Normally Linear Poincaré Complexes
And Equivariant Splittings

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Normally Linear Poincaré Complexes And Equivariant Splittings

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INTRODUCTION: The study of a number of problems in group actions on manifolds calls for explicit constructions of actions. Successful applications of surgery theory in the non-equivariant problems has been a great motivation for various generalization of surgery to the equivariant set up. However, the variety of problems which may be approached via surgery in transformation groups is quite rich. The wide range of phenomena which are to be studied in some of the traditional problems (such as existence and classification problems) has limited the range of applicability of the existing equivariant theories. As a result, it seems appropriate to device specialized surgery theories which aim at different classes of more specific problems.

In the problems which arise in conjunction with the existence and classification of actions on manifolds, it is often useful (in agreement with the general philosophy of surgery) to divide roughly the constructions to two steps. In the first step, one uses methods of algebraic topology to study the problem in the homotopy category. In the second step one passes from the homotopy category to manifolds via surgery. The objects of interest in the first step are Poincaré complexes. Since the category of Poincaré complexes plays an important role in the study of smooth manifolds, it is natural, thus, to study homotopy related problems of G-manifolds on the level of equivariant analogues of this category.

The question arises, then, as to what extent a Poincaré complex with G-action should inherit the structure of a G-manifold. In this paper, we suggest a category of Poincaré complexes with G-actions, whose objects are called "normally linear Poincaré G-complexes" and the morphisms are "isovariant normally linear maps". This category inherits

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all the homotopy aspects of the category of Poincaré complexes without G-actions, while it has a certain amount of "manifold information" (from the category of G-manifolds) built into its objects and morphisms. This is in the form of a "suitable stratification" and a linearization of the Spivak normal sphere bundles of strata.

The range of applications and usefulness of this category, of course, depends on how successfully one is able to translate "the algebraic topology" of a problem into the kind of information which would allow one to construct "homotopy models" in this category. Constructions of objects in a category of Poincaré G-complexes becomes difficult if the candidate Poincaré G-complex is required to have "too much manifold information" built into it. On the other hand, imposing "insufficient manifold-like structure" on a Poincaré G-complex makes it difficult to construct equivariant surgery problems from such complexes, (mainly due to lack of equivariant transversality.)

Thus, it appears that the nature of the problem at hand should determine the extent of manifold-like data required from homotopy models. We will illustrate this point by studying the problem of equivariant splittings of closed G-manifolds in our category. Theorem II. 1 and II. 2 give necessary and sufficient conditions for the existence of splittings up to homotopy in terms of normally linear Poincaré G-complexes. Theorems IV. 1 , IV. 3 , and IV. 7 illustrate constructions and solutions of the relevant surgery problems, using the homotopy models of Theorem II. 1 . To give concrete examples, Theorem IV. 5 considers the problem for homotopy spheres and yields a generalization of Anderson-Hambleton's theorem ([1] Theorem A) while Theorem III. 1 illustrates a shorter and different proof of their theorem. Further applications of these ideas will appear in a subsequent paper.

The contents of this paper is as follows. In Section I the category of normally linear Poincaré G-complexes is introduced, and some relevant definitions and background information is mentioned. Section II contains the construction of objects of this category which will be used to study equivariant splittings. Section III illustrates the theory applied to the special case of homotopy spheres to give another proof of the Anderson-Hambleton theorem. This section serves to motivate the generalization of this theorem in Section IV. (Theorem IV. 5), and the solution of the splitting problem up to concordance (Theorem IV. 1 , IV. 3 and IV. 7) with varying degrees of generality. We conclude the paper by a brief discussion of the algebraic obstructions which arise in the general splitting problem.

Finally, we would like to point out a few remarks and mention some features which are implicit in this particular choice of application for normally linear Poincaré complexes. First, our methods does not require "general positionality" or the so called "Gap Hypotheses" which have been used by most authors. Here, the reader will find a discussion of the problem of relaxing "general positionality" in the equivariant surgery problems in Reinhard Schultz' survey article and collection of problems [20]. Thus, the theories which use general-position-type assumptions do not apply to our situation. Secondly, we have considered non-simply-connected manifolds, not only to achieve a greater degree of generality, but also to illustrate new applications for the algebraic K-theoretic functor Wh_1^T of [8], [9] which is the relevant functor to capture such obstructions. We have postponed explicit computations of these obstructions as well as certain other surgery obstructions to a forthcoming paper. The reader, however, will find some results in this direction in [9].

The third point concerns the notion of quasisimple actions and their constructions. The homological hypotheses which are necessary in the splitting problem and "the extension problem" of [9] use $\mathbb{Z}_q\pi$-coefficients (local coefficients) where $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. When π is an infinite group, one cannot replace $\mathbb{Z}_q\pi$-coefficients with $\mathbb{Z}_{(q)}\pi$-coefficients, where $\mathbb{Z}_{(q)}$ is the integers localized at q. While the constructions of [9] are given for $\mathbb{Z}_q\pi$ (in order to provide necessary and sufficient conditions for the constructions to exist), they work as well with $\mathbb{Z}_{(q)}\pi$ replacing $\mathbb{Z}_q\pi$ everywhere. Thus, in all the homological conditions in this paper, one can replace $\mathbb{Z}_q$ by $\mathbb{Z}_{(q)}$; but the sufficient conditions obtained in this form will not be necessary anymore. S. Weinberger has independently studied "unextended homologically trivial actions" [23] (which is the analogue of our quasisimple actions for the case of $\mathbb{Z}_{(q)}\pi$-coefficients) using "Zabrodsky mixing". Weinberger's survey article in [24] contains further ideas and developments in conjunction with construction of actions. We refer the reader to [20] for articles of Schultz and Weinberger and their references for discussions of related results and problems.

Finally, to study G-actions on Poincaré complexes which are not quasisimple, one encounters completely new phenomena. The methods of constructions which assume that G acts trivially on homology do not apply to non-quasisimple actions. An alternative is to study such problems via "homotopy actions". This is the point of view of [7] (see also [4]). Construction of non-quasisimple normally linear Poincaré

G-complexes (using homotopy actions) and further applications will be discussed in a forthcoming paper of the author.

REMARK: It appears to us that the constructions of normally linear Poincaré complexes, (e.g. as in Section II) may be combined with Browder-Quinn's paper in Manifolds, Tokyo, 1973, (University of Tokyo Press 1975) to give a general set up for classification theory of quasi-simple actions. Moreover, Browder-Quinn theory can be potentially useful to analyze the G-manifold structures on normally linear Poincaré G-complexes. In this fashion, one may try to refine the results of our Section IV by analyzing the relevant surgery obstructions in the Browder-Quinn theory (instead of passing to concordance to bypass possibly non-zero obstructions).

SECTION I. PRELIMINARY NOTIONS:

Throughout this paper G is a finite group of order q, and we will work in the category of G-CW complexes, while G-actions on smooth manifolds are assumed to be smooth. The smoothness assumption is made only for convenience sake and most of the results, when appropriate, are true about more general types of action with some regulariy conditions, e.g. locally smooth PL actions, etc.

An earlier definition for a Poincaré G-complex was suggested by Frank Conolly [11], where all the homotopy analogues of the ingredients involved in a G-manifold were built into the definition of a so called "G-Poincaré complex". For our purposes, however, it is appropriate to introduce G-complexes which have inherited some linear structure on the regular neighborhoods of various strata. This restriction, in this case, makes it possible to translate the homotopy problems involving (non-free) G-manifolds into questions which involve the homotopy structure of the fixed point sets without losing the linear information naturally given for their normal bundles. Furthermore, we will discuss methods of construction for such G-complexes with this richer structure, and obtain positive answers in a variety of circumstances.

Let C be a category of Poincaré complexes (pairs). C could be the category of simple Poincaré complexes, or the category of finite Poincaré complexes, or a more general category, for instance [22]. We will fix C during the following discussion and suppress any reference to it unless it is necessary. For the applications, the context will determine the category $\mathcal{C}$.

I.1. DEFINITION: A normally linear Poincaré G-complex (pair) with one orbit type is a Poincaré complex (pair) in $\mathcal{C}$ in the ordinary sense (not necessarily connected). A normally linear Poincaré G-pair (X,Y) with $(k+1)$ orbit-types is defined inductively as follows. Let H be a maximal isotropy subgroup. Then $(G\cdot X^H, G\cdot Y^H)$ is required to be a Poincaré G-pair with one orbit type which has an equivariant regular neighborhood pair $(R,\partial_1 R)$ in (X,Y) such that:

(1) there exists a G-bundle ν over $G\cdot X^H$ such that $(R,\partial_1 R)$ is G-homeomorphic to $(D(\nu), D(\nu|G\cdot Y^H))$;

(2) there is a normally linear Poincaré G-pair $(C,\partial C)$ with k orbit types and a G-homeomorphism $f: S(\nu) \longrightarrow \partial_+ C \subset \partial C$ such that $X = C \cup_f D(\nu)$ and $Y = \partial_- C \cup D(\nu|G\cdot Y^H)$ where $\partial_- C = \overline{\partial C - \partial_+ C}$ and $f' = f|S(\nu|G\cdot Y^H)$.

REMARK: Normally linear Poincaré G-complexes defined above are different from Conolly's [11] G-Poincaré complexes in at least two different points. First, the Spivak normal fibre space of one stratum in the next is already given a linear structure. Second, the Poincaré embeddings of our definition are more manifold like in that the complement of one stratum in the next is also prescribed (subject to the appropriate identifications coming with the structure). As we shall illustrate in Section IV, this results in a great simplification of the construction of surgery problems.

ν is called the equivariant normal bundle of $G\cdot X^H$. *An isovariant normally linear map* is an equivariant map which preserves the isotropy types and the normal bundles (after the identification of regular neighborhoods and disk bundles). The G-homeomorphisms f and f' above are (G-cellular) isovariant normally linear maps of Poincaré pairs with k orbit types. It is possible to show inductively that for each subgroup $K \subseteq G$, (X^K, Y^K) is a Poincaré complex which is Poincaré embedded in (X,Y) , and its Spivak normal fibre space has a $N(K)$-linear structure. ($N(K)$ = normalizer of K in G).

Normally linear Poincaré G-complexes are constructed in [4],[5], [7],[9] in the semifree case. Smooth G-manifolds are normally linear

Poincaré G-complexes in a natural manner. We drop the prefix G whenever the context allows us to do so.

I.2. CONVENTION: All Poincaré complexes with G-actions are assumed to be normally linear Poincaré complexes. If $L \subsetneqq K \subseteq G$, $\dim X^K - \dim X^L > 2$. If X is connected, we assume that X^K is connected for all $K \subseteq G$. All manifolds are compact and all Poincaré complexes are finite.

We will study first the case of semifree actions which serve as a model for the inductive proofs of similar results for actions with several isotropy groups. However, the generalization of the results of the semifree case is not immediate, even in the case of actions on spheres (or disks) due to the fact that the fixed point sets of isotropy subgroups of composite order satisfy very little homological restrictions in general. In fact, Oliver's work [17] shows that in the case of disks, only certain Euler characteristic relationships are necessary (and sufficient). Therefore, it is inevitable to consider some restricted classes of actions where some minimal homological conditions are imposed on the fixed point sets of various isotropy subgroups.

A convenient category of G-complexes is the category of quasisimple actions.

I.2. DEFINITION: An action $\phi : G \times X \to X$ is called quasisimple if for each isotropy subgroup $K \subseteq G$, the action of $N(K)/K$ on the fundamental group of each component of X^K_α and, subsequently, $H_*(X^K_\alpha ; \mathbb{Z}_q \pi_1(X^K_\alpha))$ are trivial. Note that the triviality of the action of $N(K)/K$ on $\pi_1(X^K_\alpha)$ makes it possible to define unambiguously the action on the homology of X^K_α with local coefficients $\mathbb{Z}_q \pi_1(X^K_\alpha)$. (Recall that $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. One may also use $\mathbb{Z}_{(q)}$ systematically).

REMARKS: (1) Quasisimple actions were introduced and studied in [9].

(2) Replacing $\mathbb{Z}_q$ by $\mathbb{Z}_{(q)}$ in the above definition, for a free G-space X, quasisimplicity means that $\pi_1(X/G) \cong \pi_1(X) \times G$ and G acts trivially on the homology. This notion has been called "an unextended action" by S. Weinberger and studied in [23] independently.

<u>1.3. DEFINITION:</u> Let X be a connected G-CW complex, where G is a finite group of order q. X is called a simple G-space (and the action is called simple) if $(E_G \times_G X)_q$ is fibre homotopy equivalent to $(BG \times X)_q$. Here X_q denotes the localization of X which preserves $\pi_1(X)$ and localizes $\pi_i(X)$ for $i > 1$ at $\mathbb{Z}/q\mathbb{Z}$. Cf. [10] and [9] Section II.

In dealing with non-simply-connected complexes, it is necessary to consider simple homotopy types and simple homotopy equivalences. The equivariant generalizations of the Whitehead torsion are studied in [18],[15],[14]. To construct a G-action on a simply-connected finite complex X (up to homotopy type in the category of finite complexes) the projective class group $K_0(\mathbb{Z}G)$ and certain subgroups or subquotients play an important role (cf. [21],[17],[2],[1] etc.). If $\pi_1(X) \neq 1$, then the analogue of $K_0(\mathbb{Z}G)$ is an abelian group $Wh_1^T(\pi \subset \Gamma)$ where $\pi = \pi_1(X)$ and Γ is the extension $1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$ obtained from the action of G on $\pi_1(X)$ (whenever defined). $Wh_1^T(\pi \subset \Gamma)$ and its algebraic properties and topological applications are treated in [9], and an alternative definition in terms of the fibre of a transfer map between Whitehead spaces is given in [8].

We will briefly recall the definition and some properties of $Wh_1^T(\pi \subset \Gamma)$ when $\Gamma = \pi \times G$ (the case of quasisimple actions). Let A be the category whose objects are pairs (M, B) where M is a finitely generated $\mathbb{Z}\Gamma$-projective module which is free over π and B is a π-basis for M. Two objects are equivalent $(M,B) \sim (M',B')$ if there is a π-simple isomorphism $f : (M,B) \longrightarrow (M',B')$. Let $A' = A/\sim$ and consider the monoid structure on A' induced by direct sums (and disjoint union), taking $(0,\emptyset)$ as the neutral element. Then $(\mathbb{Z}\Gamma, G)$ generates the monoid of trivial elements T, and we define $Wh_1^T(\pi \subset \Gamma) = A'/T$. It is an abelian group which fits into a 5-term transfer exact sequence $Wh_1(\Gamma) \xrightarrow{tr} Wh_1(\pi) \xrightarrow{\beta} Wh_1^T(\pi \subset \Gamma) \xrightarrow{\alpha} \tilde{K}_0(\mathbb{Z}\Gamma) \xrightarrow{tr} \tilde{K}_0(\mathbb{Z}\pi)$. The homomorphism α is induced by the forgetful functor $(M,B) \longrightarrow M$. Furthermore, let $Wh_1^T(\pi; \mathbb{Z}_q) = K_1(\mathbb{Z}_q \pi)/\{\pm\pi\}$. Then one has a commutative diagram

$$\begin{array}{ccccc}
Wh_1(\pi) & \xrightarrow{\beta} & Wh_1^T(\pi \subset \Gamma) & \xrightarrow{\alpha} & \tilde{K}_0(\mathbb{Z}\Gamma) \\
 & \text{canon.} \searrow & \nearrow \gamma & & \\
 & & Wh_1(\pi; \mathbb{Z}_q) & &
\end{array}$$

where $\alpha \circ \gamma$ is a generalization of the Swan homomorphism (cf. [21]) $\sigma_G: (Z_q)^x \longrightarrow \widetilde{K}_0(ZG)$ (when $\pi = 1$).

A topological application of $Wh_1^T(\pi \subset \Gamma)$ is as follows. Suppose (X,Y) is a pair, $\pi_1(X) = \pi$, and X is a finite G-complex. Let $\phi : G \times Y \longrightarrow Y$ be a free quasisimple action, and let $H_*(X,Y;Z_q\pi) = 0$. Then there exists a free finite G-complex X' such that Y is an invariant subcomplex, and there exists a π-simple homotopy equivalence $f : X' \longrightarrow X$ rel Y if and only if $\gamma\tau(X,Y) = 0$ in $Wh_1^T(\pi \subset \pi \times G)$, where $\tau(X,Y)$ is the Reidemeister torsion of the pair (X,Y) (well-defined in $Wh_1(\pi;Z_q)$ due to the homological hypothesis). Cf. [9] Section I for further details.

<u>I.3. LEMMA</u>: Suppose X is a finite semifree simple G-complex. Then $H_*(X,X^G;Z_q\pi) = 0$ and $\gamma\tau(X,X^G) \in Wh_1^T(\pi \subset \pi \times G)$ vanishes, where $\pi = \pi_1(X)$.

<u>PROOF</u>: Cf. [9] Proposition II.3.

We extend the notion of admissible splittings of [1] to non-simply connected closed manifolds (Poincaré complexes). Let M^n be a closed manifold and let $M^n = M_1^n \cup M_2^n$ be a splitting so that $M_1 \cap M_2 = \partial M_2$. It is an "admissible splitting" if $\pi_1(\partial M_1) \cong \pi_1(M_i) \cong \pi_1(M) = \pi$ (similarly for Poincaré complexes).

<u>I.4. LEMMA</u>: Let $\phi : G \times M^n \longrightarrow M^n$ be semifree and suppose that $M = M_1 \cup M_2$ is an equivariant admissible decomposition of (M,ϕ) such that M_i are simple. Let $M^G = F$ and $M_i \cap F = F_i$. Then $H_*(M_i,F_i;Z_q\pi) = 0$ and $\gamma\tau(M_i,F_i) = 0$ for $i = 1,2, \pi = \pi_1(M)$.

<u>PROOF</u>: This follows from I.3.

∎

In the next section we will show how to construct normally linear Poincaré complexes to solve the equivariant splitting problem for closed G-manifolds on the level of homotopy.

SECTION II. SPLITTING UP TO HOMOTOPY:

As before, G is a finite group of order q . Let $\varphi : G \times \Sigma^n \to \Sigma^n$ be a smooth, semifree action on a homotopy sphere. In [1] Anderson and Hambleton studied criteria for the existance of equivariant homological symmetry of (Σ^n, ϕ) , i.e. $\Sigma^n = D_1^n \cup D_2^n$ where each D_i^n is an invariant disk and $H_j((D_1^n)^G) \cong H_j((D_2^n)^G)$ for all j . Roughly speaking, vanishing of a semi-characteristic type invariant characterizes (Σ^n, ϕ) which are homologically double in the above sense, provided that $n > 2 \dim \Sigma^G$. Anderson and Hambleton call this structure a (strong) balanced splitting.

Since any homotopy sphere is a twisted double, the results of [1] may be interpreted as finding obstructions to make a (given) "non-equivariant symmetry" into an equivariant one. Besides leading to the discovery of a new and interesting invariant of such semifree actions, this equivariant symmetry may be regarded as a homological regularity condition (i.e. similarity to the linear actions). From this perspective, it is natural to ask if such equivariant splittings exist for more general actions. In this section, we propose to study this question for closed manifolds under some homological restrictions which impose P.A. Smith Theoretic conditions on the fixed point sets of isotropy subgroups. Our approach is to find invariants which characterize the existence of equivariant splittings on the level of normally linear Poincaré complexes, thus reducting the problem to an equivariant surgery problem. Since the fixed-point sets of non-trivial subgroups are, in general, non-simply connected, we will study the problem with special attention to the fundamental group. The following theorem gives necessary and sufficient conditions for the existence of equivariant splittings in the category of normally linear Poincaré complexes, with semifree actions. The general case is stated separately and its proof is an elaboration of the arguments for the semifree case.

II.1. THEOREM: *Suppose $\phi : G \times X \longrightarrow X$ is a quasisimple semifree action such that (X, ϕ) is a normally linear finite Poincaré complex with $(X, \phi)^G = F$, $\nu(F \subset X) = \nu$, and a (non-equivariant) admissible splitting $X = X_1 \cup X_2$, $F \cap X_i = F_i$. Suppose (1) $H_*(X_i, F_i; Z_q\pi) = 0$ and (2) $\gamma\tau(X_i, F_i) \in Wh_1^T(\pi \subset \pi \times G)$ vanishes. Then there exists a quasi-simple semifree normally linear finite Poincaré G-complex X' with the following properties: (a) $(X')^G = F$, $\nu(F \subset X') = \nu$; (b) X' has an* equivariant admissible splitting $X' = X_1' \cup X_2'$, $X_i \cap F = F_i$ and X_i' are simple; (c) there exists a normally linear isovariant map $f : X' \longrightarrow X$ which induces a π-simple homotopy equivalence; (d) X_i' and $\partial X_i'$ are π-simple homotopy equivalent to X_i and ∂X_i rel F_i and ∂F_i respectively. Furthermore, the hypotheses (1) and (2) above are necessary for the existence of such X' .

PROOF: Since X is normally linear, there exists a Poincaré pair $(C,\partial C)$ with a free G-action, such that $\partial C = S(\nu)$ and $X = D(\nu) \cup C$ (after appropriate identifications.) Let $C_i = C \cap X_i$, and let $\partial_- C_i = \partial C \cap X_i$, $\partial_+ C_i = C \cap \partial X_i$, $\partial_0 C_i = \partial_+ C_i \cap \partial_- C_i$. Note that $\partial_0 C_1 = \partial_0 C_2$ and $\partial_+ C_1 = \partial_+ C_2$; denote them by ∂_0 and ∂_+ respectively. Thus we have the following diagram

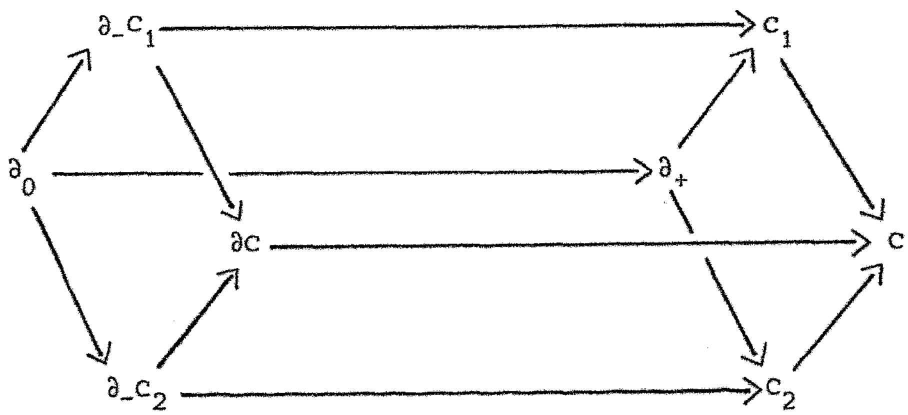

DIAGRAM (D)

in which not all maps are equivariant. If X' exists with the desired properties, we can write $X' = C' \cup D(\nu)$ and obtain a diagram (D') involving C', C_i' and the analoguous boundary decompositions in which all maps are equivariant. Furthermore, we will get a map of diagrams (D') $\longrightarrow$ (D) with the induced map $\partial_0' \longrightarrow \partial_0$, $\partial_\pm' \longrightarrow \partial_\pm$, $\partial C' \longrightarrow \partial C$, and $C' \longrightarrow C$ being the identity or an equivariant π-simple homotopy equivalence, as it is clear from the context and the requirements (a) - (d) above.

Let us use an asterisks to denote orbit spaces (e.g. $X^* = X/G$) and a bar to denote a covering with the deck transformation group G

(e.g. $\bar{C}^* = C$ in the above situation). Thus we look for a diagram (D'^*) of orbit spaces in which the spaces $C_i'^*$ and $\partial_+'^*$ as well as the dotted arrows are to be determined:

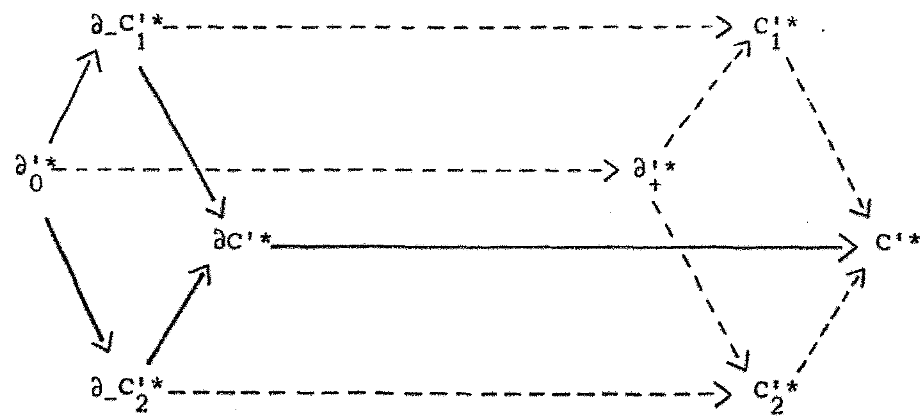

DIAGRAM (D'*)

The left side and the right side faces of the parallelograms in (D) , (D') and (D'*) are push-outs with respective push-out maps, and we denote them by (LD) , (RD), (LD') , etc. Moreover, in (D'*) we have the following equalities up to homotopy: $C'^* = C^*$, $\partial_0'^* = \partial_0^*$, $\partial_-C_i'^* = \partial_-C_i^*$ and $\partial C'^* = \partial C^*$ and the appropriate maps are induced by the corresponding maps in (D) .

In the terminology of [9] Theorem V.1, we wish "to push forward" the free action from the push out diagram of free G-spaces (LD) to the corresponding diagram (RD) after possibly replacing (RD) by homotopy equivalent complexes. Since the constructions of [9] are sufficiently functorial, they apply to this situation. Briefly, note that $H_*(C_i,\partial_+C_i;Z_q\pi) = H_*(C_i,\partial_-C_i;Z_q\pi) = H_*(X_i,F_i;Z_q\pi) = 0$ by Poincaré duality, excision and hypothesis (1) of the Theorem. Further, the quasisimplicity condition ensures that the scheme of [9] applies to construct the appropriate localizations of the diagrams

$$\begin{array}{ccc} \partial_-C_i & \longrightarrow & C_i \\ \uparrow & & \uparrow \\ \partial_0 & \longrightarrow & \partial_+ \end{array}$$

and then take push-outs. The finiteness obstruction as well as the Whitehead torsion obstruction for choosing C_i' to be finite and π-simple homotopy equivalent to C_i, is the image of the Reidemeister torsion $\tau(X_i, F_i)$ in $Wh_1^T(\pi \subset \pi \times G)$, and it vanishes by hypothesis (2). It follows from the duality of the Reidemeister torsion [16] that the corresponding obstructions for choosing ∂_+' to be finite and (equivariantly) π-simple homotopy equivalent to ∂_+ vanishes as well (cf. [9] Theorem 1.13). The existence of an equivariant π-simple homotopy equivalence $(C', \partial C') \longrightarrow (C, \partial C)$ and π-simple homotopy equivalence of C_i and C_i' follows from the constructions and the functoriality of push-outs.

The necessity of conditions (1) and (2) of the Theorem for existence of X' together with the appropriate equivariant splitting follows as in [9] Section II.

■

Equivariant splittings of actions with two isotropy types and semifree actions can be treated in a similar fashion. This observation allows one to generalize Theorem II.1 to actions with several isotropy types, provided that the fixed point sets of adjacent strata are related to each other in the same manner that the stationary point set of G and the free stratum are related in the semifree case. The condition of quasisimplicity as in Definition I.2 ensures that this is the case. (The hypotheses of the following theorem may be relaxed at the expense of introducing more complicated notions and longer statements, but we will not do this). The proof of this theorem uses an inductive argument similar to II.1 and we will omit it.

<u>II.2. THEOREM</u>: Let (X, ϕ) be a finite G-Poincaré complex. Suppose $X = X_1 \cup X_2$, and denote $F_i(K) = X^K \cap X_i$. Assume that the splittings $F_1(K) \cup F_2(K) = X^K$ are admissible for all isotropy subgroups $K \subseteq G$ such that $H_*(F_i(K), F_i(L); Z_q\pi_1(F_i(K))) = 0$ and $\gamma\tau(F_i(K), F_i(L)) \in Wh_1^T(\pi_1(F_1(K)) \subset \pi_1(F_i(K)) \times G)$ vanish. Then there exists a finite G-Poincaré complex (X', ψ) with $(X', \psi)^G = (X, \phi)^G$ and an equivariant admissible splitting $X' = X_1' \cup X_2'$ such that (a) there exists a normally linear isovariant simple homotopy equivalence $f : X' \longrightarrow X$ extending the inclusion $X'^G = X^G \subset X$; (B) X_i and X' and $X_i' \cap X'^K$ are simple homotopy equivalent to X_i and $X_i \cap X^K$ repectively and $X'^G \cap X_i' = X^G \cap X_i$.

SECTION III: A SPECIAL CASE

In the special case where M^n is a homotopy sphere, an equivariant splitting is obtained as an application of II.1, or by a direct argument. This yields another proof for a Theorem of Anderson-Hambleton [1]. We will mention this special case separately to illustrate the theory in a concrete case.

III.1. THEOREM: Let Σ^n be a homotopy sphere, $\phi : G \times \Sigma^n \longrightarrow \Sigma^n$ a semifree action and $F^k = \Sigma^G$ where $\nu = \nu(F \subset \Sigma)$ and $\dim \nu > k$. Given a splitting $F = F_1 \cup F_2$, there exists a corresponding equivariant splitting $\Sigma^n = D_1^n \cup D_2^n$ into disks such that $D_i^n \cap F = F_i$ if and only if $\bar{H}_*(F_i; \mathbb{Z}_q) = 0$ and $\mathcal{O}(F_i) = 0$ in $\tilde{K}_0(\mathbb{Z}G)$.

PROOF: Choose $X_1^n \subset \Sigma^n$ to be diffeomorphic to D^n and $X_1 \cap F = F_1$ and $\partial X_1 \cap F = \partial F_1$. This follows easily from handle body theory and general positionality, since $n > 2k$, and we are working non-equivariantly. By Theorem II.1 we have a normally linear finite Poincaré complex X_1' such that $(X_1', \partial X_1')^G = (F_1, \partial F_1)$, and $\nu(F_1 \subset X_1') = \nu|F_1$, and there is an isovariant map $f_1 : X_1' \longrightarrow \Sigma$ which extends the inclusion on $D(\nu|F_1)$. Since $H_i(F_1) = 0$ for $i \geq k-1$, it follows that X_1' is obtained from $D(\nu)$ by adding free G-cells of dimension at most k. Thus, f_1 can be deformed into an isovariant embedding extending the inclusion of $D(\nu|F_1)$. Let R be an equivariant regular neighborhood of $f_1(X_1') \cup D(\nu)$ in Σ^n. Then closure $(R - D(\nu|F_2))$ is diffeomorphic to D_1^n and Σ^n is equivariantly split as $D_1^n \cup D_2^n$ where $D_2^n = \Sigma^n - \mathrm{int}(D_1^n)$. The necessity of these conditions follows easily as in [1] or [2] Section II.

III.2. REMARK: The existence of X_1' follows from a direct argument, by attaching free G-cells of dimension $\leq k$ to $S(\nu|F_1)$ as in [2] II.V or [3] Section II. Then the equivariant map $f_1 : X_1' \longrightarrow \Sigma^n$ extending the inclusion $D(\nu|F_1) \longrightarrow \Sigma^n$ is a direct consequence of obstruction theory, and it can be deformed rel $D(\nu|F_1)$ to an isovariant map using general positionality of F.

SECTION IV. SPLITTING UP TO CONCORDANCE:

In this Section we use the existence of equivariant splittings of Section II to find equivariant splittings of a G-manifold (M,ϕ) based on a given non-equivariant splitting. This illustrates the construction of surgery problems from a given normally linear Poincaré G-complex.

When the appropriate obstructions for the existence of an equivariant splitting in the category of normally linear Poincaré complexes vanish, we obtain (X',ϕ') which is isovariantly π-simple homotopy equivalent to (M,ϕ). Next, we return to the category of G-manifolds by smoothing (X',ϕ') equivariantly, while preserving the splitting up to equivariant homotopy. The result will be (M',ψ) which is isovariantly π-simple homotopy equivalent to (M,ϕ) (relative to an equivariant regular neighborhood of $M^G = M'^G$). Rather than a detailed analysis of the relevant surgery exact sequence (leading to the surgery obstructions in order to arrange (M',ψ) to be G-diffeomorphic to (M,ϕ) and inherit the desired splitting from (X',ϕ')), we pass to a restricted concordance in order to get a positive answer. Namely, we change the action on the free part of (M,ϕ) rel $S(\nu(M^G))$ to get (M,ψ) concordant to (M,ϕ)(rel M^G) such that (M,ψ) is equivariantly split as desired.

If $\pi_1(M) = 1$, then this change in action is merely taking the equivariant connected sum of (M,ϕ) and an "almost linear" sphere (S^n,σ). Thus in this case, the G-homeomorphism type of (M,ϕ) is not changed in order to be equivariantly split. Again we give the proof in the case of semifree actions and only state the general case.

IV.1. THEOREM: Let $\phi : G \times M^n \longrightarrow M^n$ be a quasisimple smooth semifree action with $(M,\phi)^G = F^k$, $\nu(F \subset M) = \nu$ and a (non-equivariant) admissible splitting of the closed manifold $M = M_1 \cup M_2$, $M_i \cap F = F_i$. Assume that (1) $H_*(M_i,F_i;Z_q\pi) = 0$, and (2) $\gamma\tau(M_i,F_i) \in Wh_1^T(\pi \subset \pi \times G)$ vanishes. Then there exists a quasisimple semifree G-action on M, say $\psi : G \times M \longrightarrow M$, such that: (a) (M,ψ) is concordant to (M,ϕ) relative to F; (b) (M,ψ) has an equivariant splitting $M = M_1' \cup M_2'$ where M_i' are simple, $M_i' \cap F = F_i$ and M_i' and $\partial M_i'$ are π-simple homotopy equivalent to M_i and ∂M_i respectively. Furthermore, conditions (1) and (2) are necessary for the existence of (M,ψ).

PROOF: Since by Theorem II.1 the conditions (1) and (2) are necessary for the existence of equivariant splittings in the category of normally linear Poincaré complexes, (cf. II.1) we need to show only their sufficiency.

First, we construct the concordance on the level of normally linear Poincaré complexes. Thus we have an equivariantly split G-complex X' which satisfies all the stated properties if we replace X by M in Theorem II.1.

IV.2. PROPOSITION: Under the hypotheses of IV.1, there exists a normally linear G-Poincaré pair $(Y,\partial Y)$ such that: (1) $Y^G = F \times [0,1]$ and $\nu(Y^G \subset Y) = \nu \times [0,1]$; (2) $\partial Y = M \cup X'$ where the induced action on M is ϕ and on X' is the action given by Theorem II.1.

PROOF: Let $f : X \longrightarrow M$ be the isovariant map of II.1, and let Y be the mapping cylinder of f .

∎

We continue the proof of IV.1 by finding a normal invariant for $(Y,\partial Y)$ which restricts to the natural one given on $M \subset \partial Y$. Using the normal linearity, let $Y = D(\nu \times [0,1]) \cup Y'$ where $\partial Y' = C \cup S(\nu \times [0,1]) \cup C'$ using the notation of II.1, and Y' has a free quasisimple G-action.

Let BG be Stasheff's classifying space for stable spherical fibrations. As before, we denote the orbit space by an asterisk: $X^* \equiv X/G$. Let $\alpha : Y'^* \longrightarrow BG$ be the classifying map for the Spivak spherical fibration of Y'^* . Then $\alpha | C^*$ lifts to BO since C^* is a manifold. Also this lift extends over $S(\nu \times [0,1])^*$. The obstruction to extending this to a lift of α to BO is an element $\chi \in h^*(Y'^*, C^* \cup S(\nu \times [0,1])^*)$, where h^* = generalized cohomology theory of G/O . Since $H^*(Y', C \cup S(\nu \times [0,1]); Z_q) = 0$ by excision, the Cartan-Leray spectral sequence for the covering pair $(Y',C) \longrightarrow (Y'^*,C^*)$ collapses and $H^\circ(B\ , h^*(Y',C)) \cong h^*(Y'^*,C^*)$. From the hypothesis of quasisimplicity, it follows that G acts trivially on $h^*(Y',C)$ (cf. [9] II.6 and Lemma II.10) and $h^*(Y'^*,C^*) \cong h^*(Y',C)$. Thus χ is q-divisible. On the other hand the transfer $tr(\chi) \in h^*(Y',C)$ vanishes, since Y' is (non-equivariantly) homotopy equivalent to $C \times [0,1]$.

Therefore $\chi = 0$ and α lifts to BO.

This yields the desired normal invariant, say $f : (W^{n+1}, \partial W) \longrightarrow (Y'^*, \partial Y'^*)$ such that $\partial W = C^* \cup S(\nu \times [0,1])^* \cup V^n$ and $f \mid C^* \cup S(\nu \times [0,1])^*$ is the inclusion. The splitting $C'^* = C_1'^* \cup C_2'^*$ (as given in II.1) induces an equivariant decomposition $V = V_1 \cup V_2$, $V_1 \cap V_2 = V_0 = \partial V_1 = \partial V_2$. Let $f_i \mid V_i$, $i = 0,1,2$. The surgery obstruction to making $f_1 : (V_1, \partial V_1) \longrightarrow (C_1'^*, \partial C_1'^*)$ into a homotopy equivalence rel $S(\nu \times 1)^*$ such that $\overline{f}_1 : (\overline{V}_1, \partial \overline{V}_1) \longrightarrow (C_1', \partial C_1')$ is a π-simple homotopy equivalence rel $S(\nu \times 1)$ vanishes by [22] Theorem 3.3 (cf. [9] Theorem II.7). Let N_1^{n+1} be this normal cobordism, and add N_1^{n+1} to V^{n+1} along V_1 to obtain a new normal map (after smoothing corners, etc.). Then $f' : W' \longrightarrow Y'^*$ with $\partial W' = C^* \cup S(\nu \times [0,1])^* \cup V'$, $V' = V_1' \cup V_2'$, and $f' \mid V_1 : (V_1', \partial V_1') \longrightarrow (C_1'^*, \partial C_1'^*)$ is a homotopy equivalence rel $S(\nu \times 1)^* \cap \partial C_1'$ (and the induced map on the G-coverings is π-simple). Next, we can do surgery on f' rel $C^* \cup S(\nu \times [0,1])^* \cup V_1'$ to make it into a homotopy equivalence of pairs, applying again Wall's Theorem ([22] Theorem 3.3) since $\pi_1(C_2') \cong \pi_1(Y') \cong \pi$. Call the new map $f'' : W'' \longrightarrow V'$, where $\partial W'' = C^* \cup S(\nu \times [0,1])^* \cup V''$ and $V'' = V_1'' \cup V_2''$, $V_1'' = V_1'$ and $f'' \mid V_2''$ is also a homotopy equivalence (and $f'' : \overline{V}'' \longrightarrow C'$ and $f'' \mid \overline{V}_2''$ are π-simple equivalences). Adding $D(\nu \times [0,1])$ back to $\overline{W}''$ along $S(\nu \times [0,1])$ yields the desired concordance. (The reader can easily verify that $\overline{W}''$ is an s-cobordism with a free G-action, and $M_1' = V_1' \cup D(\nu \times 1 \mid F_1)$ and $M_2' = \overline{V}_2'' \cup D(\nu \times 1 \mid F_2)$ yield the equivariant splitting required by the Theorem).

■

<u>IV.3.3 THEOREM</u>: Suppose $\pi_1(M) = 1$ in IV.1. Then there exists an almost linear sphere (S^n, σ) such that the equivariant connected sum $(M, \psi) = (M, \phi) \# (S^n, \sigma)$ admits an equivariant splitting as in the conclusion of Theorem IV.1.

<u>PROOF</u>: Let $\tau \in Wh_1(G)$ be the torsion of the relative h-cobordism W'' with respect to C/G. Choose $\chi \in M^G$ and the linear sphere $S(T_\chi M \oplus R) = S^n$ where the tangent space $T_\chi M$ has the linear representation induced by ϕ. Let K^{n+1} be the concordance $S^n \times [0,1]$ obtained by adding free 2-handles and 3-handles to the free stratum of S^n so that the resulting G-h-cobordism has torsion $-\tau$. The new

equivariant concordance $M \times [0,1] \# S^n \times [0,1]$ (where the connected sum is along an arc $\{x\} \times [0,1]$ in the stationary point sets) is actually an equivariant s-cobordism, and hence a product. But $\partial(M \times [0,1] \# S^n \times [0,1]$ with the induced action is G-diffeomorphic to $(M,\phi) \cup (M,\phi \# \sigma)$ where σ is the "alomost linear" action induced on $S^n \times \{1\}$ in the concordance $S^n \times [0,1]$. (See [5]).

<u>IV.4. COROLLARY</u>: Given (M,ϕ) as in IV.1, and so that $\pi_1(M) = 1$, there exists a smooth action $\psi : G \times M \longrightarrow M$ such that (M,ψ) has an equivariant splitting as in the conclusion of Theorem IV.1, and (M,ϕ) is G-homeomorphic to (M,ψ) .

■

If M^n is a homotopy sphere, then we get an equivariant decomposition into disks, thus generalizing the Anderson-Hambleton Theorem [1] Theorem A. Note that the methods of [1] which are based on general position arguments do not apply here, since codimensions could be quite small.

<u>IV.5. THEOREM</u>: Let $\phi : G \times \Sigma^n \longrightarrow \Sigma^n$ be a semifree action with $(\Sigma^n,\phi)^G = F$ and $\nu(F \subset \Sigma) = \nu$, $\dim \nu > 2$. Assume that $\Sigma^n = D_1^n \cup D_2^n$ with $D_i^n \cap F = F_i$ is a non-equivariant splitting. Then there exists a smooth semifree G-sphere (Σ,ψ) such that (Σ,ψ) is G-homeomorphic to (Σ,ϕ) , $(\Sigma,\psi)^G = F$, $\psi | G \times \nu = \phi | G \times \nu$, and (Σ,ψ) has an equivariant splitting into disks $\Sigma = D^n \cup D^n$ with $(D^n)^G = F_i$ if and only if $\sum_{j>0} (-1)^j \sigma_G (H_j(F_i)) = 0$ in $\tilde{K}_0(ZG)$, $i = 1$ or 2 .

■

<u>IV.6. REMARK</u>: Suppose F^k is a mod q homology sphere. Then Anderson and Hambleton prove that the necessary and sufficient conditions for existence of a "balanced splitting" of F^k (i.e. F is homologically a double) is that a certain semicharacteristic type invariant vanishes (cf. [1] Theorem B). Thus Theorem IV.5 can be applied to generalize this result of Anderson-Hambleton and improve their dimension hypothesis in Theorem B of [1] from $\dim \nu \geq k+2$ to $\dim \nu > 2$.

As in Section II, we can generalize the above results to actions with many isotropy subgroups. The proofs of the semifree cases can be adapted to serve as the inductive step of the following theorem. The normally linear Poincaré G-complex which is the homotopy model in this case is provided by Theorem II.2. We omit the details.

IV.7. THEOREM: Let (X^n, ϕ) be a smooth closed G-manifold with an admissible splitting $X = X_1 \cup X_2$, satisfying all the hypotheses of Theorem II.2. Then there exists a smooth G-action $\psi : G \times X \longrightarrow X$ such that (X, ψ) is concordant to (X, ϕ) rel X^G and (X, ψ) has an equivariant splitting $X = X_1' \cup X_2'$ which satisfies the conclusions (a) and (b) of Theorem II.2.

SECTION V. REALIZATION OF OBSTRUCTIONS:

One may use normally linear Poincaré complexes to construct actions with admissible splittings which do not admit necessarily equivariant splittings. Again, the results of this section may be specialized to the situation considered by Anderson-Hambleton [1] to give an alternative proof of their Theorem C. The important algebraic calculations of the hyperbolic map in the Rothenberg-Ranicki exact sequence for the quaternionic groups are due to Anderson-Hambleton ([1] Proposition 5.2 and [13] Lemma 6.1) who applied it in their examples of actions on spheres without balanced splittings. These calculations are used to take care of the case where the 2-Sylow subgroup is the quaternion group of order 8, denoted by Q_8.

V.1. THEOREM: Let M^n be a simply-connected closed manifold, and $M^n = M_1^n \cup M_2^n$ be an admissible splitting. Suppose that $F^k \subset M$ is a closed submanifold with normal bundle ν which admits a G-bundle structure with a free representation on each fibre, where G is a subgroup of SU(2) whose 2-Sylow subgroup is either (i) cyclic or (ii) Q_8 and $K \not\equiv 1 \bmod 4$. Assume that (M_0^{n+1}, F_0^{k+1}) is a manifold pair such that $\partial(M_0, F_0) = (M, F)$ satisfying the hypotheses:
(1) for $i = 1,2$ $\pi_1(M_i) = 1$ and (2) $H_*(M_i, F_i; Z_q) = 0$, where $F_i = M_i \cap F$, $i = 1,2$. Then there exists a quasisimple semifree action $\phi : G \times M' \longrightarrow M'$ such that $M'^G = F$, where M' is homotopy equivalent to M. Further, (M', ϕ) has an equivariant splitting $M_1' \cup M_2'$, $M_i' \cap F = F_i$ if and only if $\Sigma(-1)^j \sigma_G H_j(M_1, F_1) = 0$ in $\tilde{K}_0(ZG)$.

The proof of this theorem and further applications of normally linear Poincaré complexes will appear elsewhere.

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