On the Geometry of Negative Ricci and Scalar Curvature

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Preface

This is a first collection of five papers [1] – [5] concerning existence, flexibility and approximation results for negative Ricci and scalar curvature.

[1] Ricci Curvature modulo Homotopy

[2] Metrics of Negative Ricci Curvature

[3] The Space of Negative Scalar Curvature Metrics

[4] Curvature h-principles

[5] Negative Bending of Open Manifolds

[1] is a survey/summary on the subject and can be used as an introduction to the subsequent papers.

[2] contains a series of existence theorems for metrics of negative Ricci curvature $\mathrm{Ric} < 0$.

[3] and [4] prove flexibility and approximation results, furthermore they imply h-principles for $\mathrm{Ric} < 0$ and $S < 0$.

[5] which is partially motivated from [4] presents some short proofs of general existence theorems for $\mathrm{Ric} < 0$, from a somewhat different point of view.

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J. Lohkamp — Bonn, winter 1992/93

Ricci Curvature modulo Homotopy

by Joachim Lohkamp

§1 Introduction

In this paper we want to present some results concerning metrics of negative Ricci curvature.
We will give some outline of the used arguments and describe relations between different approaches:
In §2 we will give a sketch of a simple existence proof for metrics with $Ric < 0$ on arbitrary closed manifolds, which is also motivated from our second approach to the subject presented in §3 and discussed in §6.
This second one, which was originally discovered before that in §2, starts from the technical (but also philosophical) fact that negative Ricci and scalar curvature is local in nature: There is a metric g_n on $I\!R^n$ with $Ric(g_n) < 0$ on $B_1(0)$ and $g_n \equiv g_{Eucl.}$ outside for $n \leq 3$.

A proof of this is indicated in §4 and we will use this and certain covering arguments to check several existence theorems for metrics with $Ric < 0$ or $S < 0$ on arbitrary manifolds.

Furthermore this method is strong enough to provide a framework for handling problems like the structure of spaces of metrics with restricted curvature properties and density results (cf. §3 and 5).
Finally we reformulate this using the language of h–principles in §6, which will partially motivate the approach of §2 as mentioned above.

§2 Existence of negatively curved metrics

Thurston theory [T] implies that each closed three manifold M^3 contains a link $L \subset M$ such that there are metrics g_1, g_2 with $(M \setminus L, g_1)$ is complete and hyperbolic with finite volume resp. (M, g_2) is a hyperbo- lic orbifold branched along L. Each of these results can be used to prove the existence of a regular metric with $Ric < 0$ on M:
L.Z. Gao and S.T. Yau [GY] "closed" $(M \setminus L, g_1)$ such that the resulting Riemannian manifold is M with a metric g with $g \equiv g_1$ on $M \setminus U, U \supset L$ a tubular neighborhood, and $Ric(g) < 0$ on M. R. Brooks started in [B] from (M, g_2) and smoothed the

singularities near L to get a metric g with $g \equiv g_1$ on $M \setminus U, U \supset L$ and $Ric(g) < 0$ on M.

It is unclear whether each manifold $M^n, n \geq 4$ or non-compact M^3 can admit a similar hyperbolic structures. Due to this lack the results were not extended to a broader class of manifolds.

On the other hand there are not really needed hyperbolic structure to get:

Theorem 2.1: *Each manifold $M^n, n \geq 3$ admits a complete metric with $Ric < 0$.*

This is done for closed M (the open case can be obtained by simpler approaches, see below) starting from one of the following two results:

Theorem 2.2: *Each open manifold admits a metric with negative sectional curvature*

cf. [Grl] or alternatively we can use cf. [L6] resp. Theorem (3.5.) below

Theorem 2.3: *Each open manifold admits a complete metric with $Ric < 0$.*

Sketch of proof of (2.1.): Noting the last result (which proof is much simpler than those of the other results presented in §3) we can restrict our attention to closed $M^n, n \geq 4$.

We start with the following remark: Let $B \subset M^n$ be a ball then B contains a closed submanifold N^{n-2} which admits a metric with $Ric < 0$ and which normal bundle is trivial. This is easily done in case $n = 4$ using the embedding of a hyperbolic surface in $\mathbb{R}^3 \subset \mathbb{R}^4$.
In dimension 5 we can use a result of Hirsch [H]: Each orientable closed three-manifold (in particular hyperbolic ones) admits an embedding with trivial normal bundle into $\mathbb{R}^5$.
In higher dimensions we can use induction: $S^{n-2}, n \geq 6$ admits a metric with $Ric < 0$ and we take the usual embedding $S^{n-2} \hookrightarrow \mathbb{R}^{n-1} \subset \mathbb{R}^n$
Of course these metrics are not induced metrics coming from the embedding.

Now acc. (2.2.) resp. (2.3.) we have a metric with $Ric < 0$ on the open manifold $M \setminus N$. Using conformal and warped product deformations we can get a warped product metric on a tubular neighborhood U of N with $U \setminus N =]0, r[\times S^1 \times N$ equipped with $g_{\mathbb{R}} + F^2 \cdot g_{S^1} + g_N$ for some strongly increasing $F \in C^\infty(\mathbb{R}, \mathbb{R}^{>0})$.
$(]0, r[\times S^1, g_{\mathbb{R}} + F^2 \cdot g_{S^1})$ looks similar like the spreading open end of the pseudosphere and we would be done if it was possible to "close"this with a metric with Gauß-curvature $k < 0$. But this is impossible by Gauß–Bonnet.

On the other hand we are given the additional factor (N, g_N) and this can be used to avoid this problem: We can take a singular metric $g_{sing.}$ with $k < 0$ on the disk D such that the metric near the boundary looks like $(]0, r[) \times S^1, g_{\mathbb{R}} + F^2 \cdot g_{S^1})$ with $\{0\} \times S^1 = \partial D(!)$. Now we can use $Ric(g_N) < 0$ to smooth the singularities of $g_{sing.}$ getting a warped product metric with $Ric < 0$ on $D \times N$ and glue it to $M \backslash U$. Thus we have closed M again and it is equipped with a metric with $Ric < 0$. Details and extensions are described in [L6]. □

§3 A Different Approach

In this chapter we start to describe a completely different and new method of attack. It turns out that this approach yields significantly stronger results and a deeper insight into the behaviour of such metrics in a natural way. An interesting feature is that (in conclusion, cf. §6 below) it also motivates the weaker approach of the previous chapter.

Let us begin with summarizing the results obtained in this way:

I General Existence Theorems

Theorem 3.1: *Each manifold $M^n, n \geq 3$ admits a complete metric g_M with*

$$-a(n) < r(g_M) < -b(n) \ ,$$

for some constants $a(n) > b(n) > 0$ depending only on the dimension n.

Corollary 3.2: *Each manifold $M^n, n \geq 3$ admits a complete metric with constant negative scalar curvature.*

(3.2.) was proved before by T. Aubin [A] and J. Bland, M. Kalka [BK] in a different more analytic way.

The next result is motivated from S. T. Yau's theorem [Y] that each complete, non-compact manifold with $Ric \geq 0$ has infinite volume: we get the following alternative version of (3.1.):

Theorem 3.3: *Each manifold $M^n, n \geq 3$ admits a complete metric g'_M with $r(g'_M) < -1$ and $Vol(M^n, g'_M) < +\infty$*

(3.1.)–(3.3.) are proved in [L4] (cf. also [L1]).

II Refined Existence Results

As a matter of fact closed manifolds of negative curvature cannot be embedded with "small" codimension e.g. as hypersurfaces in Euclidean spaces. But we can find a constant $c(n)$ such that (for p.i. $\equiv$ proper and isometric):

Theorem 3.4: *Let $(M^n, g_0), n \geq 3$ be p.i. embedded into (N, g) and codim $\geq c(n)$ then there is a metric g_1 on M^n with $Ric(g_1) < 0$ and a p.i. embedding of (M, g_1) into (N, g) which is isotopic to the embedding on (M, g_0) by p.i. embeddings and the isotopy (and g_1) can be chosen lying inside any prescribed neighborhood of (M, g_2)). The same conclusions hold for immersions instead of embeddings.*

Despite the fact that (3.4.) is proved in [L7] without return to Nash's isometric imbedding theory, this does not make to much sense without further knowledge of $c(n)$: Indeed one can show that the codimension is of lower order than in Nash's general theory. (For some characterizations of $c(n)$ cf. [L7].)

Our next result (cf. [L6]) is simpler than (3.1.) or (3.3.). On the other hand it contains some new information in the non compact case.

Theorem 3.5: *Let (M^n, g_0) be an open manifold, then there is a complete metric $g = e^{2f} \cdot g_0$ in the conformal class of g_0 with $Ric(g) < 0$*

(Due to non–existence results for the Yamabe problem this cannot be refined to give pinched Ricci or just scalar curvature.)

Up to now we have seen that there are no topological restrictions for the existence of metric with $Ric < 0$. But a classical result of Bochner says: If (M, g) is closed, $Ric(g) < 0$ then $Isom(M, g)$ is finite.

This is a geometric obstruction, but the only one (cf. [L4]):

Theorem 3.6: *Let $M^n, n \geq 3$ be closed, $G \subset Diff(M)$ a subgroup, then:
$G = Isom(M, g)$ for some metric g with $Ric(g) < 0 \iff G$ is finite.*

It is quite easy to prove the same for surfaces M^2 with $\chi(M) < 0$.

III Flexibility Results

In this section we present results which play a central technical role.

Theorem 3.7: *For $(M^n, g_0), n \geq 3$, let $S \subset M$ be a closed subset and $U \supset S$ an open neighborhood, and $Ric(g_0) \leq 0$ on U, then there is a metric g_1 on M with*

(i) $g_1 \equiv g_0$ *on* S

(ii) $Ric(g_1) < 0$ $M \setminus S$

As a simple application we get

Corollary 3.8: *Let $M^n, n \geq 3$ be compact with boundary $B \neq \emptyset$ and g_0 any fixed metric on B, then there is a metric g on M with $g \equiv g_0$ on B, $Ric(g) < 0$ on M and each component of B is totally geodesic w.r.t. g.*

The next theorem is the philosophical core of this approach and does make clear that similar arguments can not work for positive curvatures due to the positive energy theorem which implies non-existence of a metric looking like the one described now in the positive case:

Theorem 3.9: *On $I\!R^n, n \geq 3$ there is a metric g_n with $Ric(g_n) < 0$ on $B_1(0)$ and $g_n \equiv g_{Eucl.}$ outside.*

Perhaps it is interesting to note that for each $\epsilon > 0$ we can find a concrete metric g_n as in (3.9.) with $Vol(B_1(0), g_n) < \epsilon$ which is also included in (3.12.) below. For proofs cf. [L4].

IV Spaces of metrics

Due to results of Hitchin, Gromov, Lawson and Carr (cf. [LM]) we know that the space of metrics with positive scalar curvature on a closed manifold M $S^+(M)$ can be quite complicated:
$S^+(M)$ can be empty or $\pi_i(S^+(M)) \neq 0$.

There are no similar .problems in the negative case: More generally denote by $Ric^{<\alpha}(M)$ the space of metrics g with $r(g) < \alpha$ on $M, \alpha \in I\!R$. ($S^{<\alpha}$ is defined analogously)

Theorem 3.10: *$Ric^{<\alpha}(M)$ and $S^{<\alpha}(M)$ are non-empty, non-convex but contractible Fréchet-manifolds.*

And as an application we get using some elementary elliptic theory:

Corollary 3.11: *The space of metrics of constant negative scalar curvature is contractible*

Next recall from Bishop's comparison theorem that $Ric^{>\alpha}(M)$ cannot be dense in the space of all metrics $\mathcal{M}(M)$ w.r.t. C^0–topology: the C^0–closure in $\mathcal{M}(M)$ is ~~just~~ contained in $Ric^{\geq\alpha}(M)$. On the other hand

Theorem 3.12: *$Ric^{<\alpha}(M), S^{<\alpha}$ are C^0–dense in $\mathcal{M}(M)$ for each $\alpha \in I\!R$*

Furthermore we have using some more analytic effort if g_0 is non–flat:

Theorem 3.13: *Let $(M^n, g_0), n \geq 3$ be Ricci flat resp. scalar flat, then g_0 can be approximated by metrics in $Ric^{<0}(M)$resp. $S^{<0}(M)$ w.r.t. C^κ–norms.*

For proofs we refer to [L2], [L3] and [L5].

V On the proofs

The main ingredients for the proof of the results above are: the existence of a metric g_n in $I\!R^n, n \geq 3$ with $Ric(g_n) < 0$ on $B_1(0)$ and $g_n \equiv g_{Eucl.}$ outside and a covering arguments for arbitrary manifolds giving a "compatible" covering by negatively Ricci curved balls like $(B_1(0), g_n)$, which yields metrics with $Ric < 0$ on each manifold of dimension ≥ 3. We describe some details in the following two chapters.

§4 Producing negative curvature

Roughly speaking the existence results of the previous chapter are obtained from a suitable series of local deformations of some nearly arbitrary start metric. These deformations turn the metric inside a ball into a (stronger) negatively Ricci curved one and let it fixed outside.

The first step consists in constructing this local deformation in the flat case:

Proposition 4.1: *On $I\!R^n, n \geq 3$ there is a metric g_n^- with $r(g_n^-) < 0$ on $B_1(0)$ and $g_n^- \equiv g_{Eucl.}$ outside.*

Our (sketch of) proof will present the simplest but not the most powerful construction. The latter ones are needed to prove the results concerning the various spaces of metrics of §3, IV and we refer to [L4], [L3] (and [L5]) for these refined versions.

Proof: We start in dimension $n = 3$: It is simple to find a positive C^∞–function f on $I\!R$ with $f \equiv id$ on $I\!R^{\geq 1}$ which is symmetric in $\delta \in]0,1[$, i.e. $f(r) = f(2\delta - r)$ and fulfills $Ric(g_{I\!R} + f^2 \cdot g_{S^2}) < 0$ on $]2\delta - 1, 1[\times S^2$.
Now consider instead of the Euclidean metric the metric $g_{I\!R} + f^2 \cdot g_{S^2}$ on $I\!R^3 \setminus B_\delta(0)$: it does have two symmetries. A first one under reflections R_E along planes $E \subset I\!R^3$ with $0 \in E$ and a second "imaginary" one along $\partial B_\delta(0)$ coming from the symmetry of f in δ, in particular $\partial B_\delta(0)$ is totally geodesic. Now choose one plane E and consider the quotient space of $I\!R^3 \setminus B_\delta(0)$ under identification along $\partial B_\delta(0)$ via R_E. This is "canonically" attached with the differentiable structure of $I\!R^3$ (acc. Milnor's "smoothing of corners") and the metric on this $I\!R^3$ is smooth outside the geodesic curve γ corresponding to $\partial B_\delta(0) \cap E$, has $Ric < 0$ on $B_1(0)$ and is Euclidean outside.
The singularity along γ can be smoothed (with $Ric < 0$) using warped product techniques giving a regular metric g_3^- as claimed.
The case $n \geq 4$ can be handled in the same way as described in §2 : We choose a codim 2 submanifold $N \subset I\!R^n$ with trivial normal bundle and which admits a metric with $Ric < 0$. Next we bend $I\!R^n \setminus N$ "outwards" giving $Ric < 0$ on $B \setminus N$ for some ball $B \subset I\!R^n$ and subsequently we use the same method as in §2 to close $I\!R^n$ again and obtain the desired metric g_n^- □

Now we will give some ideas of how to derive the following result which proof is typical for many results of §3.

Proposition 4.2: *Each manifold $M^n, n \geq 3$ admits a complete metric g_M with $-a(n) < r(g_M) < -b(n)$ for constants $a(n) > b(n) > 0$ depending only on n.*

Proof: It is almost trivial to get a metric on M such that $\exp_p : B_{100}(0) \to \exp_p(B_{100}(0))$ is a diffeomorphism which is arbitrarily near w.r.t. C^k–norms to an isometry independent of $p \in M$.
Indeed we presently assume $M = (I\!R^n, g_{Eucl.})$.
Consider a covering of $I\!R^n$ by closed balls $\overline{B_5(p_i)}, p_i \in A \subset I\!R^n$ fulfilling the following conditions:

(i) $d(p,q) > 5$ for $p \neq q \in A$

(ii) $\#\{p \in A \mid z \in B_{10}(p)\} \leq c(n), c(n)$ independent of $z \in I\!R^n$.

and define $g(A,d,s) := \prod_{p \in A} \exp(2 \cdot F_{d,s} \cdot h(10 - d(p,id))) \cdot g_A$
with $g_A = g_{Eucl.}$ on $I\!R^n \setminus \bigcup_{p \in A} B_1(p), g_A = f_p^*(g_n^-)$ on $B_1(p)$ for $f_p(x) = x - p$.

Furthermore $F_{d,s} := s \cdot \exp(-d/id_{I\!R})$, $h \in C^\infty(I\!R, [0,1]), h \equiv 0$ on $I\!R^{\geq 9,6}, h \equiv 1$ on $I\!R^{\leq 9,4}$.

One can find $d, s > 0$ such that $-a < r(g(A,d,s)) < -b$ is fulfilled in each point of $I\!R^n$ and each direction for constants $a > b > 0$.
As noted above we can find a nearly flat metric $g(M)$ on each manifold, furthermore we can construct a covering fulfilling the same conditions on each of these manifolds (a "Besicovitch covering").

It is not hard to visualize that (almost) the same $d, s > 0$ and pinching constants $a > b > 0$ can be obtained for the Ricci curvature of an analogously defined metric $g(A,d,s)$ on an arbitrary manifold starting from $g(M)$. □

§5 Spaces of Metrics

The covering argument in §4 can be used to produce as many negative curvature as is necessary to "hide" each metric of some compact family of metrics behind a "veil" of negative Ricci curvature. Acc. results of Palais and Whitehead (cf. [P]) contractibility of a Fréchet-manifold F and $\pi_i(F) = 0, i = 0, 1, \cdots$ are equivalent. Thus if we want to show contractibility we will try to extend each continuous map $f : S \to Ric^{<\alpha}(M)$ resp. $S^{<\alpha}(M)$
Hence we start with some extension $F : B^{m+1} \to \mathcal{M}(M)$ and then we shift into $Ric^{<\alpha}(M)$ resp. $S^{<\alpha}(M)$ by simultaneous producing of negative Ricci resp. scalar curvature on all of the Riemannian manifolds $(M, F(x)), x \in B^{m+1}$.

In the case of scalar curvature this can be done using a more classical approach (cf. [L2]) without usage of those methods of §4. It should give some good impression of the more sophisticated case $Ric^{<\alpha}(M)$, therefore we will include a short sketch of that transparent argument:

Proposition 5.1: *$S^{<0}(M)$ is contractible.*

Proof: We will show that $S^{<0}(M)$ is path connected, $\pi_i(S^{<0}(M)) = 0$ for $i > 0$ is obtained analogously.
Note that M^n and $M^n \# S^n$ are diffeomorphic and take metric $g_1, g_2 \in S^{<0}(M)$ and $g^- \in S^{<0}(S^n)$.
Now for each pair $\lambda, \mu > 0$ the connected sum $M^n \# S^n$ can be formed such that the resulting Riemannian manifold $M(\lambda, \mu, g_i)$ is isometric to
$(M^n \setminus B_1(p), \lambda^2 \cdot g_i) \cup (]-1,1[\times S^{n-1}, g(\lambda,\mu,i)) \cup (S^n \setminus B_1(q), \mu^2 \cdot g^-)$ for $p \in M, q \in S^n$ fixed points, $i = 1, 2$,
such that the "neck" $]-1,1[\times S^{n-1}$ connecting the two main points does have bounded integral scalar curvature $\mathcal{S}_g := \int S_g d\mathrm{Vol}_g$ independent of λ, μ and i.

Hence if we choose a large λ or μ then $\mathcal{S}_{g(\lambda,\mu)} < 0$. This is due to the simple fact $\mathcal{S}_{m^2\cdot g} = m^{n-2}\cdot\mathcal{S}_g$.

Thus start with (M, g_1) and scale it by a large λ. Next take the linear path from $(M^n, \lambda^2\cdot g_1)$ to $M(\lambda,\mu)$ for some μ. If λ was large enough all metrics along this path have $\mathcal{S} < 0$.
Now take a very large $\bar{\mu}$ and take the path $M(\lambda, (t\cdot\bar{\mu} + (1-t))\cdot\mu\cdot g_1)$ (all these metrics have $\mathcal{S} < 0$).
Next connect g_1 and g_2 linearly (!), if $\bar{\mu}$ is large enough all Riemannian manifolds $M(\lambda, \bar{\mu}, t\cdot g_1 + (1-t)\cdot g_2)$ will have $S < 0$.
Finally we works backwards for g_2 and obtain a path of metrics connecting g_1 and g_2 such that each term has $S < 0$.
Now we can find a continuous family of conformal deformations of these metrics which yields a path inside of $S^{<0}(M)$. Details cf. [L2]. □

Finally we will easily see that these spaces of metrics are "highly" non–convex. For notational simplicity we restrict to $S^{<0}(M)$.

__Lemma__ **5.2:** *For any $g \in S^{<0}(M)$ and each ball $B \subset M$ there is a diffeomorphism φ with $\varphi \equiv id$ on $M\setminus B$ and $t\cdot g + (1-t)\cdot\varphi^*(g) \notin S^{<0}(M)$ for some $t \in]0,1[$.*

__Proof:__ Using scaling arguments it is enough to obtain a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi \equiv id$ on $\mathbb{R}^n \setminus B_1(0)$ and $t\cdot g_{Eucl.} + (1-t)\varphi^*(g_{Eucl}) \notin S^{\leq 0}(\mathbb{R}^n)$ for some $t \in]0,1[$.
We take φ with $\varphi(t,x) = (f(t),x), (t,x) \in \mathbb{R}^+ \times S^{n-1} \equiv \mathbb{R}^n\setminus\{0\}$ for some diffeomorphism $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f \equiv id$ on $]0, \frac{1}{10}[\cup]\frac{9}{10}, +\infty[$ and $f \equiv id + \frac{1}{10}$ on $]\frac{5}{10}, \frac{6}{10}[$.
Now it is not hard to check using warped product formulas: $\frac{1}{2}\cdot g_{Eucl.} + \frac{1}{2}\varphi^*(g_{Eucl.}) \notin S^{\leq 0}(\mathbb{R}^n)$. □

§6 h–principles

Here we want to describe some relations of those results presented in §3 and "homotopy principles" (abbr. h–principles) a concept first introduced in a broader context by M. Gromov cf. [Gr1], [Gr2].

We start with some definitions: Let $\pi : X \to M$ be a smooth fibration f over some manifold M and denote be X^κ the space of κ–jets of germs of smooth sections of π ($x^0 \equiv X$) and the induced fibration over M by $\pi_\kappa, \pi_\kappa : X^\kappa \to M$.
A section φ of π_κ is called __holonomic__ if there is a section f of π which κ–jet is φ.
A __differential relation__ $\mathcal{R}$ of order κ imposed on sections of π is just a subset $\mathcal{R} \subset X^\kappa$ and a section f of π is called __solution__ of $\mathcal{R}$ if its κ–jet lies on $\mathcal{R}$.

Finally let $\pi_{\kappa,m}$ denote the canonical projection $\pi_{\kappa,m} : X^\kappa \to X^m$ for $0 \leq m \leq \kappa$, hence a holonomic section φ lying in $\mathcal{R}$ projects to a solution $\pi_{\kappa,0}(\varphi)$ of $\mathcal{R}$.

The concept of h–principles relies on the (idea of) solving strategy to construct first a (possibly non holonomic) section of X^κ lying in $\mathcal{R}$ and then to pass (inside of $\mathcal{R}$) to a holonomic one: denote by $Sol\mathcal{R}$ the set of all solutions of $\mathcal{R}$, $C(\mathcal{R})$ the space of all sections of X^κ lying in $\mathcal{R}$ and by $J_\kappa : Sol\mathcal{R} \to C(\mathcal{R})$ the map $J_\kappa(\varphi) = \kappa$–jet of φ.

<u>Definition</u> **6.1:** *$\mathcal{R}$ fulfills the*

(i) parametric h–principle if J_κ is a weak homotopy equivalence.

(ii) h–principle for extension if for each subset $K \subset M$ and each triple of open neighborhoods $K \subset U_1 \subset \bar{U}_1 \ldots \subset U_3 \subset M$ and each section $\varphi_0 \in C(\mathcal{R} \cap \pi^{-1}(U_3))$ which is holonomic on U_2 there is a homotopy $\varphi_t \in C(\mathcal{R} \cap \pi^{-1}(U_3))$, $t \in [0,1]$ with $\varphi_1 \equiv \varphi_0$ on U and φ_1 holonomic on U_3.

<u>Definition</u> **6.2:** *$\mathcal{R}$ satisfies the*

(i) the h–principle C^m–near a section f of π, $m \leq r$, if for each $\varphi_0 \in C(\mathcal{R})$ with $\pi_{\kappa,0}(\varphi_0) = f$ and each neighborhood U of f there is a homotopy φ_t, $t \in [0,1]$ with φ_t lying in $\mathcal{R} \cap \pi_{\kappa,0}^{-1}(U)$ such that φ_1 is a holonomic section.

(ii) the C^m–dense h–principle if it fulfills the h–principle is C^m–near each section of π.

Now we specify X = the bundle of pointwise positive definite symmetric $(2,0)$ tensors and we consider differential relations $\mathcal{R} \subset X^2$ which simply restricts the curvature of a section of $\pi : X \to M$ (which is just a metric), e.g.

$$\mathcal{R} = \{\varphi \in X^2 \mid Ric(\varphi) < 0\} \equiv Ric < 0\,.$$

Next we want to relate (6.1.) and (6.2.) with those results of §3 concerning $Sol\mathcal{R}$. Therefore we must have a look at $C(\mathcal{R})$ and check the following simple result (cf. also [Gr2], (4.5.1.)):

<u>Lemma</u> **6.3:** *The fibers of the fibrations $Sec < \alpha, Ric < \alpha$ and $S < \alpha$ are non–empty and contractible. The same holds in case "$> \alpha$".*

Proof: We have to show contractibility for the space of 2–jets of germs of metrics near $0 \in I\!R^n$ with $Sec < \alpha$ etc.
These curvature relations contain the first two derivatives of the metric. Now there are two easily verified features:
For each 1–jet φ_1 of metrics there is 2–jet φ_2 with $\pi_{2,1}(\varphi_2) = \varphi_1$ and $Sec(\varphi_2) < \alpha$ etc., secondly the curvature depends linearly on the second derivatives.
This implies the fiber over each 1–jet φ_1 is non–empty and convex, furthermore the space of all 1–jets is contractible, hence the whole space is contractible. □

It a well–known results from elementary obstruction theory that fibrations with contractible fibers always have a section and the space of sections is also (weakly) contractible.

Corollary 6.4: *$C(Sec < \alpha)$ etc. are (weakly) contractible*

Hence we can reformulate (3.10.) (and (3.7.)) and with some additional but straightforward considerations (3.12.) and (3.13.) as

Theorem 6.5: *On each manifold $M^n, n \geq 3$ the differential relations $Ric < \alpha$ and $S < \alpha$, fulfill the parametric and the C^0–dense h–principle. Furthermore they fulfill the h–principle of extension and they are C^κ–dense near each Ricci flat resp. scalar flat metric.*

In contrast to (6.5.) we have a "converse" approach due to Gromov, cf. [Gr1], [Gr2], which starts from topology and arrives at geometry:

Theorem 6.6: *Each open, diffeomorphism invariant differtial relation $\mathcal{R}$ on an open manifold fulfills the parametric h–principle.*

It is obvious that $Sec < \alpha$ etc. are open and diffeomorphism invariant. Hence we obtain from (6.3.):

Corollary 6.7: *$Sec < \alpha$ resp. $> \alpha$ etc. fulfill the parametric h–principle on each open manifolds and in particular each open manifold carries a metric with $Sec < \alpha$ as well as one with $Sec > \alpha$.*

Thus we are led to two (seemingly) opposite points of view :
(6.7.) is obtained from a general topological argument namely the h–principle for open manifolds (6.6.).
On the other hand (6.5.) is proved by geometric (and analytic) techniques and does not rely on h–principles but imply them.

Thus there arises the question concerning the purely geometric status of $Ric < 0$:

For instance the existence of a metric with $Ric < 0$ on a closed manifold should be provable from some differential topological arguments resp. formal properties of the inequality $Ric < 0$ substituting (at least some of) the concrete differential geometry contained e.g. in the approach described in §4.

This cannot work perfectly since $Ric > 0$ which at first sight should fulfill similar formal conditions behaves completely different on closed and on open manifolds, cf.(6.7).

But if one takes into account at least some geometric interpretation of $Ric < 0$ one immediately gets approaches as in §2:

Here we made use of the possibility to bend (cf. [L6]) negative Ricci curvature metrics, which turns out to be a non-trivial geometric property of $Ric < 0$.

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Metrics of Negative Ricci Curvature

by Joachim Lohkamp

0. Introduction

One of the most natural and important questions in Riemannian geometry is to describe the relation between curvature and global structure of the underlying manifold. For instance complete manifolds of negative sectional curvature always are aspherical and in the compact case their fundamental group can only contain abelian subgroup which are infinite cyclic. Furthermore there was a general feeling that a (closed) manifold can not carry two metrics of different signed curvatures.
This is true for sectional curvature, but wrong for the scalar curvature S, since each manifold $M^n, n \geq 3$ admits a complete metric with $S \equiv -1$, cf. Aubin [A] and Bland, Kalka [BlK].
Hence the situation for Ricci curvature Ric, lying between sectional and scalar curvature, seemed to be quite delicate.

Up to now the most general results concerning $Ric < 0$ were proved by Gao, Yau [GY] and Brooks [Br] using Thurston's theory of hyperbolic three manifolds:
Each closed three manifold admits a metric with $Ric < 0$. This obtained from the fact that these manifolds carry hyperbolic metrics with certain singularities and Gao, Yau resp. Brooks smoothed these singularities, to get a regular metric with $Ric < 0$. These methods extend to three manifold of finite type and certain hyperbolic orbifolds.

In any case, the arguments start from an "almost regular " hyperbolic metric, which existence is neither obvious nor true in general (in the non-compact or higher dimensional case). Moreover this approach does not give any insight in the typical behaviour of metrics with $Ric < 0$ since one is led to a very special metric.

In this article we approach negative Ricci curvature in a completely different and conceptually new way which seems to be a more natural one which will be made precise in [L2].

Actually we will prove the following results; in these notes where $Ric(g)$ resp. $r(g)$ denotes the Ricci tensor resp. curvature of a smooth metric g:

Theorem A *For each manifold M^n, $n \geq 3$, there is a complete metric g_M with $-a(n) < r(g_M) < -b(n)$, with constants $a(n) > b(n) > 0$ depending only on the dimension n.*

From this one gets almost immediately.

Corollary B *Each manifold M^n, $n \geq 3$ admits a complete metric with constant negative scalar curvature.*

This Corollary was proved before in a different way by Aubin [A] resp. Bland and Kalka [BlK].

There is an interesting alternative general existence result to Theorem A : Recall from Yau [Y] that each complete manifold with non-negative Ricci curvature has infinite volume:

Theorem C *For each manifold M^n, $n \geq 3$ there is a complete metric g'_M with negative Ricci curvatures (indeed with $r(g'_M) < -1$) and finite volume $Vol(M, g'_M) < +\infty$.*

Theorem A resp. C tells us that there are no topological obstructions for negative Ricci curvature metrics. On the other hand a well-known result of Bochner (cf. [Ko]) asserts that the isometry group of a closed manifold with $r(g) < 0$ is finite. We will prove that this is the only "geometric obstruction ":

Theorem D *Let M^n, $n \geq 3$ be a closed manifold, $G \subset Diff(M)$ a subgroup, then:*

$$G \equiv Isom(M, g) \quad \text{for some metric} \quad g \quad \text{with} \quad r(g) < 0 \quad \Leftrightarrow \quad G \quad \text{is finite}$$

Next we turn to a cut-off or extension property which also extends Theorem C:

Theorem E *Let $S \subset M^n$ be a closed subset of a manifold $M^n, n \geq 3$ and $U \supset S$ an open neighborhood, g_0 any metric on U with $r(g_0) < 0$ (resp. ≤ 0), then there is a metric g on M such that:*

(i) $g \equiv g_0$ *on* S,

(ii) $r(g) < -1$ *on* $M \setminus \bar{U}$

(iii) $r(g) < 0$ *(resp.* ≤ 0*) on* M

(iv) $\lambda_n^{-1} \cdot Vol(W, g) \leq Vol(W, g_0) \leq \lambda_n \cdot Vol(W, g)$
for each measurable subset W of M and a constant $\lambda_n > 1$ depending only on n.

As a simple application we get:

Corollary F *Let be M^n, $n \geq 3$ a compact manifold with boundary $\partial M \neq \emptyset$ and g_0 any fixed metric on ∂M, then there is a metric g on M with:*

(i) $g \equiv g_0$ *on* ∂M

(ii) $r(g) < 0$ *on* M

(iii) each component of ∂M *is totally geodesic*

The paper is organized as follows: after some preliminaries in §1, we will construct and investigate deformations of foliations in §2. In §3 we prove the existence of metric g_n^- on $\mathbb{R}^n$ with $r(g_n^-) < 0$ on $B_1(0)$ and $g_n^- \equiv g_{Eucl.}$ (Euclidean metric) outside for n = 3. In §4 and 5 we assume the existence of g_n^- for $\mathbb{R}^n$, $n \geq 3$ and deduce the other results quoted above for n-dimensional manifolds. In §6 we get g_{n+1}^- on $\mathbb{R}^{n+1}$ using Theorem E for n dimensions i.e. the proof is by induction. Finally we collect some elementary technical results in an appendix.

Most of these results were announced in [L1].

There is also mentioned a result on the space of all metrics with $r(g) < 0$. Recently the author proved much stronger results on this space of metric, they will appear in [L2]. For another approach to the space of negative scalar curvature metrics cf. [L3].

1. Basic Deformations

In this chapter we perform some preliminary calculations and recall some useful deformation techniques.
We start with warped products: Let (M^m, g_M) and (N^n, g_N) be two Riemannian manifolds and $f \in C^\infty(M^m, \mathbb{R}^{>0})$, then $M \times_{f^2} N$ denotes the product manifold $M \times N$ equipped with the metric $g(f) := g_M + f^2 \cdot g_N$. The Ricci tensor $Ric(g(f))$ of $g(f)$ is calculated as follows (cf. [B], 9.106.): (note that our sign convention for Δ, $\Delta g_M \equiv tr_{g_M} Hess_{g_M}$ differs from [B])

$$Ric(g(f))(\mathbf{U},\mathbf{V}) = Ric(g_N)(\mathbf{U},\mathbf{V}) - g_N(\mathbf{U},\mathbf{V}) \cdot \left[\left(\frac{\Delta g_M f}{f} + (n-1)\frac{\| \nabla f \|^2}{f^2}\right) \circ \pi\right]$$

$$Ric(g(f))(\mathbf{X},\mathbf{Y}) = Ric(g_M)(d\pi(\mathbf{X}), d\pi(\mathbf{Y})) - \frac{n}{f}\cdot Hess_{g_M} f(d\pi(\mathbf{X}), d\pi(\mathbf{Y}))$$

$$Ric(g(f))(\mathbf{U},\mathbf{X}) = 0$$

where **U**,**V** denote vertical vectors, **X**,**Y** are horizontal vectors and π is the canonical projection onto M.

We will also make use of conformal deformations: Let (M^{m+1}, g) be a Riemannian manifold of dimension $m+1 \geq 3$ and $f \in C^\infty(\mathrm{M}, \mathbf{R})$. We are interested in $g_f = e^{2f}\cdot g$ and its related operators $Hess_{g_f}$, Δ_{g_f} and of course $Ric(g_f)$: Again (and for the last time) we cite from literature ([B], 1.J.): Let be $F \in C^\infty(M, I\!R)$ and $\nu \in T_p\mathrm{M}$ then:

$$Hess_{g_f}F(\nu,\nu) = Hess_g F(\nu,\nu) - 2\cdot df(\nu)\cdot dF(\nu) + df(\nabla^g F)\cdot g(\nu,\nu)$$

$$\Delta_{g_f}F = tr_{g_f} Hess_{g_f} F = e^{-2f}\cdot(\Delta_g F + (m-1)\cdot g(\nabla^g f, \nabla^g F))$$

Now let be $\mathrm{r}(g_f)(\nu) = \parallel \nu \parallel^{-2}\cdot Ric(g_f)(\nu,\nu)$ for $\nu \neq 0$ the Ricci curvature in direction ν,for some ν with $\parallel \nu \parallel_g = 1$ then:

$$e^{2f}\cdot r(g_f)(\nu) - r(g)(\nu) = (m-1)(\mid df(\nu)\mid^2 - \parallel \nabla^g f \parallel_g^2) - ((m-1)\cdot Hess_g f(\nu,\nu) + \Delta_g f)$$

In this chapter we make some specific assumption on (M^{n+1}, g):
$M^{n+1} \equiv I\!R^{n+1}$ and let $x_0 = t, x_1, \ldots, x_n$ be the canonical cartesian coordinates and g fulfills the following conditions for some $k > 1$.

(i) $g_{Eucl.}(\nu,\nu) \leq k^2\cdot g(\nu,\nu)$

(ii) $\parallel g \parallel_{C^3_{g_{Eucl.}}(\mathbf{R}^{n+1})} \leq k$, in particular $g(\nu,\nu) \leq k^2\cdot g_{Eucl.}(\nu,\nu)$

We adapt the usual notations : $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, $(g^{ij}) := (g_{ij})^{-1}$ and $\Gamma^k_{ij} := \frac{1}{2}\sum_{r=0}^n g^{rk}\left(\frac{\partial g_{jr}}{\partial x_i} + \frac{\partial g_{ir}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_r}\right)$ and prove a simple technical

Lemma (1.1.) *Let g be a metric which fulfills (i) and (ii), then $\mid \Gamma^k_{ij}\mid < k_1$ for some constant $k_1 = k_1(n,k)$.*

Proof: $\mid g^{rk}\mid = \frac{|det(g^{rk}_{ij})|}{|det(g_{ij})|}$, g^{rk}_{ij} is some minor of (g_{ij}). From (ii) we get a constant $k(1) = k(1)(n,k) > 0$ with $\mid det(g^{rk}_{ij})\mid, \mid \frac{\partial g_{ab}}{\partial x_c}\mid < k(1)$ and (i) implies $\mid det(g_{ij})\mid \geq k(2)$, for some constant $k(2)(n,k) > 0$, i.e.:

$$\mid \Gamma^k_{ij}\mid \leq \sum_{r=0}^n \mid g^{rk}\mid \cdot\left(\mid \frac{\partial g_{jr}}{\partial x_i}\mid + \mid \frac{\partial g_{ir}}{\partial x_j}\mid + \mid \frac{\partial g_{ij}}{\partial x_r}\mid\right) \leq 3\cdot(n+1)\frac{k(1)^2}{k(2)} =: k_1.\square$$

We consider some special conformal deformation of g which vary only with respect to t $= x_0$: Define $F_{d,s} \in C^\infty(I\!R, I\!R^{\geq 0})$ by $F_{d,s} = s \cdot \exp(-\frac{d}{t})$ on $I\!R^+$ and $F_{d,s} \equiv 0$ on $I\!R^{<0}$ and $f_{d,s}(t,\ldots) = F_{d,s}(t)$
Furthermore choose once and for all $H \in C^\infty(I\!R, [0,1])$ with $H \equiv 0$ on $I\!R^{\geq 1}, H \equiv 1$ on $I\!R^{\leq 0}$ and $H^b_\epsilon(t) := H(\frac{1}{\epsilon}(t-b)), b > 0, \epsilon > 0$ and $h^b_\epsilon \in C^\infty(I\!R^{n+1}, [0,1])$ by $h^b_\epsilon(t,\ldots) = H^b_\epsilon(t)$.
We consider $g_{d,s} := \exp(2 \cdot f_{d,s}) \cdot g$ and $g^{b,\epsilon}_{d,s} := \exp(2 \cdot h^b_\epsilon \cdot f_{d,s}) \cdot g$ for a metric g which fulfills (i) and (ii).

Lemma(1.2.)

(i) For each $b > 0$ there is a $d_0(b) > 0$, such that for $d \geq d_0(b)$ holds:

$$F^{(k)}_{d,s} > 0 \text{ on }]0,b] \text{ for } k = 0,1,2,3.$$

(ii) For each $m > 0$ and $b > a > 0$ there exists a $d_0(m,a,b) \geq d_0(b)$ such that for every $s > 0$ holds: $F''_{d,s} - m \cdot F'_{d,s} > 0$ on $]0, b[$ and $F''_{d,s} - m \cdot F'_{d,s} > s \cdot \exp(-\frac{d}{a})$ on $[a, b[$, if $d \geq d_0(m,a,b)$

(iii) There is a constant $\alpha = \alpha(\epsilon, b, k, D)$, such that for $d \geq D$ and $s \in [0,1]$ holds:
$\| g^{b,\epsilon}_{d,s} - g \|_{C^3_{g_{Eucl.}}(\mathbb{R}^{n+1})} < s \cdot \alpha$

(iv) For each $\epsilon > 0$ there exists a D_ϵ such that for every $d \geq D_\epsilon$ and each $s \in [0, 1]$ holds:
$\| g^{b,\epsilon}_{d,s} - g \|_{C^3_{g_{Eucl.}}(\mathbb{R}^{n+1})} < \epsilon$

Proof: (i), (iii) and (iv) are easily checked from the definitions. Thus we only indicate (ii): $F''_{d,s} - m \cdot F'_{d,s} = s\left(\left(-\frac{2d}{t^3} + \frac{d^2}{t^4}\right) - m \cdot \frac{d}{t^2}\right) \cdot \exp(-\frac{d}{t}) =: s \cdot \Phi_d(t) \cdot \exp(-\frac{d}{t})$
The only quadratic term with respect to d in Φ_d is $\frac{d^2}{t^4}$. This term is positive, i.e. for each $c \in]0,a]$ there is a D(m, c) with $\Phi_d > 1$ on [c, b[if $d \geq D(m,c)$
Now let be $d \geq d_0(b)$ (acc.(i)) i.e. $F'''_{d,s} > 0$ on]0, b[. Since $F'_{d,s}(0) = 0$ we obtain:
$F'_{d,s}(t_0) = \int\limits_0^{t_0} F''_{d,s} dt < F''_{d,s}(t_0) \cdot t_0$, i.e.$F''_{d,s} - m \cdot F'_{d,s} > 0$ on$]0, \frac{1}{m}]$.
Hence we choose $d_0(m,a,b) := max\{d_0(b), D(m, \frac{1}{2}min\{a, \frac{1}{m}\})\}$, then for $d \geq d_0$(m, a, b) holds:
$F''_{d,s} - m \cdot F'_{d,s} = s \cdot \Phi_d(t) \cdot \exp(-\frac{d}{t}) > s \cdot \exp(-\frac{d}{a})$ on $[a,b[$ and $'' > 0''$ on $]0,b[$. □

2. Deformation of Foliations

We start with an estimate of Ricci curvature of $g_{d,s}$ (resp. $g_{d,s}^{b,\epsilon}$) in terms of g and $f_{d,s}$. Similar results (with similar rough dependence on the background metric g) hold for analogous deformations of metrics on $\mathbb{R} \times M$ for a closed manifold M.

Let g be a metric on $\mathbb{R}^{n+1}$ which fulfills (i) and (ii) from §1 for some $k > 1$:

Proposition (2.1.) *For each $b > a > 0, \epsilon > 0$ there are constants $c_1, c_2 > 0$ which depend only on a, b, k and the dimension, such that for $d > c_2$, $s \in]0,1]$ and $0 \neq \nu \in T(\mathbb{R} \times \mathbb{R}^n)$ hold*

(i) $g_{d,s}^{b,\epsilon} \equiv g$ *on* $\mathbb{R} \backslash]0, b+\epsilon[\times \mathbb{R}^n$ *and* $\| g_{d,s}^{b,\epsilon} - g \|_{C^3_{g_{Eucl.}}} < s \cdot \alpha$
with $\alpha = \alpha(\epsilon, b, k, n) > 0$

(ii) $-s \cdot c_1 < \exp(2f_{d,s}) \cdot r(g_{d,s})(\nu) - r(g)(\nu) < 0$ *on* $]0,b] \times \mathbb{R}^n$

(iii) $-s \cdot c_1 < \exp(2f_{d,s}) \cdot r(g_{d,s})(\nu) - r(g)(\nu) < -s \cdot e^{-d/a}$ *on* $]a,b] \times \mathbb{R}^n$

(the upper estimates hold for each $s > 0$)

Proof: (i) does only collect definitions and (3.1.)(ii). Since $r(g)(\nu) = r(g)(\lambda \cdot \nu)$ for each $\lambda \neq 0$, we can assume that our ν fulfills $\| \nu \|_g = 1$ we can now use the formula for the conformal change of the Ricci curvatures from §1 to prove (ii) and (iii). We start with the simple estimate for $\left(| df_{d,s}(\nu) |^2 - \| \nabla^g f_{d,s} \|_g^2\right)$:

$$\| \nabla^g f_{d,s} \|_g^2 \;=\; | df_{d,s} |_g^2 \;=\; | \frac{\partial f_{d,s}}{\partial t} |^2 \cdot | dt |_g^2 \;=\; | F'_{d,s} |^2 \cdot | dt |_g^2 \;\leq$$

$$s^2 \cdot \frac{d^2}{t^4} \cdot \exp(-\frac{2d}{t}) \cdot k^2 \;\; (| dt |_g = sup_{\|\nu\|_g=1} | dt(\nu) | \leq sup_{\|\nu\|_{g_{Eucl.}}=k} | dt(\nu) | = k).$$

Hence (since $s^2 \leq s$ for $s \in]0,1]$):

$$-s \cdot 2 \cdot k^2 d^2/t^4 \cdot \exp(-2d/t) \leq -2 \cdot \| \nabla^g f_{d,s} \|_g^2 = | df_{d,s}(\nu) |_g^2 - \| \nabla^g f_{d,s} \|_g^2 \leq 0$$

Now we can turn to the term $(n-1) \cdot Hess_g f(\nu,\nu) + \Delta_g f$, which estimate is more complicated.

Let γ_ν be the geodesic with respect to g with $\gamma_\nu(0) = p$ and $\dot\gamma_\nu(0) = \nu$ and $\pi : I\!R \times I\!R^n \to I\!R$ the canonical projection and $h_\nu := \pi \circ \gamma_\nu$.

$$Hess_g f_{d,s}(\nu,\nu) = (f_{d,s} \circ \gamma_\nu)'(0) = (F_{d,s} \circ \pi \circ \gamma_\nu)''(0) = (F_{d,s} \circ h_\nu)''(0) =$$

$$F''_{d,s}(h_\nu(0)) \cdot (h'_\nu(0))^2 + F'_{d,s}(h_\nu(0)) \cdot h''_\nu(0)$$

$$\Delta_g f_{d,s} = tr_g Hess_g f_{d,s} = \sum_{i=0}^{n} F''_{d,s}(\pi(p)) \cdot (h'_{e_i}(0))^2 + \sum_{i=0}^{n} F'_{d,s}(\pi(p)) \cdot h''_{e_i}(0)$$

$e_0, \dots e_n$ denotes on orthonormal bases in $T_p I\!R^{n+1}$ with respect to g. This yields :

$$(n-1) \cdot Hess_g f_{d,s}(\nu,\nu) + \Delta_g f_{d,s} =$$

$$F''_{d,s} \cdot ((n-1)(h'_\nu(0))^2 + \sum_{i=0}^{n}(h'_{e_i}(0))^2) + F'_{d,s} \cdot ((n-1)h''_\nu(0) + \sum_{i=0}^{n} h''_{e_i}(0))$$

$$=: F'_{d,s}(\pi(p)) \cdot H_1(g,\nu) + F'_{d,s}(\pi(p)) \cdot H_2(g,\nu)$$

Lemma (2.2.) *There are constants $c_1^-, c_1^+ > 0$ resp $c_2^- < 0$ and $c_2^+ > 0$, which depend only on n and k such that for each $\| \nu \|_g = 1$:*

$$c_i^- \leq H_i(g,\nu) \leq c_i^+.$$

Proof: Note that $h'_\nu(0) = g(\nabla^g \pi, \nu) = d\pi(\nu) = dt(\nu)$ and $h''_\nu(0) = g(\nabla^g_\nu \nabla^g_\pi, \nu)$ since γ_ν is a geodesic. Furthermore $| dt(\nu) |^2 \leq \| \nu \|^2_{g_{Eucl.}} \leq k^2 \cdot \| \nu \|^2_g$, i.e. $| h'_\nu(0) |^2 \leq k^2$.
Let be $e_0, \dots e_n$ an orthonormal bases (w.r.t. g) with $e_0 = \lambda \cdot \frac{\partial}{\partial t}$ for some $\lambda \in [k^{-1}, k]$, then $| h'_{e_0}(0) |^2 = | d\pi(\lambda \cdot \frac{\partial}{\partial t}) |^2 \geq k^{-2}$, i.e. $k^{-2} \leq (n-1)(h'_\nu(0))^2 + \sum_{i=0}^n (h'_{e_i}(0))^2 \leq 2nk^2$, thus for

$$c_1^- =: k^{-2}, c_1^+ =: 2nk^2 \quad : \quad c_1^- \leq H_1(g,\nu) \leq c_1^+.$$

Now we will show $| H_2(g,\nu) | < C(k)$, i.e. $c_2^- =: -C(k), c_2^+ =: +C(k)$ will fulfill the claim. From $h''_\nu(0) = g(\nabla^g_\nu \nabla^g \pi, \nu)$ and the Cauchy-Schwarz inequality it suffices to get an estimate for $\| \nabla^g_\nu \nabla^g \pi \|_g$.
Let $x_o, \dots, x_n$ be the background standard Euclidean coordinates. Their induced derivatives $\frac{\partial}{\partial x_i}$ fulfill $k^{-1} \leq \| \frac{\partial}{\partial x_i} \| \leq k$, i.e. it is enough to control $\| \nabla_{\frac{\partial}{\partial x_i}} \nabla^g \pi \|_g$.

Now let $\nu_0, \dots, \nu_n$ be orthonormal bases in p w.r.t. g and $\nu_j = \sum_{k=0}^n \lambda_{jk} \frac{\partial}{\partial x_k}$ these λ_{jk} fulfills $|\lambda_{jk}| < k$ and we conclude:

$$\| \nabla_{\frac{\partial}{\partial x_i}} \nabla^g \pi \|_g^2 = \sum_{j=0}^n | g(\nabla_{\frac{\partial}{\partial x_i}} \nabla^g \pi, \nu_j) |^2 \leq \sum_{j=0}^n \sum_{k=0}^n \lambda_{jk}^2 | g(\nabla_{\frac{\partial}{\partial x_i}} \nabla^g \pi, \frac{\partial}{\partial x_k}) |^2$$

$$\leq 2 \cdot k^2 \sum_{j=0}^n \sum_{k=0}^n (| \frac{\partial}{\partial x_i} g(\nabla^g \pi, \frac{\partial}{\partial x_k}) |^2 + | g(\nabla^g \pi, \nabla^g_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k}) |^2)$$

$$= 2 \cdot k^2 \sum_{j=0}^n \sum_{k=0}^n (| \frac{\partial}{\partial x_i} (\delta_{0k}) |^2 + | \Gamma_{ik}^0 |^2) \leq 2k^2 (n+1)^2 k_1^2 \qquad k_1 \text{ from}(3.1) \square$$

This proof also implies the following statement which we note for later use (§6).

Corollary (from the proof above)(2.3.) *Let be $F \in C^\infty(\mathbb{R}, \mathbb{R})$ with $F', F'' \geq 0$ and g a metric on $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ which fulfill conditions (i) and (ii), then there are constants $\alpha_1, \alpha_2 > 0$ depending only on n and k such that: $(\alpha_1 \cdot F'' - \alpha_2 \cdot F') \circ \pi \leq \Delta_g (F \circ \pi)$*

Now we finish the

Proof of (2.1.): From (2.2.) and $F_{d,s}^{(k)} > 0$ on $]0, b]$ for k = 0, 1, 2, 3 we conclude

$$c_1^- \cdot F''_{d,s}(\pi(p)) + c_2^- \cdot F'_{d,s}(\pi(p)) \leq (n-1) \cdot Hess_g f_{d,s}(\nu, \nu) + \Delta_g f_{d,s} \leq c_1^+ \cdot F''_{d,s}(\pi(p)) + c_2^+ \cdot F'_{d,s}(\pi(p))$$

We collect our results for $d \geq d_0(| \frac{c_2^-}{c_1^-} |, a, b)$ (cf. (1.2.)):

$$(1) \qquad -s \cdot 2(n-1) \cdot k^2 \cdot \frac{d^2}{t^4} \cdot e^{-\frac{2d}{t}} - c_1^+ \cdot s \cdot (\frac{d}{t} - 2) \frac{1}{t^3} \cdot d \cdot e^{-\frac{d}{t}} - c_2^+ \cdot s \cdot \frac{d}{t^2} \cdot e^{-\frac{d}{t}} \leq \exp(2 f_{d,s}) \cdot r(g_{d,s})(\nu) - r(g)(\nu) \leq -c_1^- \cdot F''_{d,s}(\pi(p)) - c_2^- \cdot F'_{d,s}(\pi(p))$$

Since $c_1^- > 0$ and $c_2^- < 0$,we conclude from $d \geq d_0$ that the expression on the right hand side is "< 0" on $]0, b[\times \mathbb{R}^n$ and "$< -s \cdot \exp(-\frac{d}{a})$" on $[a, b[\times \mathbb{R}^n$

Now let be (t, d) $\in]0, b] \times \mathbb{R}^{\geq 1}$ and p, q $\in \mathbb{Z}^{>0}$: $\frac{d^q}{t^p} \exp(-\frac{d}{t}) \leq \max\{1, b^q\} \cdot (\frac{d}{t})^{p+q} \cdot \exp(-\frac{d}{t}) \underset{\frac{d}{t} \to \infty}{\longrightarrow} 0$, therefore we get

$$c_1 := \sup_{]0, b[, d \geq 1} (2(n-1) \cdot k^2 \cdot \frac{d^2}{t^4} \cdot e^{-\frac{2d}{t}}) + \sup_{]0, b[, d \geq 1} (c_1^+ (\frac{d}{t} - 2) \frac{1}{t^3} \cdot d \cdot e^{-\frac{d}{t}})$$

$$+ \sup_{]0,b[,\, d \geq 1} (c_2^+ \frac{d}{t^2} e^{-\frac{d}{t}}), \text{ and we conclude}$$

$0 < c_1 < +\infty$ and $-s \cdot c_1$ is a lower bound for the left hand side of (1). Thus this c_1 *and* $c_2 := \max\{1, d_0(| \frac{c_2}{c_1} |, a, b)\}$ fulfill our claimes (ii) and (iii). □

Next we generalize these results to topologically (trivial) foliations by closed manifolds. Let M^n be a closed manifold and g a fixed metric on $I\!R \times M^n$. M^n can be covered by a finite collection of charts $f_i : V_i \to I\!R^n, V_i \subset M$ open, $i = 1, \dots m$ such that there are euclidean balls $B_i \subset\subset B_i' \subset\subset f_i(V_i)$ with $M = \bigcup_i f_i^{-1}(B_i)$

Of course this also yields an atlas of $I\!R \times M$: $id \times f_i : I\!R \times V_i \to I\!R \times I\!R^n$. Choose $\tilde{h}_i \in C^\infty(I\!R^n, [0,1])$, such that $\tilde{h}_1 \equiv 1$ on B_i, $\tilde{h}_i \equiv 0$ on $I\!R^n \setminus B_i'$ and define $h_i := \tilde{h}_i \circ \pi_n$, where $\pi_n : I\!R \times I\!R^n \to I\!R^n$ is the canonical projection.
Consider the metric $g_i := h_i \cdot (id \times f_i)_*(g) + (1 - h_i) \cdot g_{Eucl.}$.
Note that g_i depends on the particular choice of charts f_i and functions h_i. To emphasize and clarify this we introduce a

Definition (2.4.) *A set $D = \{(f_i, V_i, B_i', B_i, h_i), i = 1, \dots m\}$ as above is called deformation atlas of $I\!R \times M$. If there are constants k_i (which in case are chosen minimal) such that $g_{Eucl.}(\nu, \nu) \leq k_i^2 \cdot g_i(\nu, \nu)$ and $\| g_i \|_{C^3_{g_{Eucl.}}(I\!R^{n+1})} \leq k_i$, than $k(D, g) := \max_i\{k_i\}$ is called the deformation constant of g with respect to D.*

Notice that $k(D, g)$ depends continously on the choice of g (w.r.t. uniform norms) and that if $F : (M, g) \to (M', g')$ is isometric, than $F : (M, e^{2G \circ F} \cdot g) \to (M', e^{2G} \circ g')$ for $G \in C^\infty(M', I\!R)$ is also isometric. This makes (2.4.) useful for our deformation.

Let $F_{d,s}$ and H be as in §1 (before(1.2.)) then we define again $f_{d,s} : I\!R \times M \to I\!R$ by $f_{d,s}(t, x) = F_{d,s}(t)$ and $h_\epsilon^b(t, x) = H(\frac{1}{\epsilon}(t - b))$. Furthermore we consider $g_{d,s} := \exp(2 f_{d,s}) \cdot g$ and $g_{d,s}^{b,\epsilon} := \exp(2 h_\epsilon^b f_{d,s}) \cdot g$ for a g with deformation constant $k(D, g)$ for some deformation atlas D.

Proposition (2.5.) *For $b > a > 0, \epsilon > 0$ there are constants $c_1, c_2 > 0$ depending only on a, b, k(D, g) and dim M, such that for $d > c_2$, $s \in]0,1[$ and $0 \neq \nu \in T(I\!R \times M^n)$ the following claims hold:*

(i) $g_{d,s}^{b,\epsilon} \equiv g$ *on* $I\!R \setminus]0, b + \epsilon[\times M^n$ *and*

$$\| (g_i)_{d,s}^{b,\epsilon} - g_i \|_{C^3_{g_{Eucl.}}(I\!R^{n+1})} \leq s \cdot \alpha \quad \textit{with} \quad \alpha = \alpha(\epsilon, b, k(D, g), n) > 0$$

(ii) $-s \cdot c_1 < e^{2f_{d,s}} \cdot r(g_{d,s})(\nu) - r(g)(\nu) < 0$ *on* $]0, b] \times M^n$

(iii) $-s \cdot c_1 < e^{2f_{d,s}} \cdot r(g_{d,s})(\nu) - r(g)(\nu) < -s \cdot e^{-\frac{d}{a}}$ *on* $[a, b[\times M^n$

(in (ii) and (iii) the upper estimates hold for each $s > 0$*)*

Proof: (i) is again a reformulation of the definitions and(1.2). For (ii) and (iii) we use (2.1.) and (2.4.), i.e. we use D and consider the metrics g_i. Use (3.1.) for each g_i: g_i fulfills $g_{Eucl.}(\nu,\nu) \le k^2(D,g) \cdot g_i(\nu,\nu)$ and $\| g_i \|_{C^3_{g_{Eucl.}}(\mathbb{R}^{n+1})} \le k(D,g)$ i.e. we get constants $c_1, c_2 > 0$ depending only on $k(D,g), a, b$ and n such that for $d > c_2$ and $0 \neq \nu_i \in T\mathbb{R}^{n+1}$:

$(ii)_i - s \cdot c_1 < e^{2f_{d,s}} \cdot r((g_i)_{d,s})(\nu_i) - r(g_i)(\nu_i) < 0$ on $]0, b] \times \mathbb{R}^n$

$(iii)_i - s \cdot c_1 < e^{2f_{d,s}} \cdot r((g_i)_{d,s})(\nu_i) - r(g_i)(\nu_i) < -s \cdot e^{-\frac{d}{a}}$ on $]a, b] \times \mathbb{R}^n$

From the remark after (2.4.) concerning conformal invariance we conclude that with these constants c_1, c_2 the claimed inequalities (ii) and (iii) are fulfilled for $d > c_2$ □

In applications we usually do not go into every detail as to define deformation charts etc., this is to avoid to become to technical; we just indicate the scheme: if we want to use (2.5.) for $U_R(M) \setminus U_r(M)$ where U denotes the tubular neighborhood (triviality is assumed) of some closed hypersurface $M \subset (N, g)$ then there are three steps: Define a map $F :]r, R] \times M \to U_R \setminus U_r$ and take the pull-back metric $F^*(g)$, next extend this metric using cut-off functions near r and R to some product metric on $\mathbb{R} \times M$ outside of $]r, R] \times M$ and use (2.5.), finally take push-forwards and recall the remark after (2.4.) to get the desired Ric–estimates on $U_R \setminus U_r$.

3. $Ric < 0$ on balls in three dimensions

Our first application of the techniques described in §1 and of the previous results is

Proposition (3.1.) *There exists a metric* g_3^- *on* $\mathbb{R}^3$ *with* $r(g_3^-) < 0$ *on* $B_1(0)$ *and* $g_3^- \equiv g_{Eucl.}$ *outside*

This is done in two steps: first we construct a somewhat analogous metric on $S^1 \times S^2 \# \mathbb{R}^3$ and then we get rid of the handle by a specific surgery.

Lemma (3.2.) *There is a* $f \in C^\infty(\mathbb{R}, \mathbb{R}^{>0})$ *with* $f \equiv id$ *on* $\mathbb{R}^{\ge 1}$ *and* *(i) there is a $\delta \in]0,1[$ such that $f(r) = f(2\delta - r)$ i.e. f is symmetric w.r.t. δ*

(ii) $Ric(g_R + f^2 \cdot g_{S^2}) < 0$ on $]2\delta - 1, 1[\times S^2$

Proof: From §1 we know that Ric can be calculated as follows:

$$Ric(g_{\mathbf{R}} + f^2 \cdot g_{S^2})(\nu,\nu) = \frac{1}{f} - \frac{f''}{f} - \left(\frac{f'}{f}\right)^2 \quad \text{resp.} \quad = -2 \cdot \frac{f''}{f}$$

for vertical ν resp. horizontal ν with $\| \nu \| = 1$.
Such a f as claimed above can be constructed as follows (details are left to the reader). Take any $h \in C^\infty(I\!R, I\!R^{>0})$ with $h \equiv | id |$ on $I\!R^{\leq -1} \cup I\!R^{\geq 1}$, $h(r) = h(-r)$ and $h, h'' > 0$ on] -1, 1[, $h''' < 0, 1 > h' > 0$ on]0, 1[. Calculations similar to those indicated in (2.2.) yield a $c > 0$ such that $f := \frac{1}{c+1} h((c+1)t - c)$ and $\delta := \frac{c}{c+1}$ fulfil the claims. □.

Consider $B_r^\pm := B_r(\pm 3, 0, 0) \subset I\!R^3, r > 0$ and (cf. (4.2.)) the metric $g_0 := g_{\mathbf{R}} + f^2(\| (\pm 3,0,0) - id_{\mathbf{R}^3} \|) \cdot g_{S^2}$ on $B_3^\pm \setminus B_\delta^\pm \cong [\delta, 3[\times S^2$.
It is Euclidean on $B_3^\pm \setminus B_2^\pm$, thus it can be extended on $I\!R^3 \setminus B_3^+ \cup B_3^-$ by $g_0 \equiv g_{Eucl.}$.

Now we define

$$g_1(d) := \exp\left(2(H_\epsilon^b \cdot F_{d,1})(4 - \| (-3,0,0) - id \|) + 2(H_\epsilon^b \cdot F_{d,1})(4 - \| (+3,0,0) - id \|)\right) \cdot g_0$$

with $b = 3 + \frac{1-\delta}{2}$ and $\epsilon = \frac{1-\delta}{4}$. Thus we are in standard situation to apply (2.4.): $B_4^\pm \setminus B_\delta^\pm$ interpreted as a part $[\delta, 4[\times S^2$ of $I\!R \times S^2$ and each of the two conformal deformations yields a controllable perturbation of Euclidean metric (i.e. the deformation constants stay bounded for each $d \geq 1$).
Thus we conclude from (2.4.): for d large enough $r(g_1(d)) < 0$ on $B_4^\pm \setminus B_\delta^\pm$ and $g_1(d) \equiv g_{Eucl.}$ on $I\!R^3 \setminus B_4^+ \cup B_4^-$.

Next we identify ∂B_δ^+ with ∂B_δ^- by restriction of $i : I\!R^3 \to I\!R^3, i(x,y,z) := (-x,y,z)$. The resulting manifold is diffeomorphic to $S^1 \times S^2 \# I\!R^3$ and it carries a canonical metric $\bar{g}$. $\bar{g}$ is smooth from (3.2.)(i).
The line segment from $(-3+\delta, 0, 0)$ to $(3-\delta, 0, 0)$ in $I\!R^3$ becomes (under this identification) a closed geodesic γ with trivial holonomy and length $L > 0$ which is isotopic to the S^1 - factor and there is a neighborhood U of γ in $S^1 \times S^2 \# I\!R^3$ such that $Ric(\bar{g}) < 0$ on U.

Now we want to get rid of the handle. Thus let be $U_r(\gamma) \subset U$ be a tubular neighborhood of width $r > 0$ (w.r.t to $\bar{g}$). It was proved by Gao ([G], Prop.(2.5.)) that we can choose $\Phi \in C^\infty(I\!R, [0,1])$ with $\Phi \equiv 1$ on $]-\infty, R]$, $\Phi \equiv 0$ on $[r', +\infty[$ such that for

$$g_\Phi := \Phi(L^2 \cosh^2 dt^2 + dr^2 + \sinh^2 r d\Theta^2) + (1 - \Phi)\bar{g} \text{ on } U_r(\gamma)$$

$Ric(g_\Phi) < 0$ holds again for suitable $0 < R < r' < r$.
This notation of g_Φ is with respect to Fermi-Coordinates along γ. We will prefer another notation more adequate to solve our problems:

$$(U_R(\gamma) \setminus \gamma, g_\Phi) = (\bar{S}^1 \times]0, R[\times S^1, (\tfrac{L\cdot\cosh}{2\pi})^2 \cdot g_{\bar{S}^1} + g_{\mathbb{R}} + \sinh^2 r \cdot g_{S^1})$$

(with $L_{g_{\bar{S}^1}}(\bar{S}^1) = L_{g_{S^1}}(S^1) = 2\pi$, "$-$" does only indicate a distinction between the two S^1-factors)

Now we will use the following non difficult Lemma of [L5] (which is also easily checked by the reader) which generalizes similar results by Gao, Yau [GY] and Brooks [B].

Lemma (3.3.): *Let be $F,\ G \in C^\infty(]a,b[, I\!R^{>0})$ for some $a < b$ with $F', F'', G', G'' > 0$ on $]a,b[$, then there are constants $\rho < a$, $(> 0$ and $f,\ g \in C^\infty(]\rho,b[, I\!R^{>0})$ with $f', g' > 0$ and $f'' \geq 0$ on $]\rho, b[$ and:*

$$\text{(i)}\ f \equiv \begin{cases} F \\ C \cdot \cosh(id_{\mathbb{R}} - \rho) \end{cases} \qquad g \equiv \begin{cases} G & near\ \ b \\ \sinh(id_{\mathbb{R}} - \rho) & near\ \ \rho \end{cases}$$

$$\text{(ii)}\ Ric(f^2 \cdot g_{S^1} + g_{\mathbb{R}} + g^2 \cdot g_{\bar{S}^1}) < 0 \qquad on \qquad S^1 \times]\rho, b[\times S^1$$

Now we are ready to give
<u>Proof:</u> of (3.1.): Lemma (3.3.) gives us $\rho < 0$, $c > 0$ and a metric $g_0 = g^2 \cdot g_{\bar{S}^1} + g_{\mathbb{R}} + f^2 \cdot g_{S^1}$ on $\bar{S}^1 \times]\rho, R[\times S^1$ and with $Ric(g_0) < 0$ such that for some small $\epsilon > 0$:

$$g = \begin{cases} \frac{L\cdot\cosh}{2\pi} \\ \sinh(id_{\mathbb{R}} - \rho) \end{cases} \quad , \quad f = \begin{cases} \sinh & \text{on} \quad]R-\epsilon, R[\\ L \cdot \cosh(id_{\mathbb{R}} - \rho) & \text{on} \quad]\rho, \rho+\epsilon[\end{cases}$$

i.e. $\bar{S}^1 \times \{\rho + \epsilon\} \times S^1$ can be identified in a canonical isometric way with the boundary of the following hyperbolic tube:

$$(B_\epsilon(0) \times S^1, g_{hyperbol.} + (C \cdot \cosh r)^2 \cdot g_{S^1}) \subset I\!H^2 \times S^1$$

such that $\bar{S}^1 \times \{\rho + \epsilon\} \times \{e^{it}\}$ and $\partial B_\epsilon \times \{e^{it}\}$ are identified.
This identification yields a smooth Ricmannian manifold which is diffeomorphic to $I\!R^3$: the boundary $\bar{S}^1 \times S^1$ of the complement of a tubular neighborhood $\bar{S}^1 \times B^2$ of the geodesic γ in $S^1 \times S^2 \# I\!R^3$ is identified by same identifications with the boundary of $B^2 \times S^1$ (c.f. Heegard - Splittings of $S^1 \times S^2$ and S^3, [H], §2).

Thus we obtain a metric g_1 on $I\!R^3$ with $Ric(g_1) < 0$ inside $B_4^+ \cup B_4^-$ (using the previous identifications) and $g_1 \equiv g_{Eucl.}$ outside.

Finally we use again (2.4.) for $I\!R \times S^2$. This time we interprete $B_{10}(0) \setminus B_1(0)$ as $[1, 10[\times S^2 \subset I\!R \times S^2$ and obtain a d such that
$g_2(d) := \exp(2H^{9,5}_{0,1} \cdot F_{d,1}(10- \| id \|)) \cdot g_1$ fulfills $r(g_2(d)) < 0$ on $B_{10}(0)$.
Scale $g_2(d)$ by a constant to obtain (3.1.) □

4. Main Deformation

In this paragraph we cover a given manifold by balls and perform deformations (constructed from g_n^-) on each them such that the Ricci curvature gets more and more negative. Note that the covering is a Besicovitch type covering i.e. with controlled intersecting numbers and that this is not only a trick to get pinching constants for Ric but it is essential even to prove $Ric < 0$.

Since the whole proof works by induction we can *assume* the existence of a metric g_n^- with $Ric < 0$ on $B_1(0)$ and $g_n^- \equiv g_{Eucl.}$ on $I\!R^n \setminus B_1(0)$ *in dimension n*. This was proved for $n = 3$ in §3 and will be proved for $n + 1$ in §6 using the results (that is in particular Theorem E) of this and the next chapter.

Let $A \subset I\!R^n$ be a set with $0 \in A$ and $\| p - q \| > 5$ for $p \neq q \in A$ and choose for each $p \in A$ an isometry f_p on $(I\!R^n, g_{Eucl.})$ with $f_p(p) = 0$ and define $g_A^- := f_p^*(g_n^-)$ on $B_1(p)$ for $p \in A$ and $g_A^- \equiv g_{Eucl.}$ elsewhere.
Note that g_A^- clearly depends on the choice of the isometries f_p. The effect of this choice will be studied now:

Let be F: $I\!R^n \setminus \{0\} \to I\!R^{>0} \times S^{n-1}$ the well-known diffeomorphism $F(x) = \left(\| x \|, \frac{x}{\|x\|}\right)$.
Now choose a fixed deformation atlas D_n on $I\!R \times S^{n-1}$ and consider $g^r(f_p \mid p \in A) := h_r \cdot (g_{I\!R} + g_{S^{n-1}}) + (1 - h_r) \cdot F_*(g_A^-)$ with $h_r \in C^\infty(I\!R, [0, 1])$, $h_r \equiv 1$ on $I\!R^{\leq 0,1} \cup I\!R^{\geq 12+r}$, $h_r \equiv 0$ on $[\frac{2}{10}, 11 + r]$ for some $r \geq 0$

Lemma (4.1.) *There is a constant $k_n^r > 1$ with $k_n^r > k(D_n, g^r(f_p \mid p \in A))$ for each A as described above.*

Proof: It is enough to check (4.1.) for $A = \{0, p\}$ with $p \in \overline{B_{13+r}(0)} \setminus B_3(0)$. For each p the set of possible isometries f_p is compact (isomorphic to O(n)). Define $f_p^\circ \equiv id_{I\!R^n} - p$ and consider the following mapping Q_r which is continuous by definition of deformation constants (cf.(3.4))

$$Q_r : O(n) \times \overline{B_{13+r}(0)} \setminus B_3(0) \to I\!R^{>0}, Q_r(f,p) := k(D_n, g^r(id_{I\!R^n}, f \circ f_p^\circ))$$

$O(n) \times \overline{B_{13+r}(0)} \setminus B_3(0)$ and its image ImQ_r are compact. Thus we get an upper bound k_n^r of ImQ_r. □

Definition (4.2.) *Let be (M^n, g) a Riemannian manifold without boundary and assume existence of a complete manifold $(\bar{M}^n, \bar{g})$ with $(M, g) \subset (\bar{M}, \bar{g})$ in the Riemannian sense. Furthermore assume that for each $p \in M$ holds :*
$\exp_p : B_{100+\rho}(0) \to \bar{M}$ *is a diffeomorphism with* $\| \exp_p^*(\bar{g}) - g_p \|_{C^5_{g_p}, B_{100+\rho}(0)} \leq \epsilon$ *(g_p denotes $g \mid T_pM$) for some $\epsilon \geq 0$ resp. $\rho \geq 0$ which do not depend on $p \in M$, than (M^n, g) is called (ϵ, ρ) - (resp. for $\rho = 0$ just (ϵ) -) manifold and g (ϵ, ρ) - metric.*

Remark (4.3.)

a) *Let be $(M, g) = (\epsilon, \rho)$-manifold and $\tilde{\epsilon}, \tilde{\rho}$ constants with $\epsilon \geq \tilde{\epsilon} > 0$ resp. $\epsilon = \tilde{\epsilon} = 0$ and $\tilde{\rho} \geq 0$, then $(M, \lambda^2 \cdot g)$ is $(\tilde{\epsilon}, \tilde{\rho})$ - manifold if λ is large enough. In particular one can reduce to case $\tilde{\rho} = 0$.*

b) *For each $\delta \geq 0$ there is a $\epsilon_n^\delta > 0$ such that each (m, 0) - tensor field $T, m = 0, 1, 2$ on a (ϵ_n^δ) - manifold fulfills:*

$$(1 - \delta) \| T \|_{C_g^3(B_{90}(p))} \leq \| \exp_p^*(T) \|_{C^3_{g_p}(B_{90}(p))} \leq (1 + \delta) \| T \|_{C_g^3(B_{90}(p))}$$

(in particular $\| \cdot \|_{C_g^3(B_{90}(p))}$ and $\| \exp_p^*(T)(\cdot) \|_{C_g^3(B_{90}(p))}$ are equivalent norms)

Our next (and last) preliminary result is a covering lemma which is a generalization of the well known Besicovitch covering lemma on $I\!R^n$ (cf. [F], 2.8.). We will indicate a proof in the Appendix.

Proposition (4.4.)

a) *Let be (M^n, g) a (1) - manifold, then for each $r \geq 0$ there are constants c(n, r), $m_0(n, r) \in Z\!\!Z^{>0}$ such that*

   (i) *For each $m \geq m_0$ there is a set $A = A(m, r) \subset M$ with $d_{m^2 \cdot g}(a, b) > 5 + r$ for $a \neq b, a, b \in A$*

   (ii) *$\mathcal{A} = \mathcal{A}(m, r) = \{(\overline{B_{5+r}(a)}, m^2 \cdot g) \mid a \in A\}$ is a (closed) covering of M and splits into c disjoint families $\mathcal{B}_j$ with*

$$(\overline{B_{10+r}}(a), m^2 \cdot g) \cap (\overline{B_{10+r}}(b), m^2 \cdot g) = \emptyset,$$

if : $\overline{B_{5+r}}(a)$ *and* $\overline{B_{5+r}}(b)$ *belong to the same* $\mathcal{B}_j$.

b) *Let be* M^n *compact without boundary,* $G \subset Isom(M,g)$ *a finite and non-trivial subgroup,*

$$F(G) := \{z \in M \mid f(z) = z \quad for\ some\ f \in G \setminus \{id\}\},$$

then there are constants

$$r_0(M,g,G) \quad (> 5 \quad for \quad F(G) \neq \emptyset \quad and \quad = 0 \quad for \quad F(G) = \emptyset),$$

$c(M,g,G)$, $m_0(M,g,G) \in \mathbb{Z}^{>0}$ *such that*

(i) *For each* $m \geq m_0$ *there is a* $\mathcal{A} = \mathcal{A}\,(m,r) \subset M$ *with* $G(A) = A$, $d_{m^2 \cdot g}(a,b) > 5$ *for* $a \neq b \in A$ *and if* $F(G) \neq \emptyset$: $dist_{m^2 \cdot g}(F(G), A) \geq r_0$

(ii) $\mathcal{A} = \mathcal{A}\,(m,M,g,G) = \{(\overline{B_{5+r_0}}(a), \mid a \in A\}$ *is a (closed) covering of* M *and splits into* c *disjoint families* $\mathcal{B}_j$ *with*

$(\overline{B_{10+r_0}}(a), m^2 \cdot g) \cap (\overline{B_{10+r_0}}(b), m^2 \cdot g) = \emptyset$ *if*

$\overline{B_{5+r_0}}(a)$ *and* $\overline{B_{5+r_0}}(b)$ *belong to the same* $\mathcal{B}_j$ □

Now let be (M^n, g) a (1, r)-manifold. We use the notations of (4.3.) and define for $B_i := \{z \in M \mid B_{5+r}(z) \in B_i\}$ and $A := \bigcup\limits_i B_i$ the following metric on M:

Let be $m \geq m_0, B \subset A$ an arbitary subset and $I_p : T_pM \to \mathbb{R}^n, p \in A$ a linear isometry (in case (4.3.)b) with $I^{-1}_{f(p)} \circ I_p = Df_p$ for $f \in G$, $p \in A$) and $\bar{h} \in C^\infty(\mathbb{R}, [0,1])$ with $\bar{h} \equiv 1$ on $\mathbb{R}^{\leq 1,5}$, $\bar{h} \equiv 0$ on $\mathbb{R}^{\geq 1,6}$ fixed:

$$g(B,m,r,d,s) := \prod_{p \in B} \exp(2 \cdot H^{9,5+r}_{0,1} \cdot F_{d,s}(10 + r - d_{m^2 \cdot g}(p,x))) \cdot g_A \text{ in } x \in M \text{ with}$$

$$g_A := \begin{cases} \bar{h}(d_{m^2 \cdot g}(p,x))(\exp_p^{m^2 \cdot g_p})_*(I_p^*(g_n^-)) + (1 - \bar{h}(d_{m^2 \cdot g}(p,x)) \cdot m^2 \cdot g \text{ on } \bigcup\limits_{p \in A} B_2(p) \\ m^2 \cdot g \qquad \text{elsewhere} \end{cases}$$

($\exp_p^{m^2 \cdot g}$ denotes the exponential mapping w.r.t. $m^2 \cdot g$ in $p \in M$)

Since $\mathcal{A}$ is locally finite $g(B,m,r,d,s)$ is defined and in case (4.3.)b) it fulfills:

$$f^*(g(A,m,r,d,s)) \equiv g(A,m,r,d,s) \text{ for each } f \in G.$$

Now we arrive at one of the most important results:

Proposition (4.5.) *There are $m_1 \geq m_0, d_1, s_1 > 0$ (with dependences $(m_1, d_1, s_1)(n, r)$ in case (4.3.)a), $(m_1, d_1, s_1)(M, g, G)$ in case (4.3)b)) such that for each $d \geq d_1, s^{-1} \geq s_1^{-1}$ there exists a $m \geq m_1$ with $m = m(d,\ s,\ n,\ r)$ resp. $m(d,\ s,\ M,\ g,\ G)$) such that*

$$-c_1 < r(g(A,m,r,d,s)) < -c_2$$

for constants $c_1 > c_2 > 0$ (with $c_i = c_i(n,r,d,s)$ resp. $c_i(M,g,G,d,s)$)

We will present the proof in a moment, but we first derive Theorems A and C from (4.5.):

Corollary (4.6.) *There are constants $a(n) > b(n) > 0$ and for each manifold $M^n, n \geq 3$ a complete metric g, which Ricci curvature fulfills:*

$$-a(n) < r(g) < -b(n)$$

Proof: In the appendix (A 1) it is proved that each M^n admits a complete (ϵ, ρ)-metric for $\epsilon > 0, \rho \geq 0$. Thus we can choose in (4.5.):

$$g = g(A,m,0,d_1,s_1), a(n) = c_1(n,0,d_1,s_1), b(n) = c_2(n,0,d_1,s_1) \text{ this metric fulfills}$$

$$-a(n) < r(g) < -b(n)$$

. □

Corollary (4.7.) *Let be $M^n, n \geq 3$ be a closed manifold, $G \subset Diff(M)$ a finite subgroup, then there is a metric g on M with $r(g) < 0$ and $G = Isom(M,\ g)$.*

Proof: Choose any G-invariant metric g_0 on M, again for λ large enough we can apply (4.5.) to $(M, \lambda^2 \cdot g_0)$ and we obtain a G-invariant metric g_1 with $r(g_1) < 0$ on M, i.e. $G \subset Isom(M, g_1)$. Now one can perform a G-invariant perturbation of g_1 such that the new metric g fulfills $r(g) < 0$ and $G = Isom(M, g_1)$. For such perturbations of g_1 cf.[E] (8.3.). □

Proof of (4.5.): We start with some remarks concerning g_A. Note that the situation (4.4.) a) and b) are always discussed at the same time, we also adopt the notations of (4.4.):

(1.) There is a $m^{(1)} \geq m_0$ $(m^{(1)}(n,r)$ resp. $m^{(1)}(M,g,G))$ such that for each $m \geq m^{(1)}, p \in A \subset M^n$

$$k(D_n, h_r \cdot (g_{\mathbb{R}} + g_{S^{n-1}}) + (1-h_r) \cdot F_*(I_p^*(\exp_p^{m^2 \cdot g})^*(g_A)) < k_n^r + 1$$

This is clear from (4.2.) and (4.3.) b) since given $\epsilon > 0$ and a compact $K \subset \mathbb{R}^n$ one can find a $m \geq m_0(m(n,r,\epsilon,K)$ resp. $m(M,g,G,\epsilon,K))$ a set $A \subset M^n$ (as in (5.4.)) and a second set $A(m) \subset \mathbb{R}^n$ (as before and in (5.1.)) such that for each $p \in M$

$$\| I_p^*(\exp_p^{m^2 \cdot g})^*(g_A) - g_{\bar{A}(m)} \|_{C^3_{g_{Eucl.}}(K)} < \epsilon.$$

(2.) This also implies (with (5.4.)) for $\delta > 0$
there is a $m^{(2)} \geq m^{(1)}$ with
$m^{(2)} = m^{(2)}(n,r,\delta)$ resp. $m^{(2)}(M,g,G,\delta)$ and

$$\sup_{0 \neq \nu \in TM} | r(m^2 \cdot g)(\nu) | < \delta \text{ and } \mu_n - \delta < r(g_A)(\nu) < \delta$$

for $m \geq m^{(2)}, \nu \neq 0$ and a $\mu_n < 0$ which depends only on n.

(3.) According to (4.4.) for $m \geq m_0$ each point of M is contained in at most c different balls $B_{10+r}(a), a \in A$. Thus for each $\epsilon \in]0,1[$ we have from (1), (2.5.) and (1.2.)(iii) and (iv) $\tilde{d}_0, \tilde{s}_0 > 0((\tilde{d}_0, \tilde{s}_0)(n,r,\epsilon)$ resp. $(\tilde{d}_0, \tilde{s}_0)(M,g,G,\epsilon)$ such that for $d \geq \tilde{d}_0, s^{-1} \geq \tilde{s}_0^{-1}$ and each subset $B \subset A$:

$$F_m^{d,s}(B) := \prod_{p \in B} \exp(2F_{d,s} \cdot H_{0,1}^{9,5+r}(10 + r - d_{m^2 \cdot g}(p, id_M))) \text{ fulfills}$$

(i) $\| F_m^{d,s}(B) - 1 \|_{C^3_{m^2 \cdot g}(M)} < \epsilon,$, this implies $1 \leq F_m^{d,s}(B) \leq 2$ since $\epsilon < 1$

(ii) $-\alpha_n < r(g(B,m,r,d,s))(\nu) < -\beta_n$ for $0 \neq \nu \in TB_{0,9}(a), a \in A$ and $\alpha_n > \beta_n > 0$ constants which depend only on n.

Define $r(j) := r(g(\bigcup_{j \leq i} B_j, m, r, d, s))$; now we will calculate $r(j+1)$ from $r(j)$ by (2.5.), in particular $r(g(A,m,r,d,s))$ can be from $r(g_A)$ by induction.
We start with a metric on $\mathbb{R} \times S^{n-1}$ for $m \geq m_0$:
$g(k) := h_r(g_{\mathbb{R}} + g_{S^{n-1}}) + (1-h_r)F_* \cdot (I_p^*(exp_p^{m^2 \cdot g})^*(g(\bigcup_{j \leq k} B_j, m, r, d, s)))$
this will be examined by the fixed deformation atlas D_n of $\mathbb{R} \times S^{n-1}$.

According to (3)(i) we can make $\| F_m^{d,s}(A) - 1 \|_{C^3_g(M)}$ arbitrary small (i.e. $< \epsilon$) by choosing suitable large $\tilde{d}_0, \tilde{s}_0^{-1}$. Therefore we can assume (since $k \le c$) using (1) that $k(D_n, g(k)) < k_n^r + 2$ for $m \ge m_0,\ d \ge \tilde{d}_0, s^{-1} \ge \tilde{s}_0^{-1}$ and each k.

Hence using (2.5.) and remark (4.4.) there is a $\bar{d}_0(= max\{c_2, \tilde{d}_0\})$ and $\bar{s}_0(= min\{1, \tilde{s}_0\})$ such that for $g(k)$ with $a = 1,\ b = 9,5 + r$ the following inequalities $< k + 1 >$ hold:

$$-s \cdot c_1 \le F_m^{d,s}(B_{k+1}) \cdot r(k+1)(\nu) - r(k)(\nu) \le \begin{cases} -s \cdot e^{-d} & \text{on } \bigcup_{a \in B_{k+1}} B_{9+r} \setminus B_{0,5}(a) \\ 0 & \text{on } M \setminus \bigcup\limits_{a \in B_{k+1}} B_{9+r}(a) \end{cases}$$

According to (4.4.) $B_{6+r}(a), a \in A$ covers M, in particular

$$M \setminus \bigcup_{a \in A} B_{0,6}(a) \subset \bigcup_{a \in A} B_{6+r} \setminus B_{0,5}(a)$$

Hence adding the inequalities $< 1 > - < c >$ (after multiplying $< k >$ by $(F_m^{d,s}(\bigcup\limits_{j \ge k} B_j))^{-1}$), since $\frac{1}{2} \le \sum\limits_{k=1}^{c} F_m^{d,s}(\bigcup\limits_{j \ge k} B_j)^{-1} \le c$ and $1 \le F_m^{d,s}(B) \le 2$ we obtain for $\nu \ne 0$,
$nu \in T(M \setminus \bigcup_{a \in A} B_{0,6}(a))$:

$$(*) \qquad -s \cdot c \cdot c_1 < r(c)(\nu) - F_m^{d,s}(A)^{-1} \cdot r(0)(\nu) < \tfrac{s \cdot e^{-d}}{2}$$

i.e. since $r(0) \equiv r(g_A)$ and using (2) we get:

$$-s \cdot c_1 \cdot c - \mu_n - \delta < r(g(A, m, r, d, s))(\nu) < -\tfrac{s \cdot e^{-d}}{2} + \delta$$

for any $\delta > 0, m \ge m^{(2)}(\delta), A = A(m)$ and $d \ge \bar{d}_0, s^{-1} \ge \bar{s}_0^{-1}$ note that $\bar{d}_0$ and $\bar{s}_0$ are independent of δ.
We choose $\delta := \frac{s \cdot e^{-d}}{4}$ and according to (3)(ii) we obtain on M for $\nu \ne 0$:

$$-\max\{s \cdot c_1 \cdot c + \mu_n + \frac{s \cdot e^{-d}}{4}, \alpha_n\} < r(g(A, m, r, d, s))(\nu) < -\min\{\frac{s \cdot e^{-d}}{4}, \beta_n\}$$

Hence we choose $d_1 := \bar{d}_0, s_1 := \bar{s}_0, m_1 := m^{(2)}(n, r, \delta_0)$ resp. $m^{(2)}(M, g, G, \delta_0))$ with $\delta_0 = \frac{\bar{s}_0 \cdot e^{-d_0}}{4}$ and for each $d \ge d_1, s^{-1} \ge s_1^{-1}$ (and $\delta = \frac{s \cdot e^{-d}}{4}$) there is a $m \ge m_1$ with

$$-c_1 < r(g(A, m, r, d, s) < -c_2$$

for $c_1 := \max\{s \cdot c_1 \cdot c + \mu_n + \frac{s \cdot e^{-d}}{4}, \alpha_n\}, c_2 := \min\{\frac{s \cdot e^{-d}}{4}, \beta_n\}$. □

5. Applications

The proof of (4.5.) implies most of Theorem E:

Proposition (5.1.): *Let be $S \subset M^n$ a closed subset of a manifold $M^n, n \geq 3$, $U \supset S$ an open neighborhoods, g_0 any metric on U with $r(g_0) < 0$ (resp. ≤ 0), then there is a metric g on M^n with:*

(i) $g \equiv g_0$ *on* S

(ii) $r(g) < -1$ *on* $M \setminus \bar{U}$

(iii) $r(g) < 0$ *(resp.* ≤ 0) *on* M

(iv) $\lambda_n^{-1} \cdot Vol(W, g) \leq Vol(W, g_0) \leq \lambda_n \cdot Vol(W, g)$

for each measurable set $W \subset M$ *and a constant* λ_n *depending only on* n.

Proof: We first assume that (M^n, g_0) is a compact Riemannian submanifold of a complete (1) - manifold $(\tilde{M}^n, \tilde{g})$ with $dist_{\tilde{g}}(M \setminus U, S) > 50$.

Denote $\{z \in \tilde{M} \mid dist_{\tilde{g}}(z, W) < r\}$ for some $W \subset \tilde{M}$ by $U_r(W)$ and consider the sets $A(m,0) \subset (\tilde{M}, \tilde{g})$ according to (5.4.)a)(ii) and the metrics

$$g(m,d,s) := m^{-2} \cdot g(A(m,0) \cap U_{20}(M \setminus U), A(m,0) \cap U_{20}(M \setminus U), m, 0, d, s)$$

For suitable large d, s^{-1} and m we obtain from the proof of (4.5.) adding the inequalities $< 1 > - < c >$ (in the proof of (4.5.)) for all $a \in A \cap U_{20}(M \setminus U)$: $r(g) < -c_2$ on $M \setminus \bar{U}$ and $g \equiv g_0$ on S, furthermore if m is large enough we get from (*) in that proof: $r(g) < 0$ on U.

For (iv) we note that obviously exists a constant $\Gamma_n > 1$ such that $\Gamma_n^{-1} \cdot Vol(W, g_{Eucl.}) \leq Vol(W, g_A^-) \leq \Gamma_n \cdot Vol(W, g_{Eucl.})$ for each measurable subset of $\mathbb{R}^n$, when g_A^- is defined as in the beginning of §5 and Γ_n does not depend on the choice of A. Hence $g_{A(m,0)}$ on $(\tilde{M}, \tilde{g})$ fulfills the analogous inequalities for $\Gamma_n' := \Gamma_n + 1$ if m is chosen large enough. Finally $g(A, m, 0, d, s)$ differs from $g_{A(m,0)}$ by a conformal scaling function f with $1 \leq f \leq 2^c$, i.e. choose $\lambda_n := 2^c \cdot (\Gamma_n + 1)$.

It remains to discuss the case where (M^n, g) is not a compact Ricmannian submanifold as above.
But this can be reduced to this case by an exhaustion of M by manifolds of the above type.
The new metrics will be defined by induction on the difference manifold of two elements of that exhaustion sequence. Hence we do change the metric constructed before at most two times. This also allows to prove (iv) in that case and using inductively larger scaling factors we also get $r(g) < -\text{const.} < 0$ the resulting limit metric on $M \setminus \tilde{U}$. □.

(5.1.) implies Theorem D: take $U = S = \emptyset$ and any complete metric g_M with finite volume:

Corollary (5.2.) *Each manifold M^n, $n \geq 3$ admits a complete metric g with $r(g) < -1$ and $Vol(M, g) < +\infty$.*

Furthermore we can deduce Corollary G from (5.1.):

Corollary (5.3.) *Let be M^n, $n \geq 3$ a compact manifold with boundary $\partial M \neq \emptyset$ and g_0 any fixed metric on ∂M, than there exists a metric g on M with:*

(i) $g \equiv g_0$ *on* ∂M

(ii) $r(g) < 0$ *on* M

(iii) each component of ∂M is totally geodesic.

Proof: Let be $N_1, \ldots N_m$ components of ∂M and U_i disjoint neighborhoods of these N_i each of them diffeomorphic to $]-1,0] \times N_i$. Now let be g_i the restriction of g_0 to N_i and $f_i \in C^\infty(]-1,1[, I\!R^{\geq 1})$ a function with $f_i(r) = f_i(-r), f_i'' > 0$ and $f_i(0) = \min f_i = 1$. The warped product formulas from §1 yield

$$Ric(g_{\mathbf{R}} + f_i^2 \cdot g_i)(\nu, \nu) = Ric(g_i)(\nu, \nu) - \frac{f_i''}{f_i} - (n-2)\left(\frac{f_i'}{f_i}\right)^2,$$

$$Ric(g_{\mathbf{R}} + f_i^2 \cdot g_i)(\bar{\nu}, \bar{\nu}) = -(n-1)\frac{f_i''}{f} \quad Ric(\nu, \nu') = 0$$

for vertical ν, resp. horizontal $\bar{\nu}$ with $\| \nu \| = \| \bar{\nu} \| = 1$
From this we conclude: if $f_i''(0) > 0$ is large enough, then $Ric(g_{\mathbf{R}} + f_i^2 \cdot g_i) < 0$ on $]-3\epsilon, 3\epsilon[\times N_i$ for each i and suitable small $\epsilon > 0$. Now consider a metric $\bar{g}$ on M with

$\bar{g} = g_{\mathbb{R}} + f_i^2 \cdot g_i$ on the subset of U_i diffeomorphic to $]-2\epsilon, 0] \times N_i$. Now we apply (5.1.) and get a metric g on M with $r(g) < 0$ and $g = \bar{g}$ near ∂M. Furthermore from $f_i''(0) > 0$ and $f_i'(0) = 0$ we conclude that (N_i, g_i) is totally geodesic. □

Finally we will briefly indicate a simple proof of Corollary B, which was proved in different way with more analytic effort by Bland and Kalka [BlK] resp. Aubin [A]

Corollary (5.4.) *Each manifold M^n, $n \geq 3$ admits a complete metric of constant negative scalar curvature.*

Proof: According to (4.6.) M^n admits a complete metric g_M which scalar curvature S_{g_M} is bounded by two negative constants: $-c < S_{g_M} < -k$, for some $c > k > 0$. Now we want to find a metric $g = u^{\frac{4}{n-2}} \cdot g_M$ with $S_g \equiv -1$, i.e. u must be a positive solution of the Yamabe equation (cf. [LP]):

$-\gamma \cdot \Delta_{g_M} u + S_{g_M} \cdot u = -u^\alpha$, $\gamma = 4\frac{n-1}{n-2}$, $\alpha = \frac{n-1}{n-2}$

But $-c < S_{g_M} < -k$ implies that $\bar{u} := k^{\frac{1}{\alpha-1}}$ resp. $u^\alpha := c^{\frac{1}{\alpha-1}}$ are positive Sub - resp. Supersolutions of this equation.
Thus we are left to use the usual standard procedure (cf. [K], §3) to get solutions u_n with $u^- \leq u_n \leq u^+$ on a sequence M_n of compact manifold exhausting M.
Using $C^{k,\alpha}$ a-priori-bounds on each u_n from elliptic theory a subsequence of the u_n converges to a solution u defined on M. u fulfills $u^- \leq u \leq u^+$, hence it is positive and $g = u^{\frac{4}{n-2}} \cdot g_M$ is complete. □

6. $Ric < 0$ on $B_1(0) \subset \mathbb{R}^{n+1}$

In this chapter we will conclude the existence of g_n^- on $\mathbb{R}^{n+1}$, $n \geq 3$ from Theorem E for n-dimensional manifolds:

Proposition (6.1.): *There is a metric g_{n+1}^- on $\mathbb{R}^{n+1}$, $n \geq 3$ with $r(g_{n+1}^-) < 0$ on $B_1(0)$ and $g_{n+1}^- \equiv g_{Eucl.}$ on $\mathbb{R}^{n+1} \setminus B_1(0)$.*

The proof of (6.1.) proceeds in two steps. We start with the construction of a certain metric with $Ric \leq 0$ on $S^1 \times S^1 \times B^{n-1} \subset S^1 \times S^1 \times \mathbb{R}^{n-1}$. We write $\bar{S}^1 \times S^1 \times B^{n-1}$ to distinguish the two S^1-factors. We will use again the functions $F_d(t) := \exp(-\frac{d}{t})$ which turned out to be useful in §1 and §2 and define on $\bar{S}^1 \times S^1 \times B_5^{n-1} \setminus B_1^{n-1}$

$$g(d,\tilde{d}) := ((F_d(r-2)+1)^2 \cdot g_{S^1} + \exp(2F_{\tilde{d}}(3-r)) \cdot (g_{\bar{S}^1} + g_{Eukl.}))$$

with $r = r(\bar{t},t,x) = dist((\bar{t},t,x), \bar{S}^1 \times S^1 \times \{0\}) = \| x \|_{g_{Eucl.}}, d, \tilde{d} > 0$.

Proposition (6.2.) *There are $d_0, \tilde{d}_0 > 0$ with $Ric(g(d_0,\tilde{d}_0)) < 0$ on $\bar{S}^1 \times S^1 \times B_3 \setminus \bar{B}_2$ and $Ric \leq 0$ on the complement.*

Proof: We first consider the second part of this metric:

$\tilde{g}(\tilde{d}) := \exp(2F_{\tilde{d}}(3-r)) \cdot (g_{\bar{S}^1} + g_{Eucl.}) =: \exp(2F_{\tilde{d}}(3-r)) \cdot g_e$

Let be $F : I\!R^{>0} \times \bar{S}^1 \times S^{n-2} \to \bar{S}^1 \times I\!R^{n-1} \setminus \{0\}$ the diffeomorphism defined by $F(r, e^{it}, x) := (e^{it}, r \cdot x)$ and

$g^F(\tilde{d}) = \Phi \cdot F^*(\exp(2F_{\tilde{d}}(3-r)) \cdot g_e) + (1-\Phi) \cdot (g_{I\!R} + g_{\bar{S}^1} + g_{S^{n-2}})$

with $\Phi \in C^\infty(I\!R, [0,1]), \Phi \equiv 1$ on $I\!R^{\leq 0,5} \cup I\!R^{\geq 5}$ and $g_{I\!R}, g_{\bar{S}^1}$ and $g_{S^{n-2}}$ the standard metrics on these manifolds.

Now fix a deformation atlas D of $(I\!R \times (\bar{S}^1 \times S^{n-2}), g^F(\tilde{d}))$ and consider the metrics $(g^F(\tilde{d}))_i$ on $I\!R \times I\!R^{n-1}$, $i = 1, \dots k$, cf. §2. According to (1.2.)(iv) we can find for each $\epsilon > 0$ a $\tilde{d}_0$ such that for $\tilde{d} \geq \tilde{d}_0$ and $g_0^F := \Phi \cdot F^*(g_e) + (1-\Phi) \cdot (g_{I\!R} + g_{\bar{S}^1} + g_{S^{n-2}})$ and each i holds:

$$\| (g^F(\tilde{d}))_i - g_0^F \|_{C^3_{g_{Eucl.}}(I\!R^n)} < \epsilon.$$

Hence for suitable large $\tilde{d}_0$ there is a $k > k(D, g^F(\tilde{d}))$ for all $\tilde{d} \geq \tilde{d}_0$ and we may assume that $\tilde{d}_0$ is chosen such that for each $\tilde{d} \geq \tilde{d}_0$: $exp(2F_{\tilde{d}}(3-r)) \in [1,2]$ on $[0,6]$.
Using this we obtain:

(i) from (2.3.) and (1.2.)(ii) on $]2,5] \times I\!R^n$: $\Delta_{\tilde{g}(\tilde{d})} F_{d_0}(r-2) > 0$ for some $d_0(k,n) > 0$, since d_0 can be chosen such that $F'_{d_0}(r-2), F''_{d_0}(r-2) > 0$

(ii) acc. formula (*) in §2 (proof of (2.1.)) there are constants $c_1, c_2 > 0$ depending only on k and n such that for $\| \tilde{\nu} \|_{g_0} = 1$:

$$\exp(2F_{\tilde{d}}(3-r)) \cdot r((g^F(\tilde{d}))(\tilde{\nu}) \leq (-c_1 \cdot F''_{\tilde{d}} + c_2 \cdot F'_{\tilde{d}})(3-r)$$

on $[1,4] \times \bar{S}^1 \times S^{n-2}$, since $r(g_0) \equiv 0$ on this domain. Consequently for $\tilde{d} \geq \tilde{d}_0$: $r(\tilde{g}(\tilde{d}))(\nu) \leq (-\frac{c_1}{2} \cdot F''_{\tilde{d}} + c_2 \cdot F'_{\tilde{d}})(3-r)$ on $\bar{S}^1 \times S^1 \times B_4 \setminus \bar{B}_1$ for $\| \tilde{\nu} \|_{g_0} = 1$:

Now we are ready to examine $g(d,\tilde{d})$ using warped product formulas (cf. §1): S^1 is the warped fiber, $\bar{S}^1 \times B_5 \setminus B_1$ the base space. According to (i) we obtain for vertical vectors $\nu \neq 0$ and $\tilde{d} \geq \tilde{d}_0$:

$$Ric(g(d_0,\tilde{d}))(\nu,\nu) = -\frac{\Delta_{\tilde{g}(\tilde{d})} F_{d_0}(r-2)}{F_{d_0}(r-2)+1} \cdot g_{S^1}(\nu,\nu) \begin{cases} < 0 & on \quad \bar{S}^1 \times S^1 \times B_5 \setminus \bar{B}_2 \\ < 0 & on \quad \bar{S}^1 \times S^1 \times \bar{B}_2 \setminus B_1 \end{cases}$$

The situation for horizontal $\nu \neq 0$ is more complicated:
Let be $\pi : \bar{S}^1 \times S^1 \times B_5 \setminus B_1 \to \bar{S}^1 \times B_5 \setminus B_1$ the canonical projection as in §1, then we have to show (cf. §2):

$$Ric(\tilde{g}(\tilde{d}))(d\pi(\nu), d\pi(\nu)) - \frac{Hess_{\tilde{g}(\tilde{d})} F_{d_0}(r-2)}{F_{d_0}(r-2)+1}(d\pi(\nu), d\pi(\nu)) < 0$$

From (ii) we have for ν with $g_0(d\pi(\nu), d\pi(\nu)) = 1$:

$$(**) \quad Ric(\tilde{g}(\tilde{d}))(d\pi(\nu), d\pi(\nu)) \leq (-\frac{c_1}{2} \cdot F''_{\tilde{d}} + c_2 \cdot F'_{\tilde{d}})(3-r)$$

On the other hand using the transformation formulas for conformal deformations of §1:

$$Hess_{\tilde{g}(\tilde{d})} F_{d_0}(r-2)(d\pi(\nu), d\pi(\nu)) = Hess_{g_\bullet} F_{d_0}(r-2)(d\pi(\nu), d\pi(\nu))$$

$$-2 \cdot dF_{\tilde{d}}(3-r)(d\pi(\nu)) \cdot dF_{d_0}(r-2)(d\pi(\nu)) + dF_{\tilde{d}}(3-r)(\nabla^{g_\bullet} F_{d_0}(r-2))$$

Note that $Hess_{g_\bullet} F_{d_0}(r-2)$ is positive semidefinite, since $F_{d_0}(r-2)$ is cylinder symmetric on$(S^1 \times B_5 \setminus B_1, g_{S^1} + g_{Eucl.})$ with $F'_{d_0}, F''_{d_0} \geq 0$.

The two remaining terms on the right hand side can be estimated as follows:

$$3 \cdot \| \nabla^{g_\bullet} F_{\tilde{d}}(3-r) \|_{g_\bullet} \cdot \| \nabla^{g_\bullet} F_{d_0}(r-2) \|_{g_\bullet} \leq c(d_0) \cdot | F'_{\tilde{d}} | (3-r)$$

Hence $| F_{d_0}(r-2)+1 | \geq 1$, $Hess_{g_\bullet} F_{d_0}(r-2)$ positive semidefinite and (**) imply: $Ric(g(d_0,\tilde{d})(\nu,\nu) \leq (-\frac{c_1}{2} \cdot F''_{\tilde{d}} + c_2 \cdot F'_{\tilde{d}})(3-r) + c(d_0) \cdot | F'_{\tilde{d}} | (3-r)$

Now (1.2.)(ii) (for $b = 2.5$ and $a \in]0,1[$ arbitrary and $m \geq 2 \cdot \frac{c_1 + c(d_0)}{c_1}$) yields for this inequality:
For suitable large $\tilde{d} \geq \tilde{d}_0$ the right hand side is "< 0" on $\bar{S}^1 \times S^1 \times B_3 \setminus B_1$ and "≤ 0" on $\bar{S}^1 \times S^1 \times B_5 \setminus B_3$ choose such a $\tilde{d}$ and denote it (for simplicity) again by $\tilde{d}_0$, then $g(d_0,\tilde{d}_0)$ fulfills the claims. □

The second step in the proof of (6.1.) consists in finding a suitable embedding of $S^1 \times S^1 \times B_5^{(n-1)}$:
This is exactly the point where the induction hypothesis, i.e. Theorem E (i.e. Prop.(5.1.)) for n-dimensional manifolds, enters. We get from (5.1.):

Corollary (6.3.)

a) On $B_4^n \subset I\!R^n$ there is a metric $\breve{g}$ with

(i) $\breve{g} \equiv g_{Eucl.}$ *on* $B_4^1 \setminus B_1^n$

(ii) $Ric \leq 0$

(iii) B_1^n *contains a subset* V *isometric to*

$$(S^1 \times B_5^{n-1}, (F_{d_0}(r-2)+1)^2 \cdot g_{S^1} + g_{Eucl.})$$

b) on $\bar{S}^1 \times B_5^{n-1}$ *there is a metric* $\hat{g}$ *with* $Ric \leq 0$ *and*

$$\hat{g} = \exp(2F_{\tilde{d}_0}(3-r)) \cdot (g_{\bar{S}^1} + g_{Eucl.}) \quad on \quad \bar{S}^1 \times B_5^{n-1} \setminus B_1^{n-1}$$

(6.2.) and (6.3.) imply

Lemma (6.4.) *On $\bar{S}^1 \times I\!R^n$ there is a metric g_0 with*

(i) $g_0 \equiv g_{\bar{S}^1} + g_{Eucl.}$ *on* $\bar{S}^1 \times I\!R^n \setminus B_1^n$

(ii) $Ric(g_0) < 0$ *on* $\bar{S}^1 \times B_{\frac{1}{2}}^n$

(iii) $Ric(g_0) \leq 0$ *on* $\bar{S}^1 \times B_1^n$

Proof: Let be $f : (V, g) \to (S^1 \times B_5^{n-1}, (F_{d_0}(r-2)+1)^2 \cdot g_{S^1} + g_{Eucl.})$ an isometry, then $F = id_{\bar{S}^1} \times f$ is also isometric (w.r.t. product metrics) and we consider the metric $\tilde{g} = (F_{d_0}(r-2)+1)^2 \cdot g_{S^1} + \hat{g}$ on $\bar{S}^1 \times S^1 \times B_5^{n-1}$. This metric $\tilde{g}$ equals

$g_{\bar{S}^1} + (F_{d_0}(r-2)+1)^2 \cdot g_{S^1} + g_{Eucl.}$ (in particular it has $Ric \leq 0$) on $\bar{S}^1 \times S^1 \times B_5 \setminus B_3$

$g(d_0, \tilde{d}_0)$ (with $Ric < 0$) on $\bar{S}^1 \times S^1 \times B_3 \setminus \bar{B}_2$ and
$g_{S^1} + \hat{g}$ (with Ric ≤ 0) on $\bar{S}^1 \times S^1 \times \bar{B}_2$.

$$\text{Define} \quad g^* := \begin{cases} g_{\bar{S}^1} + g_{Eucl.} & on \quad \bar{S}^1 \times I\!R^n \setminus B_4^n \\ g_{\bar{S}^1} + \breve{g} & on \quad \bar{S}^1 \times B_4^n \setminus V \\ F^*(\tilde{g}) & on \quad \bar{S}^1 \times V \end{cases}$$

and consider a diffeomorphism $h : I\!R^n \to I\!R^n$ with $h \equiv id$ on $I\!R^n \setminus B_1^n$ and $h(B_{\frac{1}{2}}^n) \subset\subset f^{-1}(S^1 \times B_3 \setminus \bar{B}_2) \subset\subset B_1^n$.
From the previous construction it is clear that $g_0 := (id_{\bar{S}^1} \times h)^*(g^*)$ fulfills our claims. □

Now we are ready to enter into the
Proof: of (6.1.): Let be $f_m : \bar{S}^1 \times I\!R^n \to \bar{S}^1 \times I\!R^n$ the m-fold covering $f_m(e^{it}, x) := (e^{imt}, x)$ and $f_\infty : I\!R \times I\!R^n \to \bar{S}^1 \times I\!R^n$, $f_\infty(t, x) = (e^{it}, x)$.
We define $g_0^m := f_m^*(g_0)$, i.e.:

$$g_0^m = \begin{cases} m^2 \cdot g_{\bar{S}^1} + g_{Eucl.} & on \quad \bar{S}^1 \times I\!R^n \setminus B_1^n \\ \text{has} \quad Ric < 0 & on \quad \bar{S}^1 \times B_{\frac{1}{2}}^n \\ \text{has} \quad Ric \leq 0 & on \quad \bar{S}^1 \times B_1^n \end{cases}$$

g_0^m and g_0 are locally isometric via f_m.
Now be $\gamma_m(t) := m(\cos t, \sin t, 0, \ldots 0) \in I\!R^{n+1}$ a circle of radius m and length $2\pi \cdot m$. We consider (for large $m \geq 5$) the diffeormorphism $F_m : \bar{S}^1 \times B_4^n \to I\!R^{n+1}$ onto $Im\, F_m = U_4(\gamma_m)$ with

$$F_m(e^{it}, x_1, \ldots x_n) = m(\cos t, \sin t, 0, \ldots 0) + (x_1 \cdot \cos t, x_1 \cdot \sin t, x_2, \ldots x_n)$$

Now we are ready to define the most important metric

$$g(m) := h(r) \cdot (F_m)_*(g_0^m) + (1 - h(r))g_{Eucl.}$$

with $r := dist(x, \gamma_m)$, $h \in C^\infty(I\!R, [0,1])$, $h \equiv 1$ on $I\!R^{\leq 2}$, $h \equiv 0$ on $I\!R^{\geq 3}$
We will check now that $\exp(2 \cdot F_{d,s} \cdot H_{0,1}^{3,8}(r-4)) \cdot g(m)$ has Ric < 0 on $U_4(\gamma_m)$ and is Euclidean outside $U_4(\gamma_m)$ for a suitable choice of m, d and s:
We can examine $(\bar{S}^1 \times I\!R^n, \exp(2F_{d,s} \cdot H_{0,1}^{3,8}(r-4) \cdot g_0)$ by (2.4.):
take $M^n = \bar{S}^1 \times S^{n-1}$, $I\!R \times M^n \supset I\!R^{>0} \times \bar{S}^1 \times S^{n-1} \cong \bar{S}^1 \times I\!R^n \setminus \{0\}$ and consider a deformation atlas for $I\!R \times M^n$. It is clear from (2.4.): there are d_1, s_0^{-1} such that for $d \geq d_1$ (since $r(g_0) \leq 0$) the following holds:

$$r(\exp(2F_{d,s_0} \cdot H_{0,1}^{3,8}(r-4)) \cdot g_0) \begin{cases} < 0 & on \quad \bar{S}^1 \times B_4 \\ < -\frac{s_0 \cdot \exp(-\frac{d}{0,1})}{2} & on \quad \bar{S}^1 \times B_{3,8} \end{cases}$$

and $\exp(2F_{d,s} \cdot H_{0,1}^{3,8}(r-4)) \cdot g_0 \equiv g_0$ otherwise.

On the other hand (2.4.) yields $d_0 \geq d_1$ such that for each $m \geq 5$, $d \geq d_0$, $s \in]0,1[$: $r(\exp(2F_{d,s} \cdot H^{3,8}_{0,1}(r-4)) \cdot g(m)) < 0$ on $U_4(\gamma_m) \setminus U_{3,5}(\gamma_m)$ (where $g(m)$ is just the Euclidean metric).

Now define $T_{m,i}(x_1, \ldots x_i, \ldots x_{n+1}) := (x_1, \ldots, x_i + m, \ldots x_{n+1})$ then $T^*_{m,2}(g(m))$ converges compactly w.r.t. any C^k-norm to $g_\infty := f^*_\infty(g_0)$, furthermore since

$$f_\infty : (\mathbb{R} \times \mathbb{R}^n, \exp(2F_{d,s} \cdot H^{3,8}_{0,1}(r-4)) \cdot g_\infty) \to (\tilde{S}^1 \times \mathbb{R}^n, \exp(2F_{d,s} \cdot H^{3,8}_{0,1}(r-4)) \cdot g_0)$$

is a Riemannian covering we obtain from this and the rotational symmetry of $g(m)$ w.r.t. rotation of angle $\frac{2\pi}{m}$ in the $x_1 - x_2$-plane: for suitable large m:

$$r(\exp(2F_{d,s} \cdot H^{3,8}_{0,1}(r-4)) \cdot g(m)) < 0 \text{ on } U_{3,8}(\gamma_m),$$

hence on $U_4(\gamma_m)$.

Now use again (2.4.) for $\mathbb{R} \times S^n$: take a point p with $B_1(p) \subset U_4(\gamma_m)$ and a ball $B_R(p)$ with $U_4(\gamma_m) \subset B_R(p)$ and take the diffeomorphism

$$f :]-2, R[\times S^n \to B_{R+2}(p) \setminus \{p\} \subset \mathbb{R}^{n+1}, \quad f(t,x) = (R-t) \cdot x + p.$$

Now consider the following metric g on $\mathbb{R} \times S^1$

$$g = h \cdot (g_R + g_{S^n}) + (1-h) \cdot f^*(\exp(2F_{d,s} \cdot H^{3,8}_{0,1}(r-4)) \cdot g(m))$$

for $h \in C^\infty(\mathbb{R}, [0,1])$, $h \equiv 1$ on $\mathbb{R}^{\leq -1} \cup \mathbb{R}^{\geq R-0,1}$, $h \equiv 0$ on $[-\frac{1}{2}, R-0,2]$.
(2.4.) yields D and § such that $r(\exp(2F_{D,S} \cdot H^{R-0,3}_{0,1}) \cdot g) < 0$ on $[-\frac{1}{2}, 2, R-0,2]$ and this implies

$$\exp(2F_{D,S} \cdot H^{R-0,3}_{0,1}(\| p \| - \| p - id \|)) \cdot \exp(2F_{d,s} \cdot H^{3,8}_{0,1}(r-4)) \cdot g(m))$$

has $Ric < 0$ on $B_R(p)$ and is Euclidean outside of $B_R(p)$. □

Appendix

In this appendix we indicate proofs of some elementary technical results used in particular in §4 and §5.

(A1:) Let be $\epsilon > 0, r \geq 0$, then each manifold M admits a complete (ϵ, r)-metric.

Proof: Once given ϵ, r and a closed Riemannian manifold (M, g) finds a large $\lambda \geq 1$, such that $(M, \lambda^2 \cdot g)$ is a (ϵ, r) manifold.
Thus we are left to prove the claim for non-compact M. Such M admits an exhaustion $\overset{\circ}{M}_{i+1} \supset \bar{M}_i$ $(M_0 := \emptyset)$ by countably many compact manifolds M_i with $\partial M_i \neq \emptyset$. Now fix metrics g_i on ∂M_i and define a complete smooth metric g(i) on $\overset{\circ}{M}_{i+1} \setminus \bar{M}_i$ which is isometric to $(I\!R^+ \times \partial M_i, g_{\mathbf{R}} + g_i)$ resp. $(I\!R^+ \times \partial M_{i+1}, g_{\mathbf{R}} + g_{i+1})$ near the boundary. We can choose $\lambda_i \geq 1$, such that $(\overset{\circ}{M}_{i+1} \setminus \bar{M}_i, \lambda_i^2 \cdot g(i))$ is a $(\frac{\epsilon}{2}, r+1)$-manifold. Finally one has to glue these parts together and it is enough to find a $\Phi_i \in C^\infty(I\!R, [0,1])$ with $\Phi_i \equiv 0$ on $I\!R^{\leq 0}, \Phi_i \equiv 1$ on $I\!R^{\geq c_i}$ for some $c_i \geq 1$ such that

$$(I\!R \times \partial M_i, g_{\mathbf{R}} + (\lambda_i^2 \cdot \Phi_i + (1 - \Phi_i) \cdot \lambda_{i+1}^2) \cdot g_i) \quad \text{is a} \quad (\epsilon, r)\text{-manifold}$$

But this is a easily done taking $\Phi_i \equiv \Phi(\Gamma_i \cdot id_{\mathbf{R}})$ for some fixed $\Phi \in C^\infty(I\!R, [0,1])$ with $\Phi \equiv 1$ on $I\!R^{\geq 1}$ and a small $\Gamma_i > 0$. □

(A2): There is an $\epsilon = \epsilon(n, r) > 0$ and a constant $c(n, r) \in Z\!\!Z^{>0}$, such that for each n-dimensional $(\epsilon, 10 \cdot r)$-manifold M, there exists a covering $\mathcal{A} = \{\overline{B_{5+r}(p)} \mid p \in A\}, A \subset M$ by closed balls such that $\mathcal{A}$ splits into $c(n, r)$ disjoint families $\mathcal{B}_j, 1 \leq j \leq c(n, r)$ with

(i) $\overline{B_{10+r}(a)} \cap \overline{B_{10+r}(b)} = \emptyset \quad \text{for} \quad \overline{B_{5+r}(a)}, \overline{B_{5+r}(b)} \in \mathcal{B}_j$

(ii) $a \notin \overline{B_{5+r}(b)} \quad \text{for} \quad a, b \quad \text{in} \quad \mathcal{A}$

Proof: We choose $\epsilon(n, r) > 0$ such that for each n-dimensional (ϵ, r)-manifold inj(M) $> 50 + 5 \cdot r$ and the sectional curvature K fulfills $k \in [-1, 1]$. Now let $S \subset M$ be a countable dense subset $(S = \{a_i \mid i \in Z\!\!Z^{\geq 0}\})$ and $\mathcal{B} := \{\overline{B_{5+r}(p)} \mid p \in S\}$. We define a map $i : S \to Z\!\!Z^{\geq 0}$ by $i(a_0) := 1$ and:

$$i(a_{n+1}) := \begin{cases} 0 & \text{if} \quad a_{n+1} \in \bigcup_{i \leq n} \overline{B_{5+r}(a_i)} \\ min\{m \mid d(a_m, a_{n+1}) > 20 + 2r\} \cup \{n+1\} & \text{otherwise} \end{cases}$$

and $\mathcal{B}_j := \{\overline{B_{5+r}(a_m)} \in \mathcal{B}_j \mid i(a_m) = j\}$, $\mathcal{A} := \bigcup_{j \geq 1} \mathcal{B}, A := \{a \in S \mid i(a) \geq 1\}$

It is obvious from this definition, that $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for $i \neq j$ and that (i) and (ii) of (A2) are fulfilled.

Thus it remains to check that there is a constant $c(n,r)$ independent of the $(\epsilon, 10\cdot r)$-manifold M^n such that $\mathcal{B}_j = \emptyset$ for $j > c$ and that $\mathcal{A}$ is a covering of M.
Since $inj(M) > 50 + 5\cdot r$ and $K \in [-1,1]$, we obtain from the comparison theorem of Bishop: there are constants $k_1(n) > 0, k_2(n,r) > 0$ with $k_1 \leq Vol_M(B_2(p))$, $k_2 \geq Vol_M(B_{30+3r}(p))$ for each $p \in M$ and we define $c(n,r) := \frac{k_2}{k_1}$.
Now assume $\overline{B_{5+r}(a)} \in \mathcal{B}_{c+1}$, then $\overline{B_{10+r}(a)} \cap \overline{B_{10+r}(p_i)} \neq \emptyset$ for c different $p_i \in A, 1 \leq i \leq c$, i.e. $\overline{B_2(p_i)} \subset B_{30+3r}(a)$, this leads to a contradiction since:

$$\begin{aligned}(c+1)\cdot k_1 &\leq \sum Vol_M(B_2(p_i)) + Vol_M(B_2(a)) \\ &= Vol_M(\bigcup B_2(p_i) \cup B_2(a)) \leq Vol_M(B_{30+3r}(a)) \leq k_2\end{aligned}$$

Finally $\mathcal{A}$ is a covering, otherwise there would be a $p \in U := M \setminus \bigcup_{a\in A} \overline{B_{5+r}(a)}$. But $\mathcal{A}$ is a locally finite, i.e. U is open and there is a $q \in S \cap U$ with $\overline{B_{5+r}(q)} \in \mathcal{A}$ and $p \in \overline{B_{5+r}(q)}$ which contradicts our assumption. □

Finally we briefly indicate how to get the "G - invariant" version of (A2), where G denotes a finite, non-trivial subgroup of $Diff(M)$ and M is a closed manifold:
Let $F(G) := \{z \in M \mid z = f(z) \text{for a } f \in G \setminus \{id\}\}$ and $(U_r, g) := \{z \in M \mid dist_g(z, F(G)) < r\}$, then some elementary considerations using Fermi coordinates yield:

<u>(A3)</u>:There is a $r_0 = r_0(M,g,G) \geq 5$ and a $m_0(M,g,G)$, such that for $m \geq m_0$

$\min\{d_{m^2\cdot g}(x, f(x)) \mid x \in (M \setminus U_{r_0}, m^2\cdot g), f \in G \setminus \{id\}\} > 5$

This and the proof of (A2) are the main ingredients to get an analogous G-invariant covering:

<u>(A4)</u>: There are constants $m_1 \geq m_0, c, m_1(M,g,G)\, c(M,g,G) \in \mathbb{Z}^{>0}$ such that for each $m \geq m_1$ there is a set $A_m \subset (M \setminus U_{r_0}, m^2\cdot g)$ with $G(A_m) = A_m$ such that A_m splits into c disjoint families $B_{i,m}$ with:

(i) $\overline{B_{10+r_0}(a)} \cap \overline{B_{10+r_0}(b)}$, if $a, b \in B_{i,m}$

(ii) $a \notin \overline{B_5(b)}$, $a, b \in A_m$

(iii) $M \setminus U_{r_0} \subset \bigcup_{a\in A_m} \overline{B_5(a)}, M = \bigcup_{a\in A_m} \overline{B_{5+r_0}(a)}$

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The space of negative scalar curvature metrics

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1 Introduction

The topology of the space of positive scalar curvature metrics $S^+(M)$ on a closed manifold M has been studied by Hitchin, Gromov, Lawson and Carr (cf. [LM, IV, §9]) and it turned out that the topology of $S^+(M)$ is quite complicated; there are manifolds M such that the ith homotopy group $\pi_i(S^+(M))$ is non-trivial for some (probably arbitrarily great) $i \geqq 0$ and even the "moduli space" $S^+(M)/\text{Diff}(M)$ can have infinitely many path components.

In this paper we will have a look at the natural counterpart: the topology of the space of negative scalar curvature metrics $S^-(M^n)$ on a closed manifold M^n of dimension $n \geqq 3$.

We will prove that $S^-(M)$ (which always is non-empty by [A] resp. [KW]) is always *connected* and *aspherical*:

Theorem 1 $\pi_i(S^-(M)) = 0, \quad i = 0, 1, 2, \ldots.$

By Theorem 1 using a general result of infinite dimensional topology due to Palais and Whitehead (cf. [P, Theorem 15 and corollary]) we get a complete insight into the topology of $S^-(M)$:

Theorem 2 $S^-(M)$ *is contractible.*

From this we get the same information for the space of metrics with constant negative scalar curvature $= -1$ denoted by $S_{-1}(M)$.

Corollary. $S_{-1}(M)$ *is contractible.*

Note that on the other hand $S^-(M)$ and $S_{-1}(M)$ are never convex (cf. [L1]).

2 Continuous extension

We are only concerned with closed C^∞-manifolds and C^∞-Riemannian metrics defined on them. Once given a manifold M we fix some reference metric g_M on M and consider the space of all C^∞-metrics $\mathcal{M}(M)$ on M equipped with the usual

C^∞-topology which is the Fréchet topology defined by all the C^k-norms $\|\cdot\|_k$ on M. $\|\cdot\|_k$ is defined with respect to g_M, but the topology does not depend on g_M.

Now let $f: S^i \to \overline{S^-(M)}$ be a continuous map, we are looking for an extension of f on $B^{i+1} \equiv \overline{B_6(0)} \subset \mathbb{R}^{i+1}$. $S^i \equiv \partial B_6(0)$, i.e. a continuous map $F: B^{i+1} \to S^-(M)$ with $F_{|S^i} \equiv f$.

We start our construction of F by some trivial extension F_1 of f defined as follows: Let g_0 be any fixed metric on M and $(x, t) \in S^i \times [0, 6]/S^i \times \{0\} \equiv B^{i+1}$ (polar coordinates) then we define

$$F_1(x, t) := \begin{cases} (1-t)\cdot g_0 + t\cdot f(x) & \text{on } S^i \times [0, 1]/S^i \times \{0\} \\ f(x) & \text{on } S^i \times [1, 6]/S^i \times \{0\}. \end{cases}$$

Obviously F_1 is a continuous map with image lying in $\mathcal{M}(M)$. Our goal will be to find deformations of F_1 inside of $S^i \times [0, 5]/S^i \times \{0\}$ such that the image of the deformed map lies in $S^-(M)$.

3 Main deformation

Let N_i^n, $i = 1, 2$ be closed manifolds of dimension $n \geqq 3$, $p_i \in N_i$ fixed base points, g_i and $\overline{g_i}$ metrics on N_i, g_i with injectivity radius $\mathrm{inj}(N_i, g_i) > 5$. Now we define for $\lambda_i \geqq 1$ new metrics on $N_i \backslash \{p_i\}$ by

$$g(\lambda_i, g_i, \overline{g_i}) = h(d_{\lambda_i^2 \cdot g_i}(p_i, \mathrm{id}_{N_i}))\cdot G_{\lambda_i} + (1 - h(d_{\lambda_i^2 \cdot g_i}(p_i, \mathrm{id}_{N_i}))\cdot \lambda_i^2 \cdot \overline{g_i}$$

$h \in C^\infty(\mathbb{R}, [0, 1])$ with $h \equiv 1$ on $\mathbb{R}^{\leqq 3}$, $h \equiv 0$ on $\mathbb{R}^{\geqq 4}$ and $G_{\lambda_i} := f^*_{\lambda_i}(g_{\mathbb{R}} + g_{S^{n-1}})$ where $f_{\lambda_i}: B_5(p_i) \backslash \{p_i\} \to]0, 5[\times S^{n-1}$ is a diffeomorphism defined as follows: Fix a linear isometry $I_i: (T_{p_i}N_i, g_i) \to (\mathbb{R}^n, g_{\mathrm{eukl.}})$ and consider the usual polar coordinates on $\mathbb{R}^n \backslash \{0\}$: $P: \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^{>0} \times S^{n-1}$, $P(z) = (\|z\|, z/\|z\|)$ and define $f_{\lambda_i}(z) := P(\lambda_i \cdot (I_i \circ (\exp^{\lambda_i}_{p_i})^{-1}(z)))$ where $\exp^{\lambda_i}_{p_i}$ denotes the exponential map in p_i for the metric $\lambda_i^2 \cdot g_i$.

By definition $\partial(N_1 \backslash B_1(p_1))$ and $\partial(N_2 \backslash B_1(p_2))$ equipped with these metrics are isometric and can be identified by the orientation preserving isometry $i(\lambda_1, \lambda_2) := f_{\lambda_2}^{-1} \circ f_{\lambda_1}$ yielding $N_1 \#_i N_2$ together with a smooth metric denoted by

$$g(\lambda_1, g_1, \overline{g_2}) \#_i g(\lambda_2, g_2, \overline{g_2}).$$

Now we specialize to $N_1 = M$, $g_1 = g_M$ (a fixed reference and base metric, with $\mathrm{inj}(M, g_M) > 5$), $\overline{g_1} = g$ (varying metrics), $\lambda_1 = \lambda$, $p_1 = p$ and $N_2 = S^n$, $g_2 = g^*$ (a fixed metric with $\mathrm{inj}(S^n, g^*) > 5$), $g_2 = g_n$ (a fixed negative scalar curvature metric on S^n, $\lambda_2 = \mu$, $p_2 = q$.

From the construction above it is clear that there is a family of diffeomorphisms $F(\lambda, \mu): M \to M \# S^n$ with $F(\lambda, \mu) \equiv \mathrm{id}$ on $M \backslash B_5(p)$ which can be chosen such that the metrics $G(g, \lambda, \mu) := F(\lambda, \mu)^*(g(\lambda, g_M, g) \#_{i(\lambda, \mu)} g(\mu, g^*, g_n))$ depend *continuously* on λ and μ.

Now we are ready to define for $\lambda_0 \geqq 1$, $\mu_1 \geqq \mu_0 \geqq 1$, $(x, t) \in B^{i+1}$

$$F_2(\lambda_0, \mu_0, \mu_1, x, t) :=
\begin{cases}
f(x) & \text{on } S^i \times [4,6]/S^i \times \{0\} \\
((4-t)\cdot\lambda_0^2 + (1-(4-t)))\cdot f(x) & \text{on } S^i \times [3,4]/S^i \times \{0\} \\
(3-t)\cdot G(f(x), \lambda_0, \mu_0) + (1-(3-t))\cdot \lambda_0^2 \cdot f(x) & \text{on } S^i \times [2,3]/S^i \times \{0\} \\
G(f(x), \lambda_0, (2-t)\cdot\mu_1 + (1-(2-t))\cdot\mu_0) & \text{on } S^i \times [1,2]/S^i \times \{0\} \\
G(F_1(x,t), \lambda_0, \mu_1) & \text{on } S^i \times [0,1]/S^i \times \{0\}.
\end{cases}$$

We claim

Proposition 1. *There are λ_0, μ_0, μ_1 such that $(F_2(x,t) :=)F_2(\lambda_0, \mu_0, \mu_1, x, t)$ is a continuous extension of f with*

(i) $F_2(x,t) \equiv f(x)$ *on* $S^i \times [4,6]/S^i \times \{0\}$

(ii) $\mathscr{S}(M, F_2(x,t)) < 0$ *on* B^{i+1} (where $\mathscr{S}(U,g) := \int_U S_g d\mathrm{Vol}_g$).

Proof. The continuity of $F_2(x,t)$ and (i) follow directly from the construction above. It remains to show (ii) for appropiate λ_0, μ_0, μ_1, which is trivial on $S^i \times [3,6]/S^i \times \{0\}$.

The following estimates are easily checked noting $\mathscr{S}(U, \lambda^2 \cdot g) = \lambda^{n-2} \cdot \mathscr{S}(U,g)$, $\lambda > 0$ ($B_r(p)$ with respect to $\lambda^2 \cdot g_M$):

(1) there is a $\mu_0 \geqq 1$, independent of $\lambda \geqq 1$, $x \in S^i$, such that $\mathscr{S}(B_4(p), G(f(x), \lambda, \mu)) < 0$ for $\mu \geqq \mu_0$

(2) given $k > 0$ there is a $\lambda(k) \geqq 1$ such that for $x \in S^i$: $\mathscr{S}(M \setminus B_5(p), \lambda^2(k)\cdot f(x)) < -k$

(3) there is $c > 0$, independent of $\lambda \geqq 1$ and $(x,t) \in S^i \times [2,3]$, such that $\mathscr{S}(B_5(p), (3-t)\cdot G(f(x), \lambda, \mu_0) + (1-(3-t))\cdot \lambda^2 f(x)) < c$

(4) given $K > 0$ there is a $\mu(K) \geqq \mu_0$ independent of $\lambda \geqq 1$ such that $\mathscr{S}(B_5(p), G(F_1(x,t), \lambda, \mu(K)) < -K$ for each $(x,t) \in B^{i+1}$.

Now we verify (ii) on $S^i \times [0,3]/S^i \times \{0\}$ for μ_0 as in (1), $\lambda_0 := \lambda(2c)$, $\mu_1 := \mu(|m|+1)$, where $m := \max_{B^{i+1}} \mathscr{S}(M \setminus B_5(p), \lambda_0^2 \cdot F_1(x,t))$: on

$$S^i \times [2,3]/S^i \times \{0\}: \mathscr{S}(M, (3-t)\cdot G(f(x), \lambda_0, \mu_0) + (1-(3-t))\cdot \lambda_0^2 f(x))$$
$$= \mathscr{S}(B_5(p), \dots) + \mathscr{S}(M \setminus B_5(p), \dots) < -c < 0, \quad \text{by (2) and (3)}$$

on

$$S^i \times [1,2]/S^i \times \{0\}: \mathscr{S}(M, G(f(x), \lambda_0, (2-t)\mu_1 + (1-(2-t))\cdot\mu_0)) < 0$$

by (1) and $f(x) \in S^-(M)$, on

$$S^i \times [0,1]/S^i \times \{0\}: \mathscr{S}(M, G(F_1(x,t), \lambda_0, \mu_1))$$
$$\leqq m + \mathscr{S}(B_5(p), G(F_1(x,t), \lambda_0, \mu_1)) < -1 \text{ by (4).}$$
□

4 Eigenvectors of the Conformal Laplacian

The scalar curvature S_g transforms under conformal deformations $g_1 = u^{4/n-2} \cdot g$, $\dim M = n \geqq 3$, according to (cf. [K, (3.2)]:

$$L_g u \equiv -\gamma \cdot \Delta_g u + S_g \cdot u = S_{g_1} \cdot u^\alpha, \quad \gamma = 4\frac{n-1}{n-2}, \quad \alpha = \frac{n+2}{n-2}.$$

We are interested in the linear operator L_g which is sometimes called "conformal Laplacian".

Recall from [K, 3.A], that the first eigenvalue $\lambda_1(g)$ of L_g, which fulfills

$$\lambda_1(g) = \inf_{u \in C^\infty(M), u \not\equiv 0} \int_M (\gamma \cdot \| \nabla u \|^2 + S_g \cdot u^2) dV_g \Big/ \int_M u^2 \cdot dV_g \equiv \inf J_g(u),$$

has a one dimensional eigenspace generated by a (unique) eigenvector $v(g) \in C^\infty(M)$ with $v(g) > 0$, $\max v(g) = 1$.

For completness we will show the following hardly surprising fact, which is hard to quote explicitly from literature:

Proposition 2 *If $g_n \to g$ with respect to the C^∞-topology, then $\lambda_1(g_n) \to \lambda_1(g)$ and $v(g_n) \to v(g)$ also with respect to the C^∞-topology.*

Proof. From the definition of $J_g(u)$, we get for $\varepsilon > 0$ some n_0, such that: $(1-\varepsilon)|J_{g_n}(u)| \leqq |J_g(u)| \leqq (1+\varepsilon)|J_{g_n}(u)|$ for $n \geqq n_0$ and each $u \in C^\infty(M)\backslash\{0\}$. This implies $\lambda_1(g_n) \to \lambda_1(g)$. Furthermore $0 < v(g_n) \leqq 1$, $g_n \to g$ in the C^∞-topology and $L_{g_n} v(g_n) = \lambda(g_n) \cdot v(g_n)$ imply by standard elliptic theory $\| v(g_n) \|_{C^k_{\infty}} < c_k$, c_k independent of n. From $\lambda_1(g_n) \to \lambda_1(g)$ and the Arzela–Ascoli-Theorem we obtain converging subsequences (by iteration) in $\| \cdot \|_k$ and we take the diagonal sequence of these subsequences. This converges in C^∞ to $\tilde{v} \in C^\infty(M)$, with $L_g \tilde{v} = \lambda_1(g) \cdot \tilde{v}$, $\tilde{v} \geqq 0$, max $\tilde{v} = 1$ (from [K, 3.A], we conclude again $\tilde{v} > 0$). But this $\tilde{v}$ has to be the unique eigenvector $v(g)$, which implies that a fortiori $v(g_n)$ converges. □

5 Final deformation

Now we are ready to complete the proof of our theorem. Since $\mathscr{S}(M, F_2(x,t)) < 0$, $(x,t) \in B^{i+1}$, we conclude from $\lambda_1(g) = \inf J_g(u)$: $\lambda_1(F_2(x,t)) < 0$ on B^{i+1}. We define

$$F(x,t) = \begin{cases} f(x) & \text{on } S^i \times [5,6]/S^i \times \{0\} \\ ((5-t) \cdot v(f(x) + (1-(5-t)))^{\frac{4}{n-2}} \cdot f(x) & \text{on } S^i \times [4,5]/S^i \times \{0\} \\ v(F_2(x,t))^{\frac{4}{n-2}} \cdot F_2(x,t) & \text{on } S^i \times [0,4]/S^i \times \{0\} \end{cases}$$

and we claim

Proposition 3. *F is a continuous extension of $f: S^i \to S^-(M)$* with *$F(B^{i+1}) \subset S^-(M)$.*

Proof. Propositions 1 and 2 imply the continuity. Now we verify $F(x,t) \in S^-(M)$: On $S^i \times [5,6]/S^i \times \{0\}$ there is nothing to prove, on $S^i \times [4,5]/S^i \times \{0\}$ we calculate:

$$\begin{aligned} S_{F(x,t)} \cdot ((5-t) \cdot v(f(x)) + (1-(5-t)))^\alpha &= L_{f(x)}((5-t) \cdot v(f(x)) + (1-(5-t))) \\ &= (5-t) \cdot \lambda_1(f(x)) \cdot v(f(x)) + S_{f(x)} \cdot (1-(5-t)) < 0 \end{aligned}$$

on $S^i \times [0,4]/S^i \times \{0\}$ we obtain:

$$S_{F(x,t)} \cdot v(F_2(x,t))^\alpha = L_{F_2(x,t)} v(F_2(x,t)) = \lambda_1(F_2(x,t)) \cdot v(F_2(x,t)) < 0.$$

Since $(\ldots)^\alpha > 0$, we conclude $S_{F(x,t)} < 0$. □

6 Constant scalar curvature

Finally we will show that $S_{-1}(M)$ is contractible (which implies $\pi_i(S_{-1}(M)) = 0$, $i = 0, 1, \ldots$), this can be deduced from:

Proposition 4. *There is a continuous map* $p: S^-(M) \to S_{-1}(M)$ *with* $p_{|S_{-1}(M)} \equiv \mathrm{id}$.

Proof. Let $g \in S^-(M)$ and u a positive solution of the Yamabe equation $-\gamma \cdot \Delta_g u + S_g \cdot u = -u^\alpha$.

We assert

(i) u is unique

(ii) $p(g) := u^{4/n-2} \cdot g$ fulfills the claims.

(i): Let v be a second positive solution, $u^{4/n-2} \cdot g$ and $v^{4/n-2} \cdot g$ have scalar curvature $\equiv -1$. write $v \equiv w \cdot u$ for some $w > 0$, $w \in C^\infty(M)$. Then w fulfills the Yamabe equation for $g_1 = u^{4/n-2} \cdot g$:

$$-\gamma \cdot \Delta_{g_1} w - w = -w^\alpha,$$

now assume $w \not\equiv 1$: since $\alpha > 1$ we get $\Delta_{g_1} w > 0$ or the maximum of w or $\Delta_{g_1} w < 0$ in the minimum of w, which yields a contradiction.

(ii): From (i) $p_{|S_{-1}(M)} \equiv \mathrm{id}$, so it remains to show $g_n \to g$ in C^∞ implies $u_n \to u$ in C^∞ (u_n, u denote the solutions of the Yamabe equation of g_n, g): $-K_1 < S_{g_n} < -K_2$ for some $K_1 > K_2 > 0$ independent of n yields

$$0 < K_2^{\frac{1}{1-\alpha}} < (\min |S_{g_n}|)^{\frac{1}{1-\alpha}} < u_n < (\max |S_{g_n}|)^{\frac{1}{1-\alpha}} < K_1^{\frac{1}{1-\alpha}}.$$

Now using *both* bounds one can proceed as in Proposition 2 to get C^k-estimates independent of n. Again uniqueness of u as shown in (i) implies convergence of u_n. □

Now let $H: S^-(M) \times [0, 1] \to S^-(M)$ be a contraction to a $g_0 \in S_{-1}(M)$, i.e. $H(\cdot, 0) \equiv \mathrm{id}$, $H(\cdot, 1) \equiv g_0$. Consider $p \circ H_{|S_{-1}(M)} \times [0, 1] \to S_{-1}(M)$. $p \circ H$ is continuous by Proposition 4 and $p \circ H(\cdot, 0)_{|S_{-1}(M)} \equiv \mathrm{id}$, $p \circ H(\cdot, 1) \equiv g_0 \in S_{-1}(M)$, i.e. $S_{-1}(M)$ is contractible.

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Note added in proof. The space of negative Ricci curvature metrics $\mathrm{Ric}^-(M)$ is also non-empty and contractible. Furthermore $\mathrm{Ric}^-(M)$ is dense in $\mathcal{M}(M)$ w.r.t. C^0-topology. This is proved in a more geometric but more intricated and conceptually different way. For details we refer to [L2].

Curvature h-principles

by Joachim Lohkamp

§ 1. Introduction

Let us start with the following intuitive question which will be subject of this paper: If g is a metric on some manifold and fulfills certain geometric conditions, to what extend will g be special in the set of all metrics ?

There are two obvious subquestions:

A. If g_1 and g_2 fulfill the same condition, will g_1 and g_2 be more or less equal or if not, does the space of these metrics have an interesting internal structure ?

B. If g is an arbitrary metric, can we find a metric g' fulfilling the condition which shares at least some geometric properties with the "model metric" g ?

We will give an answer in the cases where the conditions on g are as follows: $r(g)(\nu) < \alpha$ resp. $s(g) < \alpha$, $\alpha \in \mathbf{R}$, where $r(g)(\nu)$ resp. $s(g)$ denote the Ricci curvature in direction $\nu \neq 0$ resp. the scalar curvature of g which is defined on some manifold M^n of dimension $n \geq 3$. Usually it is assumed that M is closed, but that is only for practical not for principal reasons, since the actual problems will arise locally.

__Theorem A.__ *The space of all metrics with $r(g)(\nu) < \alpha$ resp. $s(g) < \alpha$ denoted by $\mathrm{Ric}^{<\alpha}(M)$ resp. $S^{<\alpha}(M)$ is a contractible Fréchet-manifold.*

Recall from [L2] and [L5] that these spaces are always non-empty and (highly) non-convex. Hence we can solve problem A: There is "exactly" one typical metric up to homotopy.

__Theorem B.__ *$\mathrm{Ric}^{<\alpha}(M)$ and $S^{<\alpha}(M)$ are dense in the space of all metrics $\mathcal{M}(M)$ with respect to C^0-topology and Hausdorff-topology.*

(C^0 implies Hausdorff-density, but we present different arguments for philosophical reasons. Theorem B (obviously) and Theorem D below also hold if " $<$ " is exchanged by " $\leq$ ".)

Thus (concerning question B) we can prescribe the geometric shape of our manifold and realize it in $\mathrm{Ric}^{<\alpha}(M)$ for arbitrarily strong negative α !

There are also local versions: let g_0 be a metric on a manifold M^n, $n \geq 3$, $S \subset M$ a closed subset, $U \supset S$ an open neighborhood, then we have

__Theorem C.__ *If g_1 is a second metric on M and $g_0 \equiv g_1$ on $M \setminus S$ and $r(g_i)(\nu) < \alpha$ (resp. $s(g_i) < \alpha$), then there is a continuous family of metrics g_t, $t \in [0,1]$ with $r(g_t)(\nu) < \alpha$ (resp. $s(g_t) < \alpha$) and $g_t \equiv g_0 \equiv g_1$ on $M \setminus U$.*

__Theorem D.__ *If $r(g_0)(\nu) < \alpha$ (resp. $s(g_0) < \alpha$) on $M \setminus S$, then for each $\varepsilon > 0$ we can find a metric g_ε on M with $r(g_\varepsilon) < \alpha$ (resp. $s(g_\varepsilon) < \alpha$), $g_\varepsilon \equiv g_0$ on $M \setminus U$ and*

$$\|g_\varepsilon - g_0\|_{C^0_{g_0}(M)} < \varepsilon.$$

Notice that the Theorems fail for $\mathrm{Ric}^{>\alpha}(M)$ resp. $S^{>\alpha}(M)$ (with $\alpha \geq 0$ in Theorem A and C). $S^{>\alpha}(M)$ usually has a complicated topology: $\pi_i(S^{>\alpha}(M)) \neq 0$ for certain $i \geq 0$ (cf. [LM]) and $\mathrm{Ric}^{>\alpha}(M)$ is never C^0-dense in $\mathcal{M}(M)$ acc. Bishop's comparison theorem.

There is some conceptual point of view, which can be recognized as an application as well as a motivation of the preceding results. Namely the theory of h-principles (= homotopy principles) for partial differential relations as introduced by M. Gromov (cf. [Gr1]). We roughly describe this notion (details, cf. [Gr1] Chap. 1): A solution of a differential (in)equality can be viewed as a section of some fibre bundle over the underlying manifold, which k-jet fulfills this relation imposed on the k-jet bundle of jets of germs of sections.

Now we could start from a section of this k-jet bundle which pointwise fulfills the (in)equality and try to get an actual solution from this "formal" solution by a path of sections in the k-jet bundle consisting (at least) of "formal" solutions and the endpoint is a real solution.

More generally, we could embed the space of solutions into the space of formal solutions and asked whether the first space is retract of the bigger one (with some deviation from [Gr1]), which reduces the problem to algebraic-topological obstruction theory.

If this is true we will say the relation fulfills the h-principle. This sounds fairly strange from the point of view of general P.D.E.'s and one expects that the h-principle is not fulfilled in all non-trivial cases. But indeed there are some important and geometrically significant differential relations which obey the h-principle (cf. [Gr1], Chap. 1, for examples).

In this paper we get from our Theorems an unexpected new "class" of differential relations which fulfill this principle:

Corollary E. *$r(g)(\nu) < \alpha$ and $s(g) < \alpha$ considered as differential relations imposed on the 2-jet bundle of germs of metrics fulfill (all senseful forms of) h-principles.*

There are some remarks in order: There are different versions of h-principles; here we get the parametric and the C^0-dense h-principle as well as the h-principle for extensions (cf. § 6 and [Gr1] for these notions).

Next we give an outline of the paper: It is divided into three parts. The main part (part I = § 2-6) contains the proof of Theorems. Part II = § 7-8 resp. part III = § 9-10 are devoted to the proof of some technical results used in part I.

In § 2-3 we will obtain three results: the Hausdorff-density, the C^0-density (using results of part III) and certain deformations which will prepare the metric for our main construction in § 4: Here we will produce negative (Ricci) curvature, i.e. deform the metric continuously into some more negatively Ricci curved one. This is the philosophical core of Theorem A and C which are obtained in § 5 and makes possible to derive the h-principles in § 6.

Part II contains a method to concentrate negative (sectional) curvature inside a ball of negative Ricci curvature. In Part III we construct certain metrics on $\mathbf{R}^n$ used in § 2: In § 9 we refine our construction presented in [L2] to get metrics on $\mathbf{R}^n$, $n \geq 4$. § 10 covers this construction on $\mathbf{R}^3$.

Remarks.

1. In [L3] we gave a proof of contractibility of $S^{<\alpha}(M)$ using completely different (namely analytic) techniques. It should be helpful to study that paper before reading this one, since it is very transparent and will be instructive to understand the main problems.

2. The Hausdorff-density result was already announced in [L1], the other Theorems in [L2] and [L3].

Part I. Proof of Theorems

§ 2. Standard Deformations and Density

First we recall some results from [L2] often used in the current paper. Then we will give proofs of our density results.

1. Let g some metric on $\mathbb{R} \times M^n$, $n \geq 2$, M^n closed and equipped with some fixed reference metric g_M and assume that g fulfills

(i) $(g_{\mathbb{R}} + g_M)(\nu,\nu) \leq K^2 \cdot g(\nu,\nu)$, $\nu \in T(\mathbb{R} \times M)$

(ii) $\|g\|_{C^3} \leq K$

for some constant $K > 0$ and the C^3-norm of the iterated covariant derivative w.r.t. $g_{\mathbb{R}} + g_M$ on $\mathbb{R} \times M$ denoted by $\|\cdot\|_{C^3}$. Furthermore consider $F_{d,s} : \mathbb{R} \to \mathbb{R}^{\geq 0}$ for $d,s > 0$ defined by $F_{d,s} \equiv 0$ on $\mathbb{R}^{\leq 0}$, $F_{d,s}(t) = s \cdot \exp(-d/t)$ on $\mathbb{R}^{\geq 0}$.

<u>**Proposition (2.1)**</u> *(cf. [L2]): For each $b > a > 0$ there is a constant $c > 0$ depending only on a, b, K, (M, g_M) and n such that for $d > c$ and any $s > 0$, $\nu \neq 0$:*

$$e^{2F_{d,s}} \cdot r\left(e^{2F_{d,s}} \cdot g\right)(\nu) - r(g)(\nu) < \begin{cases} 0 & \text{on }]0,a[\times M \\ -s \cdot e^{-\frac{d}{a}} & \text{on } [a,b[\times M \end{cases}$$

2. Next we will get Besicovitch type coverings not only on a fixed Riemannian manifold (M, g_0) but on a compact family $K \subset \mathcal{M}(M)$: There is a radius $r(K) > 10$ and for each $\delta > 0$ some $m_0(\delta, K)$ such that for each $m \geq m_0$ there is a discrete subset $A(m,K) \subset M$ and a constant $c(K)$ such that for each $g \in K$ and $\exp_p^{m^2 \cdot g} \equiv$ exponential map in p w.r.t. $m^2 \cdot g$:

(i) $\exp_p^{m^2 \cdot g}$ is a diffeomorphism of $B_{1000 \cdot r^2(k)}(0) \subset \left(T_pM, m^2 \cdot g_p\right)$ onto its image and $\left\| \left(\exp_p^{m^2 \cdot g}\right)^* (m^2 \cdot g) - m^2 \cdot g_p \right\|_{C^5_{m^2 \cdot g_p}(B_{100 \cdot r^2(K)}(0))} < \delta$, independent of $p \in M$ (and $g \in K$ and $m \geq m_0$)

(ii) $d_{m^2 \cdot g}(a,b) > 10 \cdot r(K)$ for $a \neq b \in A(m,K)$

(iii) $\bigcup_{a \in A(m,K)} (B_{10 \cdot r^2(K)}(a), m^2 \cdot g) = M$

(iv) $\#\{a \in A(m,K) | \left(B_{50 \cdot r^2(K)}(a), m^2 \cdot g\right) \ni z\} \leq c(K)$, independent of $z \in M$, $m \geq m_0$ and $g \in K$.

The proof relies on some simple combinatorics and is contained in the Appendix of [L2].

3. Recall from [L2] that the various existence results of metrics with Ric < 0 were sponsored from the following

Proposition (2.2) *On* $\mathbb{R}^n$, $n \geq 3$, *there exists a metric* g_n *with* $r(g_n) < 0$ *on* $B_1(0)$ *and* $g_n \equiv g_{Eucl.}$ *outside.*

This is enough to prove the Hausdorff-density (see below), but the C^0-density will rely on

Proposition (2.3) *For each* $\varepsilon > 0$ *there exists a metric* $g_{n,\varepsilon}$ *on* $\mathbb{R}^n$, $n \geq 3$ *with* $r(g_{n,\varepsilon}) < 0$ *on* $B_1(0)$, $g_{n,\varepsilon} \equiv g_{Eucl.}$ *outside and* $\|g_{n,\varepsilon} - g_{Eucl.}\|_{C^0_{g_{Eucl.}}(\mathbb{R}^n)} < \varepsilon$.

The proof is presented in part III.

4. Combining 1.-3. we can recall the construction of metrics with Ric < 0 of [L2] in a presentation designed for density as well as for contractibility results: Hence let $K \subset \mathcal{M}(M)$ be compact and $g_0 \in K$, then we can define for each subset $B \subset A(m, K)$ (from 2.):

$$G(g_0, g_{n,\varepsilon}, B, I_B, m) := \begin{cases} (1 - h(d_{m^2\cdot g_0}(p, id))) \cdot \left(\exp_p^{m^2\cdot g_0}\right)_* (I_p^*(g_{n,\varepsilon})) + h(d_{m^2\cdot g_0}(p, id)) \cdot m^2 \cdot g_0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad B_2(p), p \in B \\ m^2 \cdot g_0 \quad \text{elsewhere} \end{cases}$$

with $h \in C^\infty(\mathbb{R}, [0,1])$ with $h \equiv 1$ on $\mathbb{R}^{\geq 1,6}$, $h \equiv 0$ on $\mathbb{R}^{\leq 1,5}$ and $I_B = \{I_p | p \in B\}$ where I_p is some linear isometry $I_p : T_pM \to \mathbb{R}^n$. From this we get the central metrics $\left(r \in [0, 50 \cdot r^2(K)[\right)$:

$$g(g_0, g_{n,\varepsilon}, B, I_B, m, r, d, s) := \prod_{p\in B} \exp\left(2H_r \cdot F_{d,s}(2 + r - d_{m^2\cdot g_0}(p, id_M))\right) \cdot G(g_0, g_{n,\varepsilon}, B, I_B, m)$$

with $H_r(\mathrm{id}) = H(\mathrm{id} - r)$ for some $H \in C^\infty(\mathbb{R}, [0,1])$ and $H \equiv 0$ on $\mathbb{R}^{\geq 1,9}, H \equiv 1$ on $\mathbb{R}^{\leq 1,8}$.

Proposition (2.4) *(cf. [L2]): There are* $d_0, s_0 > 0$ *(depending on* ε *) such that for each* $d \geq d_0$, $s^{-1} \geq s_0^{-1}$ *there is a* $m_0(K, g_{n,\varepsilon}, d, s)$ *such that for each* $m \geq m_0$, $r(g(g_0, g_{n,\varepsilon}, B, I_B, m, r, d, s)) < -c$ *for some* $c = c(d,s) \geq 0$ *independent of* K,

$c > 0$ *in (i) (* $c = 0$ *in (ii)) if one of the following conditions is fulfilled:*

(i) $K \subset \mathcal{M}(M), B = A(m, K), g_0 \in K, r = R(K) := 20 \cdot r^2(K)$, *or*

(ii) $K \subset \mathrm{Ric}^{<0}(M)$, $B \subset A(m, K)$, $g_0 \in K$ *and* $r \in [0, R(K)]$.

5. (2.3) and (2.4) almost immediately imply

Corollary (2.5) $\mathrm{Ric}^{<\alpha}(M)$ *and therefore* $S^{<\alpha}(M)$ *are* C^0*-dense subsets of* $\mathcal{M}(M)$ *for each* $\alpha \in \mathbb{R}$.

Proof: Let $g_0 \in \mathcal{M}(M)$ any prescribed metric and $\delta > 0$ given. Then we can find from (2.3) (and (9.1)) some suitable small $\varepsilon > 0$ and large m such that:

$$\left\| \frac{1}{m^2} G(g_0, g_{n,\varepsilon}, A(m, \{g_0\}), I_A, m) - g_0 \right\|_{C^0_{g_M}(M)} < \frac{\delta}{2}.$$

Furthermore acc. (2.4) we can find for each $\eta > 0$ a small $s > 0$ such that for $m \geq m_0(\{g_0\}, g_{n,\varepsilon}, d_0, s)$:

(i) $1 \leq \prod_{p \in A(m,\{g_0\})} \exp\left(H_{R(\{g_0\})} \cdot F_{d_0,s}(2 + R(\{g_0\}) - d_{m^2 \cdot g_0}(p, \mathrm{id}_M))\right) \leq 1 + \eta$

(ii) $r\left(\frac{1}{m^2} \cdot g(g_0, g_{n,\varepsilon}, A(m, \{g_0\}), I_A, m, R(\{g_0\}), d_0, s)\right)(\nu) < -c \cdot m^2, \ \nu \neq 0.$

Hence for suitable small $\eta > 0$ we have

$$\left\| \frac{1}{m^2} \cdot g(g_0, g_{n,\varepsilon}, A(m, \{g_0\}), I_A, m, R(\{g_0\}), d_0, s) - g_0 \right\|_{C^0_{g_M}(M)} < \delta.$$

Furthermore the smaller $\delta > 0$ is chosen the stronger negative gets the upper bound in (ii).

□

This argument easily extends to the following local version:

Corollary (2.6): *Let g_0 be a metric on M, $r(g_0) < \alpha$ on $M \setminus S$ for some closed subset $S \subset M$, $U \supset S$ an open neighborhood and $\varepsilon > 0$ given, then we can find a metric g_ε on M with $r(g_\varepsilon) < \alpha$ on M and $g_\varepsilon \equiv g_0$ on $M \setminus U$ and $\|g_\varepsilon - g_0\|_{C^0_{g_0}(M)} < \varepsilon$.*

For the proof one roughly takes $U \cap A(m, \{g_0\})$ instead of $A(m, \{g_0\})$ and argues as in (2.5), to get the precise argument (which also works in the non-compact case by exhaustings) combine Theorem E of [L2] and its proof with (2.5).

The interesting fact is that (2.3) is not needed to prove the following weaker approximation result:

Proposition (2.7) *$\mathrm{Ric}^{<\alpha}(M)$ and $S^{<\alpha}(M)$ are dense in $\mathcal{M}(M)$ w.r.t. Hausdorff-topology for each $\alpha \in \mathbb{R}$.*

Recall that the Hausdorff-distance d_H between two metric spaces M_1, M_2 is defined as $d_H(M_1, M_2) =$ Infimum for all metric spaces M and isometric embeddings $f_i : M_i \to M$ of

$$\inf_{\varepsilon > 0} \{f_i(M_i) \subset \varepsilon - \text{neighborhood of } f_j(M_j),\ i,j = 1,2\}.$$

Now use the following lemma which obviously implies (2.7) for closed M from (2.4) (i). In the non-compact case one adds some exhausting argument.

Lemma (2.8)

(i) *Let be g some metric on $M, f \in C^\infty(M, \mathbb{R}^{\geq 0})$, then we can find for each $\varepsilon > 0$ a $\delta(\mathrm{diam}(M,g), \varepsilon) > 0$ such that for $\sup f < \delta$, $g_1 = e^{2f}$ and $g_2 = g$ fulfill $d_H((M, g_1), (M, g_2)) < \varepsilon$.*

(ii) *Let be g and $g_{R,r}$ metrics with $\mathrm{Inj}(M,g) > R > 5 \cdot r > 0$ and such that for each pair (R, r) there are finitely many balls $(B_r(p_i), g)$, $r > 0$, $p_i \in I_{R,r}$ with $\mathrm{dist}_g(B_r(p_i), B_r(p_j)) > R$ for $p_i \neq p_j$ and $\mathrm{diam}_{g_{R,r}}(B_r(p_i), g) < c \cdot r$ for some $c = c(M, g) > 2$ with $g_1 \equiv g_2$ on $M \setminus \bigcup_i B_r(p_i)$. Then there exists for each $\varepsilon > 0$ some constant $\rho > 0$ depending only on M, g, c and ε, such that for $\frac{r}{R} < \rho$ and $g_1 \equiv g_{R,r}$, $g_2 \equiv g$: $d_H((M, g_1), (M, g_2)) < \varepsilon$.*

Proof: In both cases we can use explicit embeddings into $(\mathbf{R} \times M, g_{\mathbf{R}} + (h \cdot g_1 + (1-h) \cdot g_2))$ for some $h \in C^\infty(\mathbf{R}, [0,1])$ with $h \equiv 1$ on $\mathbf{R}^{\leq \alpha}$, $h \equiv 0$ on $\mathbf{R}^{\geq 2\alpha}$. If $h \equiv h(\delta)$ resp. $h(\rho)$ and $\alpha \equiv \alpha(\delta)$ resp. $\alpha(\rho)$ are chosen suitable (namely: $\alpha^2 = (e^{2\delta} - 1) \cdot \operatorname{diam}(M,g)^2$ resp. $\alpha^2 = \beta^2 + 2 \cdot \beta \cdot (\pi + 2) \cdot \operatorname{diam}(M,g)$ for $\beta = \frac{3}{2} \cdot c \cdot \rho \cdot ((\pi+2) \cdot \operatorname{diam}(M,g) + 1)$ as is obtained from simple calculations) we can take $f_1(z) = (0,z)$, $f_2(z) = (3\alpha, z)$ and check that the f_i are isometric (in the sense of metric spaces) and $\alpha \to 0$ for δ resp. $\rho \to 0$.

(i) is quite obvious,

(ii) relies on the fact that the distance between two points on $(M, g_{R,r})$ converges to the distance on (M,g) if $\rho > 0$ is tending to zero. Details are left to the reader.

Hence: in (i) and (ii): $d_H(f_1((M,g_1)), f_2((M,g_2))) \leq 4\alpha \to 0$.

□

The philosophy of these approximation theorems is that metrics in $\operatorname{Ric}^{<\alpha}(M)$ can be crumpled (cf. also "bendings" in [L4]) such that even positively curved metrics can be approximated. It is worth to recall that the Ricci curvature gets the negativer the better we approximate.

On the other hand it is clear from Bishop's comparison theorem that metrics in $\operatorname{Ric}^{>\alpha}(M)$ cannot be "uncrumpled" to approximate negatively curved ones.

§ 3 Nest-Building

Here we will prepare given metrics with $\operatorname{Ric} < 0$ to "produce" negative Ricci curvature namely we will build some kind of nest.

Proposition (3.1) Let $\tau : S^k \to \operatorname{Ric}^{<0}(B)$ a continuous family of metrics on some n-dimensional ball B and $p \in B$. Then for each $R > 0$ we can find a homotopy $T_R : [0,1] \times S^k \to \operatorname{Ric}^{<0}(B)$ with:

(i) $T_R(0,x) = \tau(x)$ on B for each $x \in S^k$ and $T_R(t,x) = \tau(x)$ on a suitable neighborhood of ∂B

(ii) there are $\lambda, r > 0$ and a continuous family of isometries I_x, $I_x : (B_r(p), T_R(1,x)) \to (B_r(0), \lambda^{-2}(h_\lambda^*(g_{\text{hyp.}})))$ with $I_x(p) = 0$ such that $r \cdot \lambda > R$ and $\overline{B_r(p)} \subset B$. (h_λ denotes the homotety $h_\lambda(x) = \lambda \cdot x$ on $\mathbf{R}^n$).

(In other words the deformation yields $\operatorname{Ric} < 0$ - metrics on B such that B contains (after scaling by λ^2) an arbitrarily large hyperbolic ball.)

The proof is based on two main ideas:

1. Choose suitable coordinates near p and use the linearity of curvatures of g in the second derivatives of g. This mainly uses a method due to L.Z. Gao invented in [G] and yields (3.1) without estimates on the radius of the desired hyperbolic ball.

2. Therefore we make a subsequent deformation ("concentration negative curvature" cf. part II) starting from an arbitrarily small hyperbolic ball $B_{r_0}(0) \subset \mathbf{H}^n$, and get for any $R > 0$ a continuous family g_t, $0 \leq t \leq 1$ on $B_{r_0}(0)$

(i) $\mathrm{Ric}(g_t) < 0$ on $B_{r_0}(0)$

(ii) $g_0 \equiv g_{\mathrm{hyp}}$ on $B_{r_0}(0)$ and $g_t \equiv g_{hyp.}$ near $\partial B_{r_0}(0)$

(iii) $\left(B, \lambda^2 \cdot g_1\right)$ is isometric to $\left(B_R(0), g_{\mathrm{hyp.}}\right)$ for some ball $B \subset B_{\frac{r_0}{2}}(0)$.

Details are presented in part II = § 7 and 8. Thus we are left to make precise the first point: Here we mainly have to check the following simple

__Lemma (3.2)__ *Let $F : S^k \to \mathcal{M}(M^n)$ a continuous map, g_M some reference metric on M and $p \in M$, then there is a radius $r_0 = r_0(p, g_M, F(S^k))$ such that there are defined n functions $x_i : (B_{r_0}(p), g_M) \to \mathbf{R}$ with x_i depending continuously on $x \in S^k$ (for fixed F) such that:*

(i) *$x_i(p) = 0$ for each $x \in S^k$ and $i = 1, \cdots, n$*

(ii) *$g^x_{ij} = F(x)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij}$ for each $x \in S^k$ in p*

(iii) *$\frac{\partial g^x_{ij}}{\partial x_4} = 0$ in p. $1 \le i, j, k \le n$ for each $x \in S^k$.*

(In other words there are defined geodesic coordinates for each metric $F(x)$ on $B_{r_0}(p)$ depending continuously on x.)

Proof: Let U be any coordinate neighborhood, $y_i : U \to \mathbf{R}$, $1 \le i \le n$ coordinate functions with $y_i(p) = 0$ and $(B_R(p), g_M) \subset U$, then there is a $r_0 \in]0, R[$ such that $z_i := y_i + \frac{1}{2}\sum_{j,k} \Gamma^j_{ik}(x) y_i y_k$ are coordinates with $z_i(p) = 0$ on $(B_{r_0}(p), g_M)$ for each x. Here $\Gamma^j_{ik}(x)$ denotes the Christoffel-symbol w.r.t. metric $F(x)$ and the coordinates y_i.

Denote by $G^x_{ij} := g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$, then G^x_{ij} fulfills $\frac{\partial G^x_{ij}}{\partial z_k} = 0$ in p for each $x \in S^k$, $1 \le i, j, k \le n$ (as is easily checked).

Thus we are left to make linear transformations (which depend continuously on x) of these coordinates to alter G^x_{ij} to a diagonal matrix (namely the identity matrix). This is possible due to the fact that G^x_{ij} is positive definite and symmetric. These coordinates fulfill the claims.

□

Now we can combine this with the calculations of Gao in [G], to get:

__Lemma (3.3)__ Let $\tau : S^k \to \mathrm{Ric}^{<0}(B)$ continuous, $B \subset \mathbf{R}^n$ an open ball, $p \in B$. Then we can find a homotopy $T : [0,1] \times S^k \to \mathrm{Ric}^{<0}(B)$ with

(i) $T(0, x) = \tau(x)$ on B and $T(t, x) = \tau(x)$ on a suitable neighborhood of ∂B.

(ii) there exists some $r_0 = r_0(\tau(S^k)) > 0$ such that $(B_{r_0}(p), T(1, x))$ is isometric to $B_{r_0}(0) \subset \mathbf{H}^n$, the isometry depending continuously on x.

Proof: Using (3.2) we can look at the push-forward metrics $P(x)$ of $\tau(x)$ w.r.t. the coordinates $x_i = x_i(x)$ constructed in (3.2) on $B_{r_0}(p)$. Now take some Euclidean ball $B_r(0) \subset \mathbf{R}^n$ constructed in the image of $(x_1, \cdots, x_n)$ for all x and notice that the Cartesian coordinates are geodesic coordinates in $0 \in B_r(0)$ for each metric $P(x)$.

Therefore we can use the estimates of Prop. (2.5) and its proof in [G] to conclude completely analogously:

We can find some $R_1 < R_2 \in]0, \frac{r}{2}[$ and a cut-off function $\psi \in C^\infty(B_r(0),[0,1])$ with $\psi \equiv 1$ on $B_{R_1}(0)$ and $\psi \equiv 0$ on $B_r(0) \setminus B_{R_2}(0)$, R_1, R_2 and ψ independent of x, such that for each $t \in [0,1]$ and $x \in S^k$ $\operatorname{Ric}(t \cdot \psi \cdot g_{\text{hyp.}} + (1-t)\cdot\psi\cdot P(x)) < 0$ on $B_r(0)$.

Finally we take the pull-back metrics w.r.t. the coordinate charts $(x_1, \cdots, x_n)$ and obtain the desired homotopy.

□

§ 4 Producing Negative Curvature

The key ingredient in our proof of Theorems A and C is

Proposition (4.1) *Let g be any metric on $B_1(0) \subset \mathbb{R}^n$, $n \geq 3$ with $r(g) < 0$ and $R > 0$, then we find some $r \in]0, \frac{1}{2}[$ independent of R and $m > \frac{R}{r}$ and a continuous family g_t, $0 \leq t \leq 1$ of metrics on $B_1(0)$ with $r(g_t) \leq 0$ and*

(i) $g_0 \equiv g$ *on* $B_1(0)$ *and* $g_t \equiv g$ *on* $B_1(0) \setminus B_{\frac{1}{2}}(0)$

(ii) $g_1 \equiv g$ *on* $B_1(0) \setminus B_r(0)$ *and on* $B_r(0)$:

$$g_1 = \frac{1}{m^2} \exp\left(2H_R \cdot F_{d,s}(2 + R - d_{m^2 \cdot g}(0, \mathrm{id}_{B_1(0)}))\right) \cdot G(g, g_n, \{0\}, I_0, m).$$

Here we can use (3.1), (2.1) and (2.4) (ii) which obviously imply that (4.1) can be reduced to

Lemma (4.2) *There exists an $R > 3$ such that $(B_R(0), g_{\text{hyp}})$ can continuously be deformed by some family $g_t, t \in [0,1]$ with $g_0 \equiv g_{hyp.}$, $g_t \equiv g_{hyp}$ on $B_R(0) \setminus B_{R-1}(0)$ and $\operatorname{Ric}(g_t) \leq 0$, $g_1 \equiv g$ for the following metric g :*

> $g = g_{\mathbb{R}} + f^2 \cdot g_{S^{n-1}}$ *on* $B_R(0) \setminus B_1(0) \equiv [1, R[\times S^{n-1}$ *for some* $f \in C^\infty(\mathbb{R}^{\geq 1}, \mathbb{R}^{\geq 0})$, $f \equiv \mathrm{id}$ *on* $[1,2]$, $f \equiv \sin h$ *on* $[R-1, R[$ *and* $f'' \geq 0$,
>
> $g = g_n$ *on* $B_1(0) \equiv [0,1] \times S^{n-1}/\{0\} \times S^{n-1}$

Proof: Let us start with some implication for $f \in C^\infty(\mathbb{R}^{>0}, \mathbb{R}^{>0})$ which is trivially deduced from [B]:

(*) $\quad f \geq \mathrm{id}$, $f' \geq 1$, $\frac{f''}{f} \geq c \geq 0$ implies $r(g_{\mathbb{R}} + f^2 g_{S^{n-1}}) \leq -c$.

Now take some $f_k \in C^\infty(\mathbb{R}^{>0}, \mathbb{R}^{>0})$ with $f_k \equiv \mathrm{id}$ on $]0,12]$, $f_k' \geq 1$, $f_k'' \geq 0$ on $\mathbb{R}^{>0}$, $\frac{f_k''}{f_k} \geq k$ on [14, 18] and $f_k \equiv \sinh$ on $\mathbb{R}^{\geq R-1}$ for some suitable large $R = R(k) > 20$. Thus we get: $\operatorname{Ric}\left(g_{\mathbb{R}} + (t \cdot \sinh + (1-t)\cdot f_k)^2 \cdot g_{S^{n-1}}\right) \leq 0$ for each $t \in [0,1]$ and on $\mathbb{R}^{>0} \times S^{n-1}$. Next consider

$$g(k,s,d) := \begin{cases} \exp\left(2H \cdot s \cdot \exp\left(-\frac{d}{t-5}\right)\right) \cdot \left(g_{\mathbb{R}} + f_k^2 \cdot g_{S^{n-1}}\right) & \text{on} \quad \mathbb{R}^{\geq 5} \times S^{n-1} \\ g_{\mathbb{R}} + r^2 \cdot g_{S^{n-1}} \quad \text{on} \quad [0,5[\times S^{n-1} \end{cases}$$

with $H \in C^\infty(\mathbb{R}, [0,1])$, $H \equiv 1$ on $\mathbb{R}^{\leq 15}$, $H \equiv 0$ on $\mathbb{R}^{\geq 16}$.

Acc. (2.1) we can find some $d_0 > 0$ such that for each $d \geq d_0$ and $s \geq 0$:

$$\operatorname{Ric}(\exp(2H \cdot s \cdot \exp(-d/_{t-5})) \cdot g_{\text{Eucl.}}) \leq 0$$

on $\mathbf{R}^{>0} \setminus [15,16] \times S^{n-1}$.

This and (*) yield for each $d \geq d_0$ and $s \geq 0$ we can find some $k_0 = k_0(d,s) > 0$ such that $\operatorname{Ric}(g(k,d,s)) \leq 0$ on $\mathbf{R}^{>0} \times S^{n-1}$, $k \geq k_0$ on $[0,12] \cup \mathbf{R}^{\geq 14}$. On $[14,16] \times S^{n-2}$ we recall from the formulas for conformal deformations [B] (1.159 d) it is enough to note:

$$\operatorname{Hess}(g_{\mathbf{R}} + f_k^2 \cdot g_{S^{n-1}})(\exp(-d/t-5))(\nu,\nu) \geq \operatorname{Hess}(g_{\text{Eucl.}})(\exp(-d/t-5))(\nu,\nu)$$

which is clear from [L2] (3.1), since obviously $(\pi \circ \gamma_\nu^k)''(0) \geq (\pi \circ \gamma_\nu)''(0)$, $(\pi \circ \gamma_\nu^k)'(0) = (\pi \circ \gamma_\nu)'(0)$, where π is projection on $\mathbf{R}$, γ_ν^k and γ_ν are geodesics w.r.t. $g_{\mathbf{R}} + f_k^2 \cdot g_{S^{n-1}}$ resp. $g_{\text{Eucl.}}$ with $\dot\gamma_\nu^k(0) = \dot\gamma_\nu(0) = \nu$.

Now consider for $t_1, t_2 \in [0,1]$ the following continuous family $g(t_1,t_2)$ of metrics on $B_{12}(0) \subset \mathbf{R}^n$: $g(t_1,t_2) := t_1 \cdot g_{\text{Eucl.}} + (1-t_1) \cdot T_{t_2}^*(g_n)$, with $T_{t_2} := \mathrm{id}_{\mathbf{R}^n} + t_2 \cdot 10 \cdot e$ for some fixed $e \in \mathbf{R}^n$ with $\|e\|_{g_{\text{Eucl.}}} = 1$. Combine our previous result (again) with (2.1) to get: There are $\overline{d} \geq d_0$, $\overline{s} > 0$, $\overline{k} \geq k_0$ such that

$$G(t_1,t_2,\overline{k},\overline{s},\overline{d}) := \begin{cases} g(\overline{k},\overline{s},\overline{d}) & \text{on} \quad \mathbf{R}^{\geq 12} \times S^{n-1} \\ \exp\left(2H \cdot \overline{s} \cdot \exp\left(-\overline{d}/t - 5\right)\right) \cdot g(t_1,t_2) & \text{on} \quad]0,12[\times S^{n-1} \end{cases}$$

fulfills $\operatorname{Ric}(G(t_1,t_2,\overline{k},\overline{s},\overline{d})) \leq 0$ on $\mathbf{R}^{>0} \times S^{n-1}$ for $(t_1,t_2) \in \{0\} \times [0,1] \cup [0,1] \times \{1\}$.

Finally we patch together all these deformation on $B_R(0)$ for $R = R(\overline{k})$ as above:

$$g_t := \begin{cases} g_{\mathbf{R}} + \left(5t \cdot f_{\overline{k}} + (1-5t) \cdot \sinh\right)^2 \cdot g_{S^{n-1}}, & t \in [0,\frac{1}{5}] \\ g(\overline{k}, 5(t-\frac{1}{5}) \cdot \overline{s}, \overline{d}), & t \in [\frac{1}{5},\frac{2}{5}] \\ G(1 - 5 \cdot (t-\frac{2}{5}), 1, \overline{k}, \overline{s}, \overline{d}), & t \in [\frac{2}{5},\frac{3}{5}] \\ G(0, 1 - 5 \cdot (t-\frac{3}{5}), \overline{k}, \overline{s}, \overline{d}), & t \in [\frac{3}{5},\frac{4}{5}] \\ \left\{ \begin{matrix} g(\overline{k}, (1-5(t-\frac{4}{5})) \cdot \overline{s}, \overline{d}) & \text{on} & B_R(0) \setminus B_2(0) \\ g_n & \text{on} & B_2(0) \end{matrix} \right\}, & t \in [\frac{4}{5},1] \end{cases}$$

□

<u>**Remark**</u> **(4.3):** Here we will mention a different approach: Let us assume the existence of metrics $g(\varepsilon,k)$ on $\mathbf{R}^n$, $n \geq 3$ with $\|g(\varepsilon,k) - g_{\text{Eucl.}}\|_{C^k} < \varepsilon$, $g(\varepsilon,k) \equiv g_{\text{Eucl.}}$ on $\mathbf{R}^n \setminus B_1(0)$ and $\operatorname{Ric}(g(\varepsilon,n)) < 0$ for each $\varepsilon, k, n > 0$.

If we choose $k = 2$ and a suitable $\varepsilon > 0$ it seems as if we do not need to "concentrate curvature" since the above argument applies in the same way to $B_r(0)$ for small enough $\varepsilon > 0$.

But there is a drawback: r cannot be estimated from below, hence we get no lower bound for $\varepsilon > 0$ and this destroys the argument of the proof of contractibility on $\operatorname{Ric}^{<\alpha}(M)$ in § 5 below.

On the other hand we can preserve the philosophy of "concentration" using $g(\varepsilon,k)$ as follows: produce iteratively disjoint balls "$(B_1(0), g(\varepsilon,k))$" for some possibly small but fixed $\varepsilon > 0$. Scale the balls to get place enough to produce as many balls as are necesary to hide $t \cdot g_n + (1-t) \cdot g_{\text{Eucl.}}$ behind a $\operatorname{Ric} < 0$ – veil consisting of these balls, i.e. we can deform $g_{\text{Eucl.}}$ into g_n in $\operatorname{Ric}^{<0}(M)$ analogously as in (4.2) above. Subsequently (for $t = 1$) we can work backwards and remove these auxiliary balls and arrive at the same metric as in (4.1).

Of course we should say that the existence of $g(\varepsilon, k)$ is a non-trivial (but doubtless solvable) problem on itself. Therefore it is left to the taste of the reader which is the better or nicer way: In this paper we perform the concentrating argument.

§ 5 Contractibility of $\mathrm{Ric}^{<\alpha}(M)$

$\mathrm{Ric}^{<\alpha}(M)$ resp. $S^{<\alpha}(M)$ are open subsets of $\mathcal{M}(M)$ w.r.t. C^{∞}-topology. Hence they are Fréchet-manifolds. Acc. a version of Whitehead's second homotopy theorem due to Palais (cf. [P], Theorem 15 and Corollary) contractibility of Fréchet-manifolds and the vanishing of all homotopy groups are equivalent conditions.

Therefore we start with any map $f : S^k \to \mathrm{Ric}^{<\alpha}(M)$ and construct an extension, i.e. a map $F : B^{k+1} \to \mathrm{Ric}^{<\alpha}(M)$ with $F \equiv f$ on $\partial B^{k+1} = S^k$.

This problem can be reduced to the case $\alpha = 0$ as follows: Assume $\alpha > 0$ and take any extension $F : B^{k+1} \to \mathcal{M}(M)$ of f. Scale F by a large constant and join for each $x \in S^k$, $f(x)$ and $c^2 \cdot F(x)$ linearly. This yields a map $\tilde{F} : B^{k+1} \cup S^k \times [0,1]/ \sim \to \mathrm{Ric}^{<\alpha}(M)$ if c is chosen large enough ($\sim$ means identification of $\partial B^{k+1} = S^k$ and $S^k \times \{0\}$ which yields again B^{k+1}). Hence we get our desired extension by some reparametrizations. Thus the result is trivial for $\alpha > 0$. Next assume $\alpha < 0$ and presume contractibility of $\mathrm{Ric}^{<0}(M)$: Start with $f : S^k \to \mathrm{Ric}^{<\alpha}(M) \subset \mathrm{Ric}^{<0}(M)$. Thus we can find an extension $F : B^{k+1} \to \mathrm{Ric}^{<0}(M)$. But now we can use the same scaling argument as above (for some <u>small</u> $c > 0$) to get $\tilde{F} : B^{k+1} \to \mathrm{Ric}^{<\alpha}(M)$. Hence it is enough to prove

<u>Proposition (5.1)</u> *Each continuous map* $f : S^k \to \mathrm{Ric}^{<0}(M)$ *admits some continuous extension* $F : B^{k+1} \to \mathrm{Ric}^{<0}(M)$.

Proof: Let $B^{k+1} \equiv B_5(0) \subset \mathbb{R}^{k+1}$, $S^k \equiv \partial B_5(0)$ and $F_1 : B^{k+1} \to \mathcal{M}(M)$ the following trivial extension: Fix some metric $g_0 \in \mathcal{M}(M)$ and define for $z = \tau \cdot x$, $x \in S^k$, $\tau \in [0,1]$:

$$F_1(z) := \begin{cases} (1-\tau) \cdot g_0 + \tau \cdot f(x), & z \in B_1(0) \\ f(x) & \text{on} \quad B_5(0) \setminus B_1(0) \end{cases}$$

Now (4.1) and (2.4) will be used to shift F_1 into $\mathrm{Ric}^{<0}(M)$:

$$F_2(z) := \begin{cases} \frac{1}{m^2} \cdot g\Big(F_1(z), g_n, A\Big(m, F_1\big(\overline{B_2(0)}\big)\Big), I_A, m, R\Big(F_1\big(\overline{B_2(0)}\big)\Big), d, s\Big) \\ \qquad\qquad\qquad\qquad\qquad \text{on} \quad B_2(0) \\ \frac{1}{m^2} \cdot g\Big(f(x), g_n, A\Big(m, F_1\big(\overline{B_2(0)}\big)\Big), I_A, m, (\|z\| - 2) \cdot 2 + (3 - \|z\|) \cdot R\Big(F_1\big(\overline{B_2(0)}\big)\Big), d, s\Big) \\ \qquad\qquad\qquad\qquad\qquad \text{on} \quad B_3(0) \setminus B_2(0) \\ \in \mathrm{Ric}^{<0}(M) \quad \text{on} \quad B_4(0) \setminus B_3(0), \quad \text{see below} \\ f(x) \quad \text{on} \quad B_5(0) \setminus B_4(0) \end{cases}$$

The definition on $B_4(0) \setminus B_3(0)$ is acc. (4.1):

Use it on each ball $B_5(p) \subset M$, $p \in A\Big(m, F_1\big(\overline{B_2(0)}\big)\Big)$ for $f(x) \in \mathrm{Ric}^{<0}(M)$: Notice that $F_2(z) \equiv f(x)$ on $B_5(p) \setminus B_4(p) \subset M$ for $z \in \partial B_3(0) \subset B^{k+1}$ and join $F_2(z)$ with $f(x)$ acc. (4.1). Finally we have $F_2(z) \in \mathrm{Ric}^{<0}(M)$ on $B_5(0) \setminus B_4(0)$ by definition and using (2.4) we can assume d, s and m chosen such that $\mathrm{Ric}(F_2(z)) < 0$ on $B_3(0)$.

□

Again (cf. (2.6)) the corresponding local (one dimensional) version is easily obtained from partial coverings:

Corollary (5.2): *Let g_0 and g_1 be metrics on M with $r(g_i) < \alpha$ and $g_0 \equiv g_1$ outside some closed subset $S, U \supset S$ an open neighborhood, then we can find a continuous path g_t joining g_0 and g_1 with $r(g_t) < \alpha$ and $g_t \equiv g_0 \equiv g_1$ on $M \setminus U$ for each t.*

Some extensions (5.3): **1.**The same arguments work for scalar curvature, i.e. the contractibility of $S^{<\alpha}(M)$ can be derived along exactly the same lines. On the other hand this result was proved in [L3] in a much shorter analytic way. This should emphasize the importance of a (currently missing) strong analytic theory for Ricci curvature.

2. Let $G \subset \mathrm{Diff}(M)$ be a finite subgroup, then the proof above can be performed in a G-invariant way and yields: $\mathrm{Ric}^{<\alpha}(M) \cap S_G(M)$ *is contractible,*

with $S_G(M) := \{g | g \equiv f^*(g), f \in G\} \equiv$ set of G-invariant metrics for $G \neq \{\mathrm{id}\}$

and $S_{\{\mathrm{id}\}}(M) := \{g | g \equiv f^*(g) \Rightarrow f \equiv \mathrm{id}\} \equiv$ set of asymmetric metrics.

3. The above arguments can be adapted to the space $\mathrm{Ric}^{>\alpha}(M) \cap \mathrm{Ric}^{<0}(M)$, i.e. those metrics g whose Ricci curvature is pinched by α and 0 $(\alpha < r(g) < 0)$:

$\mathrm{Ric}^{>\alpha}(M) \cap \mathrm{Ric}^{<0}(M)$ *is contractible* (but not C^0-dense in $\mathcal{M}(M)$).

One can probably show the contractibility of $\mathrm{Ric}^{>\alpha}(M) \cap \mathrm{Ric}^{<\beta}(M)$ for some $\alpha < \beta < 0$ with large quotient $\frac{\alpha}{\beta} > c(n)$, on the other hand for $\frac{\alpha}{\beta} \to 1$ there definitely occur additional problems, since even the existence of such metrics does not hold in general.

§ 6 Curvature h-principles

As indicated in the introduction differential relations which fulfill the h-principle can be understood from solvability problems in algebraic topology. This unusual behaviour of analytic inequalities turns out to occur for $\mathrm{Ric} < \alpha$ and $S < \alpha$.

We start with some general remarks and notations: Let us denote for some smooth fibration $\pi : X \to M$, X^k the space of k-jets of germs of smooth sections of π and $\pi_k : X^k \to M$ the induced fibration.

A holonomic section φ of π_k is a section of π_k which is k-jet of some section f of π.

A subset $\mathcal{R} \subset X^k$ is called differential relation and a section of π is called a solution of $\mathcal{R}$ if its k-jet belongs to $\mathcal{R}$.

Definition (6.1) *Denote by* $\mathrm{Sol}(\mathcal{R})$ *resp.* $C(\mathcal{R})$ *all solutions of* $\mathcal{R}$ *resp. all sections of* π_k *lying in* $\mathcal{R}$ *(= formal solutions) and* $J_k : \mathrm{Sol}(\mathcal{R}) \to C(\mathcal{R})$, $J_k(f) = k$*-jet of* f. $\mathcal{R}$ *fulfills the (parametric)* h*-principle, if* J_k *is a weak homotopy equivalence.*

In particular if $\mathcal{R}$ fulfills the h-principle, then each formal solution can be deformed to some solution (and the path also consists of formal solutions). Sometimes it is of interest to preserve some properties of the prescribed formal solutions:

Definition (6.2) *$\mathcal{R}$ fulfills the C^0-dense h-principle if for any section f of π and any $\varphi_0 \in C(\mathcal{R})$ with $\pi_k(\varphi_0) = f$ we can find a family of sections $\varphi_t \in C(\mathcal{R})$, $t \in [0,1]$ with:*

(i) *φ_1 is holonomic*

(ii) *φ_t lies in $\mathcal{R} \cap \pi_k^{-1}(U)$ for some prescribed neighborhood U of f.*

Obviously restricting conditions on the curvature can be considered as differential relations imposed on the space of 2-jets of germs of metrics.

It is quite easy to analyze the spaces $C(\mathcal{R})$ for $\mathcal{R} \equiv$ 2-jets which (formally) fulfill curvature $< \alpha$ (resp. $> \alpha$) (cf. [Gr2]):

<u>**Lemma (6.3)**</u> *The space $C(\mathcal{R})$ is weakly contractible and non-empty.*

Hence we see from Theorem A, since each map between (weakly) contractible spaces is a weak homotopy equivalence.

<u>**Corollary (6.4)**</u> Ric $< \alpha$ *and* $S < \alpha$ *fulfill the (parametric) h-principle.*

On the other hand there are manifolds which do not admit a metric with negative sectional resp. positive scalar curvature. Hence Sec < 0, Sec > 0, Ric > 0 and $S > 0$ do <u>not</u> fulfill the h-principle acc. (6.3) on these manifolds.

But there is a striking result of Gromov [Gr2] which makes clear that these curvature relations are less restrictive in the non-compact (but non-complete) case:

<u>**Proposition (6.5)**</u> *On any open manifold all the above relations $\mathcal{R}$ fulfill the (parametric) h-principle.*

The philosophical difference between (6.4), which also holds for <u>closed</u> manifolds and (6.5) is the "local" nature of Ric $< \alpha$ resp. $S < \alpha$ which makes possible to compensate the positive curvature arising from deformations (cf. proof of (5.1)) used to obtain $\pi_i(\mathrm{Ric}^{<\alpha}(M)) = 0$. In the open case these unpleasant by-products can be shifted to "infinity", which allows to prove (and believe) (6.5).

Next we will come to a problem which even more makes use of the local nature of Ric $< \alpha$ and $S < \alpha$, namely the C^0-dense h-principle: Sec < 0, Sec > 0, Ric > 0 and $S > 0$ do not fulfill this h-principle: Indeed standard arguments like Bishop's comparison theorem prevent e.g. Ric > 0-metrics to approximate a negatively curved one, this is true for open and closed manifold. On the other hand we get

<u>**Proposition (6.6)**</u> Ric $< \alpha$ *and* $S < \alpha$ *fulfill the C^0-dense h-principle.*

Proof: Let g_0 be any metric on M, U any open neighborhood of g_0 in $\Gamma(\odot^2 T^*M)$ equipped with usual C^0-vector bundle topology. Furthermore let φ_0 be a section of $\mathcal{R}$ (i.e. $\varphi_0 \in C(\mathcal{R})$) with $\pi_2(\varphi_0) = g_0$ and consider its projection $\phi_0 = \pi_{2,1}(\varphi_0)$ into the 1-jet bundle. Acc. Theorem B we can find some $g \in \mathrm{Sol}(\mathcal{R})$ with $g \in U$ and we take $\psi_s := (1-s)\cdot\phi_0 + s \cdot$ 1-jet (g), w.l.o.g. U is convex.

It is easy to find for each (single) 1-jet j_1 of a germ of a metric near some point $p \in M$ a 2-jet j_2 with $\pi_{2,1}(j_2) = j_1$ and $j_2 \in \mathcal{R}$.

This is due to the linearity of Ric and S in the second derivatives of the metric, which also implies that the space of these 2-jets form a convex set.

In particular we can find for each $s \in [0,1]$ and each $p \in M$ a $j_2(s,p) \in \mathcal{R}$ with $\pi_{2,1}(j_2(s,p)) = \psi_s$. Indeed we can assume $j_2(s,p)$ to depend continuously on the parameters since elementary obstruction theory implies the existence of global sections of fiber bundles with contractible (namely convex) fibers. Finally we take $\Psi_t = (1-t)\cdot\varphi_0 + t\cdot j_2(0,p)$ and $\overline{\Psi}_t = (1-t)\cdot j_2(1,p) + t \cdot$ 2-jet (g) which also belong to $C(\mathcal{R})$ due to the linearity in the second derivatives. Combine the paths to get the claim.

□

Remark (6.7). We have omitted the so-called h-principle for extensions which can be checked in the same way. We refer to [Gr1], Chap. 1 for these notions and leave their verification to the reader.

Part II Concentrating Negative Curvature

In the next two chapters we will present a technique to obtain those nests used as start metrics for producing negative curvature in part I. This cannot work in dimension $n = 2$ acc. Gauß-Bonnet.

§ 7 Bending of curves in $\mathrm{Ric} < 0$

Effective deformations, as used in our context cannot be obtained by general analytic arguments. Therefore we will start with deformations leading stepwise to additional symmetrics. These structures can be used to perform concrete (inductive) constructions.

We will often come across the following situation: we are given an open set U, a small open neighborhood V of ∂U and a continuous family g_t, $t \in [0,1]$ of metrics defined on U which fulfill the following conditions:

(i) $g_t \equiv g_0$ on V for all $t \in [0,1]$

(ii) $\mathrm{Ric}(g_t) \leq 0$ (resp. < 0) on U for all $t \in [0,1]$.

In this situation we will just say:

g_t is a $\mathrm{Ric} \leq 0$ (resp. < 0)-family defined on U (and fixed on V).

Notice that a $\mathrm{Ric} \leq 0$-family g_t on U (fixed on V) with $\mathrm{Ric}(g_0) < 0$ on U can easily be deformed into a $\mathrm{Ric} < 0$-family $\overline{g}_t$ on U fixed on V and $\overline{g}_0 \equiv g_0$ on cf. [L2] § 2. This will be used here and in § 8 without further comments.

Now let us start from a circle $\gamma \in \mathrm{H}^n$, $n \geq 3$. We will bend a tube around γ making it geodesic preserving $\mathrm{Ric} < 0$:

Proposition (7.1) *Let $B_{3r}(0) \subset \mathrm{H}^n$, $n \geq 3$, $\gamma \subset \partial B_r(0), (r > 0)$ a plane closed circle, then we can find a $\mathrm{Ric} < 0$-family g_t, on $B_{3r}(0)$ fixed on $B_{3r}(0) \setminus B_{2r}(0)$ with*

(i) $g_0 \equiv g_{\mathrm{hyp.}}$ *on* $B_{3r}(0)$,

(ii) $g_1 = L^2 \cdot \cosh^2 r \cdot g_{S^1} + g_{\mathbf{R}} + \sinh^2 r \cdot g_{S^{n-2}}$ *for some* $L > 0$ *on a small tube* $U = S^1 \times [0, R_0[\times S^{n-2}/\sim$ *around* γ *(in particular* γ *is geodesic).*

The presented proof relies on a (relative) stabilization process for curves which already are geodesic. The geodesic curvature of γ above will be handled as perturbation which can be overcome if we use the process in a suitable way.

We start again from Gao's deformation already used in § 3 to deform $g_{\mathrm{hyp.}}$ restricted to a tube $V \subset B_{2r}(0) \subset \mathrm{H}^n$, $V \equiv S^1 \times B_{3\rho}(0) \subset S^1 \times \mathbf{R}^{n-1}$ of $\gamma(\equiv S^1 \times \{0\})$, $\rho \in]0, \frac{r}{4}[$ as follows:

Lemma (7.2) *For each $\varepsilon > 0$ there is a $\mathrm{Ric} < 0$-family g_t on V with $g_t \equiv g_{\mathrm{hyp.}}$ on $S^1 \times B_{3\rho}(0)$ fixed on $S^1 \times B_{3\rho}(0) \setminus B_{2\rho}(0)$ and $\|g_t - g_0\|_{C^0_{g_{\mathrm{hyp.}}}} < \varepsilon$ such that $g_1 \equiv$*

$\left(c_0 + c_1 \cdot x_1 + c_2 \cdot \sum_{i=1}^{n-1} x_i^2\right)^2 \cdot g_{S^1} + g_{\mathbf{R}} + r^2 \cdot g_{S^{n-2}}$ *on* $S^1 \times B_{3R}(0)$ *for some* $R \in]0, \frac{\varrho}{4}[$ *and some suitable constants* $c_0, c_2 > 0,\ c_1 \in \mathbf{R}$.

(7.2) clarifies our problem: Namely the presence of $c_1 \cdot x_1$ which corresponds to the (unchanged) geodesic curvature of γ. The proof of (7.2) is again easily obtained from [G].

Our actual bending starts with the base metric $g_{\text{Eucl.}}$. The following result will be useful:

Lemma (7.3): *Let* $f = c_0 + c_1 \cdot x_1 + c_2 \cdot \sum_{i=1}^{n-1} x_i^2$ *defined on* $B_{3R}(0) \subset \mathbf{R}^{n-1}$ *(i.e. Hess* $(g_{\text{Eucl.}})f = 2c_2 \cdot g_{\text{Eucl.}}$*), then there is an* $\varepsilon > 0$ *such that for each metric* g *on* $B_{3R}(0)$ *with* $\|g - g_{\text{Eucl.}}\|_{C^1_{g_{\text{Eucl.}}}} < \varepsilon$

$$\text{Hess}(g)f(\nu,\nu) \geq c_2 \cdot g_{\text{Eucl.}}(\nu,\nu).$$

Moreover there exist constants $k(n),\ \varepsilon(n) > 0$ *depending only on the dimension* n *such that if* $\varepsilon \in [0, \varepsilon(n)[$:

$$|\text{Hess}(g)(c_1 \cdot x_1)(\nu,\nu)| \leq |c_1| \cdot k(n) \cdot g_{\text{Eucl.}}(\nu,\nu) \cdot \varepsilon.$$

Proof: We can consider each monomial separately (since Hess is a linear operator)

1. $\text{Hess}(c_0) \equiv 0$ in any case, thus look at $c_1 \cdot x_1$: $\text{Hess}(g)(c_1 \cdot x_1) = (c_1 \cdot x_1 \circ \gamma_\nu)''(0) = c_1 \cdot (\pi_1(\gamma_\nu))''(0) =$ (where π_1 denotes the first projection $\mathbf{R}^n \to \mathbf{R}$, γ_ν is the geodesic w.r.t. g with $\dot{\gamma_\nu}(0) = \nu$) $= c_1 \cdot g(\nabla_\nu \nabla \pi_1)$, since γ_ν is geodesic. Now a simple calculation (cf. [L2], § 2) yields (for some constant $k_n > 0$) $|g(\nabla_\nu \nabla \pi_1, \nu)| \leq k_n \cdot \sum_{i,j,k} |\Gamma_{i,j}^k| \cdot g_{\text{Eucl.}}(\nu,\nu)$.
   For $\varepsilon(n)$ small enough we can conclude from $\|g - g_{\text{Eucl.}}\|_{C^1_{g_{\text{Eucl.}}}} < \varepsilon$ with $\varepsilon \in [0, \varepsilon(n)[$: $|\Gamma_{ij}^k| < \overline{k}_n \cdot \varepsilon$, hence define $k(n) := k_n \cdot \overline{k}_n$ and obtain $|\text{Hess}(g)(c_1 \cdot x_1)(\nu,\nu)| \leq |c_1| \cdot k(n) \cdot g_{\text{Eucl.}}(\nu,\nu) \cdot \varepsilon$.

2. Now look at $c_2 \cdot x_i^2$:
$$\begin{aligned}\text{Hess}(g)(c_2 \cdot x_i^2) &= c_2 \cdot \left((\pi_i \circ \gamma_\nu)^2\right)''(0) = \\ &= 2c_2 \cdot \left((\pi_i \circ \gamma_\nu)'(0)\right)^2 + 2c_2 \cdot (\pi_i \circ \gamma_\nu)(0) \cdot (\pi_i \circ \gamma_\nu)''(0) \geq \\ &\geq 2c_2 \cdot g(\nabla \pi_i, \nu)^2 - 2c_2 |\pi_i \circ \gamma_\nu(0)| \cdot |k(n) \cdot g_{\text{Eucl.}}(\nu,\nu) \cdot \varepsilon| \geq \\ &\geq 2c_2 \cdot |d\pi_i(\nu)|^2 - c_2 \cdot k_1(n) \cdot \varepsilon \cdot g(\nu,\nu), \quad \text{for some } k_1(n) > 0,\ (R \text{ fixed}).\end{aligned}$$
3. Now add the monomials to get for some constant $k_2(n) > 0$ $\text{Hess}(g)f(\nu,\nu) \geq 2c_2 \cdot g_{\text{Eucl.}}(\nu,\nu) - (|c_1| + |c_2|)k_2(n) \cdot g_{\text{Eucl.}}(\nu,\nu) \cdot \varepsilon$.

□

We examine a special deformation of the base metric $g_{\text{Eucl.}}$:

Lemma (7.4) *For each* $\varepsilon \in]0, \frac{R}{2}[$ *there exists a* $h = h_\varepsilon \in C^\infty(\mathbf{R}, [0,1])$ *with* $h \equiv 0$ *on* $\mathbf{R}^{\leq 0} \cup \mathbf{R}^{\geq 2R}$, $h \equiv 1$ *on* $[\frac{\varepsilon}{2}, R]$ *such that for each* $s \in [0,1]$:

(i) $\left\| g_{\mathbf{R}} + (r + s \cdot h \cdot r^2)^2 \cdot g_{S^{n-2}} - g_{\text{Eucl.}} \right\|_{C^1_{g_{\text{Eucl.}}}(B_{3R}(0))} \leq k \cdot s$ *for some constant* $k > 0$ *independent of* s *and* ε.

(ii) $(h \cdot r^2)'' \geq 0,\ 0 \leq (h \cdot r^2)' \leq 4r$ *on* $[0, R]$

(*h is only used to prevent the metric from getting singular in 0, the interesting part is* $s \cdot r^2$)

Proof: Define some function $F \in C^\infty(\mathbf{R}^{>0}, \mathbf{R}^{\geq 0})$ with the following properties: $F(r) = 2r$ on $\mathbf{R}^{\geq \frac{\varepsilon}{2}}$, $F \equiv 0$ near 0, $F' \geq 0$, $F(r) \leq 4r$, $\int_0^{\frac{\varepsilon}{2}} F(r)dr = \left(\frac{\varepsilon}{2}\right)^2$ and choose $\overline{h}(r) = \frac{1}{r^2}\left(\left(\frac{\varepsilon}{2}\right)^2 - \int_r^{\frac{\varepsilon}{2}} F(r)dr\right)$ and some fixed $\overline{h}_1 \in C^\infty(\mathbf{R}, [0,1])$ with $\overline{h}_1 \equiv 1$ on $\mathbf{R}^{\leq R}$, $\overline{h}_1 \equiv 0$ on $\mathbf{R}^{\geq 2R}$ and define $h := \overline{h} \cdot \overline{h}_1$, h fulfills (ii) by definition, thus we are left to prove (i):

$$\left\| g_{\mathbf{R}} + \left(r + s \cdot h \cdot r^2\right)^2 \cdot g_{S^{n-2}} - g_{\text{Eucl.}} \right\|_{C^1_{g_{\text{Eucl.}}}(B_{3R}(0))} =$$

$$\left\| (2shr^3 + s^2h^2r^4) g_{S^{n-2}} \right\|_{C^1_{g_{\text{Eucl.}}}(B_{3R}(0))} \leq$$

$$\sup |2shr| + \sup |s^2 \cdot h^2 r^2| + \sup|2sh| + \sup|2s^2h^2 \cdot r| + \sup |2sh'r| + \sup|2s^2h'r^2|$$

(sup = supremum on $[0, 3R]$) $\leq k \cdot s$, for some $k > 0$, since $|h| \leq 1$, $|h'r| \leq 4R$ (we can assume $R < 1$) and $\overline{h}_1$ is fixed.

□

Later on we will use some scaling by M^2 and perturbation of c_0 therefore we combine (7.3) and (7.4) as follows (note $r = \left(x_1^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$):

<u>**Corollary (7.5):**</u> $g^\beta_{s,\varepsilon,M} := \left(\beta \cdot c_0 + \frac{c_1}{M} \cdot x_1 + \frac{c_2}{M} \cdot \sum_{i=1}^{n-1} x_i^2\right)^2 \cdot g_{S^1} + g_{\mathbf{R}} + \left(r + s \cdot h_\varepsilon \cdot r^2\right)^2 \cdot g_{S^{n-2}}$ *defines for each* $\varepsilon > 0$, $M \geq 1$, $\beta \in]\frac{1}{2}, 2[$ *a continuous family of metrics (in* s *) with*

(i) $g^\beta_{s,\varepsilon,M} \equiv g_1$ *of (7.2) for* $s = 0$ *or* $s \in [0,1]$ *and* $r > 2R$

(ii) $r\left(g^\beta_{s,\varepsilon,M}\right) < -\frac{\overline{c}}{M^2}$ *on* $S^1 \times B_{3R}(0)$ *for* $s \in [0, s_0]$ *and each* $\varepsilon \in]0, \frac{R}{2}[$ *for some suitable small* $s_0 = s_0(M, c_0, c_1, c_2) > 0$ *and some* $\overline{c} = \overline{c}(c_0, c_1, c_2) > 0$ *with* $r(g_{0,\varepsilon,1}) < -6\overline{c}$ *(* s_0, $\overline{c}$ *independent of* ε *)*

Proof: (*) $r\left(g_{\mathbf{R}} + \left(r + sh_\varepsilon r^2\right)^2 \cdot g_{S^{n-2}}\right) \leq 0$ on $[0, R] \times S^{n-2}$, since $\left(r + sh_\varepsilon r^2\right)' \geq 1$, $\left(r + sh_\varepsilon r^2\right)'' \geq 0$ (cf. [B] (9.106)).

Now let us abbreviate $g_{\mathbf{R}} + \left(r + s \cdot h_\varepsilon \cdot r^2\right)^2 \cdot g_{S^{n-2}}$ by $G(s, \varepsilon)$ and $\beta \cdot c_0 + \frac{c_1}{M} \cdot x_1 + \frac{c_2}{M^2} \cdot \sum_{i=1}^{n-1} x_i^2$ by $P_\beta\left(c_0, \frac{c_1}{M}, \frac{c_2}{M^2}\right)$ or often just by P.

(**) $$\|G(s,\varepsilon) - g_{\text{Eucl.}}\|_{C^3_{g_{\text{Eucl.}}}([\frac{R}{2}, 3R] \times S^{n-2})} \underset{s \to 0}{\longrightarrow} 0$$

as is easily seen from the definition. (*), (**) and (7.4) imply (7.5) as follows:

From (7.3) we get on $B_{3R}(0)$:

(***) $$\text{Hess}(G(s,\varepsilon))(P)(\nu,\nu) \geq \frac{c_2}{M^2} \cdot g_{\text{Eucl.}}(\nu,\nu) = \frac{1}{2}\,\text{Hess}(g_{\text{Eucl.}})(P)$$

for each $s \in [0, s_\alpha]$, $\varepsilon \in]0, \frac{R}{2}[$ and suitable $s_\alpha = s_\alpha\left(\frac{c_1}{M}, \frac{c_2}{M^2}\right) > 0$.

Thus the submersion formulas (cf. [B] 9.106) for the Riemannian submersion π : $\left(S^1 \times B_{3R}(0), P^2 \cdot g_{S^1} + G(s,\varepsilon)\right) \rightarrow (B_{3R}(0), G(s,\varepsilon))$ imply for vertical resp. horizontal $\nu \neq 0$ $\left(\nu' := d\pi(\nu)/\|d\pi(\nu)\|\right)$:

(i) $r\left(g^\beta_{s,\varepsilon,M}\right)(\nu) = -\frac{\Delta(G(s,\varepsilon))(P)}{P} \leq -\frac{1}{2}\frac{\Delta(g_{\text{Eucl.}})(P)}{P} \leq -\frac{\overline{c}}{M^2}$

$$
\text{(ii)}\quad
\begin{aligned}
r\left(g^{\beta}_{s,\varepsilon,M}\right)(\nu) &= r(G(s,\varepsilon))(d\pi(\nu)) - \frac{\operatorname{Hess}(G(s,\varepsilon))(P)}{P}(\nu',\nu') \\
&\leq r(G(s,\varepsilon))(d\pi(\nu)) - \frac{1}{2}\frac{\operatorname{Hess}(g_{\text{Eucl.}})(P)}{P}(\nu',\nu') \\
&\leq r(G(s,\varepsilon))(d\pi(\nu)) - \frac{3}{2}\frac{\overline{c}}{M^2} \leq -\frac{\overline{c}}{M^2}
\end{aligned}
$$

(i) holds for $s \in [0, s_\alpha]$, (ii) holds for $s \in [0, s_0]$ for some small $s_0 \in]0, s_\alpha]$ such that $|r(G(s,\varepsilon))(d\pi(\nu))| < \frac{1}{2}\frac{\overline{c}}{M^2}$ acc. (**) if $d\pi(\nu) \in T\left(B_{3R}(0) \setminus B_{\frac{R}{2}}(0)\right)$. For this s_0 the same conclusion holds on $B_R(0)$ since $r(G(s,\varepsilon))\,(d\pi(\nu)) \leq 0$ acc. (*) and Hess fulfills (***).

□

Thus we can start from some metric $P^2 \cdot g_{S^1} + G(s,\varepsilon)$ and we will use the new base metric $G(s,\varepsilon)$ to perform deformations of the S^1-metric: we define for $c_1 = 0$ and $\delta \in]0, \frac{r_0}{2}]$ for some $r_0 \in]0, \frac{R}{4}[$ (and ln = logarithm):

$$
\widetilde{F}^M_{m,\delta}(r) := \begin{cases} c_0 + \frac{c_2}{M^2}\cdot r^2 & \text{on } \mathbf{R}^{>r_0} \\ c - \frac{r\cdot \ln r}{m} & \text{on } [\delta, r_0] \\ d + \frac{c_2}{M^2}\cdot r^2 & \text{on } [0,\delta] \end{cases}
$$

c and (afterwards) d are chosen (uniquely) such that $\widetilde{F}^M_{m,\delta}$ gets continuous. Furthermore there are some $m_0(c_0, \frac{c_2}{M^2}) > 0$ and $\delta(m) > 0$ such that for $m \geq m_0$ and $\delta \in]0, \delta(m)]$: $d, c > 0$, $(*)\ 2\frac{c_2}{M^2}\cdot\delta < -\frac{\ln\delta + 1}{m}$, $-\frac{\ln r_0 + 1}{m} < 2c_2 \cdot r_0$ and $\widetilde{F}^M_{m,\delta} \geq d$.

Now it is completely elementary to find smooth functions $F^M_{m,\delta,\mu}$, $\mu \in]0, \frac{\delta}{2}[$ such that:

(i) $F^M_{m,\delta,\mu} \equiv \widetilde{F}^M_{m,\delta}$ outside of $]\delta - \mu, \delta + \mu[\cup]r_0 - \mu, r_0 + \mu[$,

(ii) $\widetilde{F}^M_{m,\delta} + \mu \geq F^M_{m,\delta,\mu} \geq \widetilde{F}^M_{m,\delta}$ on $\mathbf{R}^{>0}$, $F''^M_{m,\delta,\mu} \geq \widetilde{F}''^M_{m,\delta}$ on $\mathbf{R} \setminus \{\delta, r_0\}$

(iii) If r_0 is small (which can always be assumed) and m_0 is large enough (both depending on μ) then: $c_0 - \mu \leq F^M_{m,\delta,\mu}$; $c - \frac{r\cdot\ln r}{m}$; c; $d \leq c_0 + \mu$.

__Lemma (7.6):__ *Additionally we can choose $F_{m,\delta,\mu}$ such that:*

$$
\operatorname{Hess}(G(s,\varepsilon))\left(F^M_{m,s,\mu}\right)(\nu,\nu) \geq \begin{cases} \operatorname{Hess}(G(s,\varepsilon))\left(c - \frac{r\cdot\ln r}{m}\right)(\nu,\nu) & \text{on }]r_0 - \mu, r_0 + \mu[\times S^{n-2} \\ \operatorname{Hess}(G(s,\varepsilon))\left(d + \frac{c_2}{M^2}\cdot r^2\right)(\nu,\nu) & \text{on }]\delta - \mu, \delta + \mu[\times S^{n-2} \end{cases}
$$

__Proof:__ Let f be a smooth radially symmetric function on $\mathbf{R}^n$, $r(x) = ||x||$ the Euclidean distance from 0, then we get for ν with $||\nu||_{G(s,\varepsilon)} = 1$: for the radial function F on $\mathbf{R}$.

$$
\begin{aligned}
\operatorname{Hess}(G(s,\varepsilon))(f)(\nu,\nu) &= \operatorname{Hess}(G(s,\varepsilon))(F\circ r)(\nu,\nu) \\
&= F''(r(x))\cdot\left((r\circ\gamma_\nu)'(0)\right)^2 + F'(r(x))\cdot(r\circ\gamma_\nu)''(0),
\end{aligned}
$$

where γ_ν denotes the geodesic w.r.t. $G(s,\varepsilon)$ with $\dot\gamma_\nu(0) = \nu$. From (7.4) (ii) we immediately get $(r\circ\gamma_\nu)''(0) \geq 0$. On the other hand we get from (*): for each $\mu \in]0, \frac{\delta}{2}[$ we can find a smoothing with $F''^M_{m,\delta,\mu} \geq \widetilde{F}''^M_{m,\delta}$ on $\mathbf{R} \setminus \{\delta, r_0\}$ and

$$
F'^M_{m,\delta,\mu} \geq \begin{cases} \left(c - \frac{r\cdot\ln r}{m}\right)' & \text{on }]r_0 - \mu, r_0 + \mu[\\ \left(d + \frac{c_2}{M^2}\cdot r^2\right)' & \text{on }]\delta - \mu, \delta + \mu[\end{cases}
$$

which implies (7.6).

□

Lemma (7.7): *On* $B_{r_0}(0) \setminus B_{\frac{s}{2}}(0)$, *for* $s \in]0,1[$ *and* $m = \frac{4}{c \cdot s}$ *we have:*

$$r\left(\left(c - \frac{r \cdot \ln r}{m}\right)^2 \cdot g_{S^1} + G(s,\varepsilon)\right)(\nu) \leq -\frac{s}{2 \cdot r}$$

Proof: The Ricci curvature will be estimated for the three eigenspaces corresponding to $\mathbb{R}$: (1), S^1 : (2) resp. S^{n-2} : (3). Abbreviate: $F \equiv c - \frac{r \cdot \ln r}{m}$, $G = r + s \cdot r^2$:

(1) $-\frac{F''}{F} - (n-2) \cdot \frac{G''}{G} \leq -\frac{F''}{F} - \frac{G''}{G}$, since $G'' \geq 0$

(2) $-\frac{F''}{F} - (n-2) \cdot \frac{G'}{G} \cdot \frac{F'}{F} \leq -\frac{F''}{F} - \frac{G'}{G} \cdot \frac{F'}{F}$, since $F',\ G' \geq 0$

(3) $-\frac{G''}{G} + (n-3) \cdot \frac{(1-(G')^2)}{G^2} - \frac{G'}{G} \cdot \frac{F'}{F} \leq -\frac{G''}{G}$, since $G' \geq 1,\ F' \geq 0$

Thus using $c_0 - \mu \leq F \leq c_0 + \mu$, $F' = -\frac{c \cdot s}{4}(\ln r + 1)$, $F'' = -\frac{c \cdot s}{4r}$, $G' = 1 + 2s \cdot r$, $G'' = 2 \cdot s$ we get:

(1) $\leq \frac{\frac{c \cdot s}{4 \cdot r}}{c_0 - \mu} - \frac{2s}{r + sr^2} \leq \frac{s}{4r} - \frac{2s}{2r} \leq -\frac{s}{2r}$

(2) $\leq \frac{\frac{c \cdot s}{4 \cdot r}}{c_0 - \mu} - \frac{-\frac{cs}{4}(\ln r + 1)}{c_0 + \mu} \cdot \frac{(1 + 2sr)}{r + sr^2} \leq \left(\frac{s}{4r} + \frac{s}{8r}(\ln r + 1)\right) \leq -\frac{s}{2r}$

(3) $\leq \frac{-2s}{r + sr^2} \leq -\frac{s}{2r}$

□

Next scale $\left(S^1 \times B_R(0), \left(c_0 + c_1 \cdot x_1 + c_2 \cdot \sum_{i=1}^{n-1} x_i^2\right) \cdot g_{S^1} + g_{\mathbb{R}} + r^2 \cdot g_{S^{n-2}}\right)$ by some constant $M > 0$ and consider $S^1 \times B_R(0)$, R as well as the Euclidean coordinates w.r.t. the new metric, i.e. $S^1 \times B_R(0)$ equipped with $M^2 \cdot \left(c_0 + \frac{c_1}{M} \cdot x_1 + \frac{c_2}{M^2} \cdot \sum_{i=1}^{n-1} x_i^2\right)^2 \cdot g_{S^1} + g_{\text{Eucl.}}$. Now we take the geodesic curvature of γ corresponding to $c_1 \cdot x_1$ into account:

Lemma (7.8) *There are some large* M *and small* $s_0 > 0$ *such that for each* $s \in]0, s_0]$, $\delta \in]0, \delta(\frac{4}{c \cdot s})[$, $\mu \in]0, \frac{\delta}{4}[$, $t \in [0,1]$:

$$r\left(\left(\frac{c_1}{M} \cdot x_1 + F(s,t)\right)^2 \cdot g_{S^1} + G(s,\varepsilon)\right) \leq 0$$

with $F(s,t) := t \cdot F^M_{\frac{4}{c \cdot s},\delta,\mu} + (1-t) \cdot \left(c_0 + \frac{c_2}{M^2} \cdot r^2\right)$.

Proof: We use again the formulas for Riemannian submersions $\pi : S^1 \times B_R(0) \to B_R(0)$ in [B]. 9.106 to calculate this Ricci curvature for vertical resp. horizontal $\nu \neq 0$ with $\|\nu\| = 1$ resp. $\|d\pi(\nu)\| = 1$ and get:

(1) $\frac{-\Delta(G(s,\varepsilon))\left(\frac{c_1}{M} \cdot x_1 + F(s,t)\right)}{\frac{c_1}{M} \cdot x_1 + F(s,t)}$, for vertical ν

(2) $r(G(s,\varepsilon))(d\pi(\nu)) - \frac{\text{Hess}(G(s,\varepsilon))\left(\frac{c_1}{M} \cdot x_1 + F(s,t)\right)(d\pi(\nu), d\pi(\nu))}{\frac{c_1}{M} \cdot x_1 + F(s,t)}$, for horizontal ν.

Now we can substitute F in the numerators by $t\cdot\left(c - \frac{r\cdot \ln r}{\frac{1}{c\cdot s}}\right) + (1-t)\cdot\left(c_0 + \frac{c_2}{M^2}\cdot r^2\right)$ and get this way (acc. (7.6)) the following upper estimates of (1) and (2):

(1) $\leq 2\cdot t\cdot \frac{\mathrm{trace}|\mathrm{Hess}(\frac{c_1}{M}\cdot x_1)|}{F(s,t)} - \frac{n-1}{2}\cdot t\cdot s - (1-t)\frac{\overline{c}}{2\cdot M^2}$ for $M \geq M_0(c_1, R)$, using (7.5) and (7.7)

$$\leq t\cdot\left(2n\cdot\frac{|c_1|\cdot k(n)\cdot k\cdot s}{(c_0-\mu)M} - \frac{n-1}{2}\cdot s\right) \leq 0 \text{ for } M \geq M_1 \text{ independent of } s$$

(2) $$\leq t\cdot\left(2\frac{|\mathrm{Hess}(\frac{c_1}{M}\cdot x_1)(d\pi(\nu), d\pi(\nu))|}{F(s,t)} - \frac{s}{4\cdot r}\right) + (1-t)\cdot r(G(s,\varepsilon))(d\pi(\nu))$$
$$-(1-t)\cdot\frac{\mathrm{Hess}\left(P\left(c_0,\frac{c_1}{M},\frac{c_2}{M^2}\right)\right)(d\pi(\nu),(d\pi(\nu)))}{\frac{c_1}{M}\cdot x_1 + F(s,t)}$$
$$\leq t\cdot\left(\frac{2|c_1|\cdot k(n)\cdot k\cdot s}{(c_0-\mu)\cdot M} - \frac{s}{4r}\right) - (1-t)\cdot\frac{\overline{c}}{M^2} \leq 0,$$

using (7.3), (7.7) and (7.5) (where F can be considered as $P(c_0, 0, \frac{c_2}{M^2})$ with perturbated c_0 (for some β)) for $M \geq M_2$, $M_2 \geq M_1$ independent of s.

□

Summarizing we obtain the

Proof of (7.1): For some large M and if $s > 0$, μ and $\delta > 0$ are small enough, then $F(s,1) + \frac{c_1\cdot x_1}{M}$ does have a minimum $z_0 \in B_R(0)$, i.e. $S^1 \times \{z_0\}$ is geodesic and we can find a S^1-equivariant isotopy $i_t : S^1 \times B_R(0) \to S^1 \times B_R(0)$ with $i_t \equiv \mathrm{id}$ near $S^1 \times \partial B_R(0)$, $i_0 \equiv \mathrm{id}$ and $i_1(S^1 \times \{0\}) = S^1 \times \{z_0\}$ such that the pull back metrics via i_t yield the final deformation necessary to make γ geodesic (namely for i_1).

To get the desired formal structure on a tube U around γ, again we use Gao's deformation which gives a deformation to $U = \left(S^1 \times [0, R_0[\times S^{n-2},\ L^2\cdot\cos h^2 r\cdot g_{S^1} + g_{\mathbb{R}} + \sin h^2 r\cdot g_{S^{n-2}}\right)$ for some small $R_0 > 0$ which completes the proof.

□

§ 8 Expanding tubes

In § 7 we have bent our metric such that γ became geodesic. This additional symmetry makes our problem much simpler and accessible for some induction scheme. The latter one uses the following remark:

$\mathbb{R}^n$, $n \geq 3$ contains a closed embedded hypersurface $N^{n-1} \subset \mathbb{R}^n$ which admits a (non induced) metric g_N with $\mathrm{Ric}(g_N) \leq 0$ and there exists a $p \in N$ such that $(B_2(p), g_N)$ is isometric to $(B_2(0), g_{\mathrm{Eucl.}}) \subset \mathbb{R}^{n-1}$:

Namely for $n = 3$: take a large flat torus T^2 and for $n \geq 4$ use Theorem E of [L2] which says that each manifold of dimension ≥ 3 admits such a metric. Thus take e.g. $S^{n-1} \subset \mathbb{R}^n$. The normal bundle ν of $N \subset \mathbb{R}^n$ is always trivial and we take the following metric on $\nu \equiv \mathbb{R} \times N$ (which has $\mathrm{Ric} < 0$): $g_{\mathbb{R}} + \cos h^2 r\cdot g_N$. Now identify $]-5, 5[\times N \subset \nu$ with some tube U around $N \subset \mathbb{R}^m$ and assume $N \subset \overline{U} \subset B_R(0) \subset \mathbb{H}^n$. Using induction

(Theorem C in dimension n) we can deform the hyperbolic metric on $B_{3R}(0)$ continuously (letting it fixed outside $B_R(0)$) such that all metric have $\mathrm{Ric} < 0$ and the final metric g_f fulfills: $g_f \equiv g_{\mathbf{R}} + \cos h^2 r \cdot g_N$ on $U \equiv]-5,5[\times N$. The existence of g_f comes from Theorem E in [L2]. In dimension $n+1$ we will use this, start from (7.1) and can easily deduce:

<u>**Lemma (8.1)**</u> *There is a* $\mathrm{Ric} \le 0$*-family* g_t *on* $S^1 \times B_R(0) \subset S^1 \times \mathbf{R}^n$, $n \ge 3$ *fixed near* $S^1 \times \partial B_R(0)$ *and*

(i) $g_0 \equiv \cos h^2 r \cdot g_{S^1} + g_{\mathrm{hyp.}}$

(ii) $g_1 = f^2 \cdot g_{S^1} + g_{\mathbf{R}} + \cos h^2 r \cdot g_N$ *on* $S^1 \times]-4,4[\times N$ *for some* $f \in C^\infty(\mathbf{R}, \mathbf{R}^{>0})$ *with* $f(r) = f(-r)$, $f'' > 0$.

These preparations make possible to handle with all dimensions ≥ 3 by the same method, cf. (8.3) and we will get:

<u>**Lemma (8.2)**</u> *For large* $m > 1$ *we can find a* $\mathrm{Ric} \le 0$*-family* g_t *on* $S^1 \times [-3,3] \times N^{n-1}$, N *as above for* $n+1 \ge 4$, *resp. on* $S^1 \times [0,3] \times S^1$ *for* $N = S^1$ *and with radial identification of* $S^1 \times \{0\}$ *in dimension 3, with:*

(i) $g_0 = f^2 \cdot g_{S^1} + g_{\mathbf{R}} + \cos h^2 r \cdot g_N$ *on* $S^1 \times]-3,3[\times N$
$g_0 = \cos h^2 r \cdot g_{S^1} + g_{\mathbf{R}} + \sin h^2 r \cdot g_{S^1}$ *on* $S^1 \times]0,3[\times S^1$

(ii) $g_t \equiv g_0$ *near* $S^1 \times \{-3,3\} \times N$ *resp.* $S^1 \times \{3\} \times S^1$

(iii) $g_1 = R^2 \cdot \frac{\cos h^2 mr}{m^2} \cdot g_{S^1} + g_{\mathbf{R}} + \frac{\sin h^2 mr}{m^2} \cdot g_{S^1}$ *on* $S^1 \times]0, \frac{2R}{m}[\times S^1$ *resp.*
$g_1 = R^2 \cdot \frac{\exp 2mr}{m^2} \cdot (g_{S^1} + g_N) + g_{\mathbf{R}}$ *on* $S^1 \times]\frac{R}{m}, \frac{4R}{m}[\times N$.

This immediately implies:

Proof of (3.1) For $\dim = 3$ (8.2) obviously implies (3.1). For $\dim \ge 4$ we recall from [BN] that g_1 (in (iii)) is hyperbolic with curvature $\equiv -m$. Furthermore the domain contains a ball of radius $\ge \frac{R}{m}$, which proves the claim.

□

The proof of (8.2) relies on the following "expanding of tubes", which is the heart of the concentration argument.

<u>**Proposition (8.3)**</u> *Let* $g_0 = \alpha^2 \cdot r^2 \cdot g_{S^1} + g_{\mathbf{R}} + \beta^2 \cdot r^2 \cdot g_N$, $\alpha, \beta > 0$ *be defined on* $S^1 \times]0, r[\times N$, *then we can find for each pair* $\gamma_1 \ge \alpha$, $\gamma_2 \ge \beta$ *a* $\mathrm{Ric} \le 0$*-family* g_t *fixed near* $S^1 \times \{r\} \times N$ *such that:*

(i) $g_t \equiv \alpha_t^2 \cdot r^2 \cdot g_{S^1} + g_{\mathbf{R}} + \beta_t^2 \cdot r^2 \cdot g_N$ *for some* $\alpha_t \in [\alpha, \gamma_1]$, $\beta_t \in [\beta, \gamma_2]$ *on* $S^1 \times]0, \varepsilon[\times N$

(ii) $g_1 \equiv \gamma_1^2 \cdot r^2 \cdot g_{S^1} + g_{\mathbf{R}} + \gamma_2^2 \cdot r^2 \cdot g_N$ *on* $S^1 \times]0, \varepsilon[\times N$, *for some suitable* $\varepsilon > 0$.

We will use an auxiliary

<u>**Lemma (8.4)**</u> *For each* $k \in \mathbf{Z}^{>0}$ *there is a function* $\varphi \equiv \varphi_k \in C^\infty(\mathbf{R}, \mathbf{R}^{\ge 0})$ *with* $\varphi \equiv 0$ *on* $\mathbf{R} \setminus]5, a[$, $\varphi > 0$ *on* $]5, a[$ *for some* $a > 6$ *with* $|\varphi| \le \frac{1}{k} |\mathrm{id}|$ *and:* $\frac{k \cdot \lambda_\delta \cdot \varphi''}{\mathrm{id} - k \cdot \lambda_\delta \cdot \varphi} < \frac{k \cdot \delta \cdot \varphi''}{\mathrm{id} + \delta \cdot \varphi}$ *in those points where* $\varphi'' \neq 0$

for $\delta \in]0,1[$, $\lambda_\delta := \delta / (\delta + 1)$.

Proof: Let $\rho, \rho_a \in C^\infty(\mathbb{R}^{>0}, \mathbb{R}^{\geq 0})$ ($a > 6$ defined below) with $\rho > 0$ on $]5, 5+\frac{1}{10}[$, $\rho_a > 0$ on $]a-\frac{1}{10}, a[$ and ρ resp. $\rho_a \equiv 0$ elsewhere, such that $\alpha(s) := \int\limits_{-\infty}^{s} \left(\int\limits_{-\infty}^{t_2} \rho(t_1) dt_1 \right) dt_2$ and $\beta(s) := \int\limits_{s}^{+\infty} \left(\int\limits_{t_2}^{+\infty} \rho_a(t_1) dt_1 \right) dt_2$ fulfill $\alpha(5.5) = \frac{1}{k+1} \cdot 5.5$, $\beta(a-\frac{1}{2}) = \frac{1}{k+1} \cdot (a - \frac{1}{2})$.

Then we have: α (resp. β)is linear with $\alpha' > 0$ on $\mathbb{R}^{>5,1}$ (resp. $\beta' < 0$ on $\mathbb{R}^{<a-\frac{1}{10}}$), $\alpha > \frac{1}{k+1}\mathrm{id}_{\mathbb{R}}$ (resp. $< \frac{1}{k+1}\mathrm{id}_{\mathbb{R}}$) on $\mathbb{R}^{>5,5}$ (resp. on $]0, 5.5[$) and $\beta > \frac{1}{k+1}\mathrm{id}_{\mathbb{R}}$ (resp. $< \frac{1}{k+1}\mathrm{id}_{\mathbb{R}}$) on $\mathbb{R}^{<a-\frac{1}{2}}$ (resp. on $\mathbb{R}^{>a-\frac{1}{2}}$). Thus we can find exactly one $p \in]5.5, a-\frac{1}{2}[$ with $\alpha(p) = \beta(p)$ and we can choose a such that $\alpha(p) = \beta(p) = \frac{p}{k}$. Take $h := \min\{\alpha, \beta\}$ and note $h \leq \frac{1}{k} \cdot \mathrm{id}$ and equality precisely holds in p (on $\mathbb{R}^{>0}$). Take the interval I around p where $h > \frac{1}{k+1} \cdot \mathrm{id}$ and define our desired φ as a C^∞-smoothing of h with:

(i) $\varphi \equiv h$ on $\mathbb{R} \setminus I$

(ii) $\frac{1}{k+1} \cdot \mathrm{id} < \varphi < h$ and $\varphi'' < 0$ on I.

Finally we briefly check (8.4): namely the claim is obviously equivalent to $\frac{t+\delta\cdot\varphi(t)}{\delta+1} < t - k \cdot \frac{\delta}{\delta+1} \cdot \varphi(t)$ for $\varphi'' > 0$ (" $>$ " for $\varphi'' < 0$) and is the same as $(k+1) \cdot \varphi < t$ resp. " $>$ " for $\varphi'' > 0$ resp. " $<$ " which is true from our definitions.

□

Proof of (8.3) It is enough to check (8.3) for $\alpha = \beta = \gamma_1 = 1$, $r = a + 5$ and prove the existence of some $\gamma_2 = (1+\eta)$, $\eta > 0$ such that (8.3) works. This is due to the scale-invariant and (essentially) symmetric (w.r.t. S^1 and N) situation: any arbitrarily large γ_2 can be obtained by iteration as $\gamma_2 = (1+\eta)^m$ subsequently one exchanges rôles of γ_1 and γ_2, details are left to the reader.

First of all $\mathrm{Ric}(f^2 \cdot g_{S^1} + g_{\mathbb{R}} + g^2 \cdot g_N) \leq 0$ is equivalent to three inequalities:

(1) $\frac{f''}{f} + (n-2)\frac{g''}{g} \geq 0,$

(2) $\frac{f''}{f} + (n-2)\frac{f'}{f} \cdot \frac{g'}{g} \geq 0$

(3) $\frac{g''}{g} + (n-3)\left(\frac{g'}{g}\right)^2 + \frac{f'}{f} \cdot \frac{g'}{g} \geq 0$

$f = \mathrm{id}$, $g = \mathrm{id}$ do obviously fulfill (1)-(3). We perturbate f and g acc. (8.4):

Consider $f_\delta := f - (n-2) \cdot \lambda_\delta \cdot \varphi$, $g_\delta := g + \delta \cdot \varphi$, $\delta \in [0, \delta_0]$, $\varphi = \varphi_{n-2}$ for some $\delta_0 \in]0, 1[$. If δ_0 is small enough f_δ, g_δ fulfill (1)-(3): (2) and (3) are clear from the fact that for small $\delta > 0$: $\frac{f'_\delta}{f_\delta} \approx \frac{g'_\delta}{g_\delta} \approx \frac{1}{\mathrm{id}}$, $\frac{f''_\delta}{f_\delta} \approx \frac{g''_\delta}{g_\delta} \approx \frac{\varepsilon_\delta}{\mathrm{id}}$ with $|\varepsilon_\delta|$ arbitrarily small for small $\delta > 0$ on $]5, a[$

(1) is obtained from (8.4):

$$\frac{f''_\delta}{f_\delta} + (n-2)\frac{g''_\delta}{g_\delta} = \frac{-(n-2) \cdot \lambda_\delta \cdot \varphi''}{\mathrm{id} - (n-2) \cdot \lambda_\delta \cdot \varphi} + (n-2)\frac{\delta \cdot \varphi''}{\mathrm{id} + \delta \cdot \varphi} \geq 0$$

and $(1)-(3) > 2c > 0$ for $\delta = \delta_0$ on some small interval $]p - \varepsilon, p + \varepsilon[\subset]I[$. This is used in our main deformation:

Define $F_\mu \in C^\infty(\mathbb{R}, \mathbb{R})$ by $F_\mu := \mu \cdot h \cdot \mathrm{id}_{\mathbb{R}} + (1-h) \cdot \mathrm{id}_{\mathbb{R}}$ for some $h \in C^\infty(\mathbb{R}, [0,1])$ with $h \equiv 0$ on $\mathbb{R}^{\geq p+\varepsilon}$, $h \equiv 1$ on $\mathbb{R}^{\leq p-\varepsilon}$. Obviously $F_\mu \to \mathrm{id}$ on $\mathbb{R}^+$ w.r.t. global C^k-norms for $\mu \to 1$. Hence acc. (1)-(3): $\mathrm{Ric}\left(f^2_{\delta_0} \cdot g_{S^1} + g_{\mathbb{R}} + (F_\mu \cdot g_{\delta_0})^2 \cdot g_N\right) \leq 0$ for each

$\mu = 1 + \eta,\ \eta \in [0, \eta_0],\ \eta_0 > 0$ small enough $\mu_0 := 1 + \eta_0$. Thus take

$$g_t := \begin{cases} f^2_{2t\cdot\delta_0} \cdot g_{S^1} + g_{\mathbf{R}} + g^2_{2t\cdot\delta_0} \cdot g_N, & t \in [0, \frac{1}{2}] \\ f^2_{\delta_0} \cdot g_{S^1} + g_{\mathbf{R}} + \left(F_{(2t-1)\mu_0} \cdot g_{\delta_0}\right)^2 \cdot g_N, & t \in [\frac{1}{2}, 1] \end{cases}$$

and notice that

$$g_t = r^2 \cdot g_{S^1} + g_{\mathbf{R}} + \beta_t^2 \cdot r^2 \cdot g_N \qquad \text{on} \qquad S^1\times]0,4[\times N$$

with $\beta_t = 1$ on $[0, \frac{1}{2}]$, $\beta_t = (1 + (2t-1)\cdot\eta_0)$ on $t \in [\frac{1}{2}, 1]$.

□

We can easily deduce (8.2) from this expanding deformation:

Proof of (8.2) To apply the preceding technique we perform some appropriate deformations:
1. We take a $\mathrm{Ric} \le 0$-family G_t fixed near $S^1 \times \{r\} \times N$ defined on $S^1\times]0,r[\times N$ with $G_0 \equiv g_1$, g_1 as in (7.1) and:

$G_t = f_t^2 \cdot g_{S^1} + g_{\mathbf{R}} + g_t^2 \cdot g_N$ on $S^1\times]0,\varepsilon[\times N$ for some $\varepsilon > 0$, $f_t, g_t \in C^\infty(\mathbf{R},\mathbf{R})$, $f_t, g_t > 0$ on $\mathbf{R}^{>0}$, $f_t', g_t' \ge 0$ on $\mathbf{R}^{\ge 0}$ and $g_t'(0) \ge 1$ for $n = 3$, $f_t'', g_t'' \ge 0$ $f_1 \equiv c\cdot\mathrm{id}$, $g_1 \equiv d\cdot\mathrm{id}$ for some $c, d > 0$ near 0.

This family G_t is obtained as follows: in dimension $n = 3$ we can use those "methods to smooth singularities" as in [L4], which already implies the existence of G_1 as above. It is not hard to adjust that construction to give the complete family G_t, details are left to the reader. In dimension $n \ge 4$ we can easily find $f_i \in C^\infty(]c,r[,\mathbf{R}^{>0})$, $i = 1,2$ for some $c < 0$ with $f_i' > 0$, $f_i'' \ge 0$ and

$$f_1 = \begin{cases} f \\ \alpha\cdot(\mathrm{id} - c) \end{cases}, \qquad f_2 = \begin{cases} \cosh & \text{on} \quad [\frac{r}{2}, r[\\ \beta\cdot(\mathrm{id} - c) & \text{on} \quad]c, \frac{r}{4}] \end{cases}$$

In particular we get $\mathrm{Ric}(f_1^2 \cdot g_{S^1} + g_{\mathbf{R}} + f_2^2 \cdot g_N) \le 0$. Thus define

$$G_t := \begin{cases} (2t\cdot f + (1-2t)\cdot f_1)^2 \cdot g_{S^1} + g_{\mathbf{R}} + (2t\cdot\cosh + (1-2t)\cdot f_2)^2 \cdot g_N, & t \in [0, \frac{1}{2}] \\ D_t^*(f_1^2 \cdot g_{S^1} + g_{\mathbf{R}} + f_2^2 \cdot g_N), & t \in [\frac{1}{2}, 1] \end{cases}$$

where $D_t, t \in [\frac{1}{2}, 1]$ is an isotopy of $S^1 \times \mathbf{R} \times N$ with $D_{\frac{1}{2}} \equiv \mathrm{id}$ and

$$D_t(s,\rho,x) = \begin{cases} (s,\rho,x) & \text{for} \quad \rho > \frac{r}{4} \\ (s, \rho + \rho_0, x) & \text{for} \quad \rho < \frac{r}{8} \end{cases}$$

for some $\rho_0 = \rho_0(t) \le 0$ with $\rho_0(\frac{1}{2}) = 0$, $\rho_0(1) = c$.

2. Thus we start from G_1 to perform the desired "concentrating of curvature" by expanding of G_1. We take the $\mathrm{Ric} \le 0$-family of (8.3): g_t on $S^1\times]0,\varepsilon[\times N$ with $g_0 = G_1$ and $g_1 = \gamma_1^2 \cdot r^2 \cdot g_{S^1} + g_{\mathbf{R}} + \gamma_2^2 \cdot r^2 \cdot g_N$ on $S^1\times]0,\delta[\times N$, for some $\delta = \delta(\gamma_1,\gamma_2) \in]0, \frac{\varepsilon}{2}[$ and arbitrarily large γ_1, γ_2.

3. Next we obtain from G_t and g_t a $\mathrm{Ric} \le 0$-family h_t of regular metrics defined on $S^1 \times B_r(0)$ for $n = 3$ resp. on $S^1\times]-5,5[\times N$ for $n \ge 4$ with

$h_t \equiv \cosh^2 r \cdot g_{S^1} + g_{\mathbf{R}} + \sinh^2 r \cdot g_{S^1}$ resp. $f^2 \cdot g_{S^1} + g_{\mathbf{R}} + \cosh^2 r \cdot g_N$ for $t = 0$ and for $t \in [0,1]$ on a neighborhood of $S^1 \times \partial B_r(0)$ resp. $S^1 \times \{-\rho, \rho\} \times N$ such that for some suitable small $\mu > 0$ and prescribed $\gamma_1, \gamma_2 \ge 1$:

$$h_1 = f^2 \cdot g_{S^1} + g_{\mathbf{R}} + g^2 \cdot g_N$$

with

$$f=\begin{cases}\gamma_1\cdot \mathrm{id}\\ \gamma_1\cdot\frac{\mu}{3}\end{cases},\quad g=\begin{cases}\gamma_2\cdot \mathrm{id} & \text{on }]\frac{\mu}{2},N[\\ \gamma_2\cdot\frac{\mu}{3} \text{ resp. } \mathrm{id}+c & \text{on }]0,\frac{\mu}{4}[\text{ resp. }]-c,\frac{\mu}{4}[\end{cases}$$

for $n\geq 4$ resp. $n=3$, $f^{(k)},g^{(k)}\geq 0$, $k=0,1,2$, $c>0$.

This is checked as follows: all the metrics G_t and g_t are of the form $f^2\cdot g_{S^1}+g_{\mathbb{R}}+g^2\cdot g_N$ on $S^1\times]0,\mu[\times N$ for μ small enough.

Thus take $F,G\in C^\infty([0,\mu[,\mathbb{R}^{>0})$ with $F^{(k)},G^{(k)}\geq 0$, $k=0,1,2$ and $F=\begin{cases}f\\ f(\frac{\mu}{3})\end{cases}$, $G=\begin{cases}g & \text{on }]\frac{\mu}{2},\mu[\\ g(\frac{\mu}{3}) \text{ resp. } g(\frac{\mu}{4})+\mathrm{id}-\frac{\mu}{4} & \text{on }]0,\frac{\mu}{4}[\text{ resp. }]\frac{\mu}{4}-g(\frac{\mu}{4}),\frac{\mu}{4}[\end{cases}$
for $n\geq 4$ resp. $n=3$.

It is obvious that F and G can be chosen depending continuously on G_t resp. g_t for varying t and yield our h_t (The domain $]\frac{\mu}{4}-g(\frac{\mu}{4}),\frac{\mu}{4}[$ is changed into $]0,\frac{\mu}{4}[$ by pull-back arguments). Furthermore we can assume for $n\geq 4$, that h_t is symmetric w.r.t. reflections along $S^1\times\{0\}\times N$.

4. Notice again the important fact that the metric of (8.3) is scale-invariant, i.e. after scaling we can assume μ to be arbitrarily large. We take $\mu:=10R$.

5. Thus we are left to deform h_1 (defined on $S^1\times B_\mu(0)$ resp. $S^1\times]-\mu,\mu[\times N$) into our claimed metric:

Consider f as extended on $\mathbb{R}^{<\mu}$ by $f\equiv\gamma_1\cdot\frac{\mu}{3}$ on $\mathbb{R}^{\leq 0}$ and notice that we can find a very large $\gamma_1=\gamma_1(R)$ and some $F\in C^\infty(]-p,\mu[,\mathbb{R}^{>0})$, $p<-3R$ with:

$$F=\begin{cases}\gamma_1\cdot\mathrm{id} & \text{near } \mu\equiv 10\cdot R\\ R\cdot\exp(\mathrm{id}+p) & \text{on }]R,4\cdot R[\\ R\cdot\cosh(\mathrm{id}+p) & \text{on }]p,p+2R[\end{cases}$$

and $F',F''\geq 0$, for some small $\delta>0$. That is also true for g $(\gamma_2=\gamma_1)$ and $n\geq 4$, and yields $G\equiv F$. Combine to get $H_t:=(t\cdot F+(1-t)\cdot f)^2\cdot g_{S^1}+g_{\mathbb{R}}+(t\cdot G+(1-t)\cdot g)^2\cdot g_N$ on $]p,p+2R[$ which "extends by reflection" to $]2p-2R,p[$ to our desired $\mathrm{Ric}\leq 0$-family deforming h_1, into the claimed metric of (8.2) (up to pull-back) in dimension $n\geq 4$.

In dimension 3 we will take $\sinh$ instead of $\cosh$:

Take $G\in C^\infty(]p,p+2R[,\mathbb{R}^{>0})$ with $G=\begin{cases}\gamma_2\cdot\mathrm{id} & \text{near } \mu\\ \sinh(\mathrm{id}+p) & \text{on }]p,p+2R[\end{cases}$

This time we will get metrics on $S^1\times]p,p+2R[\times S^1$ but again pull-back to $S^1\times]0,2R[\times S^1$ yields the desired deformation.

□

Part III. Approximation of flat metrics

Here we will construct metrics $g_{n,\varepsilon}$ on $\mathbb{R}^n$, $n\geq 3$ with $\mathrm{Ric}(g_{n,\varepsilon})<0$ on $B_1(0)$, $g_{n,\varepsilon}\equiv g_{\mathrm{Eucl.}}$ outside and $\|g_{n,\varepsilon}-g_{\mathrm{Eucl.}}\|_{C^0_{g_{\mathrm{Eucl.}}}(\mathbb{R}^1)}<\varepsilon$. In dimension $n\geq 4$ this is a refinement of our inductive construction in [L2]. For $n=3$ we adapt this idea to singular metrics which are smoothed by a "mild" procedure. Thus we begin for expository purpose with $n\geq 4$ (and presume $n=3$).

§ 9 Scale-invariance of C^0

We will use the following simple scale-invariance of C^0-norms (which was also mentioned and used in previous chapters) which is easily checked:

Lemma (9.1) *Let g_1, g_2 be two $(2,0)$-tensors (e.g. metrics) on a manifold M, g_M some fixed reference metric and $\lambda > 0$, then*

$$\|g_1 - g_2\|_{C^0_{g_M}(M)} = \left\|\lambda^2 \cdot g_1 - \lambda^2 \cdot g_2\right\|_{C^0_{\lambda^2 \cdot g_M}(M)}$$

We will assume Theorem D in dimension $n \geq 3$ to prove it as well as Theorem B in dimension $n+1$.

Denote by $g\left(s, d, \tilde{d}\right)$ the following metric on $S^1 \times \overline{S}^1 \times B_5^{n-1} \setminus B_1^{n-1} \subset S^1 \times \overline{S}^1 \times \mathbb{R}^{n-1}$

$$g_L\left(s, d, \tilde{d}\right) := L^2 \cdot \left(F_{s,d}(r-2)+1\right)^2 \cdot g_{S^1} + \exp\left(2F_{s,\tilde{d}}(3-r)\right) \cdot \left(g_{\overline{S}^1} + g_{\text{Eucl.}}\right)$$

for $s, d, \tilde{d} > 0$, $L \geq 1$ and $F_{s,d}(r) := s \cdot \exp(-d/r)$ on $\mathbb{R}^{>0}$, $F_{s,d} \equiv 0$ on $\mathbb{R}^{\leq 0}$, $r = r(t, \bar{t}, x) = \|x\|_{\text{Eucl.}}$, $(t, \bar{t}, x) \in S^1 \times \overline{S}^1 \times \mathbb{R}^{n-1}$.

Acc. [L2], § 6 we can find $d_0, \tilde{d}_0 > 1$ and $s_0 > 0$ such that for $d \geq d_0$, $\tilde{d} \geq \tilde{d}_0$, $s \in]0, s_0[$: $r\left(g_L\left(s, d, \tilde{d}\right)\right) \leq 0$ and "< 0" on $S^1 \times \overline{S}^1 \times B_3^{n-1} \setminus \overline{B}_2^{n-1}$.

Furthermore we obviously have for suitably large $d(s)$, $\tilde{d}(s)$ (for $C^3 \equiv C^3_{g_{S^1} + g_{\overline{S}^1} + g_{\text{Eucl.}}}\left(S^1 \times \overline{S}^1 \times B_5^{n-1}\right.$)

$$\text{(1)} \qquad \left\|g_L\left(s, d(s), \tilde{d}(s)\right) - \left(L^2 \cdot g_{S^1} + g_{\overline{S}^1} + g^{n-1}_{\text{Eucl.}}\right)\right\|_{C^3} \underset{s \to 0}{\to} 0$$

In particular we can compare the metric $g(L)$ on the tube $T_{L,5}$ of radius 5 in $\mathbb{R}^n$ around a plane circle of some large radius $\frac{L}{2\pi}$ with $g_L(s, d(s)) = L^2 \cdot \left(F_{s,d}(r-2)+1\right)^2 \cdot g_{S^1} + g_{\text{Eucl.}}$ (length S^1 w.r.t $g_{S^1} = 1$): For each $\varepsilon > 0$ we can find $s_1 \in]0, s_0[$, $L_0 > 0$ such that

$$\text{(2)} \qquad \|g(L) - g_L(s, d(s))\|_{C^3(S^1 \times B_5^{n-1})} < \varepsilon$$

for $s \in]0, s_1[$, $L > L_0$ (where $g(L)$ is meant to be the pull-back of the Euclidean metric via some Fermi coordinate map $S^1 \times B_5 \to T_{L,5}$). We obtain from Theorem D (for $r(g) \leq 0$) in dimension n and (2):

Corollary (9.2) *For each $\varepsilon > 0$ we can find some $s_0 > 0$ such that for each $s \in]0, s_0[$ there is a circle $\gamma \subset \mathbb{R}^n$ and a metric g_ε on $\mathbb{R}^n$ with: (i) $\text{Ric}(g_\varepsilon) \leq 0$, (ii) $\|g_\varepsilon - g_{\text{Eucl.}}\|_{C^0} < \varepsilon$, (iii) g_ε is isometric to $g_L(s, d(s))$ on $T_{L,5}(\gamma)$ and $g_\varepsilon \equiv g_{\text{Eucl.}}$ outside some ball $B_\rho(0)$.*

Next we use the n-dimensional Theorem D to get a metric with $\text{Ric} \leq 0$ C^0-near to $g_{\overline{S}^1} + g_\varepsilon$ on $S^1 \times B_{2\rho}(0)$: Take

$$g = \begin{cases} g_L\left(s, d(s), \tilde{d}(s)\right) & \text{on } \overline{S}^1 \times T_{L,5} \setminus T_{L,2} \\ g_{\text{pro}} + L^2 \cdot g_{S^1} & \text{on } \overline{S}^1 \times T_{L,2} \\ g_{\overline{S}^1} + g_\varepsilon & \text{on } \overline{S}^1 \times B_{2\rho}(0) \setminus T_{L,5} \end{cases}$$

where g_{pro} is a prolongation defined on $T_{L,3}$ with $\|g_{\text{pro}} - (g_{S^1} + g_{\text{Eucl.}})\|_{C^0} < \varepsilon$, $\text{Ric}(g_{\text{pro}}) \leq 0$ and $g_{\text{pro}} \equiv \exp\left(2F_{s,\tilde{d}}(3-r)\right) \cdot \left(g_{\overline{S}^1} + g_{\text{Eucl.}}\right)$ on $T_{L,3} \setminus T_{L,1}$ which is possible for s small enough acc. (1).

Again we can find a large plane circle $\Gamma_M \subset \mathbf{R}^{n+1}$ (of length M) such that the metric on the tube of radius $2\rho\, U_M$ around Γ_M gets C^0-near to the product metric $M^2 \cdot g_{S^1} + g_{\text{Eucl.}}$ on $\overline{S}^1 \times B_{2\rho}(0)$ via Fermi coordinates.

This time we take the Riemannian coverings of g to approximate this metric w.r.t. C^0 (since g is no warped product): namely we consider $f_k : \overline{S}^1 \times B_{2\rho}(0) \to \overline{S}^1 \times B_{2\rho}(0)$ defined by $f_k(e^{it}, z) = (e^{ikt}, z)$. Thus we can consider for integer M $f_M^*(g)$ on $S^1 \times B_{2\rho}(0)$ and acc. (6.5) in [L2] we can "embed" $(S^1 \times B_{2\rho}(0), f_M^*(g))$ as U_M into $\mathbf{R}^{n+1}$.

Again from [L2], § 6 we can find a C^0-small deformation of some ball containing U_M to get after scaling:

__Corollary (9.3):__ On $\mathbf{R}^n$, $n \geq 4$ we can find a metric $g_{n,\varepsilon}$ with $\text{Ric}\,(g_{n,\varepsilon}) < 0$ on $B_1(0)$, $g_{n,\varepsilon} \equiv g_{\text{Eucl.}}$ on $\mathbf{R}^n \setminus B_1(0)$ and $||g_{n,\varepsilon} - g_{\text{Eucl.}}||_{C^0} < \varepsilon$.

□

It is interesting to iterate this argument, i.e. choose an exhausting sequence of balls in $\mathbf{R}^n$; this yields metrics $g(n,\varepsilon)$ on $\mathbf{R}^n$ with $\text{Ric}\,(g(n,\varepsilon)) < 0$ on $\mathbf{R}^n$, $||g(n,\varepsilon) - g_{\text{Eucl.}}||_{C^0} < \varepsilon$ and $||g(n,\varepsilon) - g_{\text{Eucl.}}||_{C^k} < c(k)$ for some finite $c(k) > 0$. Now notice that C^0 is scale invariant (cf. (9.1)) but the higher derivatives decrease acc. m^{-k} if we scale by m^2, i.e. we get:

__Corollary (9.4):__ The flat metric $g_{\text{Eucl.}}$ on $\mathbf{R}^n$ can be C^∞-approximated by metrics in $\text{Ric}^{<0}(\mathbf{R}^n)$.

□

Recall that $g_{\text{Eucl.}}$ cannot even be C^0-approximated by metrics in $\text{Sec}^{<0}(\mathbf{R}^n)$, they always have infinite distance to $g_{\text{Eucl.}}$.

§ 10 Mutual Regularization

We want to prove the existence of $g_{3,\varepsilon}$ on $\mathbf{R}^3$ along the lines described in § 9. But obviously we cannot start with a metric $g_{2,\varepsilon}$ on $\mathbf{R}^2$ acc. Gauß-Bonnet. On the other hand we can find analogous metrics, singular in only one point, and the additional idea consists in using the negative (sectional = Ricci) curvature of the one of the two perpendicular planes to smooth the singularity of the other one. More precisely we can find for each $\varepsilon > 0$ a metric g_ε on $\mathbf{R}^2$ (regular outside of the origin) with:

(i) $\text{Ric}\,(g_\varepsilon) < 0$ on $B_2(0)$ and $g_\varepsilon \equiv g_{\text{Eucl.}}$ on $\mathbf{R}^2 \setminus B_2(0)$

(ii) $g_\varepsilon = g_{\mathbf{R}} + \frac{\sinh^2 r}{\alpha^2} \cdot g_{S^1}$ on $B_1(0)$ for some suitable $\alpha > 1$

(iii) $||g_\varepsilon - g_{\text{Eucl.}}||_{C^0_{g_{\text{Eucl.}}}(\mathbf{R}^2)} < \varepsilon$

Using this metric we can perform the complete construction of § 9 and get:

__Lemma (10.1):__ *For each $\varepsilon > 0$ there is a metric $g(\varepsilon)$ on $]-5,5[\times S^1 \times \overline{S}^1$ with:*

(i) $\text{Ric}\,(g(\varepsilon)) \leq 0$ *on* $]-4,4[\times S^1 \times \overline{S}^1$ *and* $g(\varepsilon) \equiv g_{\mathbf{R}} + g_{S^1} + g_{\overline{S}^1}$ *outside*

(ii) $||g(\varepsilon) - \left(g_{\mathbf{R}} + g_{S^1} + g_{\overline{S}^1}\right)||_{C^0} < \varepsilon$

$$\text{(iii) } g(\varepsilon) = \begin{cases} (\alpha+\beta\cdot r)^2\cdot g_{S^1} + g_{\mathbb{R}} + \gamma^2\cdot r^2\cdot g_{\overline{S}^1_b} & \text{on } S^1\times B_1(3,0) \\ \gamma^2\cdot r^2\cdot g_{S^1_b} + g_{\mathbb{R}} + (\alpha+\beta\cdot r)^2\cdot g_{\overline{S}^1} & \text{on } B_1(-3,0)\times \overline{S}^1 \end{cases}$$

for some $\alpha,\beta,\gamma > 0$.

(S^1_b *denotes the* S^1 *-fiber in the base-disk* $B_1(p) \equiv [0,1[\times S^1$).

□

Thus we are left to smooth those singularities of (iii) to some regular Ric < 0-metric. If we do not need C^0-estimates we can apply the techniques of [L2] and [L4] and easily get a smooth Ric<0-metric with properties (i) and (ii), which is of some use on its own in [L4]. By the way we also obtain a new existence proof of metrics on $\mathbb{R}^3$ with Ric < 0 on $B_1(0)$ which are Euclidean outside and it is the first one which completely avoids any kind of surgery.

But we also want to preserve the small C^0-deviation from the product metric (as in (ii)). Therefore we will use the following refinement of (iii) which is obtained from nearly the same construction carried out more carefully. We give an outline below and note that a generalization appears in [L6].

(iii) * for some $\gamma < 1$ near 1:

$$g(\varepsilon) = \begin{cases} ((1-3(1-\gamma)) + (1-\gamma)\cdot r)^2\cdot g_{S^1} + g_{\mathbb{R}} + \gamma^2\cdot r^2\cdot g_{S^1_b} & \text{on } S^1\times B_1(3,0) \\ \gamma^2\cdot r^2\cdot g_{S^1_b} + g_{\mathbb{R}} + ((1-3(1-\gamma)) + (1-\gamma)\cdot r)^2\cdot g_{\overline{S}^1} & \text{on } B_1(-3,0)\times \overline{S}^1 \end{cases}$$

Now it is easy to derive

Corollary (10.2): On $\mathbb{R}^3$ we can find for each $\varepsilon > 0$ a regular metric $g_{3,\varepsilon}$ with Ric $(g_{3,\varepsilon}) < 0$ on $B_1(0)$, $g_{3,\varepsilon} \equiv g_{\text{Eucl.}}$ on $\mathbb{R}^3 \setminus B_1(0)$ and $||g_{3,\varepsilon} - g_{\text{Eucl.}}||_{C^0} < \varepsilon$.

Proof: We will smooth $g(\varepsilon)$ to a metric $f^2\cdot g_{S^1} + g_{\mathbb{R}} + g^2\cdot g_{S^1_b}$ with Ric ≤ 0, i.e. with $\frac{f''}{f} + \frac{g''}{g} \ge 0$, $\frac{f''}{f} + \frac{f'}{f}\cdot\frac{g'}{g} \ge 0$, $\frac{g''}{g} + \frac{f'}{f}\cdot\frac{g'}{g} \ge 0$. Take

$$f = \begin{cases} 1-3(1-\gamma)+(1-\gamma)\cdot r \\ \text{const.} \ (\ge 1-3(1-\gamma)) \end{cases}, \quad g = \begin{cases} \gamma\cdot r & \text{on } \mathbb{R}^{\ge 1} \\ r & \text{on }]0,\delta[\end{cases}$$

and $f'' = -g'' > 0$ on $]\delta,1[$. Now notice the trivial but essential fact: $f(1) < g(1)$. This implies that for small $\delta > 0$, f and g can be chosen such that the quotients in the above inequalities, which contain f dominate those of g and yield Ric ≤ 0, furthermore the new metric is C^0-near to the old one.

Finally we take the "usual" arguments of § 9 to smoothly embed $]-5,5[\times S^1\times S^1$ into $\mathbb{R}^3$ and get the claim.

□

Remark (10.3): To put things into some perspective we mention that additional symmetries sharpen the results (cf. [L6]): In (10.1) we only took into account two singular curves related to the two S^1-factors, but if we identify the two boundary components of $]-5,5[\times S^1\times S^1$ to get T^3 we can also introduce a singularity along the new S^1-factor. Then we can form a metric analogous to (10.1) + (iii)*, but this time we get (using the argument of (10.2)) a natural C^∞-approximation (!) of g_{flat} on T^3 by metrics of $\text{Ric}^{<0}(T^3)$.

To get an idea of how to obtain those special value for α, β, γ (of (10.1) (iii)) in (iii) * we present (a sketch) of a suitable modification of that construction on T^3 just mentioned:

Consider $]-5,5[\times S^1$ with $S^1 \equiv [-1,1]/\{-1,1\}$ and cut out one of the following pieces $P^\pm_\alpha$: $P^+_\alpha := \{(x,y) \in]-3,5[\times S^1 |\, |y| < \alpha \cdot (x+3)\}$, $Q^+_\alpha :=]-5,5[\times S^1 \setminus P^+_\alpha$, $\alpha > 0$ small. P^-_α, Q^-_α analogously start at (3,0), now identify ∂Q^+_α acc. $(x,y) \sim (x,-y)$ and again analogously $Q^-_\alpha / \sim$.

This yields two cylinders $C^\pm_\alpha$ which have exactly one interior singular point at $(\pm 3, 0)$, $\partial C^\pm_\alpha$ does also have a singularity but this will be "cut-off" finally and will be ignored here. Next imitate the construction of § 9 resp. (10.1) and build some metric on $]-5,5[\times S^1 \times \overline{S}^1$ using $C^\pm_\alpha$: the metric "along" $]-5,5[\times S^1 \times \{0\}$ resp. $]-5,5[\times \{0\} \times \overline{S}^1$ is defined by some identification with C^+_α resp. C^-_α.

Now this metric already looks like that of (10.1) (iii) *, at least near the segments $]2,3[\times\{0\}\times\{0\}$ resp. $]-3,-2[\times\{0\}\times\{0\}$ of $B_1(\pm 3,0)$. But one has to "wind up" (that is in effect scaling S^1, which destroys higher C^k-estimates) the S^1-factors to make this metric radially symmetric on $B_1(\pm 3,0)$. Finally if $\alpha > 0$ is small enough we can slightly bend those cylinder-metrics to ensure $\mathrm{Ric} \le 0$ which yields (10.1) + (iii) *.

For details concerning these methods we refer to the more thorough exposition in [L6].

□

We conclude with a concrete conjecture suggested from all these results

Conjecture (10.4): Let (M^n, g_0), $n \ge 3$ be a manifold, $B \subset M$ a ball, then there is a metric g_1 and a C^∞-continuous path g_t, $t \in [0,1]$ on M with:

(i) $r(g_t)(\nu) < r(g_0)(\nu)$ on B, $t > 0$
(ii) $g_t \equiv g_0$ on $M \setminus B$

(10.4) (which is very likely to hold) implies resp. simplifies the Theorems of this paper as well as of [L2]. Furthermore it yields many new results like "Universal C^k-dense h-principles" for lower Ricci curvature approximations.

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Negative Bending of Open Manifolds

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§1 Introduction

In this paper we will have a new look on general existence theorems for metrics with negative Ricci curvature, which is motivated from several related results. We will mention the most significant ones:

1) A general feeling expresses that bending of metric yields (or preserves) negative curvature iff we bend outwards. Bending is used as an intuitive collective noun for deformations which e.g. enlarge or smaller the metric near some boundary. (Think of the growth of spheres in hyperbolic relative to Euclidean space.)

The if part will be supported by a simple construction of complete metrics of Ric < 0 on each open manifold, but we will disprove the only if part: namely we also find bendings "inwards" for Ric < 0, which yields existence results for closed manifolds.

2) The "classical" existence proof for metrics with negative scalar curvature $S < 0$ on closed manifolds (cf. [A], [KW]) starts from some metric with negative integral scalar curvature, and the integral condition suffices to find conformal deformations to get a metric with $S < 0$.

The metrics constructed here concentrate a huge amount of negative Ricci curvature in one small ball (which is the basic way to ensure analogously negative "integral" Ricci curvature). And indeed a "far-reaching" conformal diffusion yields Ric < 0 on the whole manifold.

3) In [L1] we already gave a series of existence theorems for Ric < 0 by some "covering" by local deformations which had to be made compatible.

This major technical problem disappears in this paper and we get a much shorter and new argument for general existence results.

4) In [L3] we proved by geometric arguments that Ric < 0 fulfills so-called h-principles, i.e. the partial differential inequality Ric < 0 "blows down" to a simple algebraic inequality.

This means that the geometric content of Ric < 0 is much smaller than expected.

Moreover h-principles (cf. [Gr]) are usually obtained by much less differential geometry and this led to the problem how to minimize the geometric effort to get the existence of metrics with Ric < 0 (cf. [L4]).

This originally motivated this paper, despite the fact that "bending of Ric < 0" still contains some amount of geometry.

Now we formally state the main theorems, which are obtained by "negative bendings" becoming clear in the course of the paper.

Theorem 1 *Let M^n, $n \geq 2$ be an open manifold, g_0 an arbitrary metric on M, then we can find a smooth function f with:*

$$g = e^{2f} \cdot g_0 \text{ is a complete metric with } Ric(g) < 0.$$

Notice that this cannot be refined to give pinched Ricci (or just scalar) curvature in each conformal class according to non-existence results of Ni [N] (cf. also [AM]).

While Theorem 1 is obtained using bending "outwards" we additionally use bending "inwards" to get:

Theorem 2 *Each closed manifold $M^n, n \geq 3$ admits a metric with Ric < 0.*

This can be localized:

Theorem 3 *On $\mathbb{R}^n, n \geq 3$ there exists a metric g_n with $Ric(g_n) < 0$ on $B_1(0)$ and $g_n \equiv g_{\text{Eucl.}}$ outside.*

Finally we give an outline of the paper: In §2 we construct conformal deformations of any prescribed metric on open manifolds leading to Ric < 0. In principle the conformal factor can be calculated explicitly. The next two chapters are devoted to perform a refined construction on the (open) complement of certain lower dimensional submanifolds of closed manifolds to obtain in addition a suitable structure near the boundary.

This is used in §5 in dimension $n \geq 4$: Here we close these manifolds again and get Theorems 2 and 3.

An extra argument is needed to get Theorem 2 in dimension 3 (§6), it also uses some results of [L3]. Thus the previous methods are not as adequate in this situation as in higher dimensions. But the general philosophy remains, therefore we include a sketch of proof in this case.

Finally in §7 we will briefly compare the proof of contractibility of the spaces of metrics with $S < 0$, [L2] resp. with Ric < 0, [L3] in light of the present constructions.

§2 Conformal bending

A striking differential topological (!) result of Gromov (cf. [Gr]) implies that each open manifold admits a metric with negative (as well as one with positive) sectional curvature. But these metrics are not complete.

Indeed there are known obstruction to get complete negative sectional as well as positive scalar curvature metrics in dimension $n \geq 3$ resp. $n \geq 5$ (cf. [GL]).

Therefore there is no hope to find global "outward bendings" for Sec < 0 as are now presented for Ric < 0 by conformal changes on some arbitrary open manifold M^n of dimension $n \geq 2$:

Proposition 2.1 *Let g_0 be any metric on M^n, then there exists a $f \in C^\infty(M, \mathbb{R})$ such that $g = e^{2f} \cdot g_0$ is complete and $Ric(g) < 0$.*

Proof. Let be $M(\overset{\circ}{M}_{n+1} \supset \bar{M}_n, M_0 = \emptyset)$ an exhaustion of M by compact manifolds with smooth boundaries and choose for an increasing sequence c_n of real numbers a function $F \in C^\infty(M, \mathbb{R})$ with $F \equiv c_n$ near ∂M_n and $c_n \leq F \leq c_{n+1}$ on $M_{n+1} \backslash M_n$. If the c_n are chosen suitably then $g = e^{2F} \cdot g_0$ is complete, hence we can assume g_0 to be complete.

Now using paracompactness of M we find a locally finite covering of balls $B_i, i = 1, 2, \ldots$, together with diffeomorphisms $f_i : B_6(0) \to B_i, (B_6(0) \subset \mathbb{R}^n)$ with $\bigcup_i f_i(B_4(0)) = M$ and $f_i(B_2(0)) \subset f_{i+1}(B_4(0) \backslash B_3(0))$ hence $\bigcup_i f_i(B_4(0) \backslash B_2(0)) = M$ and we define for $d_i, s_i > 0, i \geq 1$:

$$g(0) := g_0,\ g(n) := \prod_{i \leq n} \exp(2 \cdot F_i) \cdot g_0,\ g(\infty) := \prod_i \exp(2 \cdot F_i) \cdot g_0$$

with $F_i = s_i \cdot \exp(-d/5 - \|f_i^{-1}(z)\|) \cdot h(\|f_i^{-1}(z)\|)$ for z with $\|f_i^{-1}(z)\| < 5$ and $F_i \equiv 0$ otherwise.

$\|\cdot\|$ denotes the Euclidean norm on $B_6(0), h \in C^\infty(\mathbb{R},, [0, 1])$ with $h \equiv 0$ on $\mathbb{R}^{\leq 1}, h \equiv 1$ on $\mathbb{R}^{\geq 2}$.

Using lemma (2.2) below we can find for each $n \geq 1$ a $d_n > 0$ such that (for fixed, $d_i, s_i > 0, i < n$, if $n > 1$) and <u>each</u>(!) $s_n > 0$:

$$\exp(2 \cdot F_n) \cdot r(g(n))(\nu) - r(g(n))(\nu) - r(g(n-1))(\nu)$$
$$< \begin{cases} 0 & \text{on } f_n(B_5(0) \backslash B_4(0)) \\ -s_n \cdot e^{-d_n} & \text{on } f_n(B_4(0) \backslash B_2(0)), \end{cases}$$

where $g(n-1)(\nu, \nu) = 1$. Thus starting from $n = 1$ we get by induction $d_i, s_i > 0$ for each $i \geq 1$ such that $r(g(n)) < 0$ on $(\bigcup_{i \leq n} f_i(B_4(0) \backslash\ B_2(0))) \backslash f_n(B_2(0))$. This yields $r(g(\infty)) < 0$, since $B_n \cap K = \emptyset$ for each compact $K \subset M$ and $n = n(K)$ large enough. Furthermore $g(d_i, s_i \,|\, i < \infty)$ is conformal to g_0 with a pointwise conformal factor ≥ 1. Hence it is complete.

□

Lemma 2.2 *Let g_0 be any metric on $N \times \mathbb{R}$ for some closed manifold N, then there exists a $d_0 > 0$ such that for all $d \geq d_0$, $s > 0$:*

$$\exp(2 \cdot s \cdot e^{-\frac{d}{t}}) \cdot r(\exp(2 \cdot s \cdot e^{-\frac{d}{t}} \cdot g_0)(\nu) - r(g_0)(\nu) < \begin{cases} 0 & on\ N \times]0,1] \\ -s \cdot e^{-d} & on\ N \times [1,10] \end{cases}$$

for $\nu \in T(N \times \mathbb{R})$, $\|\nu\|_{g_0} = 1$ and $(x,t) \in N \times \mathbb{R}$.

The elementary proof of this technical lemma also appeared in [L1](3.5) (in a different context) for $n \geq 3$, but the (simplified) calculations also apply to $n \geq 2$. Hence to be short we omit further details.

□

§3 Opening of Manifolds

To prove the existence of metrics with Ric < 0 on closed manifolds of dimension $n \geq 4$ we first notice.

Lemma 3.1 $\mathbb{R}^n, n \geq 4$ *contains a closed manifold N^{n-2} with trivial normal bundle and which admits a metric with Ric < 0.*

Proof. $n = 4$: Each closed orientable surface F admits an embedding in $\mathbb{R}^3$ and hence in $\mathbb{R}^4$. In this situation the normal bundle is trivial since $F \subset \mathbb{R}^3$ is a hypersurface, which always fulfills this condition. Thus take a surface of genus 2, this admits a hyperbolic metric.

$n = 5$: Again each closed orientable three manifold N^3 admits an embedding in $\mathbb{R}^5$ with trivial normal bundle (cf. [H], Cor. 4). Thus take some orientable hyperbolic three manifold or alternatively take $S^3 \subset \mathbb{R}^5$ and use the existence of a Ric $<$ 0-metric on S^3 (cf. §6).

$n \geq 6$: We can use induction: In §5 we will prove that each N^{n-2} admits a metric with Ric < 0, thus we take $S^{n-2} \subset \mathbb{R}^n$.

□

Now the proof of Theorem 2 (and 3) proceeds as follows: we choose a ball $B \subset M^n, n \geq 4$ and $N^{n-2} \subset B$ acc. the previous Lemma and consider $M \backslash N$. This an open manifold and admits a metric with Ric < 0 acc. (2.1). Next (in §3-5) we use the conditions on N to bend $M \backslash N$ to get a Riemannian structure with Ric < 0 which has M as natural metric completion.

Thus let $M^n, n \geq 4$ be an arbitrary manifold, $B \subset M^n$ a ball and $N^{n-2} \subset B$ as in (3.1), denote by V, W open tubular neighborhoods of N with $\bar{V} \subset W \subset \bar{W} \subset B$. We will introduce a second bending (additionally to (2.1)), this time for standardization of the boundary structure.

Proposition 3.2 *Let g_0 be a metric on $M\backslash N$ with $Ric(g_0) \le 0$. Then there is a metric g on $M\backslash N$ with $g \equiv g_0$ on $M\backslash W$, $Ric(g) < 0$ on $W\backslash N$ and such that $(V\backslash N, g)$ is isometric to $(]0,1[\times S^1 \times N, g_{\mathbb{R}} + \frac{\sinh^2 m(r+\rho)}{m^2} \cdot g_{S^1} + \kappa^2 \cdot g_N)$, where g_N is a metric with $Ric(g_N) < 0, m \in \mathbb{Z}^{>0}, \rho \ge 0, \kappa \ge 0$.*

Proof. Using the triviality of the normal bundle of N we get a diffeomorphism Φ from $]-2,12[\times S^1 \times N$ into $B\backslash N$ with $\Phi(]0,12[\times S^1 \times N) = W\backslash N$ and $\Phi(]3,12[\times S^1 \times N) \to V\backslash N$ and which admits a continuous extension with $\Phi(\{12\} \times S^1 \times N) = N$.

Thus take the following metric on $\mathbb{R}^{>-2} \times S^1 \times N$:

$$g_1 = h \cdot \Phi^*(g_0) + (1-h) \cdot (g_{\mathbb{R}} + g_{S^1} + g_N)$$

for some $h \in C^\infty(\mathbb{R}, [0,1])$ with $h \equiv 0$ on $\mathbb{R}^{\ge 3}, h \equiv 1$ on $\mathbb{R}^{\le 2}$; acc. (2.2) we get a d_0 such that for $s > 0$, $\nu \in T(\mathbb{R} \times S^1 \times N), \|\nu\|_{g_1} = 1$, $(t,z,x) \in \mathbb{R} \times S^1 \times N$

$$\exp(2 \cdot s \cdot e^{-\frac{d_0}{t}}) \cdot r(\exp(2 \cdot s \cdot e^{-\frac{d_0}{t}}) \cdot g_1)(\nu) - r(g_1)(\nu) < \begin{cases} 0 & \text{on }]0,1] \times S^1 \times N \\ -s \cdot e^{-d_0} & \text{on } [1,10[\times S^1 \times N \end{cases}$$

Hence for s large enough we get $r(\exp(2 \cdot s \cdot e^{-\frac{d_0}{t}}) \cdot g_1)(\nu) < 0$ on $]0,10[\times S^1 \times N$ and $(]5,10[\times S^1 \times N, \exp(2 \cdot s \cdot e^{-\frac{d_0}{t}}) \cdot g_1)$ is isometric to $(]\alpha,\beta[\times S^1 \times N, g_{\mathbb{R}} + F^2 \cdot (g_{S^1} + g_N))$ (via some isometry $\varphi = (\varphi_{\mathbb{R}}, id_{S^1 \times N})$; $\varphi_{\mathbb{R}}$ is uniquely determined) for some $\alpha < \alpha + 5 < \beta$ and $F \in C^\infty(]\alpha,\beta[, \mathbb{R}^{>0})$ with $F', F'' > 0$ (obtained by rescaling $\mathbb{R}$) :
namely $F' > 0$ is independent of scaling $\mathbb{R}$ and $F'' > 0$ is clear from the warped product formula (cf. [B]):

$0 > \text{Ric}(g_{\mathbb{R}} + F^2 \cdot (g_{S^1} + g_N))(\nu,\nu) = -(n-1) \cdot \frac{F''}{F} \cdot g_{\mathbb{R}}(\nu,\nu)$, for ν tangent (i.e. horizontal) to $\mathbb{R}$).

We can assume $\max r(g_N) = -1$, then there are (acc. (4.1)(i) below) $f, g \in C^\infty(]\alpha,\beta[, \mathbb{R}^{>0})$ with $\text{Ric}(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ and $f \equiv g \equiv F$ near $\alpha, f(t) = \frac{\sinh m(t-c)}{m}$ resp. $g = \kappa > 0$ on $]\beta - 1, \beta[$ for some $m \in \mathbb{Z}^{>0}, \kappa > 0$, $c \in]\alpha, \beta - 1[$. Now define a diffeomorphism $\phi :]-2,12[\to]-2,10[$ with $\phi \equiv id$ on $]-2,1[, \phi(]2,12[) =]5,10[$ and $\varphi_{\mathbb{R}} \circ \phi$ is linear on $]3,12[$ with $\varphi_{\mathbb{R}} \circ \phi(]3,12[) =]\beta - 1, \beta[$. Now we are ready to define

$$g_2 := \begin{cases} (\varphi \circ (\phi, id_{S^1 \times N}))^*(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) & \text{on }]2,12[\times S^1 \times N \\ (\phi, id_{S^1 \times N})^*(\exp(2 \cdot s \cdot e^{-\frac{d_0}{t}}) \cdot g_1) & \text{on }]-2,2[\times S^1 \times N \end{cases}$$

In particular $g_2 \equiv g_1$ on $]-2,1] \times S^1 \times N$ and $\text{Ric}(g_2) < 0$ on $]0,12[\times S^1 \times N$. Thus define the push-forward metric $g := \Phi_*(g_2)$. It is easily checked that g fulfills the claim.

□

§4 Smoothings and Warpings

This chapter is a service-center for deformations used in §3 as well as in §5 to smooth singularities. They are done using certain warped product arguments. This also generalizes results in [GY] and [Br], cf. (5.3) below.

Thus we consider $(]a,b[\times S^1 \times N^{n-2}, g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N)$ for some $f,g \in C^\infty(]a,b[,\mathbb{R}^{>0})$. If $\max r(g_N) = -1$, then $r(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ is easily seen to be equivalent to the following three inequalities (1) - (3):

$$(n-3)\cdot\frac{1+(g')^2}{g^2}+\frac{g''}{g}+\frac{f'}{f}\cdot\frac{g'}{g}>0,\ (n-2)\cdot\frac{g''}{g}+\frac{f''}{f}>0,\ (n-2)\cdot\frac{g'}{g}\cdot\frac{f'}{f}+\frac{f''}{f}>0.$$

Proposition 4.1

(i) *Let $F \equiv G \in C^\infty(]a,b[,\mathbb{R}^{>0})$ be with $F', F'' > 0$ then there are $f,g \in C^\infty(]a,b[,\mathbb{R}^{>0})$ with $Ric(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ and for some $m \in \mathbb{Z}^{>0}, \kappa > 0$:*

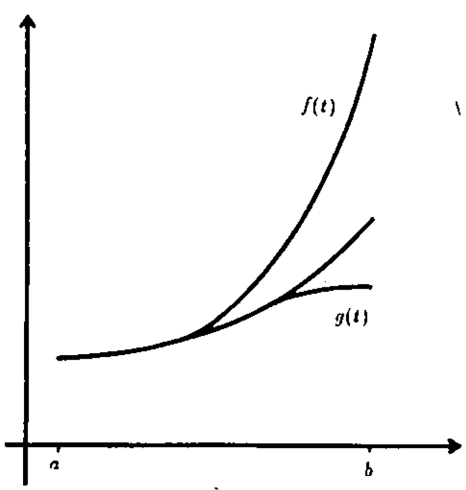
$$f(t) = \begin{cases} F(t) \\ \frac{\sinh m(t-c)}{m} \end{cases},\quad g(t) = \begin{cases} G(t) & \text{near } a \\ \kappa & \text{near } b, \text{ for some } c \in]a,b[\end{cases}$$

(ii) *Let $F \equiv \frac{\sinh}{\alpha}, \alpha > 1, G \equiv m > 0$ be defined on $]0,r_0[$ for some $r_0 > 0$. Then there are $f,g \in C^\infty(]p,r_0[,\mathbb{R}^{>0})$ for some $p < 0$ with $Ric(g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ and for some $\kappa > 0$*

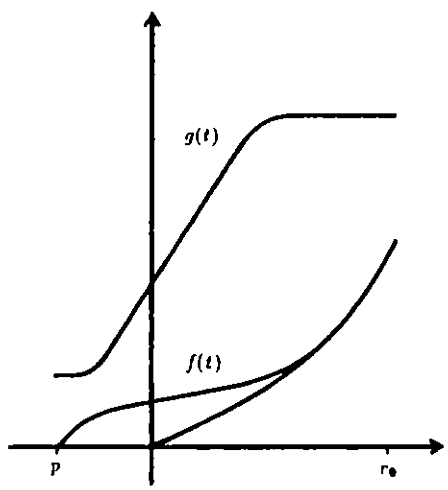
$$f(t) = \begin{cases} \frac{\sinh t}{\alpha} \\ \sinh(t-p) \end{cases} \quad g(t) = \begin{cases} m & \text{near } r_0 \\ \kappa & \text{near } p \end{cases}$$

Proof. We construct f,g which fulfill the "boundary conditions" and (1) - (3) glueing together functions defined on disjoint intervals: we will use the following simple observation:

(*) If $f_1, f_2 \in C^\infty(]\alpha,\gamma[,\mathbb{R}^{>0})$ fulfill $f_1(\beta) = f_2(\beta)$ for some $\beta \in]\alpha,\gamma[, 0 < f_1' < f_2'$, and $f_1'', f_2'' \geq 0$ (resp. > 0) then there is a function $h \in C^\infty(]\alpha,\gamma[,\mathbb{R}^{>0})$ with $h \equiv f_1$ resp. f_2 near α resp. γ and $h' > 0, h'' \geq 0$ (resp. $h'' > 0$).



(i)



(ii)

(i) We choose $\kappa := G(b) + 1$, then we can find a g with $g \equiv G$ on $]a, b - 3\epsilon[, g \equiv \kappa$ on $[b - \epsilon, b[, g' > 0$ on $]a, b - \epsilon[, g'' > 0$ on $]a, b - 2\epsilon[$ for some small $\epsilon > 0, 10 \cdot \epsilon < |b - a|$. For m large enough (say $\geq m_0$) we can find a $c_m \in]b - 4\epsilon, b - 3\epsilon[$ such that there is exactly one $t_m \in]c_m, b - 3\epsilon[$ such that $\frac{\sinh m(t_m - c_m)}{m} = F(t_m)$ and near $t_m : 0 < F^{(\kappa)} < (\frac{\sinh m(t-c_m)}{m})^{(\kappa)}, \kappa = 1, 2$.

Now we use $(*)$ to get a function $f_m \in C^\infty(]a, b[, \mathbb{R}^{>0})$ with $f'_m, f''_m > 0$ and $f_m \equiv F$ near $a, f_m \equiv \frac{\sinh m(t-c_m)}{m}$ on $]b - 3\epsilon, b[$. Thus for each $m \geq m_0$ (3) is fulfilled on $]a, b[$ for f_m and g.) (2) and (3) are fulfilled on $]a, b - 3\epsilon]$.
Furthermore $|\frac{g'}{g}|, |\frac{g''}{g}| < \kappa$ and $\frac{f''_m}{f_m} = m^2, \frac{f'_m}{f_m} \geq m$ on $]b - 3\epsilon, b[$.
Hence (2) is fulfilled on $]a, b[$ for $m^2 > (n - 2) \cdot \kappa$.

Finally to get (1) we notice $|\frac{g''}{g}| < |\frac{n-2}{g^2}|$ on $]b - \epsilon - 2\delta, b[$ for some small $\delta > 0$ and $|\frac{g'}{g}| > \kappa'$ on $]b - 3\epsilon, b - \epsilon - \delta[$, thus (1) is fulfilled on $]b - 3\epsilon, b[$ for each $m \geq m_0$ with $m \cdot \kappa' > \kappa$.

Hence define $f = f_m, c = c_m$ for some $m \geq \max\{m_0, \frac{\kappa}{\kappa'}, (n - 2) \cdot \kappa\} + 1$.

(ii) We start from $F \equiv \frac{\sinh}{\alpha}, \alpha > 1, G \equiv m > 0$ on $]0, r_0[$ which (obviously) fulfill (1) - (3). For $\gamma > 0$ define a function $G_\gamma \in C^\infty(]t_\gamma, r_0[, \mathbb{R}^{>0})$ with $t_\gamma = -\frac{m}{\gamma} + \frac{r_0}{2}$ and $G_\gamma = \gamma \cdot (t - \frac{r_0}{2}) + m$ on $]t_\gamma, \frac{r_0}{4}[, G_\gamma = m$ on $]\frac{r_0}{2}, r_0[$ and $G'_\gamma > 0$ on $]\frac{r_0}{4}, \frac{r_0}{2}[$. If $\gamma > 0$ is small enough G_γ can be defined such that F and G_γ fulfill again (1) - (3) on $]0, r_0[$.

Fix such a $\gamma > 0$ and G_γ. Now we define a function $f \in C^\infty(]t_\gamma - 3, r_0[, \mathbb{R}^{>0})$ with $f' > 0$ on $]t_\gamma - 3, r_0[, f'' > 0$ on $]t_\gamma - 1, r_0[$ and

$$f(t) = \begin{cases} \frac{\sinh t}{\alpha} & \text{on }]\frac{r_0}{4}, r_0[\\ \sinh(t - (t_\gamma - 3)) & \text{on }]t_\gamma - 3, t_\gamma - 3 + 4\delta[\quad \text{for small } \delta \in]0, \frac{1}{10}[\cdot \end{cases}$$

Next we consider $g(\kappa, m) := \max\{G_\gamma, \frac{\sinh m(t-(t_\gamma-3))}{\kappa}\}$.

For each $m \in \mathbb{Z}^{>0}$ we can find a $\kappa = \kappa(m) \in \mathbb{Z}^{>0}$ such that there is a unique $t_m \in]t_\gamma, 0[$ with $G_\gamma(t_m) = \frac{\sinh m(t_m-(t_\gamma-3))}{\kappa(m)}$ and $G'_\gamma > (\frac{\sinh m(t-(t_\gamma-3))}{\kappa(m)})'$ in t_m. Since $(\cdot)''$ of both functions is non negative we can find, using $(*)$ twice, a function $g_m \in C^\infty(]t_\gamma - 3, r_0[, \mathbb{R}^{>0})$ with $g'_m, g''_m \geq 0$ and $g'_m > 0$ on $]t_\gamma - 3 + \delta, r_0[$ and

$$g_m(t) = \begin{cases} G_\gamma & \text{on }]0, r_0[\\ \frac{\sinh m(t-(t_\gamma-3))}{\kappa(m)} & \text{on }]t_\gamma - 3 + 3\delta, t_\gamma[\\ \kappa_m & \text{on }]t_\gamma - 3, t_\gamma - 3 + \delta[\quad \text{for some suitable } \kappa_m > 0 \end{cases}$$

Now we will choose a large m such that f and g_m fulfill (1) - (3):

(1) is always fulfilled. (2) and (3) are fulfilled on $]t_\gamma - 3, t_\gamma - 3 + 3\delta[\cup]t_\gamma - 1, r_0[$. Since $\frac{g'_m}{g_m} \geq m, \frac{g''_m}{g_m} = m^2$ on $]t_\gamma - 3 + 2\delta, t_\gamma - \frac{1}{2}[$ for large m and $f' > 0$ on $]t_\gamma - 3, r_0[$ we find (3) fulfilled on $]t_\gamma - 3 + 2\delta, t_\gamma - \frac{1}{2}[$ for these m. Finally $\frac{f''}{f} > -c$ for some $c > 0$, hence $(n - 2) \cdot \frac{g''_m}{g_m} + \frac{f''}{f} > 0$ for great m, which yields (2).

Hence we choose these f and $g = g_m, \kappa = \kappa_m$ for some large m.

§5 Closing of Manifolds

We reformulate (2.1) and (3.2) in our context for closed $M^n, n \geq 4$ and $N^{n-2} \subset B \subset M^n$ and a metric g_N on N with $\mathrm{Ric}(g_N) < 0$. We obtain metrics on $M \backslash N$ with Ric < 0 and some nice behaviour near the boundary:

Corollary 5.1

(i) $M^n \backslash N^{n-2}$ *admits a metric* g *with* Ric < 0 *and there is a tube* V *of* N *such that* $V \backslash N$ *is isometric to* $(]0,1[\times S^1 \times N, g_{\mathbb{R}} + \frac{\sinh^2 m(r+\rho)}{m^2} \cdot g_{S^1} + c^2 \cdot g_N)$, *for some* $c, \rho, m > 0$.

(ii) $\mathbb{R}^n \backslash N^{n-2}, n \geq 4 (B = B_1(0))$ *admits a metric* g *with* $\mathrm{Ric}(g) < 0$ *on* $W \backslash N, (\bar{W} \subset B_1(0)), g = g_{\mathrm{Eucl.}}$ *on* $\mathbb{R}^n \backslash W$ *and there is a tube* V *of* N *such that* $V \backslash N$ *is isometric to* $(]0,1[\times S^1 \times N, g_{\mathbb{R}} + \frac{\sinh^2 m(r+\rho)}{m^2} \cdot g_{S^1} + c^2 \cdot g_N)$ *for some* $c, \rho, m > 0$.

Now we will "bend inward" = "close" $M^n \backslash N$ resp. $\mathbb{R}^n \backslash N$ to get a metric with Ric < 0 on M resp. on $B_1(0) \subset \mathbb{R}^n$ using the following

Closing Lemma 5.2 *For each pair* $m, R > 0$ *there is a metric* $g(m, R)$ *on* $S^2 \times N$ *with* Ric < 0 *and a subset* $D_R \times N$ *canonically isometric to* $(B_R(0) \times N^{n-2}, g_{\mathrm{hyp.}} + c^2 m^2 \cdot g_N) \subset (\mathbb{H}^2, g_{\mathrm{hyp.}}) \times (N^{n-2}, c^2 m^2 \cdot g_N)$.

We prove (5.2) in a moment, but we first derive the

Proof of Theorem 2 and 3: Scale the metric g of (5.1)(i) resp. (ii) by m^2. Now the tubular neighborhood $V \backslash N$ is isometric to

$(]m\rho, m + m\rho[\times S^1 \times N, g_{\mathbb{R}} + \sinh^2 r \cdot g_{S^1} + c^2 m^2 g_N) = (B_{m+m\cdot\rho}(0) \backslash B_{m\rho}(0) \times N, g_{\mathrm{hyp.}} + c^2 m^2 g_N,$

Thus choose in (5.2) $R = m + m \cdot \rho$ and glue $(S^2 \times N \backslash D_R \times N, g(m, R))$ and $M^n \backslash N$ resp. $\mathbb{R}^n \backslash N$ along their (isometric) boundaries.

This yields M^n equipped with a metric Ric < 0.

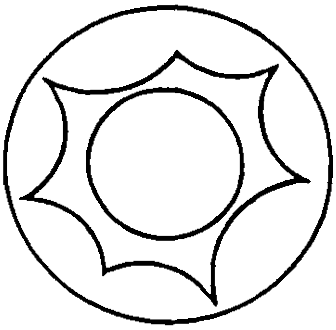
On $\mathbb{R}^n$ we obtain a metric $\bar{g}_n$ which fulfills $\mathrm{Ric}(\bar{g}_n) < 0$ on W and $\bar{g}_n \equiv g_{\mathrm{Eucl.}}$ on $\mathbb{R}^n \backslash W$. Now we consider as in (2.1) a diffeomorphism $f : B_6(0) \rightarrow B_2(0)$ with $f(B_5(0)) = B_1(0)$ and $f(B_3(0)) \subset W$ and define $g(s, d) = \exp(2 \cdot s \cdot \exp(-d/5 - \|f^{-1}(z)\|) \cdot h(\|f^{-1}(z)\|)) \cdot \bar{g}_n$ on $B_1(0), (= \bar{g}_n$ otherwise) for suitable $d > 0$ and <u>small</u> $s > 0$ we have $r(g(s, d)) < 0$ on $B_1(0), g(s, d) \equiv g_{\mathrm{Eucl.}}$ outside. Thus take $g_n \equiv g(s, d)$.

□

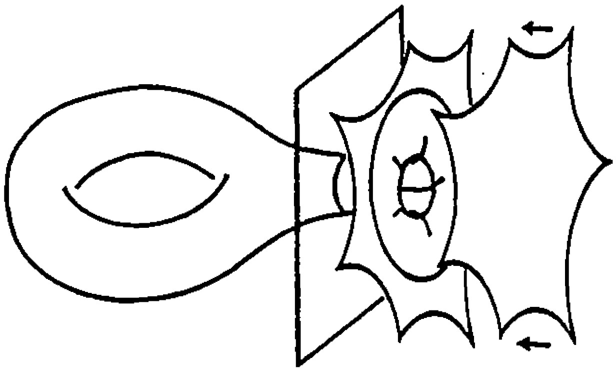
Proof of (5.2): Consider $B_{2R}(0) \subset \mathbb{H}^2$ and a compact, convex geodesic polygon P with $B_{2R}(0) \subset P$ and a second copy P'. Now we glue P and P' along their common boundary and obtain S^2 with a singular hyperbolic metric g^- : there are only finitely many singularities (corresponding to the vertices of P). Near the singular points the metric can be written w.r.t. polar coordinates:

$([0,r[\times S^1, dr^2 + \frac{\sinh^2 r}{\alpha^2} d\Theta^2)$ for some $\alpha > 1$. Take $r_0 < \frac{R}{2}$ and start with $(S^2 \times N, g^- + c^2 m^2 \cdot g_N)$:

We may assume that $B_{r_0}(p_i) \cap B_{r_0}(p_j) = \emptyset$ for $p_i \neq p_j \in S^2$, now we will smooth the metric on $B_{r_0}(p_i) \times N$ as follows:



$B_{2R}(0) \subset P \subset \mathbb{H}^2$



glue P and P'

$M^n \setminus V$

$\times N$

acc. (4.1)(ii) we can find a $\rho < 0$ and $f, g \in C^\infty(]\rho, r_0[, \mathbb{R}^{>0})$ with:

$\mathrm{Ric}(g_R + f^2 \cdot g_{S^1} + g^2 \cdot g_N) < 0$ on $]\rho, r_0[\times S^1 \times N$ and:

$$f \equiv \begin{cases} \frac{\sinh r}{\alpha} \\ \sinh(r-\rho) \end{cases}, \quad g \equiv \begin{cases} cm > 0 & \text{near } r_0 \\ \kappa > 0 & \text{near } \rho \end{cases}.$$

This is a smooth metric identical to $g(m,R)$ near $\partial B_r(p_i) \times N$ and we substitute $([0,r_0[\times S^1 \times N, dr^2 + \frac{\sinh^2 r}{\alpha^2} d\theta^2)$ by $([\delta, r_0[\times S^1 \times N, g_{\mathbb{R}} + f^2 \cdot g_{S^1} + g^2 \cdot g_N)$ and get a smooth metric with Ric < 0 on $S^2 \times N$ which contains a set canonically isometric to $(B_R(0) \times N, g_{\text{hyp.}} + c^2 m^2 \cdot g_N)$.

□

Remark 5.3

1. R. Brooks proved in [Br] that each hyperbolic orbifold of order ≥ 12 admits a metric with Ric < 0 using estimates on the width of tubes around the singular set. The proof above also extends this to arbitrary orders and without using such estimates.

2. The argument above can be used to prolongate arbitrary Ric < 0 metrics defined on a subset to the whole manifold.

§6 Closed three manifolds

The only closed codimension 2 submanifold of a three manifold is S^1, hence we cannot argue as in the higher dimensional case where we have used that the corresponding submanifold admits a metric with Ric < 0. On the other hand closed three manifold are subject to Thurston's hyperbolic Dehn surgery and this was already used to derive

Proposition 6.1 *Each closed three manifold admits a metric with Ric* < 0.

Namely Gao and Yau [GY] resp. Brooks [B] pointed out that each of these manifolds admits a hyperbolic metric which is regular outside some closed curves and they managed to smooth these singularities to get metrics with Ric < 0. Moreover the author gave an elementary proof in [L1] (but at least if we are only interested in (6.1) it is fairly lengthy).

For these reasons we will restrict to an outline of a short proof of (6.1), but the reader (familiar with [L1] and [L3]) will easily fill in the details.

The proof starts with some outward bending as before, the problem occurs if we try to close the manifold: here we mainly use two constructions of [L3].

Step 1: The proof of (3.2) also includes the following result for a three manifold M and $S^1 \times B_r(0) \subset B \subset M$, B a ball, $S^1 \times B_r(0)$ a (trivial) solid torus:

Lemma 6.2 *$M \backslash S^1 \times B_r(0)$ and $S^1 \times B_r(0)$ admit metrics g_1, g_2 with:*

(i) $g_i = g_{\mathbb{R}} + \gamma^2 \cdot r^2 \cdot (g_{S^n} + g_{S^1})$ on $]1,2[\times S^1 \times S^1 \equiv$ neck of the boundary, for some possibly large $\gamma > 0$.

(ii) $Ric(g_i) < 0$ elsewhere

Step 2: We would like to glue $M \backslash S^1 \times B_r(0)$ and $S^1 \times B_r(0)$, but these metrics do not fit together. We can reduce this problem arbitrarily well: here we will use a (non-obvious) warped product argument of [L3], §8:

Lemma 6.3 *For each $\varepsilon > 0$ there is an $a = a(\varepsilon) \geq 3$ and a metric g_ε on $]1, a[\times S^1 \times S^1$ with*

(i) Ric $(g_\varepsilon) \leq 0$

$$(ii) g_\varepsilon = \begin{cases} g_{\mathbb{R}} + \gamma^2 \cdot r^2 \cdot (g_{S^1} + g_{S^1}) & on\ ]1,2[\times S^1 \times S^1 \\ g_{\mathbb{R}} + \varepsilon^2 \cdot r^2 \cdot (g_{S^1} + g_{S^1}) & on\ ]a-1, a[\times S^1 \times S^1 \end{cases}$$

After minor modifications this can be reformulated in a more convenient way: "near the boundary" g_ε looks like $g_{\mathbf{R}} + (1 + \varepsilon \cdot r)^2 \cdot (g_{S^1} + g_{S^1})$ on $]1,2[\times S^1 \times S^1$ and we can make $\varepsilon > 0$ arbitrarily small.

Step 3: On $]-2,2[\times S^1 \times S^1$ we can easily find (cf. [L3]) a metric g^- which is a product metric near the boundary and has Ric < 0 otherwise.

Step 4: This can be used to link $M \backslash S^1 \times B_r(0)$ and $S^1 \times B_r(0)$: Take some $h \in C^\infty(\mathbb{R}, [0,1])$ with $h \equiv 0$ on $\mathbb{R} \backslash [-1-2\delta, 1+2\delta], h \equiv 1$ on $[-1-\delta, 1+\delta]$ for some small $\delta > 0$ (fixed). Consider $g(\varepsilon) = h \cdot g^- + (1-h) \cdot g_\varepsilon$ as defined on $]-2,2[\times S^1 \times S^1$ (g_ε is defined on $]-2,-1[$ by obvious reflection). Using (2.2) we can deform $g(\varepsilon)$ to have Ric < 0 if ε is small enough.

This implies (6.1) and also as in the higher dimensional cases:

Corollary 6.4 *On $\mathbb{R}^3$ there is a metric g_3 with $Ric(g_3) < 0$ on $B_1(0)$, and $g_3 \equiv g_{\text{Eucl.}}$ outside.*

§7 Concluding Remarks

In the case of scalar curvature it is possible to deform any given metric on a closed manifold $M^n, n \geq 3$ inside a ball into some metric with negative integral scalar curvature $\mathbf{S} = \int_M S_g d\text{Vol}_g < 0$, and this can be refined to show that the space of metrics with $\mathbf{S} < 0$ is contractible (cf. [L2]).

It is easily seen that a conformal change using the first eigenfunction of the conformal Laplacian yields a metric with $S < 0$ and adding some standard elliptic theory this implies:

Theorem ([L2]): *The space of metrics with $S < 0$ is contractible.*

Now look at Ric < 0 : We can change the point of view and interprete our presented constructions as deformations of arbitrary metrics inside a ball giving some metric which can be deformed by conformal change into some metric with Ric < 0.

Thus we are led to believe in a similar philosophy as above: If the "integral" or "average" Ricci curvature is negative we can make some standard (e.g. conformal) deformation leading to a Ric < 0-metric. This is additionally supported by the following result which would be provable along the argument of [L2] if we had a useful notion of average for Ric.

Theorem ([L3]): *The space of metrics with $Ric < 0$ is contractible.*

But at this point we have to note that the proof in [L3] follows different lines: namely due to the (current) lack of a sharp and senseful measure for that average of Ric we have to produce negative Ricci curvature directly by some subtle (but concrete) procedure, which allows to avoid the intermediate use of metrics which only have a kind of averaged negative Ricci curvature.

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