$, there is an inclusion $\mathcal{T}_i \bullet C_{\mathcal{V}_{\mathcal{P}}} \bullet \mathcal{P} \supseteq u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})$, because $C_{\mathcal{V}_{\mathcal{P}}} \nsubseteq \mathcal{P}$, which implies the inclusion $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_i \bullet C_{\mathcal{V}_{\mathcal{P}}}$. Taking the intersection, we obtain the inclusion

$$\widetilde{\mathcal{Q}} \subseteq \bigcap_{i \in J_{\mathcal{P}}} (\mathcal{T}_i \bullet C_{\mathcal{V}_{\mathcal{P}}}) = \Big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{T}_i\Big) \bullet C_{\mathcal{V}_{\mathcal{P}}}. \qquad (3)$$

Notice that $\widetilde{\mathcal{Q}}$ is a full subcategory of C_X closed under taking subobjects. In particular, the equality $\widetilde{\mathcal{Q}} \bigcap C_{\mathcal{V}_{\mathcal{P}}} = 0$ means precisely that every object of $\widetilde{\mathcal{Q}}$ is $C_{\mathcal{V}_{\mathcal{P}}}$-torsion free. The latter fact together with the inclusion $\widetilde{\mathcal{Q}} \subseteq \big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{T}_i\big) \bullet C_{\mathcal{V}_{\mathcal{P}}}$ (see (3)) implies that $\widetilde{\mathcal{Q}} \subseteq \bigcap_{i \in J_{\mathcal{P}}} \mathcal{T}_i \subseteq \mathcal{P}$, which contradicts to the fact that $\widetilde{\mathcal{Q}}$ is a nonzero subcategory of $\mathcal{P}^{\perp}$.

This contradiction shows that $\widetilde{\mathcal{Q}} \bigcap C_{\mathcal{V}_{\mathcal{P}}} \neq 0$, hence, by (2),

$$\widetilde{\mathcal{Q}}_0 = \bigcap_{i \in J} \widetilde{u}_i^{*^{-1}}((\mathcal{P}_0/\widetilde{\mathcal{T}}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}_0^{\perp} \neq 0.$$

(iii) Thus, the 'space' $\mathcal{V}_{\mathcal{P}}$, the cocover $\{\widetilde{\mathcal{T}}_i \mid i \in J_{\mathcal{P}}\}$, and the point $\mathcal{P}_0 = \mathcal{P} \bigcap C_{\mathcal{V}_{\mathcal{P}}}$ of the spectrum $\mathbf{Spec}^1_{\mathfrak{S}\mathfrak{c}}(\mathcal{V}_{\mathcal{P}})$ satisfy the conditions (b) with all $\widetilde{\mathcal{T}}_i$ being subcategories of $\mathcal{P}_0$. By (i) above, $\mathcal{P}_0$ belongs to the spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(\mathcal{V}_{\mathcal{P}})$, and $\mathcal{P}_0 = \langle \widetilde{\mathcal{Q}}_0 \rangle_{\mathcal{V}_{\mathcal{P}}} = \langle \mathcal{Q}_0 \rangle_{\mathcal{V}_{\mathcal{P}}}$, where $\mathcal{Q}_0$ is the smallest coreflective topologizing subcategory of $C_{\mathcal{V}_{\mathcal{P}}}$ containing $\widetilde{\mathcal{Q}}_0$. Therefore, $[\mathcal{Q}_0]_{\mathfrak{c}}$ is a point of the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ and $\langle \mathcal{Q}_0 \rangle_X = \mathcal{P}$.

$(b) \Rightarrow (c)$. The condition $\bigcap_{i \in J_{\mathcal{P}}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \neq 0$ implies that the intersection $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}$ is nonzero for every $i \in J_{\mathcal{P}}$. In particular, $(\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}$ does not coincide with $\mathcal{P}/\mathcal{T}_i$, which means that $\mathcal{P}/\mathcal{T}$ belongs to the spectrum $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i)$.

$(c) \Rightarrow (b)$ (when J is finite). For every $i \in J_{\mathcal{P}}$, let $\widetilde{\mathcal{Q}}_i$ denote the intersection $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp}$ and $\mathcal{Q}_i = [\widetilde{\mathcal{Q}}_i]_{\mathfrak{c}}$ – the smallest coreflective topologizing subcategory of C_X containing $\widetilde{\mathcal{Q}}_i$. By assumption, $\widetilde{\mathcal{Q}}_i \neq 0$ for each $i \in J_{\mathcal{P}}$, hence $\mathcal{Q}_i \nsubseteq \mathcal{P}$. The latter implies that, for every $j \in J_{\mathcal{P}}$, the coreflective topologizing subcategory spanned by $u_j^*(\mathcal{Q}_i \bullet \mathcal{P})$ contains $(\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}}$, or, equivalently, $u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}) \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i \bullet \mathcal{P}$. Therefore,

$$\widetilde{\mathcal{Q}}_j = u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \subseteq (\mathcal{T}_j \bullet \mathcal{Q}_i \bullet \mathcal{P}) \bigcap \mathcal{P}^{\perp} = (\mathcal{T}_j \bullet \mathcal{Q}_i) \bigcap \mathcal{P}^{\perp} \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i$$

which implies the inclusion $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet \mathcal{Q}_i$ for every $(i,j) \in J_{\mathcal{P}} \times J_{\mathcal{P}}$, hence the inclusion

$$\mathcal{Q}_j \subseteq \bigcap_{i \in J_{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{Q}_i) = \mathcal{T}_j \bullet \Big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{Q}_i\Big).$$

Here the equality is due to the finiteness of $J_{\mathcal{P}}$.

It follows from the inclusion $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet \big(\bigcap_{i \in J_{\mathcal{P}}} \mathcal{Q}_i\big)$ that $\bigcap_{i \in J_{\mathcal{P}}} \mathcal{Q}_i \neq 0$, because otherwise $\mathcal{Q}_j \subseteq \mathcal{T}_j \bullet 0 = \mathcal{T}_i$, which is impossible, since $\mathcal{T}_j \subseteq \mathcal{P}$ and $\mathcal{Q}_j \nsubseteq \mathcal{P}$.

If $J = J_{\mathcal{P}}$, the condition (b) is fulfilled. If $J \neq J_{\mathcal{P}}$, we need to check that $\bigcap_{i \in J^{\mathcal{P}}} \mathcal{T}_i \nsubseteq \mathcal{P}$.

In fact, if $i \in J^{\mathcal{P}} = J - J_{\mathcal{P}}$, then $\mathcal{T}_i \nsubseteq \mathcal{P}$. Therefore, for every $j \in J_{\mathcal{P}}$, the coreflective subcategory spanned by $u_j^*(\mathcal{T}_i \bullet \mathcal{P})$ contains $(\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}}$, or, equivalently, $u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}}) \subseteq \mathcal{T}_j \bullet \mathcal{T}_i \bullet \mathcal{P}$, which implies that $\widetilde{\mathcal{Q}}_j = u_j^{*^{-1}}((\mathcal{P}/\mathcal{T}_j)^{\mathfrak{c}}) \bigcap \mathcal{P}^{\perp} \subseteq (\mathcal{T}_j \bullet \mathcal{T}_i) \bigcap \mathcal{P}^{\perp}$. Taking the intersection and using the finiteness of $J^{\mathcal{P}}$, we obtain:

$$\widetilde{\mathcal{Q}}_j \subseteq \Big(\bigcap_{i \in J^{\mathcal{P}}} (\mathcal{T}_j \bullet \mathcal{T}_i)\Big) \bigcap \mathcal{P}^{\perp} = \Big(\mathcal{T}_j \bullet \big(\bigcap_{i \in J^{\mathcal{P}}} \mathcal{T}_i\big)\Big) \bigcap \mathcal{P}^{\perp}. \tag{4}$$

The inclusion $\bigcap_{i \in J^{\mathcal{P}}} \mathcal{T}_i \subseteq \mathcal{P}$ implies (together with (4)) that $\widetilde{\mathcal{Q}}_j \subseteq \mathcal{T}_j \subseteq \mathcal{P}$, which is impossible. So that $\bigcap_{i \in J^{\mathcal{P}}} \mathcal{T}_i \nsubseteq \mathcal{P}$. ■

10.4.2. Note. The reader had, probably, noticed that some parts of the proof of 10.4.1 are similar to some parts of the argument of 9.6.1. If one considers only the case of finite cocovers, one can follow the argument of 9.6.1 which is considerably shorter than the proof above.

10.4.3. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a set of Serre subcategories of the category C_X such that $\bigcap_{i \in J} \mathcal{T}_i = 0$; and let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$.*

1) The following conditions on a nonzero coreflective topologizing subcategory $\mathcal{Q}$ of C_X are equivalent:

(a) $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$,

(b) $[u_i^*(\mathcal{Q})]_{\mathfrak{c}} \in \mathbf{Spec}^0_{\mathfrak{c}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{Q} \nsubseteq \mathcal{T}_i$.

Proof. The implication $(a) \Rightarrow (b)$ follows from 10.3.1(d).

$(b) \Rightarrow (a)$. Let $J_{\langle \mathcal{Q} \rangle}$ denote the set of all $i \in J$ such that $\mathcal{Q} \nsubseteq \mathcal{T}_i$, or, equivalently, $\mathcal{T}_i \subseteq \langle \mathcal{Q} \rangle$. Notice that $J_{\langle \mathcal{Q} \rangle}$ is non-empty, because $\mathcal{Q}$ is nonzero and $\bigcap_{i \in J} \mathcal{T}_i = 0$.

Fix an $i \in J_{\langle \mathcal{Q} \rangle}$ and set $\mathcal{P}_i = u_i^{*^{-1}}(\langle u_i^*(\mathcal{Q}) \rangle)$. It follows from the formula for $\mathcal{P}_i$ that $\mathcal{Q} \nsubseteq \mathcal{P}_i$. Notice that $\mathcal{P}_i = \mathcal{P}_j$ for every $j \in J_{\langle \mathcal{Q} \rangle}$.

In fact, replacing C_X by $C_{X'} = C_X/(\mathcal{P}_i \cap \mathcal{P}_j)$ and $\mathcal{P}_k$ by $\mathcal{P}'_k = \mathcal{P}_k/(\mathcal{P}_i \cap \mathcal{P}_j)$, $k = i, j$, we can obtain that $\mathcal{P}'_i \cap \mathcal{P}'_j = 0$. It follows from 10.3.1 that the condition (b) survives this operation. By 9.6.1, the image $\mathcal{Q}'$ of $\mathcal{Q}$ in $C_{X'}$ belongs to $\mathbf{Spec}^0_{\mathfrak{c}}(X')$. Therefore, $\mathcal{P}'_i = \mathcal{P}'_j$, which implies that $\mathcal{P}_i = \mathcal{P}_j$.

So, we write $\mathcal{P}$ instead of $\mathcal{P}_i$. For every $i \in J_{\langle \mathcal{Q} \rangle}$, the subcategory $(\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}}$ contains $u_i^*(\mathcal{Q})$, hence its preimage, $u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})$, contains $\mathcal{Q}$. Since $\bigcap_{i \in J_{\langle \mathcal{Q} \rangle}} u_i^{*^{-1}}((\mathcal{P}/\mathcal{T}_i)^{\mathfrak{c}})$ contains

$\mathcal{Q}$, it is not contained in $\mathcal{P}$. Similarly, if $J^{\langle\mathcal{Q}\rangle} = J - J_{\langle\mathcal{Q}\rangle}$ is non-empty, then $\mathcal{Q} \subseteq \bigcap\limits_{i\in J^{\langle\mathcal{Q}\rangle}} \mathcal{T}_j$, hence $\mathcal{Q} \subseteq \bigcap\limits_{i\in J^{\langle\mathcal{Q}\rangle}} \mathcal{T}_j \nsubseteq \mathcal{P}$. Thus, $\mathcal{P}$ satisfies the condition (b) of 10.4.1. Therefore, by 10.4.1, $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$. It remains to show that $\mathcal{Q}$ is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ corresponding to $\mathcal{P}$.

It follows from the argument of 10.4.1 that the element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ corresponding to $\mathcal{P}$ is the coreflective topologizing subcategory $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}}$ generated by $\mathcal{Q}_{\mathcal{P}} = \mathcal{Q} \cap \mathcal{P}^{\perp}$. In particular, $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}} \subseteq \mathcal{Q}$. Let M be an object of the subcategory $\mathcal{Q}$. Since $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}}$ is a coreflective subcategory of C_X, the object M has the biggest subobject $M_{\mathcal{P}} \hookrightarrow M$ with $M_{\mathcal{P}} \in Ob[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}}$. Consider the corresponding exact sequence

$$0 \longrightarrow M_{\mathcal{P}} \xrightarrow{\ \mathfrak{j}\ } M \longrightarrow N \longrightarrow 0. \tag{5}$$

For every $i \in J_{\langle\mathcal{Q}\rangle}$, the morphism $u_i^*(\mathfrak{j})$ is an isomorphism, because the images of $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}}$ and $\mathcal{Q}$ in $C_X/\mathcal{T}_i$ coincide. This means that the object N in (5) belongs to $\mathcal{T}_i = Ker(u_i^*)$ for each $i \in J_{\langle\mathcal{Q}\rangle}$. On the other hand, $\mathcal{Q} \subseteq \mathcal{T}_\nu$ for every $\nu \in J - J_{\langle\mathcal{Q}\rangle}$ (by definition of $J_{\langle\mathcal{Q}\rangle}$); in particular, $N \in T_\nu$ for all $\nu \in J - J_{\langle\mathcal{Q}\rangle}$. Thus, N is an object of $\bigcap\limits_{i\in J} \mathcal{T}_i = 0$, i.e. $N = 0$, or, equivalently, the arrow $M_{\mathcal{P}} \xrightarrow{\ \mathfrak{j}\ } M$ in (5) is an isomorphism. This proves the inverse inclusion, $[\mathcal{Q}_{\mathcal{P}}]_{\mathfrak{c}} \supseteq \mathcal{Q}$. ∎

For every object M of C_X, we define the *support* of M in $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ by $Supp^0_{\mathfrak{c}}(M) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \subseteq [M]_{\mathfrak{c}}\}$.

10.5. Some consequences.

10.5.1. Proposition. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of an abelian category C_X such that $\bigcap\limits_{i\in J} \mathcal{T}_i = 0$ and $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for every $i \in J$. Then $\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. In particular, the canonical map*

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X), \quad \mathcal{Q} \longmapsto [\mathcal{Q}]_{\mathfrak{c}},$$

is an isomorphism.

Proof. The inclusion $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X) \subseteq \mathbf{Spec}^1_{\mathfrak{c}}(X)$ holds by 10.1.2(b). Let $\mathcal{P} \in \mathbf{Spec}^1_{\mathfrak{c}}(X)$. Then, by the implication $(a) \Rightarrow (c)$ in 10.4.1, $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$. By hypothesis, $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for all $i \in J$. Therefore, by the implication $(b) \Rightarrow (a)$ in 9.6.1, $\mathcal{P}$ belongs to $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. ∎

10.5.2. Corollary. *Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of an abelian category C_X such that $\bigcap\limits_{i\in J} \mathcal{T}_i = 0$ and for every $i \in J$, the quotient category $C_X/\mathcal{T}_i$ has enough objects of finite type. Then $\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.*

In particular, the canonical map $\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X)$ is an isomorphism.

Proof. Since each quotient category $C_X/\mathcal{T}_i$ has enough objects of finite type, it follows from 10.1.2(c) that $\mathbf{Spec}^1_{\mathfrak{c}}(X/\mathcal{T}_i) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X/\mathcal{T}_i)$ for all $i \in J$. The assertion follows now from 10.5.1. ∎

10.5.3. Affine and quasi-affine cocovers. A morphism $X \xrightarrow{f} Y$ is called *affine* if it has a conservative (i.e. reflecting isomorphisms) direct image functor, $C_X \xrightarrow{f_*} C_Y$ (– a right adjoint to f^*) which has, in turn, a right adjoint. We call a 'space' X *affine over a ring* R, if there is an affine morphism $X \longrightarrow \mathbf{Sp}(R)$, where $C_{\mathbf{Sp}(R)} = R-mod$. A 'space' X is called *affine* if it is affine over $\mathbb{Z}$. By [R4, 9.3.3], X is affine iff the category C_X has a projective generator of finite type. By a well-known theorem of Gabriel and Mitchell, the latter condition means precisely that the category C_X is equivalent to the category of modules over an associative ring.

We call set $\{T_i \mid i \in J\}$ of thick subcategories of the category C_X an *affine cocover* of the 'space' X if $\bigcap_{i\in J} \mathcal{T}_i = 0$ and $X/\mathcal{T}_i$ is affine for every $i \in J$.

10.5.4. Proposition. *Let a finite set $\{T_i \mid i \in J\}$ of Serre subcategories of C_X be a cocover of X (that is $\bigcap_{i\in J} \mathcal{T}_i = 0$) such that every quotient category $C_X/\mathcal{T}_i$ has a family of generators of finite type. Then $\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$.*

In particular, $\mathbf{Spec}^1_{\mathfrak{c}}(X) = \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, if $\{T_i \mid i \in J\}$ is an affine cocover of X.

Proof. In fact, quotients of an object of finite type is an object of finite type. Therefore, if a category C_Y has a family of generators of finite type, then every nonzero object of C_Y has a subobject of finite type. The assertion follows now from 10.5.3. ∎

10.5.5. Remark. If C_X is a Grothendieck category and $\mathcal{S}$ is a Serre subcategory of C_X, then the localization functor $C_X \longrightarrow C_X/\mathcal{S}$ has a right adjoint, hence $C_X/\mathcal{S}$ is Grothendieck category (see [BD, Ch.6]). In particular, all $C_X/\mathcal{T}_i$ are Grothendieck categories. One can regard Grothendieck categories with a generator of finite type as a noncommutative version of a quasi-affine scheme.

Recall that quasi-affine commutative schemes are, by definition, quasi-compact open subschemes of affine commutative schemes.

10.6. Geometric centers of a 'space' X associated with $\mathbf{Spec}^0_{\mathfrak{c}}(X)$.

10.6.1. The geometric center associated with a topology on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. Let τ be a topology on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. For every open subset $\mathcal{U}$, let $\widetilde{\mathcal{O}}_X(\mathcal{U})$ denote the *center* of the quotient category $C_X/\langle\mathcal{U}\rangle$, where $\langle\mathcal{U}\rangle = \bigcap_{\mathcal{Q}\in\mathcal{U}} \langle\mathcal{Q}\rangle$. Recall that the center of the category C_Y is the (commutative) ring of the endomorphisms of the identical functor $C_Y \longrightarrow C_Y$.

The correspondence $\mathcal{U} \longmapsto \widetilde{\mathcal{O}}_X(\mathcal{U})$ is a presheaf of commutative rings on the topological space $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \tau)$. We denote by $\mathcal{O}_X$ the associated sheaf. The ringed space $((\mathbf{Spec}^0_{\mathfrak{c}}(X), \tau), \mathcal{O}_X)$ is called the *geometric center* of X associated with $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \tau)$.

10.6.2. Zariski geometric center. A topologizing subcategory $\mathbb{T}$ of C_X is called a *Zariski topologizing subcategory* if it is bireflective, i.e. the inclusion functor $\mathbb{T} \hookrightarrow C_X$ has right and left adjoints. Subsets $U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q} \not\subseteq \mathbb{T}\}$, where $\mathbb{T}$ runs through the preorder $\mathfrak{T}_{\mathfrak{z}}(X)$ of Zariski topologizing subcategories, are open sets of the Zariski topology on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ (cf. A1.2.4.1). We call the corresponding geometric center the *Zariski geometric center* of X.

10.6.3. The topologies $\tau_{\mathfrak{c}}$ and $\tau^{\mathfrak{c}}$ on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. The Zariski topology might be trivial or too coarse in the noncommutative case. For instance, it is trivial if $C_X = R-mod$, where the ring R is simple (i.e. does not have nonzero proper two-sided ideals), like, for instance, any Weyl algebra. Following A1.7, we introduce other topologies on $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ by fixing a set Ξ of coreflective topologizing subcategories of C_X and then taking subsets $U^0_{\mathfrak{c}}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \ \mathcal{Q} \not\subseteq \mathbb{T}\}$, $\mathbb{T} \in \Xi$, as a base of open sets of a topology τ_{Ξ}.

Taking $\Xi = \mathbf{Spec}^0_{\mathfrak{c}}(X)$, we obtain the topology $\tau_{\mathfrak{c}}$ (compare with A1.7.2). Thus the sets $V^0_{\mathfrak{c}}(\mathcal{Q}) = \{\mathcal{Q}' \in \mathbf{Spec}^0_{\mathfrak{c}}(X) \mid \mathcal{Q}' \subseteq \mathcal{Q}\}$, $\mathcal{Q} \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$, form a base of closed subsets of this topology.

The topology $\tau^{\mathfrak{c}}$ is determined by Ξ consisting of the subcategories $[M]_{\mathfrak{c}}$ spanned by objects which are *locally* of finite type. Here 'locally of finite type' means that the localization of M at every point $\mathcal{Q}$ of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$ (i.e. its image in the quotient category $C_X/\langle\mathcal{Q}\rangle$) is an object of finite type. It seems that $\tau^{\mathfrak{c}}$ is an appropriate version of Zariski topology for the spectrum $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. If C_X has enough objects of locally finite type, then the topology $\tau^{\mathfrak{c}}$ is finer than the topology $\tau_{\mathfrak{c}}$.

10.7. Reconstruction of commutative schemes.

10.7.1. Proposition. *Let C_X be the category of quasi-coherent sheaves on a scheme $\mathbf{X} = (\mathcal{X}, \mathcal{O})$. Suppose that there is an affine cover $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ of the scheme $\mathbf{X}$ such that all immersions $\mathcal{U}_i \hookrightarrow \mathcal{X}$, $i \in J$, have a direct image functor. Then the geometric center $(\mathbf{Spec}^0_{\mathfrak{c}}(X), \mathcal{O}_X)$ is isomorphic to the scheme $\mathbf{X}$.*

Proof. The argument follows the lines of the proof of 9.8.2.

(a) The underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$ is isomorphic to $\mathbf{Spec}^0_{\mathfrak{c}}(X)$.

Let $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ be an affine cover of the scheme $\mathbf{X}$. For each $i \in J$, we denote by C_{U_i} the category of quasi-coherent sheaves on the affine scheme $(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$ and by $\mathcal{T}_i$ the kernel of the inverse image functor $C_X \xrightarrow{u_i^*} C_{U_i}$. This inverse image functor uniquely determines the equivalence of the quotient category $C_X/\mathcal{T}_i$ and C_{U_i}. The fact that $\{\mathcal{U}_i \hookrightarrow \mathcal{X} \mid i \in J\}$ is a cover means precisely that $\bigcap_{i\in J} \mathcal{T}_i = 0$. The existence of a direct image functor, $C_{U_i} \xrightarrow{u_{i*}} C_X$, of the embedding $\mathcal{U}_i \hookrightarrow \mathcal{X}$ implies that the subcategory $\mathcal{T}_i$ is coreflective: a right adjoint to the inclusion functor $\mathcal{T}_i \longrightarrow C_X$ assigns to every object M of C_X the kernel of the adjunction morphism $M \longrightarrow u_{i*}u_i^*(M)$.

(a1) Let x be a point of the underlying space $\mathcal{X}$ of the scheme $\mathbf{X}$. Let $\mathcal{I}_{\bar{x}}$ be the defining ideal of the closure $\bar{x}$ of the point x and $\mathcal{M}_{\bar{x}}$ the quotient sheaf $\mathcal{O}/\mathcal{I}_{\bar{x}}$. Set $J_x = \{i \in J \mid \mathcal{M}_{\bar{x}} \notin Ob\mathcal{T}_i\}$. We claim that $\mathcal{Q}_x = [\mathcal{M}_{\bar{x}}]_{\mathfrak{c}}$ is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$.

For every $i \in J_x$, the object $u_i^*(\mathcal{M}_{\bar{x}})$ of the category C_{U_i} belongs to $Spec(U_i)$ and $Spec(U_i) = Spec^0_{\mathfrak{c}}(U_i)$, because C_{U_i} is (equivalent to) the category of modules over a ring. Therefore, $[u_i^*(\mathcal{Q}_x)]_{\mathfrak{c}} = [u_i^*(\mathcal{M}_{\bar{x}})]_{\mathfrak{c}}$ is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(U_i)$. By 10.4.3, $\mathcal{Q}_x \in \mathbf{Spec}^0_{\mathfrak{c}}(X)$.

(a2) Conversely, let $\mathcal{Q}$ be an element of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. Let $\mathcal{Q} \not\subseteq \mathcal{T}_i$, or, equivalently, $\mathcal{T}_i \subseteq \langle\mathcal{Q}\rangle$. By 10.4.3, $[u_i^*(\mathcal{Q})]_{\mathfrak{c}}$ is an element of $\mathbf{Spec}^0_{\mathfrak{c}}(U_i)$. Since U_i is affine, $\mathbf{Spec}^0_{\mathfrak{c}}(U_i)$

is in bijective correspondence with the underlying space $\mathcal{U}_i$ of the subscheme $(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$; in particular, to the element $[u_i^*(\mathcal{Q})]_{\mathfrak{c}}$ there corresponds a point x of $\mathcal{U}_i$ which we identify with its image in $\mathcal{X}$. Notice that the point x does not depend on the choice of $i \in J_{\langle \mathcal{Q} \rangle} = \{j \in J \mid \mathcal{T}_j \subseteq \langle \mathcal{Q} \rangle\}$. This gives a map $\mathbf{Spec}_{\mathfrak{c}}^0(X) \longrightarrow \mathcal{X}$ which is inverse to the map $\mathcal{X} \longrightarrow \mathbf{Spec}_{\mathfrak{c}}^0(X)$ constructed in (a1) above. These maps are homeomorphisms in the case if the cover consists of one element, i.e. the scheme is affine. The general case follows from the commutative diagrams

$$\begin{array}{ccc} \mathbf{Spec}_{\mathfrak{c}}^0(U_i) & \xrightarrow{\sim} & \mathcal{U}_i \\ \big\downarrow & & \big\downarrow \\ \mathbf{Spec}_{\mathfrak{c}}^0(X) & \longrightarrow & \mathcal{X}, \qquad i \in J. \end{array} \tag{6}$$

in which vertical arrows are open immersions and the upper horizontal arrow is a homeomorphism; hence the lower horizontal arrow is a homeomorphism.

(b) The diagrams (6) extend to the commutative diagrams of ringed spaces

$$\begin{array}{ccc} (\mathbf{Spec}_{\mathfrak{c}}^0(U_i), \mathcal{O}_{U_i}) & \xrightarrow{\sim} & (\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i}) \\ \big\downarrow & & \big\downarrow \\ (\mathbf{Spec}_{\mathfrak{c}}^0(X), \mathcal{O}_X) & \longrightarrow & (\mathcal{X}, \mathcal{O}) \qquad i \in J. \end{array} \tag{7}$$

in which $\mathbf{Spec}_{\mathfrak{c}}^0(U_i), \mathcal{O}_{U_i})$ and $(\mathbf{Spec}_{\mathfrak{c}}^0(X), \mathcal{O}_X)$ are Zariski geometric centra of resp. U_i and X, vertical arrows are open immersions and upper horizontal arrow is an isomorphism. Therefore the lower horizontal arrow is an isomorphism. ■

Appendix 1. The noncommutative cosite of topologizing subcategories and topologies on spectra.

A1.1. The noncommutative finite cosite of topologizing subcategories. We regard topologizing subcategories of an abelian category C_X as 'closed sets' of a 'finite topology' $\tau_X^{\mathfrak{f}}$ defined as follows. We call a set of inclusions $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ of topologizing subcategories a *cocover* if there exists a finite subset J_0 of J such that $\bigcap_{i \in J_0} \mathbb{T}_i = \mathbb{T}$.

Two of the three standard properties of cocovers follow immediately:

(a) $\mathbb{T} \xrightarrow{id} \mathbb{T}$ is a cocover;

(b) the composition of cocovers is a cocover: if $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ is a cocover and $\{\mathbb{T}_i \hookrightarrow \mathbb{T}_{ij} \mid j \in J_i\}$ is a cocover for every $i \in J$, then $\{\mathbb{T} \hookrightarrow \mathbb{T}_{ij} \mid i \in J,\ j \in J_i\}$ is a cocover.

The third standard property – the invariance under the base change, acquires the following form:

(c) If $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ is a cocover, then, for any $\mathbb{S} \in \mathfrak{T}(X)$, both $\{\mathbb{T}\bullet\mathbb{S} \hookrightarrow \mathbb{T}_i\bullet\mathbb{S} \mid i \in J\}$ and $\{\mathbb{S} \bullet \mathbb{T} \hookrightarrow \mathbb{S} \bullet \mathbb{T}_i \mid i \in J\}$ are cocovers.

The property (c) follows from [R4, 4.2.1]. Its proof is also contained in the argument of A1.2.3(b) below.

We call the triple $(\mathfrak{T}(X), \bullet; \tau_X^{\mathfrak{f}})$ the *noncommutative finite cosite* of topologizing subcategories of C_X.

A1.1.1. Note. One can define a finer topological structure on $\mathfrak{T}(X)$ by taking as cocovers all sets of inclusions $\{\mathbb{T} \hookrightarrow \mathbb{T}_i \mid i \in J\}$ such that $\bigcap_{i \in J} \mathbb{T}_i = \mathbb{T}$. The family τ_X of such cocovers satisfies the conditions (a) and (b) above, but fails, in general, the invariance with respect to a base change. The situation improves when one considers instead of all topologizing subcategories coreflective or reflective topologizing categories. This is made precise below.

A1.2. Coreflective and reflective topologizing categories. Let $\mathfrak{T}_{\mathfrak{c}}(X)$ (resp. $\mathfrak{T}^{\mathfrak{c}}(X)$) denote the preorder of all coreflective (resp. reflective) topologizing subcategories of the category C_X. Recall that a subcategory $\mathcal{B}$ of C_X is called *coreflective* (resp. *reflective*) if the inclusion functor $\mathcal{B} \hookrightarrow C_X$ has a right (resp. left) adjoint. By [R, III.6.2.1], both $\mathfrak{T}_{\mathfrak{c}}(X)$ and $\mathfrak{T}^{\mathfrak{c}}(X)$ are monoidal subcategories of the monoidal category (preorder) $(\mathfrak{T}(X), \bullet)$. We shall use the same notation – $\tau_X^{\mathfrak{f}}$, for the restrictions of the topological structure $\tau_X^{\mathfrak{f}}$ defined in A1.1 to $\mathfrak{T}_{\mathfrak{c}}(X)$ and to $\mathfrak{T}^{\mathfrak{c}}(X)$.

Notice that $(\mathfrak{T}_{\mathfrak{c}}(X), \bullet; \tau_X^{\mathfrak{f}})$ is naturally anti-isomorphic to $(\mathfrak{T}^{\mathfrak{c}}(X^o), \bullet; \tau_{X^o}^{\mathfrak{f}})$.

The term 'anti-isomorphic' refers to the monoidal structure $(\mathbb{S} \bullet \mathbb{T})^o = \mathbb{T}^o \bullet \mathbb{S}^o$, where $\mathbb{S}^o$ denotes the subcategory of $C_{X^o} = C_X^{op}$ corresponding to the subcategory $\mathbb{S}$ of C_X.

A1.2.1. Lemma. *Suppose that C_X is an abelian category with supremums of sets of subobjects (for instance, if C_X has infinite coproducts). Then*

(a) The intersection of any set of reflective topologizing subcategories is a reflective topologizing subcategory.

(b) $\big(\bigcap_{i\in J}\mathbb{T}_i\big)\bullet\mathbb{S} = \bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{S})$ *for any set* $\{\mathbb{T}_i \mid i\in J\}$ *of topologizing subcategories and any coreflective subcategory* $\mathbb{S}$.

Proof. (a) See [R, III.6.2.2].

(b) The inclusion $\big(\bigcap_{i\in J}\mathbb{T}_i\big)\bullet\mathbb{S} \subseteq \bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{S})$ is obvious. On the other hand, let M be an object of the subcategory $\bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{S})$; that is, for every $i\in J$, there exists an exact sequence $0\longrightarrow M_i\longrightarrow M\longrightarrow L_i\longrightarrow 0$ such that $M_i\in Ob\mathbb{S}$ and $L_i\in Ob\mathbb{T}_i$. Since $\mathbb{S}$ is a coreflective subcategory, the supremum M_J of the set of subobjects $\{M_i \mid i\in J\}$ is an object of $\mathbb{S}$. The canonical epimorphism $M\longrightarrow M/M_J$ factors through $M\longrightarrow L_i$ for each $i\in J$. Therefore, the object M/M_J, being a quotient of the object L_i, belongs to the subcategory $\mathbb{T}_i$ for each $i\in J$, hence it belongs to $\bigcap_{i\in J}\mathbb{T}_i$. ■

A1.2.2. Corollary. *Let C_X be an abelian category with infinums of sets of quotient objects (i.e. the dual category $C_{X^o} = C_X^{op}$ has maximums of sets of subobjects). Then*

(a) The intersection of any set of coreflective topologizing subcategories is a coreflective topologizing subcategory.

(b) $\mathbb{S}\bullet\big(\bigcap_{i\in J}\mathbb{T}_i\big) = \bigcap_{i\in J}(\mathbb{S}\bullet\mathbb{T}_i)$ *for any set* $\{\mathbb{T}_i \mid i\in J\}$ *of topologizing subcategories and any reflective subcategory* $\mathbb{S}$.

Proof. The assertion is dual to the assertion of A1.2.1. ■

A1.2.3. Corollary. *Let C_X be an abelian category with supremums of sets of subobjects and infinums of sets of quotient objects. Then*

(a) The intersection of any set of reflective (resp. coreflective) topologizing subcategories is a reflective (resp. coreflective) topologizing subcategory.

(b) If $\mathbb{S}$ is a reflective and $\mathbb{U}$ a coreflective topologizing subcategory of C_X, then

$$\mathbb{S}\bullet\big(\bigcap_{i\in J}\mathbb{T}_i\big) = \bigcap_{i\in J}(\mathbb{S}\bullet\mathbb{T}_i) \quad \textit{and} \quad \big(\bigcap_{i\in J}\mathbb{T}_i\big)\bullet\mathbb{U} = \bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{U})$$

for any set $\{\mathbb{T}_i \mid i\in J\}$ of topologizing subcategories.

A1.2.3.1. Note. The conditions on C_X in A1.2.3 hold if the category C_X has infinite products and coproducts. In particular, they hold for any Grothendieck category, or the category of quasi-coherent sheaves on an arbitrary scheme.

A1.2.4. Interpretations. Let τ_X denote the family of cocovers on $\mathfrak{T}(X)$ in the sense of A1.1.1; that is $\{\mathbb{T}\hookrightarrow\mathbb{T}_i \mid i\in J\}$ is a cocover iff the intersection of all $\mathbb{T}_i$ coincides with $\mathbb{T}$. Corollary A1.2.3 can be spelled as follows:

If the category C_X has supremums of sets of subobjects and infinums of sets of quotient objects, then τ_X induces the structure of a *right cosite* on the monoid $(\mathfrak{T}_{\mathfrak{c}}(X),\bullet)$ of coreflective topologizing subcategories and the structure of a *left cosite* on the monoid $(\mathfrak{T}^{\mathfrak{c}}(X),\bullet)$ of reflective topologizing subcategories.

A1.2.4.1. Zariski cosite. We denote by $\mathfrak{T}_{\mathfrak{z}}(X)$ the intersection $\mathfrak{T}_{\mathfrak{c}}(X)\bigcap\mathfrak{T}^{\mathfrak{c}}(X)$. Objects of $\mathfrak{T}_{\mathfrak{z}}(X)$ – *bireflective* subcategories of C_X, are interpreted as Zariski closed subspaces. By this reason, we shall call them sometimes *Zariski* topologizing subcategories. Under conditions of A1.2.3 (i.e. if C_X and C_{X^o} have supremums of sets of subobjects), $(\mathfrak{T}_{\mathfrak{z}}(X),\bullet;\tau_X)$ is a *two-sided cosite.* The latter means that for any set $\{\mathbb{S},\ \mathbb{T}_i\mid i\in J\}$ of Zariski topologizing subcategories, the intersection $\bigcap\limits_{i\in J}\mathbb{T}_i$ is a Zariski topologizing subcategory and

$$\mathbb{S}\bullet\big(\bigcap_{i\in J}\mathbb{T}_i\big)=\bigcap_{i\in J}(\mathbb{S}\bullet\mathbb{T}_i),\qquad \big(\bigcap_{i\in J}\mathbb{T}_i\big)\bullet\mathbb{S}=\bigcap_{i\in J}(\mathbb{T}_i\bullet\mathbb{S}). \tag{1}$$

We call $(\mathfrak{T}_{\mathfrak{z}}(X),\bullet;\tau_X)$ *the noncommutative Zariski finite cosite of the 'space' X.* One of the reasons for these interpretaions comes from the following example.

A1.2.5. Example. Let $C_X = R-mod$ for an associative ring R. For every two-sided ideal α in R, let $\mathbb{T}_\alpha$ denote the full subcategory of $R-mod$ whose objects are modules annihilated by the ideal α. By [R, III.6.4.1], the map $\alpha\longmapsto\mathbb{T}_\alpha$ is an isomorphism of the preorder $(I(R),\supseteq)$ of two-sided ideals of the ring R onto $(\mathfrak{T}^{\mathfrak{c}}(X),\subseteq)$. Moreover, $\mathbb{T}_\alpha\bullet\mathbb{T}_\beta=\mathbb{T}_{\alpha\beta}$ for any pair of two-sided ideals $\alpha,\ \beta$. This means that the map $\alpha\longmapsto\mathbb{T}_\alpha$ is an isomorphism of monoidal categories (preorders), where the monoidal structure on $I(R)$ is the multiplication of ideals.

It follows from this description that every reflective topologizing subcategory of $C_X = R-mod$ is coreflective, that is $\mathfrak{T}^{\mathfrak{c}}(X)=\mathfrak{T}_{\mathfrak{z}}(X)$.

One can see that $\bigcap\limits_{i\in J}\mathbb{T}_{\alpha_i}=\mathbb{T}_{\alpha_{_J}}$, where $\alpha_{_J}=\sup(\alpha_i|i\in J)$. Thus, the cotopology τ_X on $\mathfrak{T}^{\mathfrak{c}}(X)=\mathfrak{T}_{\mathfrak{z}}(X)$ induces a (noncommutative) Zariski topology on $I(R)$: the set of inclusions of two-sided ideals $\{\alpha_i\hookrightarrow\alpha\mid i\in J\}$ is a *cover* if $\alpha=\sup(\alpha_i|i\in J_0)$.

The invariance with respect to base change in $I(R)$ is expressed by the equalities

$$\beta\sup(\alpha_i|\ i\in J)=\sup(\beta\alpha_i|\ i\in J)\quad\text{ and }\quad\sup(\alpha_i|\ i\in J)\beta=\sup(\alpha_i\beta|\ i\in J)$$

for any set of two-sided ideals $\{\beta,\ \alpha_i\mid i\in J\}$. One can deduce directly from these equalities the base change invariance on $\mathfrak{T}_{\mathfrak{z}}(X)$ (in the case when $C_X = R-mod$). In fact, we have

$$\bigcap_{i\in J}(\mathbb{T}_{\alpha_i}\bullet\mathbb{T}_\beta)=\bigcap_{i\in J}\mathbb{T}_{\alpha_i\beta}=\mathbb{T}_{\sup(\alpha_i\beta|\ i\in J)}=\mathbb{T}_{\sup(\alpha_i|\ i\in J)\beta}=$$
$$\mathbb{T}_{\sup(\alpha_i|\ i\in J)}\bullet\mathbb{T}_\beta=\big(\bigcap_{i\in J}\mathbb{T}_{\alpha_i}\big)\bullet\mathbb{T}_\beta.$$

Similar calculation shows that $\bigcap\limits_{i\in J}(\mathbb{T}_\beta\bullet\mathbb{T}_{\alpha_i})=\mathbb{T}_\beta\bullet\big(\bigcap\limits_{i\in J}\mathbb{T}_{\alpha_i}\big)$.

A1.2.6. Example: reflective topologizing subcategories of the category of quasi-coherent sheaves on a scheme. Let C_X be the category $Qcoh_{\mathbf{X}}$ of quasi-coherent sheaves on a scheme $\mathbf{X}=(\mathcal{X},\mathcal{O}_{\mathcal{X}})$. Then elements of $\mathfrak{T}^{\mathfrak{c}}(X)$ are in one to one correspondence with quasi-coherent ideals of the structure sheaf $\mathcal{O}_{\mathcal{X}}$, or, equivalently, with closed subschemes of the scheme $\mathbf{X}$.

A1.3. Example: coreflective topologizing subcategories of an affine 'space'. Let C_X be the category $R - mod$ of left modules over an associative ring R. We denote by $I_\ell(R)$ the set of left ideals of R.

Recall that a set $\mathfrak{F}$ of left ideals of R is called a *topologizing filter* if it is closed under finite intersections, contains with every left ideal $\mathfrak{m}$ left ideals $(\mathfrak{m} : r) = \{a \in R \mid ar \in \mathfrak{m}\}$ for all $r \in R$ and all left ideals containing $\mathfrak{m}$.

Topologizing filters of left ideals form a monoidal category (a preorder) with respect to the *Gabriel multiplication* defined as follows.

$$\mathfrak{F} \circ \mathfrak{G} = \bigcup_{\mathfrak{m} \in \mathfrak{G}} \mathfrak{F} \circ \{\mathfrak{m}\}, \quad \text{where} \quad \mathfrak{F} \circ \{\mathfrak{m}\} = \{\mathfrak{n} \in I_\ell(R) \mid (\mathfrak{n} : r) \in \mathfrak{F} \text{ for all } r \in \mathfrak{m}\}.$$

There is a natural bijective correspondence between topologizing filters of left ideals and coreflective topologizing subcategories of the category $R - mod$. Namely, to each coreflective topologizing subcategory $\mathbb{T}$ of $R - mod$, there corresponds the filter $\mathfrak{F}_{\mathbb{T}}$ formed by annihilators of elements of modules from $\mathbb{T}$. The inverse map assigns to each topologizing filter $\mathfrak{F}$ the full subcategory $\mathbb{T}_{\mathfrak{F}}$ whose objects are all R-modules M such that each element of M is annihilated by some left ideal from $\mathfrak{F}$. These maps are mutually inverse isomorphisms between the monoidal preorder $(\mathfrak{T}\mathfrak{F}_\ell(R), \circ)$ of topologizing filters of left ideals of the ring R and the monoidal preorder $(\mathfrak{T}_{\mathfrak{c}}(X), \bullet)$ of coreflective topologizing subcategories of $C_X = R - mod$.

To every left ideal $\mathfrak{m}$ in R, one can assign the smallest topologizing filter $[\mathfrak{m}]$ containing $\mathfrak{m}$. It is easy to see that $[\mathfrak{m}]$ consists of all left ideals $\mathfrak{n}$ which contain $(\mathfrak{m} : x) = \{r \in R \mid rx \subset \mathfrak{m}\}$ for some finite set x of elements of R. The corresponding coreflective topologizing subcategory is formed by all R-modules M such that every element of M is annihilated by the left ideal $(\mathfrak{m} : x)$ for some finite set x of elements of R.

Notice that if $\mathfrak{m}$ is a two-sided ideal, then $\mathfrak{m} \subseteq (\mathfrak{m} : x)$ for any $x \subset R$. In this case the filter $[\mathfrak{m}]$ consists of all left ideals of R containing $\mathfrak{m}$ and the corresponding topologizing subcategory coincides with the subcategory $\mathbb{T}_{\mathfrak{m}}$ whose objects are modules annihilated by $\mathfrak{m}$ (see A1.2.1). If α and β are two-sided ideals, then $[\alpha] \circ [\beta] = [\alpha\beta]$. This shows that the map $I(R) \longrightarrow \mathfrak{T}\mathfrak{F}_\ell(R),\ \alpha \longmapsto [\alpha]$ is an embedding of monoidal preorders.

Thus, we have a commutative diagram of morphisms of monoidal preorders

$$\begin{array}{ccc}
(I(R), \cdot) & \longrightarrow & (\mathfrak{T}\mathfrak{F}_\ell(R), \circ) \\
\wr\big\downarrow & & \big\downarrow\wr \\
(\mathfrak{T}^{\mathfrak{c}}(X), \bullet) & \longrightarrow & (\mathfrak{T}_{\mathfrak{c}}(X), \bullet)
\end{array}$$

where the lower horizontal arrow is the inclusion functor and $C_X = R - mod$.

A1.4. The cosites of thick and Serre subcategories. Consider the preorder $\mathfrak{Th}(X)$ of thick subcategories and the preorder $\mathfrak{Se}(X)$ of Serre subcategories of the category C_X together with cocovers induced from $\mathfrak{T}(X)$.

A1.4.1. Proposition. *(a) $\mathfrak{Th}(X)$ and $\mathfrak{Se}(X)$ are Grothendieck precosites.*

(b) The map $\mathfrak{T}(X) \longrightarrow \mathfrak{Th}(X)$ which assigns to every topologizing subcategory $\mathbb{T}$ the thick subcategory $\mathbb{T}^\infty$ generated by $\mathbb{T}$ is a morphism from the noncommutative precosite $(\mathfrak{T}(X), \bullet)$ of topologizing subcategories to the Grothendick precosite $\mathfrak{Th}(X)$ of thick subcategories of C_X.

(c) The map $\mathfrak{T}(X) \longrightarrow \mathfrak{Se}(X)$, $\mathbb{T} \longmapsto \mathbb{T}^-$, is a morphism from the noncommutative precosite $(\mathfrak{T}(X), \bullet)$ to the Grothendick precosite $\mathfrak{Se}(X)$ of Serre subcategories.

Proof. For any pair of topologizing subcategories $\mathbb{S}$ and $\mathbb{T}$, the thick subcategory $(\mathbb{S}\bullet\mathbb{T})^\infty$ generated by $\mathbb{S}\bullet\mathbb{T}$ coincides with the coproduct $\mathbb{S}^\infty \sqcup \mathbb{T}^\infty$ of the thick subcategories $\mathbb{S}^\infty$ and $\mathbb{T}^\infty$ generated respectively by $\mathbb{S}$ and $\mathbb{T}$.

Similarly, $(\mathbb{S} \bullet \mathbb{T})^-$ coincides with the coproduct $\mathbb{S}^- \vee \mathbb{T}^-$ of Serre subcategories generated respectively by $\mathbb{S}$ and $\mathbb{T}$.

In other words, the maps

$$\mathfrak{T}(X) \longrightarrow \mathfrak{Th}(X),\ \mathbb{T} \longmapsto \mathbb{T}^\infty, \quad \text{and} \quad \mathfrak{T}(X) \longrightarrow \mathfrak{Se}(X),\ \mathbb{T} \longmapsto \mathbb{T}^-, \tag{1}$$

are morphisms of monoidal preorders resp.

$$(\mathfrak{T}(X), \bullet) \longrightarrow (\mathfrak{Th}(X), \sqcup) \quad \text{and} \quad (\mathfrak{T}(X), \bullet) \longrightarrow (\mathfrak{Se}(X), \vee).$$

It remains to verify that the maps (1) transfer cocovers to cocovers, or, equivalently, for any finite set $\{\mathbb{T}_i \mid i \in J\}$ of topologizing subcategories of C_X, there are equalities

$$\big(\bigcap_{i\in J} \mathbb{T}_i\big)^\infty = \bigcap_{i\in J} \mathbb{T}_i^\infty \quad \text{and} \quad \big(\bigcap_{i\in J} \mathbb{T}_i\big)^- = \bigcap_{i\in J} \mathbb{T}_i^-.$$

The first equality is proven in [R4, 4.6.1] and the second equality is the assertion [R4, 4.1].

Altogether proves (b) and (c). The assertion (a) is a consequence of (b) and (c).

In fact, for any finite set $\{\mathbb{S},\ \mathbb{T}_i \mid i \in J\}$ of thick subcategories, we have by (b)

$$\bigcap_{i\in J}(\mathcal{T}_i \sqcup \mathcal{S}) = \bigcap_{i\in J}(\mathcal{S} \bullet \mathcal{T}_i)^\infty = \big(\bigcap_{i\in J} \mathcal{S} \bullet \mathcal{T}_i\big)^\infty = \big(\mathcal{S} \bullet (\bigcap_{i\in J} \mathcal{T}_i)\big)^\infty = \mathcal{S} \sqcup \big(\bigcap_{i\in J} \mathcal{T}_i\big),$$

Similarly, if $\{\mathbb{S},\ \mathbb{T}_i \mid i \in J\}$ are Serre subcategories, then it follows from (c) that

$$\bigcap_{i\in J}(\mathcal{T}_i \vee \mathcal{S}) = \bigcap_{i\in J}(\mathcal{S} \bullet \mathcal{T}_i)^- = \big(\bigcap_{i\in J} \mathcal{S} \bullet \mathcal{T}_i\big)^- = \big(\mathcal{S} \bullet (\bigcap_{i\in J} \mathcal{T}_i)\big)^- = \mathcal{S} \vee \big(\bigcap_{i\in J} \mathcal{T}_i\big),$$

hence the assertion. ∎

A1.5. Monoidal subcategories of $(\mathfrak{T}(X), \bullet)$ and topologies on spectra. Any full monoidal subcategory $\mathfrak{G}$ of $(\mathfrak{T}(X), \bullet)$ closed under arbitrary intersections defines a topology $\tau_{\mathfrak{G}}$ on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ (hence on $\mathbf{Spec}(X)$) by taking $V^0_{\mathfrak{t}}(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}^0_{\mathfrak{t}}(X) \mid \mathcal{P} \subseteq \mathbb{T}\}$ (resp. $V(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}(X) \mid \mathcal{P} \subseteq \mathbb{T}\}$), $\mathbb{T} \in \mathfrak{G}$, as the set of closed subsets.

The map $\mathfrak{G} \longmapsto \tau_{\mathfrak{G}}$ is a surjective map from the family of full monoidal subcategories of $(\mathfrak{T}(X), \bullet)$ closed under arbitrary intersections onto the set of topologies on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ which are coarser than the topology $\tau^0_{\mathfrak{t}}$ corresponding to $\mathfrak{T}(X)$.

A1.6. Zariski topology. Suppose that the category C_X has supremums of sets of subobjects (for instance, C_X has infinite coproducts). Then, by [R, III.6.2.2], the intersection of any set of reflective topologizing subcategories is a reflective topologizing subcategory. Taking as $\mathfrak{G}$ the subcategory $\mathfrak{T}^{\mathfrak{c}}(X)$ of reflective topologizing subcategories, we obtain the *Zariski* topology on $\mathbf{Spec}_{\mathfrak{t}}^0(X)$ which we denote by $\tau_{\mathfrak{z}}^0$. Its restriction to $\mathbf{Spec}(X)$ will be denoted by $\tau_{\mathfrak{z}}$.

A1.6.1. Proposition. *Suppose C_X has the property (sup) and a generator of finite type. Then the topological space $(\mathbf{Spec}(X), \tau_{\mathfrak{z}})$ is quasi-compact.*

Proof. See [R, III.6.5.2.1]. ■

A1.6.2. Example. Example A1.2.6 shows that if C_X is the category of quasi-coherent sheaves on a (commutative) scheme $\mathbf{X}$, then the elements of $\mathfrak{T}^{\mathfrak{c}}(X)$ are categories of quasi-coherent sheaves on closed subschemes of $\mathbf{X}$. Suppose that the scheme $\mathbf{X}$ is quasi-compact and quasi-separated (more generally, quasi-compact and the embeddings of every point has a direct image functor). Then $\mathbf{Spec}(X)$ is the set of points of the underlying space of the scheme $\mathbf{X}$ and closed sets of the Zariski topology on $\mathbf{Spec}(X)$ are spectra of closed subschemes. So that the Zariski topology on $\mathbf{Spec}(X)$ coincides with the Zariski topology in the conventional sense.

A1.6.3. Example: Zariski topology on an affine noncommutative scheme. Let $C_X = R - mod$ for an associative unital ring R. It follows from A1.6.1 that the topological space $(\mathbf{Spec}(X), \tau_{\mathfrak{z}})$ is quasi-compact.

This fact is a special case of a much stronger assertion: the open subset $\mathcal{U}$ of the space $(\mathbf{Spec}(X), \tau_{\mathfrak{z}})$ is quasi-compact iff $\mathcal{U} = U(\mathbb{T}_\alpha) = \mathbf{Spec}(X) - V(\mathbb{T}_\alpha)$ for a finitely generated two-sided ideal α of the ring R (cf. A1.2.1).

Two different proofs of this theorem can be found in [R]: I.5.6 and III.6.5.3.1. One of its consequences is that quasi-compact open sets form a base of the Zariski topology on $\mathbf{Spec}(X)$. In fact, every two-sided ideal α is the supremum of a set $\{\alpha_i \mid i \in J\}$ of its two-sided subideals, so that $U(\mathbb{T}_\alpha) = U(sup(\mathbb{T}_{\alpha_i} \mid i \in J)) = \bigcup_{i \in J} U(\mathbb{T}_{\alpha_i})$ (see A1.2.1).

A1.6.4. Note. Unlike the commutative case, the Zariski topology is trivial or too coarse in many important examples of noncommutative affine schemes. It follows from the previous discussion that if $C_X = R - mod$, then the Zariski topology on $\mathbf{Spec}(X)$ is trivial iff R is a simple ring (i.e. it does not have non-trivial two-sided ideals). In particular, the Zariski topology on $\mathbf{Spec}(X)$ is trivial if C_X is the category of D-modules on the affine space $\mathbb{A}^n$, because the algebra A_n of differential operators on $\mathbb{A}^n$ is simple. It is not sufficiently rich in the case when C_X is the category of representations of a semisimple Lie algebra over a field of characteristic zero.

A1.7. Some other canonical topologies. A way to define a topology on $\mathbf{Spec}_{\mathfrak{t}}^0(X)$ (and on $\mathbf{Spec}(X)$) is to single out a class of topologizing subcategories, Ξ, of C_X, take the smallest monoidal subcategory $\mathfrak{G}_\Xi$ of $(\mathfrak{T}(X), \bullet)$ which contains Ξ and is closed under arbitrary intersections (which are products in $(\mathfrak{T}(X), \subseteq)$) and obtain this way the topology $\tau_{\mathfrak{G}_\Xi}^0$. This is the same as taking the smallest topology on $\mathbf{Spec}_{\mathfrak{t}}^0(X)$ for which the sets $V_{\mathfrak{t}}^0(\mathbb{T})$, $\mathbb{T} \in \Xi$, are closed.

A1.7.1. The topology τ^*. For instance, taking as Ξ the class of all topologizing subcategories $[M]$, where M is an object of finite type, we obtain a topology τ^* on $\mathbf{Spec}(X)$ which in the case when C_X is the category of modules over a commutative ring (more generally, the category of quasi-coherent sheaves on a quasi-compact quasi-separated scheme; see A1.2.6 above) coincides with the Zariski topology. It is drastically different in most of noncommutative cases. For any simple ring R (in particular, for any Weyl algebra A_n), the Zariski topology is trivial, while the topology τ^* separates distinct points of the spectrum in Kolmogorov's sense, i.e. $(\mathbf{Spec}(X), \tau^*)$ is a Kolmogorov's space.

A1.7.2. The topology $\tau_{\mathfrak{s}}$. We take as Ξ the set $\mathbf{Spec}(X)$ and denote the corresponding topology on $\mathbf{Spec}(X)$ by $\tau_{\mathfrak{s}}$. This means that finite unions of sets $V(\mathcal{P})$ form a base of the closed sets of the topology $\tau_{\mathfrak{s}}$.

Notice that if the category C_X has enough objects of finite type (i.e. every nonzero object of C_X has a nonzero subobject of finite type), then the topology $\tau_{\mathfrak{s}}$ is coarser than the topology τ^*. In fact, in this case every element $\mathcal{P}$ of $\mathbf{Spec}(X)$ is of the form $[M]$ for some object M of finite type.

A1.8. Functorialities.

A1.8.1. Proposition. *Let C_X and C_Y be abelian categories, and let $X \xrightarrow{f} Y$ be a continuous morphism such that adjunction arrows $f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}$ and $Id_{C_Y} \xrightarrow{\eta_f} f_*f^*$ are monomorphisms. Then the map $\mathbb{T} \longmapsto [f^{*^{-1}}(\mathbb{T})]$ defines a morphism of monoids $(\mathfrak{T}(X), \bullet) \xrightarrow{\mathfrak{T}(f)} (\mathfrak{T}(Y), \bullet)$.*

Proof. (a) Let $X \xrightarrow{f} Y$ be a morphism such that f^* is *semi-exact*; i.e. f^* maps any exact sequence $M' \longrightarrow M \longrightarrow M''$ to an exact sequence (for instance, f^* is right, or left exact). Then $f^{*^{-1}}(\mathbb{T}) \bullet f^{*^{-1}}(\mathbb{S}) \subseteq f^{*^{-1}}(\mathbb{T} \bullet \mathbb{S})$ for any pair $\mathbb{T}$, $\mathbb{S}$ of subcategories of C_X. In particular, $[f^{*^{-1}}(\mathbb{T})] \bullet [f^{*^{-1}}(\mathbb{S})] \subseteq [f^{*^{-1}}(\mathbb{T} \bullet \mathbb{S})]$.

In fact, if $M' \longrightarrow M \longrightarrow M''$ is an exact sequence with $f^*(M') \in Ob\mathbb{S}$ and $f^*(M'') \in Ob\mathbb{T}$, then $f^*(M) \in Ob\mathbb{T} \bullet \mathbb{S}$, because the sequence $f^*(M') \longrightarrow f^*(M) \longrightarrow f^*(M'')$ is exact, due to the semi-exactness of the functor f^*.

(a1) Notice that the inverse image functor of a continuous morphism is right exact, hence semi-exact.

(b) In order to prove the inverse inclusion, $[f^{*^{-1}}(\mathbb{T})] \bullet [f^{*^{-1}}(\mathbb{S})] \supseteq [f^{*^{-1}}(\mathbb{T} \bullet \mathbb{S})]$, it suffices to show that $[f^{*^{-1}}(\mathbb{T})] \bullet [f^{*^{-1}}(\mathbb{S})] \supseteq f^{*^{-1}}(\mathbb{T} \bullet \mathbb{S})$.

Let $f^*(M) \in Ob\mathbb{T} \bullet \mathbb{S}$; i.e. there is an exact sequence $L' \longrightarrow f^*(M) \longrightarrow L''$ with $L' \in Ob\mathbb{S}$ and $L'' \in Ob\mathbb{T}$. Consider the commutative diagram

$$\begin{array}{ccccc}
f^*f_*(L') & \longrightarrow & f^*f_*f^*(M) & \longrightarrow & f^*f_*(L') \\
{\scriptstyle \epsilon_f(L')}\downarrow & & {\scriptstyle \epsilon_f f^*(M)}\downarrow & & \downarrow{\scriptstyle \epsilon_f(L'')} \\
L' & \longrightarrow & f^*(M) & \longrightarrow & L''
\end{array}$$

Since ϵ_f is a monomorphism and $\epsilon_f f^*(M)$ is a strict epimorphism (coretraction), $\epsilon_f f^*(M)$ is an isomorphism. The monomorphness of $\epsilon_f(L')$ and $\epsilon_f(L'')$ imply that $f^*f_*(L') \in Ob\mathbb{S}$ and $f^*f_*(L'') \in Ob\mathbb{T}$. Thus, we have an exact sequence $f_*(L') \longrightarrow f_*f^*(M) \longrightarrow f_*(L'')$

with $f_*(L') \in Obf^{*^{-1}}(\mathbb{S})$ and $f_*(L'') \in Obf^{*^{-1}}(\mathbb{T})$, hence $f_*f^*(M) \in Obf^{*^{-1}}(\mathbb{T}) \bullet f^{*^{-1}}(\mathbb{S})$. If the adjunction morphism $M \longrightarrow f_*f^*(M)$ is a monoarrow, the object M belongs to the subcategory $[f^{*^{-1}}(\mathbb{T}) \bullet f^{*^{-1}}(\mathbb{S})] = [f^{*^{-1}}(\mathbb{T})] \bullet [f^{*^{-1}}(\mathbb{S})]$. ■

A1.8.2. Note. The conditions of A1.8.1 hold if C_Y is a coreflective full subctegory of C_X and f^* is the inclusion functor $C_Y \hookrightarrow C_X$. In this case, the adjunction arrow $Id_{C_Y} \xrightarrow{\eta_f} f_*f^*$ is an isomorphism, and the second adjunction arrow, $f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}$, is a monomorphism.

Appendix 2. Complements on $\mathbf{Spec}_\mathfrak{t}^0(X)$ and $\mathbf{Spec}(X)$.

A2.1. The difference between $\mathbf{Spec}_\mathfrak{t}^0(X)$ and $\mathbf{Spec}(X)$. The following two assertions provide examples of elements of $\mathbf{Spec}_\mathfrak{t}^0(X)$ which do not belong to $\mathbf{Spec}(X)$.

A2.1.1 Proposition. *Let M be a semisimple object of the category C_X. Then the topologizing subcategory $[M]$ spanned by M belongs to $\mathbf{Spec}_\mathfrak{t}^{0,0}(X) = \mathbf{Spec}_\mathfrak{t}^0(X) - \mathbf{Spec}(X)$ iff M is isomorphic to an infinite coproduct of copies of a simple object L.*

The topologizing subcategory $[M]$ belongs to $\mathbf{Spec}(X)$ iff M is isomorphic to a finite coproduct of copies of a simple object L.

Proof. (a) Let M be isomorphic to the coproduct L^J of J copies of a simple object L.

(a1) If J is finite, then $[M] = [L]$ is a closed point of $\mathbf{Spec}(X)$.

Conversely, if $[M]$ belongs to $\mathbf{Spec}(X)$, then $[M] = [L]$, or, equivalently, $L \succ M$. The latter means that M is a subquotient of a finite coproduct of copies of the simple object L, hence M is isomorphic to a finite coproduct of copies of L.

(a2) Suppose that J is infinite and the object M belongs to the Gabriel product $\mathbb{S}_2 \bullet \mathbb{S}_1$ of two topologizing subcategories of C_X; that is there exists an exact sequence $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ such that $M_i \in Ob\mathbb{S}_i,\ i = 1, 2$. The objects M_1 and M_2, being subquotients of the semisimple object L^J are isomorphic to coproducts of resp. J_1 and J_2 copies of L. Since J is infinite, then either J_1 or J_2 should be isomorphic to J. If $J_i \simeq J$, then $M_i \simeq M$, hence $M \in Ob\mathbb{S}_i$.

(b) If M is a semisimple object which has non-isomorphic simple components, then $M \simeq M_1 \oplus M_2$, where $M_1,\ M_2$ are nonzero objects and $Hom(M_1, M_2) = 0$. This implies that $[M] = [M_1] \bullet [M_2]$ and M does not belong to $[M_1] \bigcup [M_2]$. By 4.3(a), $[M]$ is not an element of $\mathbf{Spec}_\mathfrak{t}^0(X)$. ■

A2.1.2. Corollary. *If M is isomorphic to the product of infinite set of copies of a simple object of the category C_X, then $[M]$ belongs to $\mathbf{Spec}_\mathfrak{t}^{0,0}(X)$.*

Proof. For every object N of C_X, let N^o denote the corresponding object of the dual category $C_{X^o} = C_X^{op}$. The object M^o is isomorphic to the coproduct of an infinite set of copies of a simple object L^o; hence, by A2.1.1, the topologizing subcategory $[M^o]$ belongs to $\mathbf{Spec}_\mathfrak{t}^{0,0}(X^o)$. But, as it was observed in 4.2, $\mathbf{Spec}_\mathfrak{t}^0(X)$ is naturally isomorphic to $\mathbf{Spec}_\mathfrak{t}^0(X^o)$; in particular, $[M]$ belongs to $\mathbf{Spec}_\mathfrak{t}^0(X)$. It remains to show that $[M]$ does not belong to $\mathbf{Spec}(X)$.

Suppose that, on the contrary, $[M]$ belongs to $\mathbf{Spec}(X)$, i.e. $\widehat{[M]} = \langle M \rangle$ is a Serre subcategory. Then M has a nonzero subquotient, M', which is $\langle M \rangle$-torsion free. It follows that $[M] = [M']$. Every subquotient of a product of a family of simple objects is isomorphic to the product of its subfamily. In particular, M' is isomorphic to the product of J copies of the simple object L; hence the simple object L is a subobject of M'. Since M' belongs to $Spec(X)$, this implies that $L \succ M'$, hence M' is isomorphic to a finite product of copies of L. But then the equality $[M] = [M']$ implies that $L \succ M$, hence M is isomorphic to a finite product of copies of L, which contradicts to the initial hypothesis. ■

A2.1.3. Note. Given an object L and a set J, let $L^{\oplus J}$ (resp. $L^{\Pi J}$) denote the direct sum, (resp. product) of J copies of the object L. Let L be a simple object of the category C_X, and let J and I be infinite sets such that $Card(I) < Card(J)$. Then there are proper inclusions $[L^{\oplus J}] \subsetneq [L^{\Pi J}] \subsetneq [L^{\Pi I}] \supsetneq [L^{\oplus I}] \supsetneq [L^{\oplus J}]$. In particular, $[L^{\oplus J}]$, $[L^{\oplus I}]$, $[L^{\Pi J}]$, and $[L^{\Pi I}]$ are four distinct elements of $\mathbf{Spec}_{\mathfrak{t}}^{0,0}(X)$.

A2.2. Functorialities and topologies. For any topologizing subcategory $\mathbb{T}$ of the category C_X, set

$$U_{\mathfrak{t}}^1(\mathbb{T}) = \{\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^1(X) \mid \mathbb{T} \subseteq \mathcal{P}\} \quad \text{and} \quad U_{\mathfrak{t}}^0(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}_{\mathfrak{t}}^0(X) \mid \mathcal{Q} \not\subseteq \mathbb{T}\}.$$

A2.2.1. Proposition. *(a) If $\mathbb{T}$, $\mathbb{S}$ are topologizing categories, then*

$$U_{\mathfrak{t}}^0(\mathbb{T} \bullet \mathbb{S}) = U_{\mathfrak{t}}^0(\mathbb{T}) \bigcap U_{\mathfrak{t}}^0(\mathbb{S}) \quad \textit{and} \quad U_{\mathfrak{t}}^1(\mathbb{T} \bullet \mathbb{S}) = U_{\mathfrak{t}}^1(\mathbb{T}) \bigcap U_{\mathfrak{t}}^1(\mathbb{S}).$$

In particular, $U_{\mathfrak{t}}^i(\mathbb{T}) = U_{\mathfrak{t}}^i([\mathbb{T}]_\bullet)$, where $[\mathbb{T}]_\bullet$ is the thick subcategory generated by $\mathbb{T}$ and $i = 0, 1$.

(b) $U_{\mathfrak{t}}^0(\bigcap_{j \in J} \mathbb{T}_j) = \bigcup_{j \in J} U_{\mathfrak{t}}^0(\mathbb{T}_j)$ for any family $\{\mathbb{T}_j \mid j \in J\}$ of topologizing categories.

(c) Suppose that $\mathbb{T}$ is a thick subcategory of C_X. Then there is a commutative diagram

$$\begin{array}{ccccc} \mathbf{Spec}_{\mathfrak{t}}^0(X) & \longleftarrow & U_{\mathfrak{t}}^0(\mathbb{T}) & \longrightarrow & \mathbf{Spec}_{\mathfrak{t}}^0(X/\mathbb{T}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Spec}_{\mathfrak{t}}^1(X) & \longleftarrow & U_{\mathfrak{t}}^1(\mathbb{T}) & \longrightarrow & \mathbf{Spec}_{\mathfrak{t}}^1(X/\mathbb{T}) \end{array} \qquad (4)$$

of injective maps.

Proof. (a) The equality $U_{\mathfrak{t}}^0(\mathbb{T} \bullet \mathbb{S}) = U_{\mathfrak{t}}^0(\mathbb{T}) \bigcap U_{\mathfrak{t}}^0(\mathbb{S})$ follows from 4.3(a).

The inclusions $\mathbb{S} \subseteq \mathbb{T} \bullet \mathbb{S} \supseteq \mathbb{T}$ imply that $U_{\mathfrak{t}}^1(\mathbb{T} \bullet \mathbb{S}) \subseteq U_{\mathfrak{t}}^1(\mathbb{T}) \bigcap U_{\mathfrak{t}}^1(\mathbb{S})$. The inverse inclusion follows from the fact that elements of $U_{\mathfrak{t}}^1(-)$ are thick subcategories: if $\mathcal{P}$ is a thick subcategory containing $\mathbb{S}$ and $\mathbb{T}$, then $\mathcal{P}$ contains $\mathbb{T} \bullet \mathbb{S}$.

(b) The equality $U_{\mathfrak{t}}^0(\bigcap_{j \in J} \mathbb{T}_j) = \bigcup_{j \in J} U_{\mathfrak{t}}^0(\mathbb{T}_j)$ is straightforward.

(c) Let $\mathbb{T}$ be a thick subcategory of C_X, and let $\mathcal{P} \in U_{\mathfrak{t}}^1(\mathbb{T})$, i.e. $\mathbb{T} \subseteq \mathcal{P} \neq \mathcal{P}^{\mathfrak{t}}$. Then the topologizing subcategory $(\mathbb{T} \bullet \mathcal{P}^{\mathfrak{t}} \bullet \mathbb{T})/\mathbb{T}$ of $C_X/\mathbb{T}$ coincides with $(\mathcal{P}/\mathbb{T})^{\mathfrak{t}}$. One can see that $(\mathbb{T} \bullet \mathcal{P}^{\mathfrak{t}} \bullet \mathbb{T})/\mathbb{T} \neq \mathcal{P}/\mathbb{T}$, because $\mathcal{P}^{\mathfrak{t}} \neq \mathcal{P}$; that is $\mathcal{P}/\mathbb{T}$ is an element of $\mathbf{Spec}_{\mathfrak{t}}^1(X/\mathbb{T})$. The map $U_{\mathfrak{t}}^1(\mathbb{T}) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^1(X/\mathbb{T})$ assigns to each $\mathcal{P}$ its quotient $\mathcal{P}/\mathbb{T}$.

The map $U_{\mathfrak{t}}^0(\mathbb{T}) \longrightarrow \mathbf{Spec}_{\mathfrak{t}}^0(X/\mathbb{T})$ assigns to each $\mathcal{Q} \in U_{\mathfrak{t}}^0(\mathbb{T})$ the topologizing subcategory $\mathcal{Q}_{\mathbb{T}}$ generated by the image of $\mathcal{Q}$ in $C_X/\mathbb{T}$, which coincides with $(\mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T})/\mathbb{T}$ (see

5.1.1). Since $\mathcal{Q} \not\subseteq \mathbb{T}$, the topologizing subcategory $\mathbb{T}$ is contained in the thick subcategory $\widehat{\mathcal{Q}}$; hence $\widehat{\mathcal{Q}}/\mathbb{T}$ is a thick subcategory of the quotient category $C_X/\mathbb{T}$. One can see that $\widehat{\mathcal{Q}}/\mathbb{T} = \widehat{\mathcal{Q}_{\mathbb{T}}}$; so that $\mathcal{Q}_{\mathbb{T}}$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^0(X/\mathbb{T})$. ■

A2.2.1.1. Corollary. *Suppose that $\mathbb{T}$ is a thick subcategory of C_X. Then there is a commutative diagram*

$$\begin{array}{ccccc}
\mathbf{Spec}(X) & \longleftarrow & U_{\mathfrak{t}}^{0,1}(\mathbb{T}) & \longrightarrow & \mathbf{Spec}(X/\mathbb{T}) \\
\wr\downarrow & & \downarrow & & \downarrow\wr \\
\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) & \longleftarrow & U_{\mathfrak{t}}^{1,1}(\mathbb{T}) & \longrightarrow & \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathbb{T}),
\end{array} \qquad (5)$$

where $U_{\mathfrak{t}}^{1,1}(\mathbb{T}) = \{\mathcal{P} \in U_{\mathfrak{t}}^1(\mathbb{T}) \mid \mathcal{P} = \mathcal{P}^-\}$, $U_{\mathfrak{t}}^{0,1}(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}(X) \mid \mathcal{Q} \not\subseteq \mathbb{T}\}$, *and all horizontal arrows are injective maps.*

Proof. The assertion is a consequence of A2.2.1(c). ■

A2.2.2. The 'specialization' topology on $\mathbf{Spec}_{\mathfrak{t}}^0(X)$. It follows from the assertions A2.2.1 (a) and (b) that $\{U_{\mathfrak{t}}^0(\mathbb{T}) \mid \mathbb{T} \in \mathfrak{T}(X)\}$ is the set of open sets of a topology on $\mathbf{Spec}_{\mathfrak{t}}^0(X)$ which we denote by $\tau_{\mathfrak{t}}^0$. Let $V_{\mathfrak{t}}^0(\mathbb{T})$ be the complement to $U_{\mathfrak{t}}^0(\mathbb{T})$ in $(\mathbf{Spec}_{\mathfrak{t}}^0(X)$, i.e.

$$V_{\mathfrak{t}}^0(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}_{\mathfrak{t}}^0(X) \mid \mathcal{Q} \subseteq \mathbb{T}\}.$$

Recall that a topological space $\mathfrak{X}$ is called *sober* if every irreducible closed subset of $\mathfrak{X}$ has a unique generic point.

A topological space is called a *Kolmogorov's space* if for every pair of its distinct points, one of the points has an open neighborhood which does not contain the other point.

A2.2.3. Proposition. $(\mathbf{Spec}_{\mathfrak{t}}^0(X), \tau_{\mathfrak{t}}^0)$ *is a sober Kolmogorov's topological space.*

Proof. (a) Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be distinct elements of $(\mathbf{Spec}_{\mathfrak{t}}^0(X), \tau_{\mathfrak{t}}^0)$. If $\mathcal{Q}_1 \not\subseteq \mathcal{Q}_2$, then $\mathcal{Q}_1 \in U_{\mathfrak{t}}^0(\mathcal{Q}_2) \not\ni \mathcal{Q}_2$. This shows that $(\mathbf{Spec}_{\mathfrak{t}}^0(X), \tau_{\mathfrak{t}}^0)$ is a Kolmogorov's space.

(b) For every topologizing subcategory $\mathbb{T}$, let $\mathbb{T}_{\mathfrak{red},0}$ denote the supremum of all $\mathcal{Q} \in V_{\mathfrak{t}}^0$. Here $\mathfrak{red}$ stands for 'reduced'. It follows that $\mathbb{T}_{\mathfrak{red},0}$ is the smallest topologizing subcategory of C_X such that $V_{\mathfrak{t}}^0(\mathbb{T}_{\mathfrak{red},0}) = V_{\mathfrak{t}}^0(\mathbb{T})$.

The closed set $V_{\mathfrak{t}}^0(\mathbb{T})$ is irreducible iff $\mathbb{T}_{\mathfrak{red},0}$ is an element of $\mathbf{Spec}_{\mathfrak{t}}^0(X)$.

In fact, suppose that $V_{\mathfrak{t}}^0(\mathbb{T})$ is irreducible, and let $\mathbb{S}_1$, $\mathbb{S}_2$ be a pair of topologizing subcategories such that $\mathbb{T}_{\mathfrak{red},0} \subseteq \mathbb{S}_1 \bullet \mathbb{S}_2$. Then

$$V_{\mathfrak{t}}^0(\mathbb{T}) = V_{\mathfrak{t}}^0(\mathbb{T}_{\mathfrak{red},0}) \subseteq V_{\mathfrak{t}}^0(\mathbb{S}_1 \bullet \mathbb{S}_2) = V_{\mathfrak{t}}^0(\mathbb{S}_1) \bigcup V_{\mathfrak{t}}^0(\mathbb{S}_2).$$

Since $V_{\mathfrak{t}}^0(\mathbb{T})$ is irreducible, it is contained in $V_{\mathfrak{t}}^0(\mathbb{S}_i)$ for $i = 1$, or $i = 2$. But then $\mathbb{T}_{\mathfrak{red},0} \subseteq \mathbb{S}_i$ for the same i. This shows that $\mathbb{T}_{\mathfrak{red},0}$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^0(X)$.

Conversely, if $\mathcal{Q}$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^0(X)$ and $\mathbb{S}_1$, $\mathbb{S}_2$ are topologizing subcategories such that $V_{\mathfrak{t}}^0(\mathcal{Q}) \subseteq V_{\mathfrak{t}}^0(\mathbb{S}_1) \bigcup V_{\mathfrak{t}}^0(\mathbb{S}_2) = V_{\mathfrak{t}}^0(\mathbb{S}_1 \bullet \mathbb{S}_2)$. Then $\mathcal{Q} \subseteq \mathbb{S}_1 \bullet \mathbb{S}_2$, hence $\mathcal{Q} \subseteq \mathbb{S}_i$ for $i = 1$ or 2, which, in turn, means that $V_{\mathfrak{t}}^0(\mathcal{Q}) \subseteq V_{\mathfrak{t}}^0(\mathbb{S}_i)$.

Altogether proves that the topological space $(\mathbf{Spec}_{\mathfrak{t}}^0(X), \tau_{\mathfrak{t}}^0)$ is sober. ■

A2.3. The spectrum $\mathbf{Spec}(X)$ as a subspace of $(\mathbf{Spec}^0_{\mathfrak{t}}(X), \tau^0_{\mathfrak{t}})$. The topology $\tau^0_{\mathfrak{t}}$ on $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ (cf. A2.2.2) induces the topology $\tau^{\succ}$ on $\mathbf{Spec}(X)$ (see 1.6). Its closed sets are $V(\mathbb{T}) = \{\mathcal{Q} \in \mathbf{Spec}(X) \mid \mathcal{Q} \subseteq \mathbb{T}\}$, where $\mathbb{T}$ runs though the family $\mathfrak{T}(X)$ of topologizing subcategories of C_X.

For every topologizing subcategory $\mathbb{T}$, let $\mathbb{T}_{\mathfrak{red}}$ denote the supremum of all $\mathcal{Q} \in V(\mathbb{T})$. It follows that $\mathbb{T}_{\mathfrak{red}}$ is the smallest topologizing subcategory of C_X such that $V(\mathbb{T}_{\mathfrak{red}}) = V(\mathbb{T})$. We call a topologizing subcategory $\mathbb{T}$ *reduced* if $\mathbb{T} = \mathbb{T}_{\mathfrak{red}}$.

One can see that the map $\mathbb{T} \longmapsto V(\mathbb{T})$ is a bijection between the preorder $\mathfrak{T}_{\mathfrak{red}}(X)$ of reduced topologizing subcategories and the preorder of closed subsets of $(\mathbf{Spec}(X), \tau^{\succ})$ such that $V(sup(\mathbb{T}, \mathbb{S})) = V(\mathbb{T}) \bigcup V(\mathbb{S})$.

A2.3.1. Proposition. *(a) The subspace $V(\mathbb{T})$ is irreducible iff $\mathbb{T}_{\mathfrak{red}}$ belongs to $\mathbf{Spec}(X)$.*

Proof. (a) Let $\mathbb{T}_{\mathfrak{red}} \subseteq \mathbb{S}_1 \bullet \mathbb{S}_2$. Then $V(\mathbb{T}_{\mathfrak{red}}) \subseteq V(\mathbb{S}_1 \bullet \mathbb{S}_2) = V(\mathbb{S}_1) \bigcup V(\mathbb{S}_2)$. Since $V(\mathbb{T}_{\mathfrak{red}})$ is irreducible, $V(\mathbb{T}_{\mathfrak{red}}) \subseteq V(\mathbb{S}_i)$ for $i = 1$, or 2. Therefore $\mathbb{T}_{\mathfrak{red}} \subseteq (\mathbb{S}_i)_{\mathfrak{red}} \subseteq \mathbb{S}_i$. By 4.3(a), this means that $\mathbb{T}_{\mathfrak{red}}$ is an element of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$.

(b) If $\mathbb{T}$ is reduced, i.e. $\mathbb{T} = \mathbb{T}_{\mathfrak{red}}$, then $\mathbb{T} \in \mathbf{Spec}^0_{\mathfrak{t}}(X)$ iff $\mathbb{T} \in \mathbf{Spec}(X)$.

In fact, since $\mathbb{T} = \mathbb{T}_{\mathfrak{red}}$, there exists an object $\mathcal{Q} \in V(\mathbb{T})$ such that $\mathcal{Q} \nsubseteq \widehat{\mathbb{T}}$. This means that $\mathbb{T} = \mathcal{Q}$. ∎

A2.3.2. Proposition. $(\mathbf{Spec}(X), \tau^{\succ})$ *is a sober Kolmogorov's space.*

Proof. (a) It follows from A2.3.1 that every irreducible closed subset of $\mathbf{Spec}(X)$ is of the form $V(\mathcal{Q})$ for some $\mathcal{Q} \in \mathbf{Spec}(X)$, i.e. $(\mathbf{Spec}(X), \tau^{\succ})$ is a sober topological space.

(b) Let $\mathcal{Q}$ and $\mathcal{Q}'$ be elements of $\mathbf{Spec}(X)$ such that $\mathcal{Q} \nsubseteq \mathcal{Q}'$. This means that the closure of $\mathcal{Q}'$ in the topology $\tau^{\succ}$ does not contain $\mathcal{Q}$; or, equivalently, there is an open neighborhood of $\mathcal{Q}$ which does not contain the point $\mathcal{Q}'$. ∎

Appendix 3. Supports and specializations. Krull filtrations.

A3.1. Support in $\mathbf{Spec}(X)$. Let M be an object of an abelian category C_X. The *support* of M in $\mathbf{Spec}(X)$ is the set $Supp(M)$ of all $[P] \in \mathbf{Spec}(X)$ such that $M \succ P$, or, equivalently, $[P] \subseteq [M]$.

A3.2. Supports in $\mathbf{Spec}^1(X)$ and in $\mathbf{Spec}^-(X)$. The *support* of an object M of C_X in $\mathbf{Spec}^1(X)$ is the set $Supp^1(X)$ of all $\mathcal{P} \in \mathbf{Spec}^1(X)$ such that $M \notin Ob\mathcal{P}$.

The support of M in the S-spectrum is the set

$$Supp^-(M) = Supp^1(M) \bigcap \mathbf{Spec}^-(X) = Supp^1(M) \bigcap \mathfrak{Se}(X).$$

A3.3. Lemma. *Let M be an object of C_X.*

(a) The following conditions are equivalent:

(a1) $\mathcal{P} \in Supp^1(M)$;

(a2) $M \succ L$ for some nonzero object L of $\mathcal{P}^{\circledast} - \mathcal{P}$.

(b) The following conditions are equivalent:

(b1) $\mathcal{P} \in Supp^-(M)$;

(b2) $M \succ L$ for some nonzero object L of $\mathcal{P}_{\circledast} = \mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$.

Proof. Let $C_X \xrightarrow{q^*_{\mathcal{P}}} C_{X/\mathcal{P}}$ be the localization functor at $\mathcal{P} \in \mathbf{Spec}^1(X)$.

(a1)⇒(a2). The condition $\mathcal{P} \in Supp^-(M)$ means precisely that $q^*_{\mathcal{P}}(M) \neq 0$. On the other hand, $q^*_{\mathcal{P}}(L')$ is a quasi-final object of $C_{X/\mathcal{P}}$ for every nonzero object L' of $\mathcal{P}^{\circledast} - \mathcal{P}$. Therefore, $q^*_{\mathcal{P}}(M) \succ q^*_{\mathcal{P}}(L')$. The latter means that there exists a diagram

$$M^{\oplus n} \xleftarrow{\mathfrak{j}'} K' \xrightarrow{\mathfrak{e}''} L'' \xrightarrow{\mathfrak{s}} L' \tag{9}$$

such that $Ker(\mathfrak{j}')$ and $Cok(\mathfrak{s})$ are objects of $\mathcal{P}$, $\mathfrak{e}''$ is an epimorphism and $\mathfrak{s}$ is a monomorphism. Replacing K' by $K = K/Ker(\mathfrak{j}')$ and L' by the cokernel of the composition $Ker(\mathfrak{j}') \longrightarrow K' \xrightarrow{\mathfrak{e}''} L''$, we obtain the diagram $M^{\oplus n} \xleftarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}} L$ in which $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}$ is an epimorphism; i.e. $M \succ L$. Since $q^*_{\mathcal{P}}(L)$ is isomorphic to $q^*_{\mathcal{P}}(L')$ and L' is an object of $\mathcal{P}^{\circledast} - \mathcal{P}$, the object L belongs to $\mathcal{P}^{\circledast} - \mathcal{P}$ too.

(a2)⇒(a1) & (b2)⇒(b1). If $M \succ L$ and $L \notin Ob\mathcal{P}$, then $M \notin Ob\mathcal{P}$, i.e. $\mathcal{P} \in Supp^1(X)$.

(b1)⇒(b2). If $\mathcal{P} \in Supp^-(M)$ and L' is a nonzero object of $\mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$, then $q^*_{\mathcal{P}}(M) \succ q^*_{\mathcal{P}}(L')$ which is expressed by the diagram (9). Since this time L' is a $\mathcal{P}$-torsion object, the composition of $Ker(\mathfrak{j}') \longrightarrow K' \xrightarrow{\mathfrak{e}''} L''$ is zero. Therefore, replacing K' by $K = K'/Ker(\mathfrak{j}')$, we obtain a diargram $M^{\oplus n} \xleftarrow{\mathfrak{j}} K \xrightarrow{\mathfrak{e}''} L'' \hookrightarrow L'$ in which $\mathfrak{j}$ is a monomorphism and $\mathfrak{e}''$ is an epimorphism. So that $M \succ L''$, where L'' is a subobject of an object of $\mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$, hence L'' belongs to $\mathcal{P}^{\circledast} \cap \mathcal{P}^{\perp}$. ■

A3.4. Proposition. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be elements of* $\mathbf{Spec}^-(X)$. *Then the following conditions are equivalent:*

(a) $\mathcal{P}_2 \subseteq \mathcal{P}_1$;

(b) for every nonzero object M_1 of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp}$, there exists a nonzero object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ such that $M_1 \succ M_2$.

(c) There exists a nonzero object M_1 of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp}$ with the following property: for any nonzero subobject L_1 of M_1, there is an object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ such that $L_1 \succ M_2$.

Proof. (a)⇒(b). If $\mathcal{P}_2 \subseteq \mathcal{P}_1$ and M_1 is a nonzero object of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp}$), then $\mathcal{P}_2 \in Supp^-(M_1)$. By A3.3(b), there exists an object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ such that $M_1 \succ M_2$.

(b)⇒(a). Suppose that $\mathcal{P}_2 \not\subseteq \mathcal{P}_1$; and let N be an object of $\mathcal{P}_2 - \mathcal{P}_1$. In particular, $\mathcal{P}_1 \in Supp^-(N)$. By A3.3(b), there exists a nonzero object M_1 of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp}$ such that $N \succ M_1$. By condition (b), $M_1 \succ M_2$ for some nonzero object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ which implies that $N \succ M_2$. The latter is impossible, because $N \in Ob\mathcal{P}_2$ and $M_2 \notin Ob\mathcal{P}_2$. Therefore $\mathcal{P}_2 \subseteq \mathcal{P}_1$.

Obviously, (b)⇒(c).

(c)⇒(a). Replacing X by $X/(\mathcal{P}_1 \cap \mathcal{P}_2)$ and the objects M_1 and M_2 by their images in $C_{X/(\mathcal{P}_1 \cap \mathcal{P}_2)}$, we can assume that $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$. Suppose that $\mathcal{P}_2 \neq 0$. Then $\mathcal{P}_2$ is a local subcategory. If $\mathcal{P}_1 = 0$, then C_X is local too, and nonzero objects of $\mathcal{P}_1^{\circledast} \cap \mathcal{P}_1^{\perp} = 0^{\mathfrak{t}}$ are precisely quasi-final objects of C_X. Since $\mathcal{P}_2 \neq 0$, it contains $0^{\mathfrak{t}}$; in particular, $M_1 \in Ob\mathcal{P}_2$. This contradicts to the condition (c) according to which $M_1 \succ M_2$ for some $M_2 \in \mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$.

Suppose now that both $\mathcal{P}_1$ and $\mathcal{P}_2$ are nonzero, hence both of them are local. There exists a quasi-final object L_2 of $\mathcal{P}_2$ and a monomorphism $L_2 \longrightarrow M_1$ such that $M_1/L_2 \in$

$Ob\mathcal{P}_1$. By condition (c), there exists a nonzero object M_2 of $\mathcal{P}_2^{\circledast} \cap \mathcal{P}_2^{\perp}$ such that $L_2 \succ M_2$. Since $L_2 \in \mathcal{P}_2$, we run into a contradiction again. Altogether shows that $\mathcal{P}_2 = 0$. ■

A3.5. The Krull filtration of $\mathbf{Spec}^-(X)$ and the associated filtration of X. Fix an abelian category C_X. For every cardinal α, we define a subset $\mathfrak{S}_\alpha^-(X)$ of $\mathbf{Spec}^-(X)$ as follows.

$\mathfrak{S}_0^-(X) = \emptyset$;

if α is not a limit cardinal, then $\mathfrak{S}_\alpha^-(X)$ consists of all $\mathcal{P} \in \mathbf{Spec}^-(X)$ such that any $\mathcal{P}' \in \mathbf{Spec}^-(X)$ properly contained in $\mathcal{P}$ belongs to $\mathfrak{S}_{\alpha-1}^-(X)$;

if α is a limit cardinal, then $\mathfrak{S}_\alpha^-(X) = \bigcup\limits_{\beta<\alpha} \mathfrak{S}_\beta^-(X)$.

It follows from this definition (borrowed from [R, VI.6.3]) that $\mathfrak{S}_1^-(X)$ consists of all closed points of $\mathbf{Spec}^-(X)$.

We denote by $\mathfrak{S}_\omega^-(X)$ the union of all $\mathfrak{S}_\alpha^-(X)$. The filtration $\{\mathfrak{S}_\alpha^-(X)\}$ determines a filtration

$$C_{X_0} \hookrightarrow C_{X_1} \hookrightarrow \dots C_{X_\alpha} \hookrightarrow \dots \tag{5}$$

of the category C_X (or the 'space' X) by taking as C_{X_α} the full subcategory of C_X generated by objects M such that $Supp^-(M) \subseteq \mathfrak{S}_\alpha^-(X)$. Recall that $Supp^-(M) = \{\mathcal{P} \in \mathbf{Spec}^-(X) \mid M \notin Ob\mathcal{P}\}$. In particular, C_{X_ω} is the full subcategory of C_X generated by all $M \in ObC_X$ such that $Supp^-(M) \subseteq \mathfrak{S}_\omega^-(X)$.

It follows from the general properties of supports that C_{X_α} is a Serre subcategory of C_X and $\mathbf{Spec}^-(X_\alpha)$ is naturally identified with $\mathfrak{S}_\alpha^-(X)$; in particular, $\mathbf{Spec}^-(X_\omega)$ is identified with $\mathfrak{S}_\omega^-(X)$.

A3.6. The Krull dimension. For every element $\mathcal{P}$ of $\mathbf{Spec}^-(X_\omega)$, there is the biggest cardinal, $\mathfrak{ht}^-(\mathcal{P})$, among all the cardinals α such that $\mathcal{P} \notin \mathfrak{S}_\alpha^-(X)$. The cardinal $\mathfrak{ht}^-(\mathcal{P})$ is called the hight of $\mathcal{P}$ ([R, VI.6.3]).

The *Krull dimension* of X is the supremum of all $\mathfrak{ht}^-(\mathcal{P})$, where $\mathcal{P}$ runs through $\mathbf{Spec}^-(X_\omega)$ (in [R] it is called the *flat* dimension).

An object M of C_X is said to have a Krull dimension if it belongs to the subcategory C_{X_ω}. Finally, the 'space' X (or the category C_X) has a Krull dimension if $X = X_\omega$ (that is $C_X = C_{X_\omega}$) and every nonzero object of C_X has a nonempty support, i.e. $C_{X_0} = \mathbb{O}$.

A3.7. The Krull dimension and the Gabriel-Krull dimension. We recall the notion of the *Gabriel filtration* of an abelian category as it is defined in [R, 6.6]. Let C_X be an abelian category. The *Gabriel filtration of* X assigns to every cardinal α a Serre subcategory $C_{X_\alpha^-}$ of C_X which is constructed as folows:

Set $C_{X_0^-} = \mathbb{O}$.

If α is not a limit cardinal, then $C_{X_\alpha^-}$ is the smallest Serre subcategory of C_X containing all objects M such that the localization $q_{\alpha-1}^*(M)$ of M at $C_{X_{\alpha-1}^-}$ has a finite length.

If β is a limit cardinal, then $C_{X_\beta^-}$ is the smallest Serre subcategory containing all subcategories $C_{X_\alpha^-}$ for $\alpha < \beta$.

Let $C_{X_\omega^-}$ denote the smallest Serre subcategory containing all the subcategories $C_{X_\alpha^-}$. It follows that the quotient category C_{X/X_ω^-} has no simple objects.

An object M is said to have the *Gabriel-Krull dimension* β, if β is the smallest cardinal such that M belongs to $C_{X_\beta^-}$.

The 'space' X has a Gabriel-Krull dimension if $X = X_\omega^-$.

Every locally noetherian abelian category (e.g. the category of quasi-coherent sheaves on a noetherian scheme, or the category of left modules over a left noetherian associative algebra) has a Gabriel-Krull dimension.

It is argued in [R, VI.6] that if X has a Gabriel-Krull dimension, then the filtration (5) coincides with the Gabriel filtration of the category C_X. In particular, X has a Krull dimension: $X = X_\omega = X_\omega^-$. Thus, the Krull dimension is an extension of the Gabriel-Krull dimension to a wider class of 'spaces'.

A3.8. A description of $\mathbf{Spec}_\rtimes(X_\omega)$. The filtration $\{\mathfrak{S}_\alpha^-(X)\}$ of $\mathbf{Spec}^-(X)$ induces, via the isomorphism $\mathbf{Spec}_\rtimes(X) \xrightarrow{\sim} \mathbf{Spec}^-(X)$ (defined in 6.4), a filtration $\{\mathfrak{S}_\alpha^\rtimes(X)\}$ of the spectrum $\mathbf{Spec}_\rtimes(X)$. We call it the *Krull* filtration of $\mathbf{Spec}_\rtimes(X)$.

A3.8.1. Proposition. *The spectrum $\mathbf{Spec}_\rtimes(X_\omega)$ of X_ω is naturally isomorphic to $\bigcup_\alpha \mathbf{Spec}_\rtimes(X_\alpha/X_{\alpha-1})$, and $\mathbf{Spec}^-(X_\omega)$ is isomorphic to $\bigcup_\alpha \mathbf{Spec}^-(X_\alpha/X_{\alpha-1})$, where α runs through non-limit cardinals. These isomorphisms are compatible with the isomorphisms $\mathbf{Spec}_\rtimes(X_\omega) \xrightarrow{\sim} \mathbf{Spec}^-(X_\omega)$ and $\mathbf{Spec}_\rtimes(X_\alpha/X_{\alpha-1}) \xrightarrow{\sim} \mathbf{Spec}^-(X_\alpha/X_{\alpha-1})$.*

Proof. More precisely,

$$\mathbf{Spec}_\rtimes(X_\omega) = \bigcup_\alpha (\mathfrak{S}_\alpha^\rtimes(X) - \mathfrak{S}_{\alpha-1}^\rtimes(X)),$$

where α runs though non-limit cardinals, and for every non-limit cardinal α, there is a natural isomorphism

$$\mathfrak{S}_\alpha^\rtimes(X) - \mathfrak{S}_{\alpha-1}^\rtimes(X) \xrightarrow{\sim} \mathbf{Spec}_\rtimes(X_\alpha/X_{\alpha-1}). \tag{6}$$

The isomorphism (6) is given by the map $\mathfrak{S}_\alpha^\rtimes(X) \longrightarrow \mathfrak{T}(X/X_{\alpha-1})$ which assigns to every element $\mathcal{P}_\circledast$ of $\mathfrak{S}_\alpha^\rtimes(X)$ the smallest topologizing subcategory $[q_{\alpha-1}^*(\mathcal{P}_\circledast)]$ of $C_{X/X_{\alpha-1}}$ spanned by the image of $\mathcal{P}_\circledast$.

Let $\mathcal{P} \in \mathbf{Spec}^-(X_\omega)$, i.e. $\mathcal{P} \in \mathfrak{S}_\alpha(X)$ for some α. Consider all cardinals β such that $C_{X_\beta} \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a Serre subcategory, the smallest Serre subcategory spanned by all C_{X_β} coincides with $C_{X_{\alpha-1}}$ for a non-limit cardinal α. The image $\mathcal{P}_\circledast = \mathcal{P}^\circledast \cap \mathcal{P}^\perp$ of $\mathcal{P}$ in $\mathbf{Spec}_\rtimes(X)$ is an element of $\mathfrak{S}_\alpha^\rtimes(X) - \mathfrak{S}_{\alpha-1}^\rtimes(X)$. ∎

A3.9. The Krull filtrations and equivalences of categories.

A3.9.1. Proposition. *Let C_X and C_Y be abelian categories. Any category equivalence $C_X \xrightarrow{\Theta} C_Y$ induces equivalences $C_{X_\alpha} \xrightarrow{\Theta_\alpha} C_{Y_\alpha}$ for all cardinals α. In particular, Θ induces a category equivalence $C_{X_\omega} \xrightarrow{\Theta_\omega} C_{Y_\omega}$*

Proof. The argument is by (transfinite) induction. The assertion is, obviously, true for $\alpha = 0$. It is also true for $\alpha = 1$: if $\mathcal{P}$ is a closed point of $\mathbf{Spec}^-(X)$, then $[\Theta(\mathcal{P})]$ is a closed point of $\mathbf{Spec}^-(Y)$.

Suppose now that Θ induces equivalences $C_{X_\alpha} \xrightarrow{\Theta_\alpha} C_{Y_\alpha}$ for all cardinals $\alpha < \beta$. We claim that then it induces a category equivalence $C_{X_\beta} \xrightarrow{\Theta_\beta} C_{Y_\beta}$.

(a) If β is a limit cardinal, then it follows from the definition of the filtration (cf. A3.5), that $C_{X_\beta} = \big(\bigcup\limits_{\alpha<\beta} C_{X_\alpha}\big)^-$. By the induction hypothesis, Θ induces a category equivalence $\bigcup\limits_{\alpha<\beta} C_{X_\alpha} \longrightarrow \bigcup\limits_{\alpha<\beta} C_{Y_\alpha}$. It is easy to show that if Θ induces an equivalence between a subcategory $\mathbb{T}$ of C_X and a subcategory $\mathbb{S}$ of C_Y, then Θ induces an equivalence $\mathbb{T}^- \longrightarrow \mathbb{S}^-$. In particular, Θ induces a category equivalence from $C_{X_\beta} = \big(\bigcup\limits_{\alpha<\beta} C_{X_\alpha}\big)^-$ to $C_{Y_\beta} = \big(\bigcup\limits_{\alpha<\beta} C_{Y_\alpha}\big)^-$.

(b) Suppose now that β is not a limit cardinal. By the induction hypothesis, Θ induces a category equivalence $C_{X_{\beta-1}} \longrightarrow C_{Y_{\beta-1}}$; hence Θ induces an equivalence between quotient categories $C_{X/X_{\beta-1}} \xrightarrow{\widehat{\Theta}_{\beta-1}} C_{Y/Y_{\beta-1}}$. The equivalence $\widehat{\Theta}_{\beta-1}$ induces an equivalence $C_{(X/X_{\beta-1})_1} \longrightarrow C_{(Y/Y_{\beta-1})_1}$. Notice that C_{X_β} is the preimage of $C_{(X/X_{\beta-1})_1}$ in C_X. Similarly for C_{Y_β}. Therefore Θ induces a functor $C_{X_\beta} \xrightarrow{\Theta_\beta} C_{Y_\beta}$ and its quasi-inverse, Θ^*, induces a functor $C_{Y_\beta} \xrightarrow{\Theta^*_\beta} C_{X_\beta}$. Since Θ is an equivalence, Θ_β is an equivalence with a quasi-inverse Θ^*_β. ∎

A3.9.2. Proposition. *Any category equivalence $C_X \xrightarrow{\Theta} C_Y$ induces isomorphisms*

$$\mathfrak{S}^-_\alpha(X) \xrightarrow{\sim} \mathfrak{S}^-_\alpha(Y) \quad \textit{and} \quad \mathfrak{S}^\rtimes_\alpha(X) \xrightarrow{\sim} \mathfrak{S}^\rtimes_\alpha(Y)$$

for all cardinals α. In particular, Θ induces an isomorphisms

$$\mathfrak{S}^-_\omega(X) \xrightarrow{\sim} \mathfrak{S}^-_\omega(Y) \quad \textit{and} \quad \mathfrak{S}^\rtimes_\omega(X) \xrightarrow{\sim} \mathfrak{S}^\rtimes_\omega(Y).$$

Proof. The assertion follows from A3.9.1 and from the fact that the natural isomorphisms

$$\mathfrak{S}^-_\alpha(X) \simeq \mathbf{Spec}^-(X_\alpha), \quad \mathfrak{S}^\rtimes_\alpha(X) \simeq \mathbf{Spec}_\rtimes(X_\alpha)$$

are compatible with category equivalences for all cardinals α. In particular, we have commutative diagrams

$$\begin{array}{ccc} \mathfrak{S}^-_\omega(X) & \xrightarrow{\sim} & \mathfrak{S}^-_\omega(Y) \\ \wr\big\downarrow & & \big\downarrow\wr \\ \mathbf{Spec}^-(X_\omega) & \xrightarrow{\sim} & \mathbf{Spec}^-(Y_\omega) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{S}^\rtimes_\omega(X) & \xrightarrow{\sim} & \mathfrak{S}^\rtimes_\omega(Y) \\ \wr\big\downarrow & & \big\downarrow\wr \\ \mathbf{Spec}_\rtimes(X_\omega) & \xrightarrow{\sim} & \mathbf{Spec}_\rtimes(Y_\omega) \end{array}$$

Details are left to the reader. ∎

A3.9.3. Corollary. *Let C_X be an abelian category and $C_X \xrightarrow{\Theta} C_X$ an autoequivalence.*

(a) If $\mathcal{P} \in \mathbf{Spec}^-(X_\omega)$, then $\Theta(\mathcal{P}) \subseteq \mathcal{P} \Leftrightarrow [\Theta(\mathcal{P})] = \mathcal{P} \Leftrightarrow \mathcal{P} \subseteq [\Theta(\mathcal{P})]$.

(b) If $\mathcal{P}_\circledast \in \mathbf{Spec}_\rtimes(X_\omega)$, then $\Theta(\mathcal{P}_\circledast) \subseteq \mathcal{P}_\circledast \Leftrightarrow [\Theta(\mathcal{P}_\circledast)) = \mathcal{P}_\circledast \Leftrightarrow \mathcal{P}_\circledast \subseteq [\Theta(\mathcal{P}_\circledast))$.

(a) If $\mathcal{P} \in \mathbf{Spec}(X_\omega)$, then $\Theta(\mathcal{P}) \subseteq \mathcal{P} \Leftrightarrow [\Theta(\mathcal{P})] = \mathcal{P} \Leftrightarrow \mathcal{P} \subseteq [\Theta(\mathcal{P})]$.

Here $[\Theta(\mathcal{P})]$ and $[\Theta(\mathcal{P}_\circledast))$ coincide with strictly full subcategories of C_X generated by resp. $\Theta(\mathcal{P})$ and $\Theta(\mathcal{P}_\circledast)$.

Proof. (a) (i) Let $\Theta(\mathcal{P}) \subseteq \mathcal{P}$. If $\mathcal{P} \nsubseteq [\Theta(\mathcal{P})]$, then $\mathfrak{ht}^-([\Theta(\mathcal{P})]) < \mathfrak{ht}^-(\mathcal{P})$. By A3.9.1, this implies that $\mathfrak{ht}^-([\Theta^*\Theta(\mathcal{P})]) \leq \mathfrak{ht}^-([\Theta(\mathcal{P})]) < \mathfrak{ht}^-(\mathcal{P})$. But, since Θ^* is a quasi-inverse to Θ, $[\Theta^*\Theta(\mathcal{P})] = \mathcal{P}$. Therefore $\mathcal{P} = [\Theta(\mathcal{P})]$.

(ii) The implication $\mathcal{P} \subseteq [\Theta(\mathcal{P})] \Rightarrow [\Theta(\mathcal{P})] = \mathcal{P}$ follows from (i), because the inclusion $\mathcal{P} \subseteq [\Theta(\mathcal{P})]$ is equivalent to the inclusion $\Theta^*(\mathcal{P}) \subseteq \mathcal{P}$.

(b) The assertion (b) follows from (a) and the observation that the isomorphism $\mathbf{Spec}^-(X) \xrightarrow{\sim} \mathbf{Spec}_\rtimes(X)$ (cf. 6.4) is compatible with the actions of autoequivalences on resp. $\mathbf{Spec}^-(X)$ and $\mathbf{Spec}_\rtimes(X)$.

(c) The assertion (c) follows from (b) and an observation that the canonical embedding

$$\mathbf{Spec}(X) \longrightarrow \mathbf{Spec}_\rtimes(X), \quad \mathcal{P} \longmapsto \mathcal{P} \cap \langle \mathcal{P} \rangle^\perp$$

is compatible with the actions of autoequivalences on resp. $\mathbf{Spec}(X)$ and $\mathbf{Spec}_\rtimes(X)$. Details are left to the reader. ∎

A4. Local properties of spectra and closed points.

A4.1. Closed points od spectra and Gabriel-Krull dimension. If X has a Gabriel-Krull dimension, then the set $\mathbf{Spec}_\mathfrak{t}^{1,1}(X)_1$ of the closed points of $\mathbf{Spec}_\mathfrak{t}^{1,1}(X)$ coincides with the set $\mathbf{Spec}^-(X)_1$ of the closed points of $\mathbf{Spec}^-(X)$. Since in this case $\mathbf{Spec}^-(X) = \mathbf{Spec}^1_{\mathfrak{Ge}}(X)$, the spectra $\mathbf{Spec}^1_{\mathfrak{Ge}}(X)$, $\mathbf{Spec}^-(X)$, and $\mathbf{Spec}_\mathfrak{t}^{1,1}(X)$ have the same sets of closed points.

A4.2. Lemma. *Suppose that every nonzero object of C_X has a non-empty support in $\mathbf{Spec}(X)$ (for instance, C_X has enough objects of finite type; cf. A3.1). Then for any closed point $\mathcal{P}$ in $\mathbf{Spec}_\mathfrak{t}^{1,1}(X)$ and for any thick subcategory $\mathbb{T}$ of C_X such that $\mathbb{T} \subseteq \mathcal{P}$, the subcategory $\mathcal{P}/\mathbb{T}$ of $C_X/\mathbb{T}$ is a closed point in $\mathbf{Spec}_\mathfrak{t}^{1,1}(X/\mathbb{T})$.*

Proof. By 3.2(ii), $\mathcal{P} = \widehat{\mathcal{Q}}$ for a uniquely determined by this equality element $\mathcal{Q}$ of $\mathbf{Spec}(X)$, which is a closed point in $\mathbf{Spec}(X)$, since $\mathcal{P}$ is a closed point in $\mathbf{Spec}_\mathfrak{t}^{1,1}(X)$. We claim that image $[q^*(\mathcal{Q})]$ of $\mathcal{Q}$ is $\mathcal{T}(X/\mathbb{T})$ is a closed point of $\mathbf{Spec}(X/\mathbb{T})$.

In fact, let $\mathcal{Q}'$ be a nonzero topologizing subcategory of $C_X/\mathbb{T}$ contained in $[q^*(\mathcal{Q})]$. This means that the preimage $\mathcal{Q}'' = q^{*^{-1}}(\mathcal{Q}')$ of $\mathcal{Q}'$ in C_X is a topologizing subcategory of C_X which does not contain $\mathbb{T}$ and is contained in $q^{*^{-1}}([q^*(\mathcal{Q})])$. By 5.1.1, $q^{*^{-1}}([q^*(\mathcal{Q})]) =$

$\mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T}$. Any object M of the subcategory $\mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T}$ can be described by the diagram

$$\begin{array}{ccccccccc} & & M_1' & \longleftarrow & 0 & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & M_1'' & \longrightarrow & 0 & & & & \end{array} \qquad (1)$$

which incorporates two short exact sequences such that the objects M_1' and M_2, belong to $\mathbb{T}$, and $M_1'' \in Ob\mathcal{Q}$. One can see from this description that M is an object of $\mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T} - \mathbb{T}$ iff M_1'' is an object of $\mathcal{Q} - \mathbb{T}$. It follows from the diagram (1) that $M_1'' \in Ob[M]$. Since $\mathbb{T} \not\supseteq \mathcal{Q}'' \subseteq \mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T}$, the topologizing subcategory $\mathcal{Q}'' \cap \mathcal{Q}$ is not contained in $\mathbb{T}$. In particular, it is nonzero. Let M be a nonzero element of $\mathcal{Q}'' \cap \mathcal{Q}$. By hypothesis, $Supp(M)$ is non-empty; i.e. there exists an element $\widetilde{\mathcal{Q}}$ of $\mathbf{Spec}(X)$ such that $\widetilde{\mathcal{Q}} \subseteq [M]$. Thus, we have inclusions $\widetilde{\mathcal{Q}} \subseteq [M] \subseteq \mathcal{Q}'' \cap \mathcal{Q} \subseteq \mathcal{Q}$. Since $\mathcal{Q}$ is a closed point of $\mathbf{Spec}(X)$, the inclusion $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ implies that $\widetilde{\mathcal{Q}} = \mathcal{Q}$. Therefore the inclusions above can be replaced by equalities. In particular, $\mathcal{Q}'' \cap \mathcal{Q} = \mathcal{Q}$, that is $\mathcal{Q} \subseteq \mathcal{Q}''$ which means that $\mathcal{Q}'$ coincides with $[q^*(\mathcal{Q})] = (\mathbb{T} \bullet \mathcal{Q} \bullet \mathbb{T})/\mathbb{T}$. ∎

A4.2.1. Corollary. *Suppose that every nonzero object of C_X has a non-empty support in $\mathbf{Spec}(X)$. Then every closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ is a closed point of $\mathbf{Spec}^1(X)$.*

Proof. Let $\mathcal{P}$ be a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$; and let $\mathcal{P}_1$ be an element of $\mathbf{Spec}^1(X)$ such that $\mathcal{P}_1 \subseteq \mathcal{P}$. By A4.2, $\mathcal{P}/\mathcal{P}_1$ is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{P}_1)$. But, $X/\mathcal{P}_1$ is a local 'space', hence it has a unique closed point – 0. This shows that $\mathcal{P}/\mathcal{P}_1 = 0$, i.e. $\mathcal{P} = \mathcal{P}_1$. ∎

A4.3. Proposition. *Suppose that C_X is an abelian category with the property (sup). Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of C_X such that $\bigcap_{i \in J} \mathcal{T}_i = 0$. Then*

(a) A point $\mathcal{P}$ of $\mathbf{Spec}^-(X)$ is closed iff $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

(b) Suppose that every nonzero object of C_X has a nonzero support in $\mathbf{Spec}(X)$. Then a point $\mathcal{P}$ of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ is closed iff $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

Proof. (a) If $\mathcal{P} \in \mathbf{Spec}^-(X)$, then $\mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}^-(X/\mathcal{T}_i)$ for all i such that $\mathcal{T}_i \subseteq \mathcal{P}$. And if $\mathcal{P}$ is a closed point, then $\mathcal{P}/\mathcal{T}_i$ is a closed point.

In fact, if $\mathcal{T}_i \subseteq \mathcal{P}$, then $\mathcal{P}/\mathcal{T}_i$ is an element of $\mathbf{Spec}^1(X/\mathcal{T}_i)$; and $\mathcal{P}/\mathcal{T}_i$ is a Serre subcategory of $C_X/\mathcal{T}_i$ (due to the reflectivity of the Serre subcategory $\mathcal{T}_i$ which is a consequence of the property (sup)). Therefore, it belongs to $\mathbf{Spec}^-(X/\mathcal{T}_i)$. If $\mathcal{P}' \in \mathbf{Spec}^-(X/\mathcal{T}_i)$ and $\mathcal{P}' \subseteq \mathcal{P}$, then the preimage $\mathcal{P}''$ of $\mathcal{P}'$ in C_X is a Serre subcategory which belongs to $\mathbf{Spec}^-(X)$ and is contained in $\mathcal{P}$. Thus, if $\mathcal{P}$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$, then $\mathcal{P}'' = \mathcal{P}$, hence $\mathcal{P}' = \mathcal{P}/\mathcal{T}_i$.

(a1) Conversely, let $\mathcal{P}/\mathcal{T}_i$ be closed for all $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$. Then we claim that $\mathcal{P}$ is closed. If the number $Card(J) = 1$, then the statement is true by a trivial reason. In the general case, let $\mathcal{P}'$ be an element of $\mathbf{Spec}^-(X)$ such that $\mathcal{P}' \subseteq \mathcal{P}$. And let $J^{\mathcal{P}'}$ denote the set $\{i \in J \mid \mathcal{T}_i \not\subseteq \mathcal{P}'\}$. Since J is finite, by 9.3, there exists $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}'$. Therefore $Card(J^{\mathcal{P}'}) < Card(J)$. By (the end of the argument of) A1.4.1(c) (or [R4, 4.2]),

$$\mathcal{P}' = (\bigcap_{i\in J} \mathcal{T}_i) \vee \mathcal{P}' = \bigcap_{i\in J}(\mathcal{T}_i \vee \mathcal{P}') = \bigcap_{i\in J^{\mathcal{P}'}} (\mathcal{T}_i \vee \mathcal{P}').$$

So that $\{\mathcal{T}_i' = (\mathcal{T}_i \vee \mathcal{P}')/\mathcal{P}',\ i \in J^{\mathcal{P}'}\}$ is a set of Serre subcategories of $C_X/\mathcal{P}'$ whose intersection is zero. The point $\widetilde{\mathcal{P}} = \mathcal{P}/\mathcal{P}'$ of $\mathbf{Spec}^-(X/\mathcal{P}')$ is such that that $\widetilde{\mathcal{P}}/\mathcal{T}_i'$ is a closed point of $\mathbf{Spec}^-(X/(\mathcal{T}_i \vee \mathcal{P}')$ for all $i \in J^{\mathcal{P}'}$ such that $\mathcal{T}_i' \subseteq \widetilde{\mathcal{P}}$. Since $Card(J^{\mathcal{P}'}) < Card(J)$, by induction hypothesis, $\widetilde{\mathcal{P}}$ is a closed point of $X/\mathcal{P}'$. The latter 'space' being local, this means that $\widetilde{\mathcal{P}} = 0$, or, equivalently, $\mathcal{P} = \mathcal{P}'$.

(b) If $\mathcal{P}$ is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$, then, by A4.2, $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ for every $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}$.

Conversely, suppose that $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ is such that $\mathcal{P}/\mathcal{T}_i$ is a closed point of the spectrum $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ if $\mathcal{T}_i \subseteq \mathcal{P}$. Let $\mathcal{P}'$ be an element of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ such that $\mathcal{P}' \subseteq \mathcal{P}$. By 9.3, there exists $i \in J$ such that $\mathcal{T}_i \subseteq \mathcal{P}'$; in particular, $\mathcal{T}_i \subseteq \mathcal{P}$. Since $\mathcal{P}'/\mathcal{T}_i$ is a point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ and $\mathcal{P}/\mathcal{T}_i$ is a closed point, the inclusion $\mathcal{P}'/\mathcal{T}_i \subseteq \mathcal{P}/\mathcal{T}_i$ implies that $\mathcal{P}'/\mathcal{T}_i$ and $\mathcal{P}/\mathcal{T}_i$ coincide. Therefore, $\mathcal{P}' = \mathcal{P}$. ■

A4.3.1. Corollary. *Suppose that C_X is an abelian category with the property (sup). Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories such that $\bigcap_{i\in J} \mathcal{T}_i = 0$ and for every $i \in J$, any element of $\mathbf{Spec}^-(X/\mathcal{T}_i)$ contains a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$. Then every element of $\mathbf{Spec}^-(X)$ contains a closed point of $\mathbf{Spec}^-(X)$.*

Proof. Let $\mathcal{P} \in \mathbf{Spec}^-(X)$. Since $\bigcap_{i\in J} \mathcal{T}_i = 0$, there exists $J_{\mathcal{P}} = \{i \in J \mid \mathcal{T}_i \subseteq \mathcal{P}\}$ is non-empty. Fix an $i \in J_{\mathcal{P}}$. By hypothesis, $\mathcal{P}_i \subseteq \mathcal{P}$, where $\mathcal{P}_i/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$. If $J_{\mathcal{P}_i} = \{i\}$, then, by A4.3(a), $\mathcal{P}_i$ is a closed point of $\mathbf{Spec}^-(X)$. If $J_{\mathcal{P}_i}$ contains more than one element, we take $j \in J_{\mathcal{P}_i} - \{i\}$ and repeat the argument replacing $\mathcal{P}$ by $\mathcal{P}_i$; and so on. Since J is finite, the process stabilizes. As a result, we find an element $\mathcal{P}'$ of $\mathbf{Spec}^-(X)$ such that $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{P}'/\mathcal{T}_j$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_j$ for every $j \in J_{\mathcal{P}'}$. By A4.3(a), the latter means that $\mathcal{P}'$ is a closed point of $\mathbf{Spec}^-(X)$. ■

A4.3.2. Corollary. *Suppose that C_X is an abelian category with the property (sup). Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories such that $\bigcap_{i\in J} \mathcal{T}_i = 0$ and for every $i \in J$, the set $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)_1$ of the closed points of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ contains the set $\mathbf{Spec}^-(X/\mathcal{T}_i)_1$ of the closed points of $\mathbf{Spec}^-(X/\mathcal{T}_i)$. Then $\mathbf{Spec}^-(X)_1 \subseteq \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1$.*

Suppose that, in addition, one of the following conditions holds:

(a) For all $i \in J$, every element of $\mathbf{Spec}^-(X/\mathcal{T}_i)$ contains a closed point.

(b) Every nonzero object of C_X has a non-empty support in $\mathbf{Spec}(X)$.

Then $\mathbf{Spec}^-(X)_1$ and $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1$ coincide.

Proof. Let $\mathcal{P}$ be a closed point of $\mathbf{Spec}^-(X)$. By A4.3, $\mathcal{P}/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$ for all $i \in J_{\mathcal{P}} = \{j \in J \mid \mathcal{T}_j \subseteq \mathcal{P}\}$. By hypothesis, $\mathcal{P}/\mathcal{T}_i$ is a closed point of

$\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$ for all $i \in J_{\mathcal{P}}$. By 9.6.1, $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Since $\mathcal{P}$ is a closed point of the space $\mathbf{Spec}^-(X)$, it is, definitely, a closed point of its subspace $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. This shows the inclusion $\mathbf{Spec}^-(X)_1 \subseteq \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1$.

(a) Let $\mathcal{P} \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Since $\mathcal{P}$ is an element of $\mathbf{Spec}^-(X)$, by A4.3.1, $\mathcal{P} \supseteq \mathcal{P}'$, where $\mathcal{P}'$ is a closed point of $\mathbf{Spec}^-(X)$. By A4.3(a), for every $i \in J_{\mathcal{P}'} = \{j \in J \mid \mathcal{T}_j \subseteq \mathcal{P}'\}$, the quotient subcategory $\mathcal{P}'/\mathcal{T}_i$ is a closed point of $\mathbf{Spec}^-(X/\mathcal{T}_i)$, hence, by hypothesis, it belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X/\mathcal{T}_i)$. By 9.6.1, the latter implies that $\mathcal{P}'$ belongs to $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Since it $\mathcal{P}'$ is a closed point of $\mathbf{Spec}^-(X)$, it is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$.

(b) If every nonzero object of C_X has a non-empty support, then, by A4.2.1, we have the inverse inclusion: $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)_1 \subseteq \mathbf{Spec}^-(X)_1$. ∎

A4.4. Proposition. *Let C_X be an abelian category and $\{\mathcal{T}_i \mid i \in J\}$ a finite set of thick subcategories such that $\bigcap_{i \in J} \mathcal{T}_i = 0$. Suppose that each category $C_{X/\mathcal{T}_i}$ has enough objects of finite type. Then closed points of $\mathbf{Spec}(X)$ are in a natural bijective correspondence with the isomophism classes of simple objects of C_X.*

Proof. Let u_i^* denote the localization functor $C_X \longrightarrow C_X/\mathcal{T}_i$. Let M be an object of $Spec(X)$ such that $[M]$ be a closed point of $\mathbf{Spec}(X)$. Since $\bigcap_{i \in J} \mathcal{T}_i = 0$, there is an $i \in J$ such that $M \notin \mathcal{T}_i$. Therefore, $[u_i^*(M)]$ is a closed point of $\mathbf{Spec}(X/\mathcal{T}_i)$. Since the category $C_{X/\mathcal{T}_i} = C_X/\mathcal{T}_i$ has enough objects of finite type, all closed points of $\mathbf{Spec}(X/\mathcal{T}_i)$ correspond to simple objects. In particular, $u_i^*(M)$ is the direct sum of a finite number of copies of a simple object $u_i^*(L)$, and there is a monomorphism $u_i^*(L) \longrightarrow u_i^*(M)$. This monomorphism is described by a diagram $L \xleftarrow{s} L' \xrightarrow{\mathfrak{j}'} M$ such that $u_i^*(s)$ is an isomorphism and $Ker(\mathfrak{j}')$ belongs to $\mathcal{T}_i$. Since the object M is $\mathcal{T}_i$-torsion free, the object $L_i = L'/Ker(\mathfrak{j}')$ is $\mathcal{T}_i$-torsion free too. It follows that $u_i^*(L_i)$ is isomorphic to $u_i^*(L)$. In particular, L_i is a nonzero subobject of M. Since $M \in Spec(X)$, the object L_i also belongs to $Spec(X)$ and $[L_i] = [M]$. So, we replace M by L_i. Repeating this procedure consecutively for all $j \in J$ such that M does not belong to $\mathcal{T}_j$, we replace M by its subobject, N such that for any $j \in J$, the object $u_j^*(N)$ is either zero, or simple. Since N belongs to $Spec(X)$, it follows that for every nonzero monomorphism $N' \xrightarrow{h} N$, its image $u_i^*(h)$ is an isomorphism for every $i \in J$. The condition $\bigcap_{i \in J} \mathcal{T}_i = 0$ means that the family of localization functors $\{C_X \xrightarrow{u_i^*} C_X/\mathcal{T}_i \mid i \in J\}$ is conservative; hence h is an isomorphism. This shows that N is a simple object. Therefore, M is isomorphic to the coproduct of a finite number of copies of N. ∎

The following proposition is a refinement of 1.6.2.

A4.5. Proposition. *Suppose that C_X is an abelian category with the property (sup). Let $\{\mathcal{T}_i \mid i \in J\}$ be a finite set of Serre subcategories of C_X such that $\bigcap_{i \in J} \mathcal{T}_i = 0$, and for every $i \in J$, the category $C_{X/\mathcal{T}_i}$ has enough objects of finite type. Then*

(a) The intersection $\mathbf{Spec}(X) \bigcap \mathbf{Spec}(X^o)$ coincides with the set $\mathbf{Spec}(X)_1$ of closed points of $\mathbf{Spec}(X)$, and closed points of $\mathbf{Spec}(X)$ are of the form $[M]$, where M runs

through simple objects of C_X*.*

(b) Closed points of $\mathbf{Spec}^-(X)$*) are in bijective correspondence with the isomorphism classes of simple objects of* C_X*.*

Proof. (a) By A4.4, closed points of $\mathbf{Spec}(X)$ are of the form $[M]$, where M runs through simple objects of C_X. Since simple objects of C_X and $C_X^{op} = C_{X^o}$ are the same, the set $\mathbf{Spec}(X)_1$ of closed points of $\mathbf{Spec}(X)$ is contained in $\mathbf{Spec}(X) \cap \mathbf{Spec}(X^o)$.

(a1) Let $\mathfrak{U}$ denote the finite cover $\{U_i = X/\mathcal{T}_i \xrightarrow{u_i} X \mid i \in J\}$ associated with $\{\mathcal{T}_i \mid i \in J\}$. And let $\mathbf{Spec}_{\wp}^{1,1}(\mathfrak{U}) = \{\mathcal{P} \in \mathfrak{Th}(X) \mid \mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i) \text{ if } \mathcal{T}_i \subseteq \mathcal{P}\}$. By 9.6.1, the natural map $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) \longrightarrow \mathbf{Spec}_{\wp}^{1,1}(\mathfrak{U})$ is an isomorphism. This isomorphism and the embedding $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X^o) \longrightarrow \mathbf{Spec}_{\wp}^{1,1}(\mathfrak{U}^o)$ induce an injective map

$$\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X) \bigcap \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X^o) \longrightarrow \mathbf{Spec}_{\wp}^{1,1}(\mathfrak{U}) \bigcap \mathbf{Spec}_{\wp}^{1,1}(\mathfrak{U}^o), \qquad (2)$$

where

$$\mathbf{Spec}_{\wp}^{1,1}(\mathfrak{U}) \bigcap \mathbf{Spec}_{\wp}^{1,1}(\mathfrak{U}^o) =$$
$$\{\mathcal{P} \in \mathbf{Spec}^-(X) \mid \mathcal{P}/\mathcal{T}_i \in \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i) \bigcap \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i^o) \text{ if } \mathcal{T}_i \subseteq \mathcal{P}_i\}.$$

Since each category $C_{U_i} = C_X/\mathcal{T}_i$, $i \in J$, has enough objects of finite type, it follows from 1.6.2 and the isomorphism $\mathbf{Spec}(U_i) \xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i)$ (see 3.2(ii)) that the intersection $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i) \bigcap \mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i^o)$ coincides with the set of closed points of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(U_i)$ and these closed points are in bijective correspondence with isomorphism classes of simple objects of the category C_{U_i}. It follows now from (the argument of) A4.4 and the isomorphism $\mathbf{Spec}(X) \xrightarrow{\sim} \mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ (see 3.2(ii)) that the map (2) above is bijective.

(b) Notice that the conditions of this proposition imply the conditions (a) and (b) of A4.3.2. In particular, by A4.3.2, the spectra $\mathbf{Spec}^-(X)$ and $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ have the same closed points. The assertion follows this fact and from (a) above. ■

A4.6. Semilocal 'spaces'.

A4.6.1. Proposition. *Suppose that there is a finite subset* $\{\mathcal{P}_i \mid i \in J\}$ *of* $\mathbf{Spec}^-(X)$ *such that* $\bigcap_{i\in J} \mathcal{P}_i = 0$. *Then* $\mathcal{P} \in \mathbf{Spec}^-(X)$ *is a closed point iff it is a closed point of* $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$*, i.e. it is of the form* $\mathcal{P} = \langle L \rangle$ *for an object* L *of* $Spec(X)$*.*

The set of closed points of $\mathbf{Spec}^-(X)$ *coincides with the set of minimal elements of* $\{\mathcal{P}_i \mid i \in J\}$*.*

Proof. Let $\mathcal{P}$ be a closed point of $\mathbf{Spec}^-(X)$). By 9.3, the set $J_{\mathcal{P}} = \{i \in J \mid \mathcal{P}_i \subseteq \mathcal{P}\}$ is not empty. Since $\mathcal{P}$ is a minimal element of $\mathbf{Spec}^-(X)$, the set $J_{\mathcal{P}}$ consists of all $i \in J$ such that $\mathcal{P}_i = \mathcal{P}$. Thus, $\mathcal{P}/\mathcal{P}_i$ is the zero subcategory of $C_X/\mathcal{P}_i$ which is the only closed point of the local space $X/\mathcal{P}_i = X/\mathcal{P}$. By 9.6.1, $\mathcal{P}$ is an element of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$. Since $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$ is a subset of the spectrum $\mathbf{Spec}^-(X)$ and $\mathcal{P}$ is a closed point of the latter, it is a closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$.

This argument shows that the set of closed points of $\mathbf{Spec}^-(X)$ is a subset of the set of minimal elements of $\{\mathcal{P}_i \mid i \in J\}$, and that it is a subset of closed point of $\mathbf{Spec}_{\mathfrak{t}}^{1,1}(X)$.

Notice that every minimal element of $\{\mathcal{P}_i \mid i \in J\}$ is a closed point of $\mathbf{Spec}^-(X)$.

In fact, let $\mathcal{P}_j$ be a minimal element of the set $\{\mathcal{P}_i \mid i \in J\}$, and let $\mathcal{P}' \in \mathbf{Spec}^-(X)$ be a subcategory of $\mathcal{P}_j$. The set $J_{\mathcal{P}'} = \{i \in J \mid \mathcal{P}_i \subseteq \mathcal{P}'\}$ is non-empty, and $\mathcal{P}_m \subseteq \mathcal{P}' \subseteq \mathcal{P}_j$ for every $m \in J_{\mathcal{P}'}$. Since $\mathcal{P}_j$ is a minimal element, this implies that $\mathcal{P}_m = \mathcal{P}' = \mathcal{P}_j$.

Let $\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$. The set $J_{\mathcal{P}} = \{i \in J \mid \mathcal{P}_i \subseteq \mathcal{P}\}$ is not empty. Let $\mathcal{P}_i$ be a minimal element of $\{\mathcal{P}_j \mid j \in J_{\mathcal{P}}\}$. Then, by the argument above, $\mathcal{P}_i$ is a closed point of $\mathbf{Spec}^-(X)$, hence it is a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ which is contained in $\mathcal{P}$. Thus, if $\mathcal{P}$ is a closed point of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$, then $\mathcal{P}_i = \mathcal{P}$. ∎

A4.6.2. Corollary. *Let C_X be an abelian category. The following conditions are equivalent:*

(a) There is a finite subset $\{\mathcal{P}_i \mid i \in J\}$ of $\mathbf{Spec}^-(X)$ such that $\bigcap\limits_{i \in J} \mathcal{P}_i = 0$.

(b) The set $\mathbf{Spec}^-(X)_1$ of closed points of $\mathbf{Spec}^-(X)$ is finite, and the intersection $\bigcap\limits_{\mathcal{P} \in \mathbf{Spec}^-(X)_1} \mathcal{P}$ is zero.

(c) The set $\mathbf{Spec}^-(X)_1$ is finite, and the support in $\mathbf{Spec}^-(X)$ of any nonzero object of C_X contains a closed point.

(d) The set $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)_1$ of closed points of $\mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)$ is finite, and the support in $\mathbf{Spec}(X)$ of every nonzero object of C_X contains a closed point.

Proof. Obviously, $(b) \Rightarrow (a)$. The implication $(a) \Rightarrow (b)$ follows from A4.6.1.

$(b) \Leftrightarrow (c)$. If $\bigcap\limits_{\mathcal{P} \in \mathbf{Spec}^-(X)_1} \mathcal{P} = 0$, then for every nonzero object M of C_X, there exists a closed point $\mathcal{P}$ of $\mathbf{Spec}^-(X)$ such that $M \notin Ob\mathcal{P}$, which means precisely that $\mathcal{P} \in Supp^-(M)$. Conversely, if every nonzero object of C_X has an element of $\mathbf{Spec}^-(X)_1$ in its support, then $\bigcap\limits_{\mathcal{P} \in \mathbf{Spec}^-(X)_1} \mathcal{P} = 0$.

$(d) \Rightarrow (a)$. The support in $\mathbf{Spec}(X)$ of a nonzero object M contains a point $\mathcal{Q}$, that is $\mathcal{Q} \subseteq [M]$, means precisely that $[M] \not\subseteq \widehat{\mathcal{Q}}$, or, equivalently, $M \notin Ob\widehat{\mathcal{Q}}$. By 3.2(ii), $\mathcal{Q}$ is a closed point of $\mathbf{Spec}(X)$ iff $\widehat{\mathcal{Q}} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)_1$. Therefore, the condition (d) implies that $\bigcap\limits_{\mathcal{P} \in \mathbf{Spec}^{1,1}_{\mathfrak{t}}(X)_1} \mathcal{P} = 0$.

The implication $(a) \Rightarrow (b)$ follows from A4.6.1. ∎

A4.6.3. Definition. Let C_X be an abelian category. We call the 'space' X *semilocal* if the equivalent conditions of A4.6.2 hold.

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