ITEP-TH-20/05

Painlevé VI, Rigid Tops and Reflection Equation

A.M.Levin
Max Planck Institute of Mathematics, Bonn, Germany,
Institute of Oceanology, Moscow, Russia,
e-mail alevin@wave.sio.rssi.ru
M.A.Olshanetsky
Max Planck Institute of Mathematics, Bonn, Germany,
Institute of Theoretical and Experimental Physics, Moscow, Russia,
e-mail olshanet@itep.ru
A.V.Zotov
Institute of Theoretical and Experimental Physics, Moscow, Russia,
e-mail zotov@itep.ru

Abstract

We show that the Painlevé VI equation has an equivalent form of the non-autonomous Zhukovsky-Volterra gyrostat. This system is a generalization of the Euler top in $\mathbb{C}^3$ and include the additional constant gyrostat momentum. The quantization of its autonomous version is achieved by the reflection equation. The corresponding quadratic algebra generalizes the Sklyanin algebra. As by product we define integrable XYZ spin chain on a finite lattice with new boundary conditions.

January 27, 2006

Contents

1 Introduction

In this paper we discuss a few issues related to isomonodromy problems on elliptic curves, integrable systems with the spectral parameter on the same curves and the XYZ spin-chain on a finite lattice. Our main object is the Painlevé VI equation (PVI). It is a second order ODE depending on four free parameters $(\alpha,\beta,\gamma,\delta)$

$$\frac{d^2X}{dt^2}=\frac{1}{2}\left(\frac{1}{X}+\frac{1}{X-1}+\frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{X-t}\right)\frac{dX}{dt}+$$
$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{X^2}+\gamma\frac{t-1}{(X-1)^2}+\delta\frac{t(t-1)}{(X-t)^2}\right). \qquad (1.1)$$

PVI was discovered by B.Gambier [9] in 1906. He accomplished the Painlevé classification program of the second order differential equations whose solutions have not movable critical points. This equation and its degenerations $PV-PI$ have a lot of applications in Theoretical and Mathematical Physics. (see, for example [40]).

We prove here that PVI can be write down in a very simple form as ODE with a quadratic non-linearity. It is a non-autonomous version of the SL$(2,\mathbb{C})$ *Zhukovsky-Volterra gyrostat* (ZVG) [41, 38]. The ZVG generalizes the standard Euler top in the space $\mathbb{C}^3$ by adding an external constant rotator momentum. The ZVG equation describes the evolution of the momentum vector $\vec{S}=(S_1,S_2,S_3)$ lying on a SL$(2,\mathbb{C})$ coadjoint orbit. We consider Non-Autonomous Zhukovsky-Volterra gyrostat (NAZVG)

$$\partial_\tau\vec{S}=\vec{S}\times(\vec{J}(\tau)\cdot\vec{S})+\vec{S}\times\vec{\nu}' \qquad (1.2)$$

where $\vec{J}(\tau)\cdot\vec{S}=(J_1S_1,J_2S_2,J_3S_3)$. Three additional constants $\vec{\nu}'=(\nu_1',\nu_2',\nu_3')$ form the gyrostat momentum vector, and the vector $\vec{J}=\{J_\alpha(\tau)\}$, $(\alpha=1,2,3)$ is the inverse inertia vector depending on the "time" τ in the following way. Let $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ be the elliptic curve, and $\wp(x,\tau)$ is the Weierstrass function. Then $J_\alpha(\tau)=\wp(\omega_\alpha,\tau)$, where ω_α are the half-periods $(\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2})$. The constants ν_α along with the value of the Casimir function $\sum_\alpha S_\alpha^2=(\nu'_0)^2$ are expressed through the four constants $(\alpha,\beta,\gamma,\delta)$ of the PVI. If the gyrostat momentum vanishes, $\vec{\nu}'=0$ the ZVG is simplified to *the non-autonomous Euler Top* (NAET) equation

$$\partial_\tau \vec{S} = \vec{S}\times(\vec{J}(\tau)\cdot\vec{S})\,. \tag{1.3}$$

To establish the connection between the PVI eq. (1.1) and (1.2) we start with the elliptic form of (1.1) [28, 22]

$$\frac{d^2u}{d\tau^2} = -\sum_{a=0}^{3}\nu_a^2\wp'(u+\omega_a,\tau)\,,\quad (\omega_a=(0,\omega_\alpha))\,. \tag{1.4}$$

The interrelation between two sets of the constants ν_a and ν'_a is explained in Sections 5.3 and 5.4. We call this equation EPVI to distinguish it from (1.1). By replacing τ on the external time t we come to the BC_1 *Calogero-Inozemtsev system* (CI)

$$\frac{d^2u}{dt^2} = -\sum_{a=0}^{3}\nu_a^2\wp'(u+\omega_a,\tau)\,. \tag{1.5}$$

This equation is the simplest case of the integrable BC_N CI hierarchy [13]. When the constants are equal, (1.5) describes the two-body elliptic Calogero-Moser (CM) system in the center of the mass frame

$$\frac{d^2u}{dt^2} = -\nu^2\wp'(2u,\tau)\,. \tag{1.6}$$

In this case EPVI (1.4) assumes the form

$$\frac{d^2u}{d\tau^2} = -\nu^2\partial_u\wp(2u,\tau)\,. \tag{1.7}$$

The genuine interrelations between integrable and isomonodromy hierarchies arise on the level of the corresponding linear systems. The linear equations of the isomonodromy problem look like a quantization (the Whitham quantization) of the linear problem for the integrable hierarchy [19]. While for the integrable systems the Lax matrices are sections of the Higgs bundles (one forms on the spectral curve), they become the holomorphic components of the flat connections for the monodromy preserving equations. Recently, the Lax representation for (1.5) was proposed in Ref. [42]. It allows us to construct the linear system for EPVI (1.4).

In our previous work [21] we proposed a transformation of the linear system for the N-body CM system to the linear system for the integrable $\mathrm{SL}(N,\mathbb{C})$ elliptic Euler-Arnold top [29]. It is the so-called *Hecke correspondence* of the Higgs bundles. It is accomplished by a singular gauge transform (*the modification*) of the Lax equations. This action on the dynamical variables is a symplectomorphisms. In the simplest case it provides a change of variables from the two particle CM (1.6) to the $\mathrm{sl}(2,\mathbb{C})$ autonomous Euler Top (ET)

$$\partial_t\vec{S} = \vec{S}\times(\vec{J}\cdot\vec{S})\,,\quad (\vec{J}\cdot\vec{S})=(J_\alpha S_\alpha)\,. \tag{1.8}$$

It opens a way to define the Lax matrix $L^{ET}(z)$ of the ET from L^{CM}. The both models contain a free constant which is the coupling constant for CM and the value of the Casimir function of the $\mathrm{SL}(2,\mathbb{C})$ coadjoint orbit for ET. It should be mentioned that the analogous transformation was used in [11, 35, 37] for other purposes.

In the similar way we prove that the autonomous $\mathrm{SL}(2,\mathbb{C})$ ZVG

$$\partial_t \vec{S} = \vec{S} \times \hat{J}\vec{S} + \vec{S} \times \vec{\nu}' \,. \tag{1.9}$$

is derived from the BC_1 CI eq. (1.5) via the same modification. In this way we obtain the Lax matrix $L^{ZVG}(z)$ from the Lax matrix of the CI system. The change of variables is given explicitly. For SO(3) ZVG the Lax pair with the rational spectral parameter was constructed in Ref. [8] and discussed in Ref. [4]. In our case the Lax pair depends on the elliptic spectral parameter.

In fact, the modification can be applied to the isomonodromy problem [1, 26]. It acts on connections and, in particular, transforms (1.7) to NAET (1.3) and the generic EPVI (1.4) to the NAZVG (1.2). We present the explicit dependence $\vec{S}(u, \partial_\tau u)$. In this way we establish the equivalence between PVI (1.1) and NAZVG (1.2).

There exists another way to define the Lax matrix $L^{ZVG}(z)$ and thereby to derive ZVG (1.9) that will be used in this paper. The starting point is a special Elliptic Garnier-Gaudin system (EGG). EGG is an example of the Hitchin systems [21]. It is derived from the rank two quasi-parabolic Higgs bundle [30] over Σ_τ of degree one. We assume that the Higgs field has simple poles at the half-periods ω_a, $(a = 0, \dots, 3)$. The invariant part of the Higgs field with respect to the involution $z \to -z$ leads to $L^{ZVG}(z)$.

Finally, we can derive ZVG starting with the quantum reflection equation. Consider first the quantization of $\mathrm{SL}(2,\mathbb{C})$ ET. It can be performed by the quantum exchange relations with the Baxter R-matrix [2]:

$$R(z,w)\hat{L}_1^{ET}(z)\hat{L}_2^{ET}(w) = \hat{L}_2^{ET}(w)\hat{L}_1^{ET}(z)R(z,w)\,, \tag{1.10}$$

where $\hat{L}^{ET}(z)$ is the quantum Lax matrix of the elliptic top. This equation is equivalent to the Sklyanin algebra [33]. We show that the quantum Lax matrix for ZVG satisfies the reflection equation introduced in [31]:

$$R^-(z,w)\hat{L}_1^{ZVG}(z)R^+(z,w)\hat{L}_2^{ZVG}(w) = \hat{L}_2^{ZVG}(w)R^+(z,w)\hat{L}_1^{ZVG}(z)R^-(z,w)\,. \tag{1.11}$$

The corresponding quadratic algebra generalizes the Sklyanin algebra. In the classical limit (1.11) yields two Poisson structures for ZVG:

$$\{L_1(z), L_2(w)\}_2 = \frac{1}{2}[L_1(z)L_2(w), r^-(z,w)] - \frac{1}{2}L_1(z)r^+(z,w)L_2(w) + \frac{1}{2}L_2(w)r^+L_1(z)\,, \tag{1.12}$$

$$\{L_1(z), L_2(w)\}_1 = \frac{1}{2}[L_1(z) + L_2(w), r^-(z,w)] - \frac{1}{2}[L_1(z) - L_2(w), r^+(z,w)]$$

which are compatible as in the case of the Sklyanin algebra [16]. The first type of brackets generalizes the classical Sklyanin algebra while the second is just Poisson-Lie brackets. The coefficient $\frac{1}{2}$ in (1.12) comes from the statement that these brackets are derived from the standard brackets

$$\{L_1(z), L_2(w)\}_2 = [L_1(z)L_2(w), r^-(z,w)]\,, \tag{1.13}$$

$$\{L_1(z), L_2(w)\}_1 = [L_1(z) + L_2(w), r^-(z,w)]$$

by the Poisson reduction procedure for the constraints $L(z) + L^{-1}(-z)\det L(-z) = 0$ and $L(z)+L(-z) = 0$ for the quadratic and linear brackets correspondingly. This procedure however will not be discussed here.

In [31] the reflection equation was used to construct an integrable version of the XYZ spin-chain on a finite lattice. Following this recipe we obtain the XYZ spin-chain with the quantum ZVG on the boundaries. Three additional constants here combine into the vector of magnetic field. The classical Hamiltonian is presented in Proposition 8.3. As far as we know the obtained model was not discussed earlier.

Acknowledgments.
The work is supported by the grants NSh-1999-2003.2 of the scientific schools, RFBR-03-02-17554 and CRDF RM1-2545. The work of A.Z. was also partially supported by the grant MK-2059.2005.2. We are grateful for hospitality to the Max Planck Institute of Mathematics at Bonn where two of us (A.L. and M.O.) were staying during preparation of this paper. We would like to thank a referee for valuable remarks, that allows us to improve the paper.

2 Isomonodromic deformations and Elliptic Calogero-Moser System

We consider differential equations related to the N-body integrable elliptic Calogero-Moser system with spin (CM) [5, 23, 10, 39]. They are defined as monodromy preserving equations of some linear system on an elliptic curve and generalize (1.6) to N dependent variables [19]. It is a Hamiltonian non-autonomous system that describes dynamics of N particles with internal degrees of freedom (spin) in a time-depending potential. We call this system Non-autonomous Calogero-Moser system (NACM). In section 2.4 we consider the interrelations between autonomous equations, corresponding to integrable hierarchies and non-autonomous (monodromy preserving) equations.

2.1 Phase space of NACM

The phase space of NACM system is the same as of CM. Let $\Sigma_\tau = \mathbb{C}/\mathbb{Z}^{(2)}_\tau$, $\mathbb{Z}^{(2)}_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$, $(\Im m\, \tau > 0)$ be the elliptic curve. The coordinates of the particles lie in Σ_τ:

$$\mathbf{u} = (u_1, \dots, u_N)\,, \quad u_j \in \Sigma_\tau$$

with the constraint on the center of mass $\sum u_j = 0$. Let $\mathbb{Z}^{(2)}_\tau \ltimes W_N$ be the semi-direct product of the two-dimensional lattice group and the permutation W_N. The coordinate part of the phase space is the quotient

$$\Lambda = (\mathbb{C}^N/(\mathbb{Z}^{(2)}_\tau \ltimes W_N))/\Sigma_\tau\,. \tag{2.1}$$

The last quotient respects the constraint on the center of mass. Let

$$\mathbf{v} = (v_1, \dots, v_N)\,, \quad v_j \in \mathbb{C}\,, \ \sum v_j = 0\,.$$

be the momentum vector dual to $\mathbf{u}$: $\{v_j, u_k\} = \delta_{jk}$. The pair $(\mathbf{v}, \mathbf{u})$ describes the "spinless" part of the phase space.

The additional phase variables, describing the internal degrees of freedom of the particles, are the matrix elements of the N-order matrix $\mathbf{p}$. More exactly, we consider $\mathbf{p}$ as an element of the Lie coalgebra $\mathrm{sl}(N, \mathbb{C})^*$. The linear (Lie-Poisson) brackets on $\mathrm{sl}(N, \mathbb{C})^*$ for the matrix elements have the form

$$\{p_{jk}, p_{mn}\} = p_{jn}\delta_{km} - p_{mk}\delta_{jn}\,. \tag{2.2}$$

Let $\mathcal{O}$ be a coadjoint orbit

$$\mathcal{O} = \{\mathbf{p} \in \mathrm{sl}(N,\mathbb{C})^* \mid \mathbf{p} = \mathrm{Ad}^*_h \mathbf{p}^0,\ h \in \mathrm{SL}(N,\mathbb{C}),\ \mathbf{p}^0 \in \mathfrak{H}^*\}, \tag{2.3}$$

where $\mathfrak{H}$ is the Cartan subalgebra of $\mathrm{sl}(N,\mathbb{C})$. The phase space $\mathcal{R}^{CM} = T^*\Lambda \times \mathcal{O}$ is a symplectic manifold with the symplectic form

$$\omega = \langle d\mathbf{v} \wedge d\mathbf{u}\rangle - \langle \mathbf{p}^0 dhh^{-1} dhh^{-1}\rangle, \tag{2.4}$$

where the brackets stand for the trace. The form ω is invariant with respect to the action of the diagonal subgroup $D = \exp \mathfrak{H}$: $h \to hh_1$, $h_1 \in D$. Therefore, we can go further and pass to the symplectic quotient

$$\tilde{\mathcal{O}} = \mathcal{O}//D. \tag{2.5}$$

It implies the following constraints:
i) the moment constraint $p_{jj} = 0$,
ii) the gauge fixing, for example, as $p_{j,j+1} = p_{j+1,j}$.
Example. Let $\mathbf{p}^0 = \nu\mathrm{diag}(N-1, -1, \ldots, -1)$. Then $\dim \mathcal{O} = 2N - 2$. It is the most degenerate non-trivial orbit. It leads to the spinless model, since in this case $\dim \tilde{\mathcal{O}} = 0$. We should represent $\mathbf{p}^0$ in the special form that takes into account the moment constraint (i):

$$\mathbf{p}^0 = \mathbf{J}^C = \nu \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}. \tag{2.6}$$

For $N = 2$ these orbits are generic.

In this way we come to the phase space of the CM

$$\mathcal{R}^{CM}_{red} = \{T^*(\Lambda) \times \tilde{\mathcal{O}}\}, \tag{2.7}$$

Note that

$$\dim(\mathcal{R}^{CM}_{red}) = 2N - 2 + \dim \mathcal{O} - 2\dim(D) = \dim \mathcal{O}. \tag{2.8}$$

2.2 Equations of motion and Painlevé VI

The CM Hamiltonian has the form

$$H^{CM,spin} = \frac{1}{2}\sum_{j=1}^{N} v_j^2 - \sum_{j>k} p_{jk}p_{kj} E_2(u_j - u_k; \tau), \tag{2.9}$$

where $E_2(x;\tau) = \wp(x;\tau) + 2\eta_1(\tau)$ is the second Eisenstein function (A.4) and τ plays the role of time. [1] For the orbit, corresponding to (2.6), the spinless Hamiltonian is

$$H^{CM} = \frac{1}{2}\sum_{j=1}^{N} v_j^2 - \nu^2 \sum_{j>k} E_2(u_j - u_k; \tau). \tag{2.10}$$

[1] In what follows we replace the Weierstrass function $\wp(x;\tau)$ used in Introduction by the Eisenstein function $E_2(x;\tau)$. It does not affect the equations of motion.

For general non-autonomous Hamiltonian systems it is convenient to work with the extended phase space by including the time. Here we deal with the extended space $\mathcal{R}^{ext} = (\mathcal{R}^{\mathcal{CM}}, \mathcal{T})$, where $\mathcal{T} = \{\tau \in \mathbb{C} \,|\, \Im m\tau > 0\}$. Equip it with the degenerate two-form

$$\omega^{ext} = \omega - \frac{1}{\kappa} dH^{CM,spin}(\mathbf{v}, \mathbf{u}, \tau) \wedge d\tau \,,$$

where $\kappa \geq 0$ is the so-called *classical level.* Note that ω^{ext} is invariant with respect the modular transformations $\mathrm{PSL}_2(\mathbb{Z})$ of $\mathcal{T}$ [19]. It means that ω^{ext} can be restricted on the moduli space $\mathcal{M} = \mathcal{T}/\mathrm{PSL}_2(\mathbb{Z})$. The vector field

$$\mathcal{V}_\tau = \sum_{j,k,l} \left(\frac{\partial H^{CM,spin}}{\partial u_j} \partial_{v_j} - \frac{\partial H^{CM,spin}}{\partial v_j} \partial_{u_j} + \sum_{mn} \frac{\partial H^{CM,spin}}{\partial p_{mn}} (p_{ml}\delta_{nk} - p_{kn}\delta_{lm}) \partial_{p_{kl}} \right) + \kappa \partial_\tau$$

annihilate ω^{ext} and define the equations of motion of NACM system

$$\frac{df(\mathbf{v}, \mathbf{u}, \mathbf{p}, \tau)}{d\tau} = \mathcal{V}_\tau f(\mathbf{v}, \mathbf{u}, \mathbf{p}, \tau) \,.$$

In particular,

$$\kappa \partial_\tau u_j = v_j \,, \tag{2.11}$$

$$\kappa \partial_\tau v_n = - \sum_{j \neq n} p_{jk} p_{kj} \partial_{u_n} E_2(u_j - u_n; \tau) \,, \tag{2.12}$$

$$\kappa \partial_\tau \mathbf{p} = 2[\mathbf{J}_\mathbf{u}(\tau) \cdot \mathbf{p}, \mathbf{p}] \,, \tag{2.13}$$

where the operator $\mathbf{J}_\mathbf{u} \cdot \mathbf{p}$ is defined by the diagonal action

$$\mathbf{J}_\mathbf{u}(\tau) \cdot \mathbf{p} \;:\; p_{jk} \;\rightarrow\; E_2(u_j - u_k; \tau) p_{jk} \,, \tag{2.14}$$

i.e. each matrix element p_{jk} is multiplied on $E_2(u_j - u_k; \tau)$. For $N = 2$ we put $u_1 = -u_2 = u$, $v_1 = -v_2 = v = \kappa \partial_\tau u$ and come to the two body NACM model

$$\partial_\tau^2 u = -\nu^2 \partial_u E_2(2u) \,. \tag{2.15}$$

It coincides with (1.7) since $\partial_u \wp(u) = \partial_u E_2(u)$.

2.3 Lax representation

The goal of this subsection is the Lax representation of (2.11) – (2.13) [19].

2.3.1 Deformation of elliptic curve

Let $T^2 = \{(x, y) \in \mathbb{R} \,|\, x, y \in \mathbb{R}/\mathbb{Z}\}$ be a torus. Complex structure on T^2 is defined by the complex coordinates

$$\Sigma_{\tau_0} = \{z = x + \tau_0 y \,,\; \bar{z} = x + \bar{\tau}_0 y\} \,, \quad \Im m \,\tau_0 > 0 \,, \quad \Sigma_{\tau_0} \sim \mathbb{C}/(\mathbb{Z} + \tau_0 \mathbb{Z}) \,.$$

Consider the deformation of the complex structure that preserves the point $z = 0$. Let $\chi(z, \bar{z})$ be the characteristic function of a neighborhood of $z = 0$. For two neighborhoods $\mathcal{U}' \supset \mathcal{U}$ of $z = 0$ define smooth function

$$\chi(z, \bar{z}) = \begin{cases} 1, & z \in \mathcal{U} \\ 0, & z \in \Sigma_{\tau_0} \setminus \mathcal{U}'. \end{cases} \tag{2.16}$$

Consider new complex coordinates $(w,\bar{w})$ defined by the chiral deformation of the complex coordinates $(z,\bar{z})$

$$\left\{\begin{array}{l} w = z - \frac{\tau-\tau_0}{\rho}(\bar{z}-z)(1-\chi(z,\bar{z}))\,, \\ \bar{w} = \bar{z} \end{array}\right. \qquad (\rho = \tau_0 - \bar{\tau}_0)\,. \tag{2.17}$$

New coordinates define the deformed elliptic curve $\Sigma_\tau = \{w,\bar{w}\}$.

In the new coordinates the partial derivatives assume the form

$$\left\{\begin{array}{l} \partial_w = \partial_z\,, \\ \partial_{\bar{w}} = \partial_{\bar{z}} + \mu\partial_z\,, \end{array}\right. \qquad \mu = \frac{\tau-\tau_0}{\tau-\bar{\tau}_0}(1-\chi(z,\bar{z}))\,,$$

where μ – is the *Beltrami differential*.

2.3.2 Flat bundles of degree zero

Let V_N^0 be a flat vector bundle of rank N and degree 0 over the deformed elliptic curve Σ_τ. Consider the connections

$$\left\{\begin{array}{l} \kappa\partial_w + L^{(0)}(w,\bar{w},\tau)\,, \\ \partial_{\bar{w}} + \bar{L}^{(0)}(w,\bar{w},\tau)\,, \end{array}\right.$$

$$L^{(0)}(w,\bar{w},\tau)\,,\ \bar{L}^{(0)}(w,\bar{w},\tau) \in C^\infty Map(\Sigma_\tau, \mathrm{sl}(N,\mathbb{C}))\,.$$

The flatness of the bundle V_N^0 means

$$\partial_{\bar{w}}L^{(0)} - \kappa\partial_w\bar{L}^{(0)} + [\bar{L}^{(0)}, L^{(0)}] = 0\,. \tag{2.18}$$

By means of the gauge transformations $f(w,\bar{w}) \in C^\infty\mathrm{Map}\,(\Sigma_\tau \to \mathrm{GL}(N,\mathbb{C}))$

$$\bar{L}^{(0)} \to f^{-1}\partial_{\bar{w}}f + f^{-1}\bar{L}^{(0)}f$$

the connections of generic bundles of degree zero can be put in the following form:
1. $\bar{L}^{(0)} = 0$. Then from the flatness (2.18) we have

$$\partial_{\bar{w}}L^{(0)}(w,\bar{w}) = 0\,.$$

2. The connection of generic bundles of $\deg(V_N^0) = 0$ have the following quasi-periodicity:

$$L^{(0)}(w+1) = L^{(0)}(w)\,,\quad L^{(0)}(w+\tau) = \mathbf{e}(\mathbf{u})L^{(0)}(w)\mathbf{e}(-\mathbf{u})\,,$$

where the diagonal elements of $\mathbf{e}(\mathbf{u}) = \mathrm{diag}(\exp(2\pi i u_1),\ldots,\exp(2\pi i u_N))$ define *the moduli of holomorphic bundles*. We identify $\mathbf{u}$ with the coordinates of particles. In fact, u_j, $j=1,\ldots,N$ belong to the dual to Σ_τ elliptic curve (*the Jacobian*), isomorphic to Σ_τ.
3. We assume that $L^{(0)}$ has a simple pole at $w=0$ and

$$Res|_{w=0}\,L^{(0)}(w) = \mathbf{p}\,.$$

The conditions $1,2,3$ fix $L^{(0)}$ up to a diagonal matrix P

$$L^{(0)} = P + X\,, \tag{2.19}$$

$$P = \mathrm{diag}(v_1,\ldots,v_N)\,,$$

$$X = \{X_{jk}\}\,,\ (j\neq k)\,,\quad X_{jk} = p_{jk}\phi(u_j-u_k,w)\,.$$

The function ϕ is determined by (A.8). The quasi-periodicity of $L^{(0)}$ is provided by (A.17). The free parameters $\mathbf{v} = (v_1, \dots, v_N)$ of P can be identified with the momenta.

The flatness of the bundle upon the gauge transform amounts the consistency of the system

$$\begin{cases} i. & (\kappa\partial + L^{(0)}(w,\tau))\Psi = 0\,, \\ ii. & \bar{\partial}\Psi = 0\,. \end{cases} \tag{2.20}$$

To come to the monodromy preserving equation, we assume that the Baker-Akhiezer vector Ψ satisfies an additional equation. Let $\mathcal{Y}$ be a monodromy matrix of the system (2.20) corresponding to homotopically non-trivial cycles $\Psi \to \Psi\mathcal{Y}$. The equation

$$iii. \quad (\kappa\partial_\tau + M^{(0)}(w))\Psi = 0 \tag{2.21}$$

means that $\partial_\tau \mathcal{Y} = 0$, and thereby the monodromy is independent on the complex structure of T^2. The consistency of i. and iii. is the monodromy preserving equation

$$\partial_\tau L^{(0)} - \partial_w M^{(0)} - \frac{1}{\kappa}[L^{(0)}, M^{(0)}] = 0\,. \tag{2.22}$$

In contrast with the standard Lax equation it has additional term $\partial_w M^{(0)}$.

Proposition 2.1 *The equation (2.22) is equivalent to the monodromy preserving equations (2.11), (2.12), (2.13) for $L^{(0)}$ (2.19) and*

$$M^{(0)} = \{Y_{jk}\}\,,\ (j \neq k)\,,\quad \mathrm{diag}\, M^{(0)} = 0$$

$$Y_{jk} = p_{jk} f(u_j - u_k, w)\,,\quad f(u,w) = \partial_u \phi(u,w)\,.$$

Proof is based on the Calogero functional equation (A.20) and the heat equation (A.13).

2.4 Isomonodromic deformations and integrable systems

We can consider the isomonodromy preserving equations as a deformation (*Whitham quantization*) of integrable equations [36]. The level κ plays the role of the deformation parameter. Here we investigate the particular example - the integrable limit of the vector generalization of PVI (2.11) – (2.13) [19].

Introduce the independent time t as $\tau = \tau_0 + \kappa t$ for $\kappa \to 0$ and some fixed τ_0. It means that t plays the role of a local coordinate in an neighborhood of the point τ_0 in the moduli space $\mathcal{M}$ of elliptic curves. It follows from (2.17) that the limiting curve is $\Sigma_{\tau_0} = \{z, \bar{z}\}$. In this limit we come to the equations of motion of CM (2.11) – (2.13):

$$\begin{gathered} \partial_t u_j = v_j\,, \\ \partial_t v_n = -\sum_{j \neq n} p_{jk} p_{kj} \partial_{u_n} E_2(u_j - u_n; \tau_0)\,, \\ \partial_t \mathbf{p} = 2[\mathbf{J}_\mathbf{u}(\tau_0) \cdot \mathbf{p}, \mathbf{p}]\,. \end{gathered} \tag{2.23}$$

The linear problem for this system is obtained from the linear problem for the Isomonodromy problem (2.20), (2.21) by the analog of the quasi-classical limit in Quantum Mechanics. Represent the Baker-Akhiezer function in the WKB form

$$\Psi = \Phi \exp(\frac{\mathcal{S}^{(0)}}{\kappa} + \mathcal{S}^{(1)})\,. \tag{2.24}$$

Substitute (2.24) in the linear system (2.20), (2.21). If $\partial_{\bar{z}}\mathcal{S}^{(0)} = 0$ and $\partial_t\mathcal{S}^{(0)} = 0$, then the terms of order κ^{-1} vanish. In the quasi-classical limit we put $\partial\mathcal{S}^{(0)} = \lambda$. In the zero order approximation we come to the linear system for CM

$$\left\{\begin{array}{ll} i. & (\lambda + L^{(0)}(z,\tau_0))Y = 0\,, \\ ii. & \partial_{\bar{z}}Y = 0\,, \\ iii. & (\partial_t + M^{(0)}(z,\tau_0))Y = 0\,. \end{array}\right.$$

The consistency condition of this linear system

$$\partial_t L^{(0)}(z) - [L^{(0)}(z), M^{(0)}(z)] = 0\,,$$

is equivalent to the Calogero-Moser equations (2.23) [17].

The Baker-Akhiezer function Y takes the form

$$Y = \Phi e^{t\frac{\partial}{\partial\tau_0}\mathcal{S}^{(0)}}\,.$$

The same quasi-classical limit can be applied for the monodromy preserving equations that will be considered in next Section.

3 Isomonodromic deformations and Elliptic Top

3.1 Euler-Arnold top on $\mathrm{SL}(N,\mathbb{C})$.

Let $\mathbf{S} \in \mathrm{sl}(N,\mathbb{C})^*$. Expand it in the basis (B.4) $\mathbf{S} = \sum_{\alpha\in\tilde{\mathbb{Z}}_N^{(2)}} S_\alpha T_\alpha$.

According with (B.6) the Lie-Poisson brackets on $\mathrm{sl}(N,\mathbb{C})^*$ assume the form

$$\{S_\alpha, S_\beta\}_1 = \mathbf{C}(\alpha,\beta)S_{\alpha+\beta}\,.$$

The Lie-Poisson brackets are degenerated on $\mathfrak{g}^* = \mathrm{sl}(N,\mathbb{C})^*$ and their symplectic leaves are coadjoint orbits of $\mathrm{SL}(N,\mathbb{C})$. To descend to a particular coadjoint orbit $\mathcal{O}$ one should fix the values of the Casimirs for the linear bracket. The phase space is a coadjoint orbit

$$\mathcal{R}^{ET} = \{\mathbf{S} \in \mathfrak{g}^* \,|\, \mathbf{S} = gS_0g^{-1}\,,\ g \in \mathrm{SL}(N,\mathbb{C})\,,\ S_0 \in \mathfrak{g}^*\}\,. \tag{3.1}$$

The Hamiltonian of the Euler-Arnold top has a special form. It is a quadratic functional on $\mathfrak{g}^*$

$$H = -\frac{1}{2}\mathrm{tr}(\mathbf{S},\mathbf{J}(\mathbf{S}))\,,\quad \mathbf{S} \in \mathfrak{g}^*\,,$$

where $\mathbf{J}$ is an invertible symmetric operator

$$\mathbf{J} \,:\, \mathfrak{g}^* \to \mathfrak{g}\,.$$

It is called *the inverse inertia tensor.* The equations of motion assume the form

$$\partial_t\mathbf{S} = \{\mathbf{J}(\mathbf{S}),\mathbf{S}\}_1 = [\mathbf{J}(\mathbf{S}),\mathbf{S}]\,. \tag{3.2}$$

3.2 Non-autonomous Elliptic top (NAET).

Consider a special form of the inverse inertia tensor. Let

$$E_2(\alpha) = E_2(\frac{\alpha_1 + \alpha_2\tau}{N}|\tau)\,, \quad \alpha = (\alpha_1, \alpha_2) \in \tilde{\mathbb{Z}}_N^{(2)}\,,$$

where $\tilde{\mathbb{Z}}_N^{(2)}$ is defined by (B.3). Define as above the diagonal action

$$\mathbf{J}(\tau, \mathbf{S}) \ : \ S_\alpha \to E_2(\alpha) S_\alpha\,, \quad \mathbf{J}(\tau, \mathbf{S}) = \mathbf{J}(\tau) \cdot \mathbf{S}\,.$$

Then the Hamiltonian of ET assumes the form

$$H^{NAET}(\mathbf{S}, \tau) = \frac{\pi\imath}{N^2}\mathrm{tr}(\mathbf{S}\mathbf{J}(\tau) \cdot \mathbf{S}) = -\frac{1}{2}\sum_{\gamma \in \tilde{\mathbb{Z}}_N^{(2)}} S_\gamma E_2(\gamma) S_{-\gamma}\,.$$

The equation of motion of NAET is similar to (3.2)

$$\kappa\partial_\tau \mathbf{S} = [\mathbf{J}(\tau) \cdot \mathbf{S}, \mathbf{S}]\,, \quad (\,\kappa\partial_\tau S_\alpha = \sum_{\gamma \in \tilde{\mathbb{Z}}_N^{(2)}} S_\gamma S_{\alpha-\gamma} E_2(\gamma) \mathbf{C}(\alpha, \gamma)\,)\,. \tag{3.3}$$

As in the previous case one can consider the limit $\kappa \to 0$, ($\tau \to \tau_0$) to the integrable elliptic top

$$\partial_t \mathbf{S} = [\mathbf{J}(\tau_0) \cdot \mathbf{S}, \mathbf{S}]\,. \tag{3.4}$$

3.3 Lax representation

3.3.1 Flat bundles of degree one

Let V_N^1 be a flat bundle over the deformed elliptic curve Σ_τ of rank N and degree 1 with the connections

$$\left\{ \begin{array}{l} \kappa\partial_w + L^{(1)}(w, \bar{w}, \tau)\,, \\ \partial_{\bar{w}} + \bar{L}^{(1)}(w, \bar{w}, \tau)\,, \end{array} \right. \tag{3.5}$$

where $L^{(1)}(w, \bar{w}, \tau)$, $\bar{L}^{(1)}(w, \bar{w}, \tau)$ are meromorphic maps of Σ_τ in $\mathrm{sl}(N, \mathbb{C})$.

For generic flat bundles of degree one the connections can be chosen in the form
1. $\bar{L}^{(0)} = 0$. From the flatness one has

$$\bar{\partial} L^{(1)} = 0\,.$$

2. The Lax matrices satisfy the quasi-periodicity conditions

$$\begin{array}{c} L^{(1)}(w+1) = Q L^{(1)}(w) Q^{-1}\,, \\ L^{(1)}(w+\tau) = \tilde{\Lambda} L^{(1)}(w) \tilde{\Lambda}^{-1} + \frac{2\pi i\kappa}{N}\,, \end{array}$$

$$\tilde{\Lambda}(z, \tau) = -\mathbf{e}_N(-z - \frac{\tau}{2})\Lambda$$

for Q, Λ (B.1), (B.2). It means that there are no moduli parameters for E_N^1.
3. $L^{(1)}$ has a simple pole at $w = 0$ and all degrees of freedom come from the residue

$$Res|_{w=0}\, L^{(1)}(w) = \mathbf{S}\,.$$

The Lax matrix is fixed by these conditions:

Lemma 3.1 *The connection assumes the form*

$$L^{(1)}(w) = -\frac{\kappa}{N}\partial_w \ln \vartheta(w;\tau)Id + \sum_{\alpha\in\tilde{\mathbb{Z}}_N^{(2)}} S_\alpha \varphi_\alpha(w)T_\alpha \,. \tag{3.6}$$

where $\varphi(\alpha, w)$ is defined by (B.10), and T_α are the basis elements (B.4).

Fixing the connections we come from (3.5) to the linear system

$$\left\{\begin{array}{ll} i. & (\kappa\partial_w + L^{(1)}(w))\Psi = 0\,, \\ ii & \partial_{\bar{w}}\Psi = 0\,. \end{array}\right. \tag{3.7}$$

As above, the independence of the monodromy of (3.7) means that the Baker-Akhiezer vector satisfies the additional linear equation

$$iii.\ (\kappa\partial_\tau + M^{(1)})\Psi = 0\,. \tag{3.8}$$

Lemma 3.2 *The equation of motion of the non-autonomous top (3.3) is the monodromy preserving equation for (3.7) with the Lax representation*

$$\partial_\tau L^{(1)} - \partial_w M^{(1)} + \frac{1}{\kappa}[M^{(1)}, L^{(1)}] = 0\,, \tag{3.9}$$

where $L^{(1)}$ is defined by (3.6),

$$M^{(1)} = -\frac{\kappa}{N}\partial_\tau \ln \vartheta(w;\tau)Id + \sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}} S_\gamma f_\gamma(w)T_\gamma\,,$$

and $f_\gamma(w)$ is defined by (B.11).

The proof of the equivalence of (3.3) and (3.9) is based on the addition formula (A.20) and the heat equation (A.13) as in the case of CM.

In the quasi-classical limit $\kappa \to 0$ we come to the integrable top on $\mathrm{SL}(N,\mathbb{C})$ (3.4).

4 Symplectic Hecke correspondence

We construct here a map from the phase space of CM (2.7) to the phase space of ET (3.1)

$$\Xi^+ : \mathcal{R}^{CM} \to \mathcal{R}^{ET}\,, \quad ((\mathbf{v},\mathbf{u},\mathbf{p}) \mapsto \mathbf{S})\,, \tag{4.1}$$

such that Ξ^+ is the symplectic map

$$\Xi^{+*}\omega(\mathbf{S}) = \omega(\mathbf{v},\mathbf{u},\mathbf{p})\,.$$

To construct it we define the map of the sheaves of sections $\Gamma(V_N^{(0)}) \to \Gamma(V_N^{(1)})$ such that it is an isomorphism on the complement to w and it has one-dimensional cokernel at $w \in \Sigma_\tau$:

$$0 \longrightarrow \Gamma(V_N^{(0)}) \xrightarrow{\Xi^+} \Gamma(V_N^{(1)}) \longrightarrow \mathbb{C}|_w \longrightarrow 0\,.$$

It is the so-called *upper modification* of the bundle $E_N^{(0)}$ at the point w.

On the complement to the point w consider the map

$$\Gamma(V_N^{(1)}) \stackrel{\Xi^-}{\longleftarrow} \Gamma(V_N^{(0)}) \,,$$

such that $\Xi^-\Xi^+ =$Id. It defines *the lower modification* at the point w.

In general case the modifications lead to the *Hecke correspondence* between the moduli spaces of the holomorphic bundles of degree k and $k+1$

$$\mathcal{M}^k \to \mathcal{M}^{k+1} \,. \tag{4.2}$$

The modifications can be lifted to the Higgs bundle. They act as singular gauge transformations on the Lax matrices and provide the symplectomorphisms between Hitchin systems (*symplectic Hecke correspondence*).

The modifications on the Higgs bundles can be applied for the monodromy preserving equation as well. The action of the upper modification on the Lax matrices has the form

$$L^{(1)} = \Xi^+\kappa\partial\Xi^{+-1} + \Xi^+L^{(0)}\Xi^{+-1} \,. \tag{4.3}$$

This form of transformation provides its symplectic action.

Consider in details the map (4.1). According with its definition the upper modification $\Xi^+(z)$ is characterized by the following properties:

• *Quasi-periodicity*:

$$\Xi^+(z+1,\tau) = -Q \times \Xi^+(z,\tau) \,, \tag{4.4}$$

$$\Xi^+(z+\tau,\tau) = \tilde{\Lambda}(z,\tau) \times \Xi^+(z,\tau) \times \mathrm{diag}(\mathbf{e}(u_j)) \,, \quad \tilde{\Lambda}(z,\tau) = -\mathbf{e}_N(-z-\frac{\tau}{2})\Lambda \tag{4.5}$$

•• $\Xi^+(z)$ has a simple pole at $z=0$. Let $\mathbf{r}_i = (r_{i,1},\dots,r_{i,N})$ be an eigen-vector of the matrix $\mathbf{p}\in\tilde{\mathcal{O}}$ (2.5), $\mathbf{p}\mathbf{r}_i = p_i^0\mathbf{r}_i$. Then $Res(\Xi)_{z=0}\mathbf{r}_i = 0$.

The former condition provides that the quasi-periods of the transformed Lax matrix corresponds to the bundle of degree one. The latter condition implies that $L^{(1)}$ has only a simple pole at $z=0$. The residue at the pole is identified with $\mathbf{S}$.

First, we construct $(N\times N)$- matrix $\tilde{\Xi}(z,\mathbf{u};\tau)$ that satisfies (4.4) and (4.5) but has a special one-dimensional kernel:

$$\tilde{\Xi}_{ij}(z,\mathbf{u};\tau) = \theta\left[\begin{array}{c} \frac{i}{N}-\frac{1}{2} \\ \frac{N}{2} \end{array}\right](z-Nu_j,N\tau), \tag{4.6}$$

where $\theta\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau)$ is the theta function with a characteristic

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau) = \sum_{j\in\mathbb{Z}}\mathbf{e}\left((j+a)^2\frac{\tau}{2}+(j+a)(z+b)\right) \,.$$

It can be proved that the kernel of $\tilde{\Xi}$ at $z=0$ is generated by the following column-vector :

$$\left\{(-1)^l\prod_{j<k;j,k\neq l}\vartheta(u_k-u_j,\tau)\right\}, \quad l=1,2,\cdots,N \,.$$

Then the matrix $\Xi(z,\mathbf{u},\mathbf{r}_i)$, $(\mathbf{r}_i=(r_{1,i},\dots,r_{N,i}))$ assumes the form

$$\Xi^+(z,\mathbf{u},\mathbf{r}_i) = \tilde{\Xi}(z)\times\mathrm{diag}\left(\frac{(-1)^l}{r_{l,i}}\prod_{j<k;j,k\neq l}\vartheta(u_k-u_j,\tau)\right) \,. \tag{4.7}$$

The transformation (4.3) with Ξ (4.7) leads to the map $\mathcal{R}^{CM} \to \mathcal{R}^{ET}$ (4.1).

For the spinless CM, defined by the coupling constant ν^2, this transformation leads to the degenerate orbit of the NAET.

Note that the equation for the spin variables of CM (2.13) reminds the equation of motion for the NAET with the coordinate-dependent operator $\mathbf{J_u}$ (2.14). The only difference is the structure of the phase spaces $\mathcal{R}^{CM}$ (2.7) and $\mathcal{R}^{ET}$ (3.1). The upper modification Ξ^+ carries out the pass from $\mathcal{R}^{CM}$ to $\mathcal{R}^{ET}$. It depends only on the part of variables on $\mathcal{R}^{CM}$, namely on $\mathbf{u}$ and $\mathbf{p}$ through the eigenvector $\mathbf{r}_j$. For the rank two bundles it is possible to write down the explicit dependence $\mathbf{S}(u, v, \nu)$. We postpone this example to the general case of PVI in Sect.6.

5 Elliptic Garnier-Gaudin models

5.1 General construction

Here we present another way to construct the elliptic form of PVI (1.4) that does not appeal to the transformation (6.3) of its rational form (1.1). The advantage of this approach is a simple derivation of the linear system and the corresponding Lax operator.

The CM and ET hierarchies are particular case of the following construction. The main objects are the Lax matrices and the corresponding integrable systems. Let $\Delta = \{z_a,\ a = 1...m\}$ be the divisor of marked points on Σ_τ and $V_N^k(\Sigma_\tau \backslash \Delta)$ is a holomorphic vector bundle of rank N and degree k over Σ_τ with quasi-parabolic structure at the marked points [30]. This data allows us to define an elliptic analog of the Garnier-Gaudin system.

The moduli space $\mathcal{M}^{(0|k|N)}$ of holomorphic vector bundles of degree k and rank N over Σ_τ is parameterized by k (mod$N$) elements $\mathbf{u} = (u_1, \dots, u_k)$, $u_j \in \Sigma_\tau$ and

$$\dim(\mathcal{M}^{(0|k|N)}) = \text{g.c.d.}(N, k)\,.$$

The transition matrices for a generic holomorphic vector bundle can be chosen as follows. For $d = \text{g.c.d.}(N, k)$ define the diagonal matrix $\mathbf{U}$ containing d blocks $(u_1, \dots, u_k)$

$$\mathbf{U} = \text{diag}\left((u_1, \dots, u_k), \dots, (u_1, \dots, u_k)\right)\,.$$

Then the transition matrices corresponding to the two basic non-contractible cycles take the form

$$g_1 = Q\,, \quad g_\tau = -\mathbf{e}_N(-kz - \frac{1}{2}k\tau)\exp(2\pi\imath\mathbf{U})\Lambda^k\,.$$

The quasi-parabolic structure means the fixing a flag structure Fl_a at $z = z_a$. One can act on the set of flags by the group $D_k = Centr(g_1, g_\tau)$. It easy to see that D_k is a diagonal matrix ($D_k \subseteq D \subset \mathrm{SL}(N, \mathbb{C})$) and $\dim D_k = d$. The moduli space of the quasi-parabolic bundles is defined as

$$\mathcal{M}^{(m|k|N)} \sim \mathcal{M}^{(0|k|N)} \times (\sqcap_a Fl_a)/D_k\,.$$

Then

$$\dim \mathcal{M}^{(m|k|N)} = \text{g.c.d.}(N, k) + \sum_{a=1}^{m} \dim Fl_a - \text{g.c.d.}(N, k) = \sum_{a=1}^{m} \dim Fl_a\,.$$

In what follows we assume that all flag varieties are non-degenerate. Thereby

$$\dim \mathcal{M}^{(m|k|N)} = \frac{1}{2}mN(N-1)\,.$$

The elliptic Garnier-Gaudin models $EGG(m|k|N)$ are particular examples of Hitchin systems [12] related to holomorphic vector bundles of degree k and rank N. Particular cases of the elliptic Garnier-Gaudin models were considered in Ref. [25] ($k=0$) and in Ref. [29, 34] ($k=1$).

The Lax matrix $L^{(m|k|N)}(z)dz$ is a meromorphic section of the bundle $\mathrm{End}V_N^k \otimes \Omega^{(1,0)}(\Sigma_\tau \backslash \Delta)$ with simple poles at $\{z_a\} \in \Delta$ and fixed residues

$$Res\, L^{(m|k|N)}(z_a) = \mathbf{S}^a \in \mathrm{sl}^*(N, \mathbb{C})\,. \tag{5.1}$$

More exactly, we assume that $\mathbf{S}^a \in \mathcal{O}^a$ is an element of a non-degenerate coadjoint orbit. The Lax matrix $L^{(m|k|N)}(z)dz$ plays the role of the quasi-parabolic Higgs field in the Hitchin construction [2].

The Higgs field respects the lattice action $\mathbb{Z} \oplus \tau\mathbb{Z}$:

$$L^{(m|k|N)}(z+1) = g_1^{-1} L^{(m|k|N)}(z) g_1\,, \qquad L^{(m|k|N)}(z+\tau) = g_\tau^{-1} L^{(m|k|N)}(z) g_\tau\,. \tag{5.2}$$

It follows from (5.2) that the diagonal part of $L^{(m|k|N)}(z)$ is a double-periodic meromorphic function. It depends on the additional variables $(v_1, \dots, v_k)$. The conditions (5.1), (5.2) determine the Lax matrix $L^{(m|k|N)}(z)$. Below we consider particular appropriated cases.

The whole phase

$$\mathcal{R}^{(m|k|N)} = T^*\mathcal{M}^{(0|k|N)} \times (\mathcal{O}^1, \dots, \mathcal{O}^m) = \{\mathbf{v}, \mathbf{u}; \mathbf{p}^1, \dots, \mathbf{p}^m\}$$

is equipped with the Poisson brackets

$$\{v_j, u_k\} = \delta_{jk}\,, \quad \{p^a_{il}, p^b_{jk}\} = \delta^{ab}(p^a_{ik}\delta_{lj} - p^a_{jl}\delta_{ik})\,.$$

In the case when $\dim(\mathcal{M}^{(0|k|N)}) \neq 0$ one can go further and consider the symplectic quotient with respect to the defined above group D_k

$$\mathcal{R}^{(m|k|N)}_{red} = \mathcal{R}^{(m|k|N)}//D_k\,. \tag{5.3}$$

The dimension of the reduced space

$$\dim \mathcal{R}^{(m|k|N)}_{red} = \sum_{a=1}^m \dim \mathcal{O}^a = mN(N-1)$$

is independent on the degree of the bundles. In fact, all the spaces $\mathcal{R}^{(m|k|N)}_{red}$ for $k = 0, \dots, N-1$ are symplectomorphic [21]. The symplectomorphisms is provided by the symplectic Hecke correspondence placed between the Higgs bundles of degree k and $k+1$ (4.2).

The commuting Hamiltonians of $EGG(m|k|N)$ can be extracted from the expansion over the basis of the double-periodic functions $E_i(z-z_a)$ (A.2), (A.4), (A.5)

$$\frac{1}{j}\mathrm{tr}(L^{(m|k|N)}(z))^j = \begin{cases} \sum_a H^a_{11} E_1(z-z_a)\,, & (j=1) \\ H^0_{0j} + \sum_a \sum_{i=1}^j H^a_{ij} E_i(z-z_a)\,, & (j=2,\dots,N)\,, \end{cases}$$

where $\sum_a H^a_{1j} = 0$. Equivalently, the commuting Hamiltonians can be defined by the integral representations [12]

$$H^a_{ij} = \frac{1}{j}\int_{\Sigma_\tau} \mathrm{tr}(L^{(m|k|N)}(z))^j \mu^a_{ij}\,.$$

[2] In what follows we omit the differential dz in the notation of the Lax matrices and work with $L(z)$ instead of $L(z)dz$

Here $\mu^a_{ij} = \mu^a_{ij}(z, \bar{z})$ are $(1-j, 1)$ differentials on Σ_τ. They are chosen to be dual to the basis of the elliptic functions

$$\frac{1}{j}\int_{\Sigma_\tau} \mu^a_{ij} E_l(z - z_b) = \delta^{ab}_{il}\,.$$

They have the following form [19, 20]. For a marked point z_a define a smooth function $\chi_a(z, \bar{z})$ as for $z = 0$ (2.16) Then for $a \neq 0$, $i \neq 0$

$$\mu^a_{ij} = \frac{j}{2\pi\imath}(z - z_a)^{i-1}\bar{\partial}\chi_a(z, \bar{z})$$

and

$$\mu^0_{02} = \frac{1}{\tau - \bar{\tau}}\bar{\partial}(\bar{z} - z)(1 - \sum_{a=1}^{m}\chi_a(z, \bar{z}))\,.$$

The higher order differentials μ^0_{0j}, $j > 2$ are related to the so-called W-structures on Σ_τ [20] and will not be discussed here.

The commuting Hamiltonians generate the hierarchy of the Lax equations

$$\partial_{(a,i,j)}L^{(m|k|N)}(z)) = [L^{(m|k|N)}(z)), M^{(m|k|N)}_{(a,i,j)}(z)]\,, \quad \partial_{(a,i,j)} = \partial_{t_{(a,i,j)}}\,. \tag{5.4}$$

Here the matrices $M^{(m|k|N)}_{(a,i,j)}$ have the same quasi-periodicity properties as $L^{(m|k|N)}$ and satisfy the equation

$$[M^{(m|k|N)}_{(a,i,j)}, \mathbf{U}] - \bar{\partial}M^{(m|k|N)}_{(a,i,j)} = (L^{(m|k|N)})^{j-1}\mu^a_{ij} - \partial_{(a,i,j)}\mathbf{U}\,.$$

The Lax equations (5.4) as well as the form of the M-matrices follow from the equations of motion for the Hitchin systems. Their derivation can be found, for example, in [27].

5.2 Involution of the Higgs bundles

Let ς be an involution ($\varsigma^2 = Id$) acting on the space of sections of the Higgs bundle. Then we have decomposition

$$\Gamma(\mathrm{End}V^k_N) = \Gamma^+(\mathrm{End}V^k_N) \oplus \Gamma^-(\mathrm{End}V^k_N)\,.$$

Let

$$L^{(m|k|N)\,\pm} = \frac{1}{2}(L^{(m|k|N)} \pm \varsigma(L^{(m|k|N)}))\,.$$

Assume that the involution preserves the hierarchy

$$\varsigma\{M^{(m|k|N)}_{a,i,j}\} = \{M^{(m|k|N)}_{a,i,j}\}\,,$$

and we choose the invariant subset $\{M^{(m|k|N)+}_{a,i,j}\}$. Then the constraint

$$L^{(m|k|N)\,-}(t) \equiv 0 \tag{5.5}$$

is consistent with the involutions (5.4)

$$\partial_{a,i,j}L^{(m|k|N)\,+}(z)) = [L^{(m|k|N)\,+}(z)), M^{(m|k|N)+}_{a,i,j}(z)]\,.$$

Consider the case of the special divisor

$$\Delta = (\omega_0 = 0\,, \omega_\alpha\,,\ \alpha = 1, 2, 3)\,. \tag{5.6}$$

Then the transformation $z \to -z$ preserves $\Sigma_\tau \setminus \Delta$. It can be accompanied with an involution of $\mathrm{sl}(N, \mathbb{C})$ to generate ς. In next subsections we consider two examples of the rank two bundles with the involutions of such type.

5.3 Degree zero bundles

We have derived the space $\mathcal{M}^{(0|0|N)}$ in 2.3.2. Remind that it is described by N parameters $\mathbf{u}=(u_1,\ldots,u_N)$, $u_i\in\Sigma_\tau$, such that $u_1+...+u_N=0$ and the multiplicators are:

$$g_1=Id_N\,,\qquad g_\tau=\mathbf{e}(-\mathbf{u})=\mathrm{diag}(\mathbf{e}(-u_1),...,\mathbf{e}(-u_N))\,,\quad (\mathbf{e}(u)=\exp 2\pi\imath u)\,. \tag{5.7}$$

These conditions fix the Lax matrix of $EGG(m|0|N)$ up to a diagonal matrix diag $\mathbf{v}$, $\mathbf{v}=(v_1,\ldots,v_N)$. It follows from (5.2) and (5.7) that the Lax matrix assumes the form

$$L^{(m|0|N)}_{ij}(z)=\delta_{ij}(v_i+\sum_{a=1}^{m}p^a_{ii}E_1(z-z_a))+(1-\delta_{ij})\sum_{a=1}^{m}p^a_{ij}\phi(u_i-u_j,z-z_a)\,. \tag{5.8}$$

The particular important case corresponding to a single marked point was considered in Section 2.

Now consider the special form of (5.8) with Δ (5.6) and $N=2$. The Lax matrix (5.8) can be gauge transform to the form

$$L^{(4|0|2)}(z)=(v+\sum_{a=0}^{3}p^a_3E_1(z+\omega_a))\sigma_3+ \tag{5.9}$$

$$\sum_{a=0}^{3}(p^a_+\mathbf{e}(2u\partial_\tau\omega_a)\phi(2u,z+\omega_a)\sigma_++p^a_-\mathbf{e}(-2u\partial_\tau\omega_a)\phi(-2u,z+\omega_a)\sigma_-\,.$$

Define the involution as

$$\begin{array}{l}\varsigma^{(0)}(L^{(4|0|2)}(z))=-\sigma_1L^{(4|0|2)}(-z)\sigma_1\,,\ ^3\\ \varsigma^{(0)}(M^{(4|0|2)}(z))=\sigma_1M^{(4|0|2)}(-z)\sigma_1\,.\end{array} \tag{5.10}$$

Proposition 5.1 ([42]) *The invariant Lax matrix $L^{4|0|2\,+}$ with respect to $\varsigma^{(0)}$ assumes the form*

$$L^{(4|0|2)\,+}(z)=\tilde{L}^{CI}(z)\,,$$

$$\tilde{L}^{CI}=\begin{pmatrix}v&0\\0&-v\end{pmatrix}+\sum_{a=0}^{3}\tilde{L}^{CI}_a\,,\qquad \tilde{L}^{CI}_a=\begin{pmatrix}0&\tilde{\nu}_a\varphi_a(2u,z+\omega_a)\\ \tilde{\nu}_a\varphi_a(-2u,z+\omega_a)&0\end{pmatrix}\,, \tag{5.11}$$

where $\tilde{\nu}^2_a=\frac{1}{2}\mathrm{tr}(\mathbf{p}^a)^2$, $\varphi_a(2u,z+\omega_a)=\mathbf{e}(-2u\partial_\tau\omega_a)\phi(2u,z+\omega_a)$.

Proof.
The projection of the Lax matrix on its anti-invariant part should vanish. It follows from (A.9) and (A.17) that

$$\mathbf{e}(2u\partial_\tau\omega_a)\phi(2u,z+\omega_a)\stackrel{(z\to -z)}{\longrightarrow}-\mathbf{e}(-2u\partial_\tau\omega_a)\phi(-2u,z+\omega_a)\,.$$

Since $\sigma_1\sigma_3\sigma_1=-\sigma_3$, $\sigma_1\sigma_+\sigma_1=\sigma_-$ the vanishing of $L^{(4|0|2)\,-}(z)$ implies

$$p^a_3=0\,,\quad p^a_+=p^a_-\,,\ (a=0,\ldots,3)\,. \tag{5.12}$$

Thus one can put $p^a_+=p^a_-=\tilde{\nu}_a$ and the invariant part $L^{(4|0|2)\,+}(z)$ coincides with (5.11). $\square$

[3]The difference of the ς action on L and M is due to $L\in\Omega^{(1,0)}(\Sigma_\tau)$ while $M\in\Omega^{(0)}(\Sigma_\tau)$.

Remark 5.1 *Remind that upon the involution the phase space (5.3) is the symplectic quotient $\mathcal{R}^{(4|0|2)}_{red} = \mathcal{R}^{(4|0|2)}//D_2$. The conditions (5.12) imply the moment constraint $\sum_a p^a_3 = 0$ and the gauge fixing $\sum_a p^a_+ = \sum_a p^a_-$ and thereby provide the pass to the symplectic quotient.*

According with Ref. [42] the new Hamiltonian can be read off from the decomposition

$$\frac{1}{4}\mathrm{tr}(\tilde{L}^{CI})^2 = H^{CI} + \frac{1}{2}\sum_{a=0}^{3}\tilde{\nu}_a^2 E_2(z-\omega_a)\,, \tag{5.13}$$

where

$$H^{CI} = \frac{1}{2}v^2 - \frac{1}{2}\sum_{a=0}^{3}\nu_a^2 E_2(u-\omega_a)\,,$$

and

$$\begin{pmatrix}\tilde{\nu}_0\\ \tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}\nu_0\\ \nu_1\\ \nu_2\\ \nu_3\end{pmatrix} \tag{5.14}$$

The canonical bracket $\{v,u\} = 1$ and the Hamiltonian H^{CI} yields the BC_1 CI system (1.5).

Proposition 5.2 ([42]) *There exists the matrix $M^{4|0|2\,+} = \tilde{M}^{CI}$,*

$$\tilde{M}^{CI} = \sum_{a=0}^{3}\tilde{M}^{CI}_a, \qquad \tilde{M}^{CI}_\alpha = \begin{pmatrix} 0 & \tilde{\nu}_\alpha\varphi'_a(2u, z+\omega_\alpha)\\ \tilde{\nu}_a\varphi'_a(-2u, z+\omega_\alpha) & 0\end{pmatrix},$$

such that the Lax equation is equivalent to the BC_1 CI equation (1.5).

5.4 Degree one bundles

The Lax matrices in this case was considered in [29]. The multiplicators of a bundle V_N^1 are:

$$g_1 = -Q, \qquad g_\tau = -\mathbf{e}_N\left(-\frac{\tau}{2} - z\right)\Lambda\,, \quad \mathbf{e}_N = \exp\frac{2\pi\imath}{N}$$

where Q and Λ are the matrices (B.1), (B.2). There are no moduli parameters in this case. The phase space is a direct sum of the coadjoint orbits $\mathcal{O}^1 \times ... \times \mathcal{O}^m$ and $\mathcal{R}^{(m|1|N)} = \mathcal{R}^{(m|1|N)}_{red}$. In the basis $\{T_\alpha\}$, $(\alpha \in \tilde{\mathbb{Z}}^{(2)}_N)$ (B.4) the Poisson brackets are

$$\{S^a_\alpha, S^b_\beta\}_1 = \mathbf{C}(\alpha,\beta)\delta^{ab}S^a_{\alpha+\beta}\,. \tag{5.15}$$

The Lax matrix is completely fixed by the quasi-periodicity conditions

$$L^{(m|1|N)}(z) = \sum_{a=0}^{m-1}\sum_{\alpha\in\tilde{\mathbb{Z}}^{(2)}_N} T_\alpha S^a_\alpha\varphi_\alpha(z-z_a)\,, \quad \varphi_\alpha(z) = \mathbf{e}(z\partial_\tau\omega_\alpha)\phi(z,\omega_\alpha)\,, \ \omega_\alpha = \frac{\alpha_1+\alpha_2\tau}{N}\,. \tag{5.16}$$

Consider the case $N = 2$ and Δ (5.6)

$$L^{(4|1|2)}(z) = \sum_{a=0}^{3}\sum_{\alpha=1}^{3} S^a_\alpha\varphi_\alpha(z-\omega_a)\sigma_\alpha\,. \tag{5.17}$$

Define the involution

$$\varsigma^{(1)}(L^{4|1|2}(z)) = -L^{4|1|2}(-z)\,, \ \ \varsigma^{(1)}(M^{4|1|2}(z)) = M^{4|1|2}(-z)\,. \tag{5.18}$$

Proposition 5.3 *The invariant Lax matrix $L^{(4|1|2)\,+}$ with Δ (5.6) assumes the form*

$$L^{(4|1|2)\,+}(z) = L^{ZVG}(z)\,.$$

$$L^{ZVG} = \sum_{\alpha=1}^{3}\left(S_\alpha\varphi_\alpha(z) + \tilde{\nu}_\alpha\varphi_\alpha(z-\omega_\alpha)\right)\sigma_\alpha = \sum_{\alpha=1}^{3}\left(S_\alpha\varphi_\alpha(z) + {\nu'}_\alpha\frac{1}{\varphi_\alpha(z)}\right)\sigma_\alpha\,, \tag{5.19}$$

where $S_\alpha = S^0_\alpha$, $\tilde{\nu}_\alpha = S^\alpha_\alpha = const$ and

$$\nu'_\alpha = -\tilde{\nu}_\alpha \mathbf{e}(-\omega_\alpha\partial_\tau\omega_\alpha)\left(\frac{\vartheta'(0)}{\vartheta(\omega_\alpha)}\right)^2\,. \tag{5.20}$$

Proof.
It follows from (5.18) and (B.18) that

$$L^{(4|1|2)\,-} = \frac{1}{2}(L^{(4|1|2)}(z) - L^{4|1|2}(-z)) = \sum_{\alpha\neq a} S^\alpha_\alpha\varphi_\alpha(z-\omega_a)\sigma_\alpha\,.$$

If $L^{(4|1|2)\,-} = 0$, then $S^a_\alpha = 0$ for $\alpha \neq a$. Fixing the Casimir functions before the involution as $\sum_\alpha (S^a_\alpha)^2 = \tilde{\nu}_a^2$ we conclude that $S^\alpha_\alpha = \tilde{\nu}_\alpha$. On the other hand

$$L^{(4|1|2)\,+} = \sum_{\alpha=1}^{3}\left(S^0_\alpha\varphi_\alpha(z) + S^\alpha_\alpha\varphi_\alpha(z-\omega_\alpha)\right)\sigma_\alpha\,.$$

and we come to the statement. □

Again one can construct Hamiltonians from the Lax matrix L^{ZVG} as in (5.13). Then the Hamiltonian

$$H^{ZVG} = -\frac{\pi^2}{2}\mathrm{tr}\left(\mathbf{S}, (\mathbf{J}(\tau)\cdot\mathbf{S} + \nu')\right) \tag{5.21}$$

with the (5.15) Lie-Poisson brackets leads to the ZVG equation (1.9).

It is worthwhile to mention that we deal with three types of constants ν, $\tilde{\nu}$ and ν'. The latter two are expressed through the former by (5.14) and (5.20).

Proposition 5.4 *There exists the matrix $M^{(4|1|2)\,+} = M^{ZVG}$,*

$$M^{ZVG}(z) = \sum_\alpha\left[-S_\alpha\varphi_\alpha(z)(E_1(z+\omega_\alpha) - E_1(\omega_\alpha)) + \tilde{\nu}_\alpha\varphi(z-\omega_\alpha)E_1(z)\right]\sigma_\alpha = \tag{5.22}$$

$$= \sum_\alpha -S_\alpha\frac{\varphi_1(z)\varphi_2(z)\varphi_3(z)}{\varphi_\alpha(z)}\sigma_\alpha + E_1(z)L^{ZVG}(z)\,.$$

such that the Lax equation is equivalent to the autonomous $\mathrm{SL}(2,\mathbb{C})$ *ZVG (6.1).*

The proof of this statement essentially is the same as Proposition 5.2.

6 Non-autonomous systems

Consider the non-autonomous systems NAZVG (1.2), and EPVI (1.4)

$$\partial_\tau \vec{S} = \vec{S} \times \vec{J}(\tau, \vec{S}) + \vec{S} \times \vec{\nu}' \,, \tag{6.1}$$

$$\frac{d^2u}{d\tau^2} = -\sum_{a=0}^{3} \nu_a^2 \wp'(u + \omega_a, \tau) \,. \tag{6.2}$$

We remind the relation between the last equation and PVI (1.1), established in [22]. Assume that the parameters of the equations are related as follows

$$\nu_0 = \alpha \,, \ \nu_1 = -\beta \,, \ \nu_2 = \gamma \,, \ \nu_3 = \frac{1}{2} - \delta \,.$$

Then the substitution in (1.1)

$$(u, \tau) \to \left(X = \frac{E_2(u|\tau) - e_1}{e_2 - e_1} \,, \ t = \frac{e_3 - e_1}{e_2 - e_1} \right) \,, \quad e_a = E_2(\omega_a) \tag{6.3}$$

brings it in the form (6.2).

The Lax matrices for the non-autonomous systems can be obtained from the Lax matrices for integrable systems, constructed in previous section, by replacing the coordinates $z \to w$ (2.17). In this way we come to the following two statements.

Proposition 6.1 ([42]) *The equation of motion of EPVI (6.2) has the Lax form*

$$\partial_\tau L^{CI} - \partial_w M^{CI} + \frac{1}{\kappa}[M^{CI}, L^{CI}] = 0 \tag{6.4}$$

with

$$L^{CI} = P + X \,, \quad P = \mathrm{diag}(v, -v) \,, \tag{6.5}$$

$$X_{12}(u, z) = \sum_a \tilde{\nu}_a \varphi_a(2u, w + \omega_a) \,, \quad X_{21}(u, w) = X_{12}(-u, w) \,,$$

$$M^{CI} = Y_{jk} \,, \quad (j \neq k \,, \ j, k = 1, 2) \tag{6.6}$$

$$Y_{12}(u, z) = \sum_a \tilde{\nu}_a \varphi_a'(2u, w + \omega_a) \,, \quad Y_{21}(u, w) = Y_{12}(-u, w) \,.$$

Proposition 6.2 *The equation of motion of NAZVG (6.1) has the Lax form with*

$$L^{NAZVG} = -\frac{\kappa}{2} \partial_w \ln \vartheta(w; \tau) \sigma_0 + \sum_\alpha (S_\alpha \varphi_\alpha(w) + \nu_\alpha \varphi_\alpha(w - \omega_\alpha)) \sigma_\alpha \,. \tag{6.7}$$

$$M^{NAZVG} = -\frac{\kappa}{2} \partial_\tau \ln \vartheta(w; \tau) \sigma_0 + \sum_\alpha -S_\alpha \frac{\varphi_1(w)\varphi_2(w)\varphi_3(w)}{\varphi_\alpha(w)} \sigma_\alpha + E_1(w) L^{NAZVG}(\kappa = 0) \,. \tag{6.8}$$

The proofs of the both statements repeat the autonomous cases. The only new ingredient is the heat equation (A.13), that should be used.

Now we establish the interrelation between these systems. It is provided by the same modification transform as above. Namely

$$L^{NAZVG}(\mathbf{S}, w) = \Xi^+ \kappa \partial (\Xi^+)^{-1} + \Xi^+ L^{(CI)} \Xi^{+-1} \,,$$

where Ξ^+ is defined by (4.7).

The upper modification leads to the following relations between the coordinates of the phase spaces

$$
\begin{array}{l}
S_1 = -v\frac{\theta_2(0)}{\vartheta'(0)}\frac{\theta_2(2u)}{\vartheta(2u)} - \frac{\kappa}{2}\frac{\theta_2(0)}{\vartheta'(0)}\frac{\theta_2'(2u)}{\vartheta(2u)} + \\
\tilde{\nu}_0\frac{\theta_2^2(0)}{\theta_3(0)\theta_4(0)}\frac{\theta_3(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_1\frac{\theta_2^2(2u)}{\vartheta^2(2u)} + \tilde{\nu}_2\frac{\theta_2(0)}{\theta_4(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_3\frac{\theta_2(0)}{\theta_3(0)}\frac{\theta_2(2u)\theta_3(2u)}{\vartheta^2(2u)}\,, \\
iS_2 = v\frac{\theta_3(0)}{\vartheta'(0)}\frac{\theta_3(2u)}{\vartheta(2u)} + \frac{\kappa}{2}\frac{\theta_3(0)}{\vartheta'(0)}\frac{\theta_3'(2u)}{\vartheta(2u)} - \\
\tilde{\nu}_0\frac{\theta_3^2(0)}{\theta_2(0)\theta_4(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} - \tilde{\nu}_1\frac{\theta_3(0)}{\theta_2(0)}\frac{\theta_3(2u)\theta_2(2u)}{\vartheta^2(2u)} - \tilde{\nu}_2\frac{\theta_3(0)}{\theta_4(0)}\frac{\theta_3(2u)\theta_4(2u)}{\vartheta^2(2u)} - \tilde{\nu}_3\frac{\theta_3^2(2u)}{\vartheta^2(2u)}\,, \\
S_3 = -v\frac{\theta_4(0)}{\vartheta'(0)}\frac{\theta_4(2u)}{\vartheta(2u)} - \frac{\kappa}{2}\frac{\theta_4(0)}{\vartheta'(0)}\frac{\theta_4'(2u)}{\vartheta(2u)} + \\
\tilde{\nu}_0\frac{\theta_4^2(0)}{\theta_2(0)\theta_3(0)}\frac{\theta_2(2u)\theta_3(2u)}{\vartheta^2(2u)} + \tilde{\nu}_1\frac{\theta_4(0)}{\theta_2(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_2\frac{\theta_4^2(2u)}{\vartheta^2(2u)} + \tilde{\nu}_3\frac{\theta_4(0)}{\theta_3(0)}\frac{\theta_4(2u)\theta_3(2u)}{\vartheta^2(2u)}\,.
\end{array}
\tag{6.9}
$$

The proof is based on the direct usage of (B.19), (B.20) and Riemann identities for theta functions [24].

7 Quadratic brackets and NAET

7.1 r-matrix structure

The classical r-matrix is the quasi-periodic map from Σ_τ to $\mathrm{End}(E_N^{(1)})\otimes\mathrm{End}(E_N^{(1)})$ [3]

$$
r(w) = \sum_\gamma \varphi_\gamma(w) T_\gamma \otimes T_{-\gamma}\,, \tag{7.10}
$$

where φ_γ is defined by (B.10). It satisfies the classical Yang-Baxter equation

$$
[r^{(12)}(z-w), r^{(13)}(z)] + [r^{(12)}(z-w), r^{(23)}(w)]
$$

$$
+[r^{(13)}(z), r^{(23)}(w)] = 0\,.
$$

By means of the r-matrix one can define the linear brackets. Let $L^{(1)}(z)$ be the Lax matrix (3.6) for $\kappa = 0$

$$
L^{(1)}(z) = \sum_\gamma S_\alpha \varphi_\alpha(z) T_\alpha\,.
$$

Proposition 7.1 ([16]) *The Lie-Poisson brackets on* $\mathrm{sl}(N,\mathbb{C})$

$$
\{S_\alpha, S_\beta\}_1 = \mathbf{C}(\alpha,\beta) S_{\alpha+\beta}
$$

in terms of the Lax operator $L^{(1)}(\mathbf{S}, z)$ *are equivalent to the following relation for the Lax operator*

$$
\{L_1^{(1)}(z), L_2^{(1)}(z')\}_1 = [r(z-z'), L^{(1)}(z)\otimes Id + Id\otimes L^{(1)}(z')]
$$

$$
L_1^{(1)} = L^{(1)}\otimes Id\,,\quad L_2^{(1)} = Id\otimes L^{(1)}\,.
$$

The proof is based on the Fay three-section formula (A.19).

7.2 Quadratic Poisson algebra

In addition to the N^2-1 variables $\mathbf{S} = \{S_\alpha,\, \alpha \in \tilde{\mathbb{Z}}_N^{(2)}\}$ introduce a new variable S_0 and the GL$(N,\mathbb{C})$-valued Lax operator

$$\tilde{L} = -S_0 Id + L^{(1)}(\mathbf{S}, w)\,.$$

It satisfies the classical exchange algebra:

$$\{\tilde{L}_1(w), \tilde{L}_2(w')\}_2 = [r(w-w'), \tilde{L}_1(w) \otimes \tilde{L}_2(w')]\,. \tag{7.11}$$

These brackets are Poisson, since the Jacobi identity is provided by the classical Yang-Baxter equation.

Proposition 7.2 *The quadratic Poisson algebra (7.11) in the coordinates* $(S_0, \mathbf{S})$ *takes the form*

$$\{S_\alpha, S_0\}_2 = \sum_{\gamma \neq \alpha} S_{\alpha-\gamma} S_\gamma (E_2(\frac{\gamma_1+\gamma_2\tau}{N}) - E_2(\frac{\alpha_1-\gamma_1+(\alpha_2-\gamma_2\tau)}{N}))\mathbf{C}(\alpha,\gamma)\,, \tag{7.12}$$

$$\{S_\alpha, S_\beta\}_2 = S_0 S_{\alpha+\beta}\mathbf{C}(\alpha,\beta) + \sum_{\gamma\neq\alpha,-\beta} S_{\alpha-\gamma}S_{\beta+\gamma}\mathbf{f}(\alpha,\beta,\gamma)\mathbf{C}(\gamma,\alpha-\beta)\,,$$

where

$$\mathbf{f}(\alpha,\beta,\gamma) = E_1(\frac{\gamma_1+\gamma_2\tau}{N}) + E_1(\frac{\beta_1+\gamma_1-\alpha_1+(\beta_2+\gamma_2-\alpha_2)\tau}{N}) -$$
$$E_1(\frac{\beta_1+\gamma_1+(\beta_2+\gamma_2)\tau}{N}) + E_1(\frac{\alpha_1-\gamma_1+(\alpha_2-\gamma_2)\tau}{N})\,.$$

It is the classical Sklyanin-Feigin-Odesski (SFO) algebra [32, 7]. These brackets are extracted from (7.11) by means of (A.21), (A.22).

Two Poisson structures are called *compatible* (or, form *a Poisson pair*) if their linear combinations are Poisson structures as well. It turns out that the linear and quadratic Poisson brackets are compatible, namely, there exists the one-parametric family of the Poisson brackets

$$\{\mathbf{S},\mathbf{S}\}_\lambda = \{\mathbf{S},\mathbf{S}\}_2 + \lambda\{\mathbf{S},\mathbf{S}\}_1\,.$$

In the case of the ET integrable hierarchy these compatible brackets provide the hierarchy with the bihamiltonian structure [16]. The hierarchies of the monodromy preserving equations are more intricate [20] and we do not consider here the hierarchy of NAET. However, we have the following manifestation of the bihamiltonian structure.

Proposition 7.3 *In terms of the quadratic brackets the equation of motion of NAET (3.3) assumes the form*

$$\kappa\partial_\tau S_\alpha = \{S_0, S_\alpha\}_2\,.$$

The proof follows immediately from (7.12). We replace the linear brackets on quadratic and simultaneously the quadratic Hamiltonian H^{NAET} on the linear S_0.

8 Reflection equation and generalized Sklyanin algebra

In this Section we present another Hamiltonian form of NAZVG (1.2). It is based on the quadratic Poisson brackets. The quantization of these brackets is described by quantum reflection equation.

8.1 Quantum reflection equation

Let R^- be the quantum vertex R-matrix, that arises in the XYZ model. We introduce also the matrix R^+

$$R^{\pm}(z,w) = \sum_{a=0}^{3} \varphi_{\alpha}^{\frac{\hbar}{2}}(z \pm w)\sigma_\alpha \otimes \sigma_\alpha \tag{8.1}$$

where $\varphi_a^{\frac{\hbar}{2}}$ is defined by (C.1). Define the quantum Lax operator

$$\hat{L}(z) = \hat{S}_0 \phi^{\hbar}(z)\sigma_0 + \sum_\alpha (\hat{S}_\alpha \varphi_\alpha^{\hbar}(z) + \tilde{\nu}_\alpha \varphi_\alpha^{\hbar}(z - \omega_\alpha))\sigma_\alpha \,. \tag{8.2}$$

Proposition 8.1 *The Lax operator satisfies the quantum reflection equation*

$$R^-(z,w)\hat{L}_1(z)R^+(z,w)\hat{L}_2(w) = \hat{L}_2(w)R^+(z,w)\hat{L}_1(z)R^-(z,w)\,. \tag{8.3}$$

iff its components S_a generate the associative algebra with relations

$$[\tilde{\nu}_\alpha, \tilde{\nu}_\beta] = 0\,, \quad [\tilde{\nu}_\alpha, \hat{S}_a] = 0\,, \tag{8.4}$$

$$i[\hat{S}_0, \hat{S}_\alpha]_+ = [\hat{S}_\beta, \hat{S}_\gamma]\,, \tag{8.5}$$

$$[\hat{S}_\gamma, \hat{S}_0] = i\frac{K_\beta - K_\alpha}{K_\gamma}[\hat{S}_\alpha, \hat{S}_\beta]_+ - 2i\frac{1}{K_\gamma}(\tilde{\nu}_\alpha \rho_\alpha \hat{S}_\beta - \tilde{\nu}_\beta \rho_\beta \hat{S}_\alpha)\,, \tag{8.6}$$

where

$$K_\alpha = E_1(\hbar + \omega_\alpha) - E_1(\hbar) - E_1(\omega_\alpha)\,, \quad \rho_\alpha = -\exp(-2\pi\imath\omega_\alpha\partial_\tau\omega_\alpha)\phi(\omega_\alpha + \hbar, -\omega_\alpha)\,.$$

For the proof see Appendix D.

If all $\tilde{\nu}_\alpha = 0$ the algebra (8.4) – (8.6) coincides with the Sklyanin algebra. Therefore, it is a three parametric deformation of the Sklyanin algebra.

Two elements

$$C_1 = \hat{S}_0^2 + \sum_\alpha \hat{S}_\alpha^2\,,$$

$$C_2 = \sum_\alpha \hat{S}_\alpha^2 K_\alpha (K_\alpha - K_\beta - K_\gamma) + 2\tilde{\nu}_\alpha \rho_\alpha K_\alpha \hat{S}_\alpha$$

belong to the center of the generalized Sklyanin algebra (8.4), (8.5). They are the coefficients of the expansion of the quantum determinant

$$\det_{\hbar} \mathrm{tr} P(\hat{L}(z,\hbar) \otimes \hat{L}^+(z,-\hbar))\,,$$

where

$$P = \sigma_0 \otimes \sigma_0 + \sum_\alpha \sigma_\alpha \otimes \sigma_\alpha\,,$$

$$\hat{L}^+(z,\hbar) = \hat{S}_0 \phi^{\hbar}(z)\sigma_0 - \sum_\alpha (\hat{S}_\alpha \varphi_\alpha^{\hbar}(z) + \tilde{\nu}_\alpha \varphi_\alpha^{\hbar}(z - \omega_\alpha))\sigma_\alpha \,.$$

over the basis of the elliptic functions. However, we do not know how to derive the expression for the quantum determinant directly from the reflection equation.

8.2 Classical reflection equations

Consider (8.3) in the limit $\hbar \to 0$. The classical $r^{\pm}$-matrices are defined from the expansion

$$R^{\pm}(z,w) = \left(\frac{\hbar}{2}\right)^{-1} \sigma_0 \otimes \sigma_0 + r^{\pm}(z,w) + O(\hbar)\,,$$

$$r^{\pm}(z,w) = \sum_{\alpha} \varphi_\alpha(z \pm w)\sigma_\alpha \otimes \sigma_\alpha\,.$$

For the Lax operator one has

$$\hat{L}(z) = \hbar^{-1}\hat{S}_0\sigma_0 + \sum_{\alpha}(\hat{S}_\alpha\varphi_\alpha(z) + \tilde{\nu}_\alpha\varphi_\alpha(z-\omega_\alpha))\sigma_\alpha + O(\hbar)\,.$$

Define the classical variables

$$\hat{S}_\alpha \to S_\alpha\,, \quad \hat{S}_0 \to \hbar S_0\,,$$

and the corresponding classical Lax operator

$$\tilde{L}(z) = S_0\sigma_0 + \sum_{\alpha=1}^{3}(S_\alpha\varphi_\alpha(z) + \nu_\alpha\varphi_\alpha(z-\omega_\alpha))\sigma_\alpha\,.$$

Then, taking into account that $[L_1, L_2] = \hbar\{\tilde{L}_1, \tilde{L}_2\} + O(\hbar^2)$ one finds the classical reflection equation in the first order of $\hbar^{-1}$

$$\{\tilde{L}_1(z), \tilde{L}_2(w)\}_2 = \frac{1}{2}[\tilde{L}_1(z)\tilde{L}_2(w), r^-(z,w)] + \tag{8.7}$$

$$\frac{1}{2}\tilde{L}_2(w)r^+(z,w)\tilde{L}_1(z) - \frac{1}{2}\tilde{L}_1(z)r^+(z,w)\tilde{L}_2(w)\,.$$

by passing from the group-valued element $\tilde{L}$. For the Lie-algebraic element L

$$L(z) = \sum_{\alpha=1}^{3}(S_\alpha\varphi_\alpha(z) + \tilde{\nu}_\alpha\varphi_\alpha(z-\omega_\alpha))\sigma_\alpha$$

we come to the linear brackets

$$\{L_1(z), L_2(w)\}_1 = -\frac{1}{2}[r^-(z,w), L_1(z) + L_2(w)] + \frac{1}{2}[r^+(z,w), L_1(z) - L_2(w)]\,. \tag{8.8}$$

Proposition 8.2 *The classical reflection equations (8.7) leads to the quadratic Poisson structure on $\mathbb{C}^4$ generalizing the classical Sklyanin algebra*

$$\{S_\alpha, S_\beta\}_2 = 2i\varepsilon_{\alpha\beta\gamma}S_0S_\gamma\,, \tag{8.9}$$

$$\{S_0, S_\alpha\}_2 = i\varepsilon_{\alpha\beta\gamma}S_\beta S_\gamma(E_2(\omega_\beta) - E_2(\omega_\gamma)) + 2i\varepsilon_{\alpha\beta\gamma}S_\beta\nu'_\gamma\,, \tag{8.10}$$

where ν' is defined as (5.20).

Proof.
The only dissimilarity from the classical Sklyanin algebra is the linear term in (8.10). It comes from the last term in the right hand side of (8.6) in the limit $\hbar \to 0$. □

There are two Casimir elements of the Poisson algebra (8.9), (8.10)

$$c_1 = \sum_\alpha S_\alpha^2 \,,$$

$$c_2 = S_0^2 + \sum_\alpha (\wp(\omega_\alpha) S_\alpha^2 + 2\nu'_\alpha S_\alpha)\,.$$

They are the coefficients of the expansion of $\det(\tilde{L}(z)$ over the basis of elliptic functions.

It is easy to see that the brackets (8.9) are compatible with the linear sl_2 Lie-Poisson brackets

$$\{S_\alpha, S_\beta\}_1 = 2i\varepsilon_{\alpha\beta\gamma} S_\gamma \,.$$

The equation of motion of NAZVG (1.2) is written in terms of the linear brackets:

$$\partial_\tau S_\alpha = \{H^{ZVG}, S_\alpha\}_1 \,.$$

The straightforward calculations shows that (1.2) can be written in the form

$$\partial_\tau S_\alpha = \{H_0, S_\alpha\}_2 \,, \quad H_0 = S_0 \,. \tag{8.11}$$

In this way for the generic form of PVI we have the same analog of the bihamiltonian property as in the degenerate case (Proposition 5.4).

8.3 Spin chain with boundaries

Quantum reflection equation allows us to define XYZ model on a finite lattice with boundary conditions [33]. The Lax operator (8.2) can be considered as a new solution of the reflection equation.

Consider a pair of matrices $K^\pm(z)$ with Poisson brackets

$$\{K_1^\pm, K_2^\pm\} = [K_1^\pm(z) K_2^\pm(w), r(z-w)] + K_2^\pm(w) r(z+w) K_1^\pm(z) - K_1^\pm(z) r(z+w) K_2^\pm(w) \tag{8.12}$$

and $L^i(z),\ i = 1 ... N$ with brackets

$$\{L_1^i(z), L_2^j(w)\} = \delta^{ij} [r(z-w), L_1^i(z) L_2^j(w)] \,. \tag{8.13}$$

Then, according with [33]

$$h(z) = tr\left[K^+(z) L^N(z) ... L^1(z) K^-(z) \left(L^1(-z)\right)^{-1} ... \left(L^N(-z)\right)^{-1}\right] \tag{8.14}$$

is the generating function of commuting Hamiltonians

$$\{h(z), h(w)\} = 0 \,. \tag{8.15}$$

Choosing $L^{ZVG_\pm}(z) = L^{ZVG_\pm}(z, \pm\hbar)$ for $K^\pm(z)$ we construct a spin chain with boundaries. But in (8.7) we have a factor $\frac{1}{2}$ which comes from (8.1) on the quantum level. Thus, we should put the brackets on boundaries for $L^{ZVG_\pm}(z)$ to be two times more than in (8.9). In other words R-matrices for $\hat{L}^{ZVG_\pm}(z)$ should depend on the same Planck constant $\hbar$ as for all other $\hat{L}^i(z)$.

Proposition 8.3 *Spin chain involving N internal vertices $L^i(z)$ with boundaries $L^{ZVG_\pm}(z)$ is integrable if*

$$\begin{array}{l} \{S^\pm_\alpha, S^\pm_\beta\} = 4\imath\varepsilon_{\alpha\beta\gamma}S^\pm_0 S^\pm_\gamma\,, \\ \{S^\pm_0, S^\pm_\alpha\} = 2\imath\varepsilon_{\alpha\beta\gamma}(S^\pm_\beta S^\pm_\gamma\,, (E_2(\omega_\beta) - E_2(\omega_\gamma)) + S^\pm_\beta \nu'_\gamma) \end{array} \tag{8.16}$$

and for $i, j = 1...N$

$$\begin{array}{c} \{S^i_\alpha, S^j_\beta\} = 2\delta^{ij}\imath\varepsilon_{\alpha\beta\gamma}S^i_0 S^i_\gamma\,, \\ \{S^i_0, S^j_\alpha\} = \delta^{ij}\imath\varepsilon_{\alpha\beta\gamma}S^i_\beta S^i_\gamma(E_2(\omega_\beta) - E_2(\omega_\gamma))\,. \end{array} \tag{8.17}$$

The nearest-neighbor interaction is described by the Hamiltonian:

$$\begin{array}{c} H = \ln\left(S^-_0 S^1_0 + \sum\limits_\alpha S^-_\alpha S^1_\alpha(C - E_2(\omega_\alpha)) + (\nu')^-_\alpha S^1_\alpha\right) + \\ \ln\left(S^N_0 S^+_0 + \sum\limits_\alpha S^N_\alpha S^+_\alpha(C - E_2(\omega_\alpha)) + (\nu')^+_\alpha S^N_\alpha\right) + \\ \sum\limits_{i=1}^{N-1} \ln\left(S^i_0 S^{i+1}_0 + \sum\limits_\alpha S^i_\alpha S^{i+1}_\alpha(C - E_2(\omega_\alpha))\right)\,, \end{array} \tag{8.18}$$

where C is a constant, equal to the fraction of the values of the Casimir functions for Sklyanin bracket (8.17).

The proof of (8.18) is similar to those one given in [6] (Chapter III, § 5). Consider a special point $z_0 \in \Sigma$, such that $E_2(z_0) = \frac{S^i_0 S^i_0 + \sum_{\alpha=1}^3 S^i_\alpha S^i_\alpha E_2(\omega_\alpha)}{\sum_{\alpha=1}^3 S^i_\alpha S^i_\alpha} = C_i$ and assume that this relation is independent on a point of the lattice $C_i = C_j$ $\forall i, j = 1...N$. In this case all $L^i(z_0)$ are degenerated and have the form $L^i(z_0) = \alpha^i(z_0) \times \beta^i(z_0)$ for some vectors $\alpha^i(z_0)$ and covectors $\beta^i(z_0)$. To finish the proof we should notice that all $\det L^i(z)$ are Casimirs for (8.13) and in our case $L^i(z)L^i(-z) = 1 \cdot \det L^i(z)$ $\forall i$.

Some solutions of the reflection equation corresponding to the spin chains with dynamical boundary conditions are constructed in Ref. [18, 14].

9 Appendix

9.1 Appendix A. Elliptic functions.

We assume that $q = \exp 2\pi i\tau$, where τ is the modular parameter of the elliptic curve E_τ.

The basic element is the theta function:

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n \mathbf{e}(\frac{1}{2}n(n+1)\tau + nz) = \quad (\mathbf{e} = \exp 2\pi\imath) \tag{A.1}$$

$$q^{\frac{1}{8}} e^{-\frac{i\pi}{4}}(e^{i\pi z} - e^{-i\pi z}) \prod_{n=1}^\infty (1 - q^n)(1 - q^n e^{2i\pi z})(1 - q^n e^{-2i\pi z})\,.$$

The Eisenstein functions

$$E_1(z|\tau) = \partial_z \log\vartheta(z|\tau), \quad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{A.2}$$

where

$$\eta_1(\tau) = \frac{3}{\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(m\tau+n)^2} = \frac{24}{2\pi i} \frac{\eta'(\tau)}{\eta(\tau)}\,, \tag{A.3}$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0} (1-q^n)\,.$$

is the Dedekind function.

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log \vartheta(z|\tau), \quad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1\,. \tag{A.4}$$

The higher Eisenstein functions

$$E_j(z) = \frac{(-1)^j}{(j-1)!} \partial^{(j-2)} E_2(z)\,, \quad (j>2)\,. \tag{A.5}$$

Relation to the Weierstrass functions

$$\zeta(z,\tau) = E_1(z,\tau) + 2\eta_1(\tau) z\,, \tag{A.6}$$

$$\wp(z,\tau) = E_2(z,\tau) - 2\eta_1(\tau)\,. \tag{A.7}$$

The next important function is

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}\,. \tag{A.8}$$

$$\phi(u,z) = \phi(z,u)\,, \quad \phi(-u,-z) = -\phi(u,z)\,. \tag{A.9}$$

It has a pole at $z=0$ and

$$\phi(u,z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + \dots\,. \tag{A.10}$$

Let

$$f(u,z) = \partial_u \phi(u,z)\,. \tag{A.11}$$

Then

$$f(u,z) = \phi(u,z)(E_1(u+z) - E_1(u))\,. \tag{A.12}$$

Heat equation

$$\partial_\tau \phi(u,w) - \frac{1}{2\pi i} \partial_u \partial_w \phi(u,w) = 0\,. \tag{A.13}$$

Quasi-periodicity

$$\vartheta(z+1) = -\vartheta(z)\,, \quad \vartheta(z+\tau) = -q^{-\frac{1}{2}} e^{-2\pi i z} \vartheta(z)\,, \tag{A.14}$$

$$E_1(z+1) = E_1(z)\,, \quad E_1(z+\tau) = E_1(z) - 2\pi i\,, \tag{A.15}$$

$$E_2(z+1) = E_2(z)\,, \quad E_2(z+\tau) = E_2(z)\,, \tag{A.16}$$

$$\phi(u,z+1) = \phi(u,z)\,, \quad \phi(u,z+\tau) = e^{-2\pi \imath u} \phi(u,z)\,. \tag{A.17}$$

$$f(u,z+1) = f(u,z)\,, \quad f(u,z+\tau) = e^{-2\pi \imath u} f(u,z) - 2\pi \imath \phi(u,z)\,. \tag{A.18}$$

The Fay three-section formula:

$$\phi(u_1,z_1)\phi(u_2,z_2)-\phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1)-\phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2)=0\,. \tag{A.19}$$

Particular cases of this formula is the Calogero functional equation

$$\phi(u,z)\partial_v\phi(v,z)-\phi(v,z)\partial_u\phi(u,z)=(E_2(v)-E_2(u))\phi(u+v,z)\,, \tag{A.20}$$

Another important relation is

$$\phi(v,z-w)\phi(u_1-v,z)\phi(u_2+v,w)-\phi(u_1-u_2-v,z-w)\phi(u_2+v,z)\phi(u_1-v,w)= \tag{A.21}$$

$$\phi(u_1,z)\phi(u_2,w)f(u_1,u_2,v)\,,$$

where

$$f(u_1,u_2,v)=\zeta(v)-\zeta(u_1-u_2-v)+\zeta(u_1-v)-\zeta(u_2+v)\,. \tag{A.22}$$

One can rewrite the last function as

$$f(u_1,u_2,v)=-\frac{\vartheta'(0)\vartheta(u_1)\vartheta(u_2)\vartheta(u_2-u_1+2v)}{\vartheta(u_1-v)\vartheta(u_2+v)\vartheta(u_2-u_1+v)\vartheta(v)}\,. \tag{A.23}$$

Theta functions with characteristics:
For $a,b\in\mathbb{Q}$ by definition:

$$\theta\left[\begin{array}{c}a\\b\end{array}\right](z,\tau)=\sum_{j\in\mathbb{Z}}\mathbf{e}\left((j+a)^2\frac{\tau}{2}+(j+a)(z+b)\right)\,.$$

In particular, the function ϑ (A.1) is a theta function with characteristics:

$$\vartheta(x,\tau)=\theta\left[\begin{array}{c}1/2\\1/2\end{array}\right](x,\tau)\,. \tag{A.24}$$

Properties:

$$\theta\left[\begin{array}{c}a\\b\end{array}\right](z+1,\tau)=\mathbf{e}(a)\theta\left[\begin{array}{c}a\\b\end{array}\right](z,\tau)\,,$$

$$\theta\left[\begin{array}{c}a\\b\end{array}\right](z+a'\tau,\tau)=\mathbf{e}\left(-a'^2\frac{\tau}{2}-a'(z+b)\right)\theta\left[\begin{array}{c}a+a'\\b\end{array}\right](z,\tau)\,,$$

$$\theta\left[\begin{array}{c}a+j\\b\end{array}\right](z,\tau)=\theta\left[\begin{array}{c}a\\b\end{array}\right](z,\tau),\quad j\in\mathbb{Z}\,.$$

The following notations are used: $\theta\left[\begin{array}{c}a/2\\b/2\end{array}\right]=\theta_{ab}$ and $\vartheta=\theta_{11}$.

9.2 Appendix B. Lie algebra $\mathrm{sl}(N,\mathbb{C})$ and elliptic functions

Introduce the notation

$$\mathbf{e}_N(z)=\exp(\frac{2\pi i}{N}z)$$

and two matrices

$$Q=\mathrm{diag}(\mathbf{e}_N(1),\ldots,\mathbf{e}_N(m),\ldots,1) \tag{B.1}$$

$$\Lambda=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{B.2}$$

Let

$$\mathbb{Z}^{(2)}_N=(\mathbb{Z}/N\mathbb{Z}\oplus\mathbb{Z}/N\mathbb{Z})\,,\quad \tilde{\mathbb{Z}}^{(2)}_N=\mathbb{Z}^{(2)}_N\setminus(0,0) \tag{B.3}$$

be the two-dimensional lattice of order N^2 and N^2-1 correspondingly. The matrices $Q^{a_1}\Lambda^{a_2}$, $a=(a_1,a_2)\in\mathbb{Z}^{(2)}_N$ generate a basis in the group $\mathrm{GL}(N,\mathbb{C})$, while $Q^{\alpha_1}\Lambda^{\alpha_2}$, $\alpha=(\alpha_1,\alpha_2)\in\tilde{\mathbb{Z}}^{(2)}_N$ generate a basis in the Lie algebra $\mathrm{sl}(N,\mathbb{C})$. Consider the projective representation of $\mathbb{Z}^{(2)}_N$ in $\mathrm{GL}(N,\mathbb{C})$

$$a\to T_a=\frac{N}{2\pi i}\mathbf{e}_N(\frac{a_1a_2}{2})Q^{a_1}\Lambda^{a_2}\,, \tag{B.4}$$

$$T_aT_b=\frac{N}{2\pi i}\mathbf{e}_N(-\frac{a\times b}{2})T_{a+b}\,,\quad (a\times b=a_1b_2-a_2b_1) \tag{B.5}$$

Here $\frac{N}{2\pi i}\mathbf{e}_N(-\frac{a\times b}{2})$ is a non-trivial two-cocycle in $H^2(\mathbb{Z}^{(2)}_N,\mathbb{Z}_{2N})$. It follows from (B.5) that

$$[T_\alpha,T_\beta]=\mathbf{C}(\alpha,\beta)T_{\alpha+\beta}\,, \tag{B.6}$$

where $\mathbf{C}(\alpha,\beta)=\frac{N}{\pi}\sin\frac{\pi}{N}(\alpha\times\beta)$ are the structure constants of $\mathrm{sl}(N,\mathbb{C})$.

Introduce the following constants on $\tilde{\mathbb{Z}}^{(2)}$:

$$\vartheta(\gamma)=\vartheta(\frac{\gamma_1+\gamma_2\tau}{N})\,, \tag{B.7}$$

$$E_1(\gamma)=E_1(\frac{\gamma_1+\gamma_2\tau}{N})\,,\quad E_2(\gamma)=E_2(\frac{\gamma_1+\gamma_2\tau}{N})\,, \tag{B.8}$$

and the quasi-periodic functions on Σ_τ

$$\phi_\gamma(z)=\phi(\frac{\gamma_1+\gamma_2\tau}{N},z)\,, \tag{B.9}$$

$$\varphi_\gamma(z)=\mathbf{e}_N(\gamma_2 z)\phi_\gamma(z)\,, \tag{B.10}$$

$$f_\gamma(z)=\mathbf{e}_N(\gamma_2 z)\partial_u\phi(u,z)|_{u=\frac{\gamma_1+\gamma_2\tau}{N}}\,. \tag{B.11}$$

$$f_\gamma(z)=\mathbf{e}_N(\gamma_2 z)\phi_\gamma(z)(E_1(\frac{\gamma_1+\gamma_2\tau}{N}+z)-E_1(\frac{\gamma_1+\gamma_2\tau}{N}))\,. \tag{B.12}$$

It follows from (A.8) that

$$\varphi_\gamma(z+1)=\mathbf{e}_N(\gamma_2)\varphi_\gamma(z)\,,\quad \varphi_\gamma(z+\tau)=\mathbf{e}_N(-\gamma_1)\varphi_\gamma(z)\,. \tag{B.13}$$

$$f_\gamma(z+1)=\mathbf{e}_N(\gamma_2)f_\gamma(z)\,,\quad f_\gamma(z+\tau)=\mathbf{e}_N(-\gamma_1)f_\gamma(z)-2\pi\imath\varphi_\gamma(z)\,. \tag{B.14}$$

α	(1,0)	(0,1)	(1,1)
σ_α	σ_3	σ_1	σ_2
half-periods	$\omega_1=\frac{1}{2}$	$\omega_2=\frac{\tau}{2}$	$\omega_3=\frac{1+\tau}{2}$
$\varphi_\alpha(z)$	$\frac{\theta_2(z)\theta'_1(0)}{\theta_2(0)\theta_1(z)}$	$\frac{\theta_4(z)\theta'_1(0)}{\theta_4(0)\theta_1(z)}$	$\frac{\theta_3(z)\theta'_1(0)}{\theta_3(0)\theta_1(z)}$

SL$(2,\mathbb{C})$ *case*
For SL$(2,\mathbb{C})$ instead of T_α we use the basis of sigma-matrices

$$\sigma_0 = Id\,,\quad \sigma_1 = i\pi T_{0,1}\,,\quad \sigma_2 = i\pi T_{1,1}\,,\quad \sigma_3 = -i\pi T_{1,0}\,, \tag{B.15}$$

$$\{\sigma_a\} = \{\sigma_0,\sigma_\alpha\}\,,(a=0,\alpha)\,,(\alpha=1,2,3)$$

$$\sigma_+ = \frac{\sigma_1 - \imath\sigma_2}{2}\,,\quad \sigma_- = \frac{\sigma_1+\imath\sigma_2}{2}\,.$$

The standard theta-functions with the characteristics are

$$\theta_{0,0} = \theta_3\,,\quad \theta_{1,0}=\theta_2\,,\quad \theta_{0,1}=\theta_4\,,\quad \theta_{1,1}=\theta_1\,. \tag{B.16}$$

$$\varphi_\alpha(z)\varphi_\alpha(z-\omega_\alpha) = -\mathbf{e}_1(-\omega_\alpha\partial_\tau\omega_\alpha)\left(\frac{\vartheta'(0)}{\vartheta(\omega_\alpha)}\right)^2 \tag{B.17}$$

$$\varphi_a(-z-\omega_b) = -\varphi_a(z-\omega_a)\delta_{ab} + (1-\delta_{ab})\varphi_a(z-\omega_b)\,. \tag{B.18}$$

Formulae with doubled modular parameter:

$$\begin{array}{l}
\theta_4(x,\tau)\theta_3(y,\tau)+\theta_4(y,\tau)\theta_3(x,\tau) = 2\theta_4(x+y,2\tau)\theta_4(x-y,2\tau)\\
\theta_4(x,\tau)\theta_3(y,\tau)-\theta_4(y,\tau)\theta_3(x,\tau) = 2\vartheta(x+y,2\tau)\vartheta(x-y,2\tau)\\
\theta_3(x,\tau)\theta_3(y,\tau)+\theta_4(y,\tau)\theta_4(x,\tau) = 2\theta_3(x+y,2\tau)\theta_3(x-y,2\tau)\\
\theta_3(x,\tau)\theta_3(y,\tau)-\theta_4(y,\tau)\theta_4(x,\tau) = 2\theta_2(x+y,2\tau)\theta_2(x-y,2\tau)
\end{array} \tag{B.19}$$

$$\begin{array}{l}
2\vartheta(x,2\tau)\theta_4(y,2\tau) = \vartheta(\frac{x+y}{2},\tau)\theta_2(\frac{x-y}{2},\tau)+\theta_2(\frac{x+y}{2},\tau)\vartheta(\frac{x-y}{2},\tau)\\
2\theta_3(x,2\tau)\theta_2(y,2\tau) = \vartheta(\frac{x+y}{2},\tau)\vartheta(\frac{x-y}{2},\tau)+\theta_2(\frac{x+y}{2},\tau)\theta_2(\frac{x-y}{2},\tau)\\
2\theta_3(x,2\tau)\theta_3(y,2\tau) = \theta_3(\frac{x+y}{2},\tau)\theta_3(\frac{x-y}{2},\tau)+\theta_4(\frac{x+y}{2},\tau)\theta_4(\frac{x-y}{2},\tau)\\
2\theta_2(x,2\tau)\theta_2(y,2\tau) = \theta_3(\frac{x+y}{2},\tau)\theta_3(\frac{x-y}{2},\tau)-\theta_4(\frac{x+y}{2},\tau)\theta_4(\frac{x-y}{2},\tau)
\end{array} \tag{B.20}$$

9.3 Appendix C. Deformed elliptic functions

$$\varphi_a^\eta(z) = \mathbf{e}_N(a_2 z)\phi(\frac{a_1+a_2\tau}{N}+\eta,z)\,,\ a\in\mathbb{Z}_N^{(2)}\,,\ \eta\in\Sigma_\tau\,. \tag{C.1}$$

It follows from (B.13) that $\varphi_a^\eta(z)$ is well defined on $\mathbb{Z}_N^{(2)}$:

$$\varphi_{a+c}^\eta(z) = \varphi_a^\eta(z)\,,\ \text{for } c_{1,2}\in\mathbb{Z}\bmod N\,. \tag{C.2}$$

$$\varphi_a^\eta(z+1) = \mathbf{e}_N(a_2)\varphi_a^\eta(z)\,,\quad \varphi_a^\eta(z+\tau) = \mathbf{e}_N(-a_1-N\eta)\varphi_a^\eta(z)\,. \tag{C.3}$$

The following formulae can be proved directly by checking the structure of poles and quasi-periodic properties:

$$\phi(w,\eta)\varphi_\alpha^\eta(z-w)+\phi(-w,\eta)\varphi_\alpha^\eta(z+w) = \phi(z,\eta)(\varphi_\alpha(z-w)+\varphi_\alpha(z+w)) \tag{C.4}$$

$$\begin{aligned}\phi^\eta(z-w)\phi^\eta(z+w)+\varphi^\eta_\alpha(z-w)\varphi^\eta_\alpha(z+w)+\varphi^\eta_\beta(z-w)\varphi^\eta_\beta(z+w)+\\ \varphi^\eta_\gamma(z-w)\varphi^\eta_\gamma(z+w)=2\phi^{2\eta}(z-w)\phi^w(2\eta)+2\phi^{2\eta}(z+w)\phi^{-w}(2\eta)\end{aligned} \tag{C.5}$$

$$\begin{aligned}\phi^\eta(z-w)\phi^\eta(z+w)+\varphi^\eta_\alpha(z-w)\varphi^\eta_\alpha(z+w)-\varphi^\eta_\beta(z-w)\varphi^\eta_\beta(z+w)-\\ \varphi^\eta_\gamma(z-w)\varphi^\eta_\gamma(z+w)=2\phi^{2\eta}(z-w)\varphi^w_\alpha(2\eta)+2\phi^{2\eta}(z+w)\varphi^{-w}_\alpha(2\eta)\end{aligned} \tag{C.6}$$

$$\begin{aligned}\varphi^\eta_\beta(z-w)\phi^\eta(z+w)-\phi^\eta(z-w)\varphi^\eta_\beta(z+w)-\varphi^\eta_\alpha(z-w)\varphi^\eta_\gamma(z+w)+\\ \varphi^\eta_\gamma(z-w)\varphi^\eta_\alpha(z+w)=2\varphi^{2\eta}_\beta(z-w)\varphi^w_\alpha(2\eta)-2\varphi^{2\eta}_\beta(z+w)\varphi^{-w}_\alpha(2\eta)\end{aligned} \tag{C.7}$$

$$\begin{aligned}\varphi^\eta_\beta(z-w)\varphi^\eta_\gamma(z+w)+\varphi^\eta_\gamma(z-w)\varphi^\eta_\beta(z+w)-\varphi^\eta_\alpha(z-w)\phi^\eta(z+w)-\\ \phi^\eta(z-w)\varphi^\eta_\alpha(z+w)=2\varphi^{2\eta}_\alpha(z-w)\varphi^w_\alpha(2\eta)+2\varphi^{2\eta}_\alpha(z+w)\varphi^{-w}_\alpha(2\eta)\end{aligned} \tag{C.8}$$

$$\varphi^\eta_\beta(z+w)\varphi^\eta_\gamma(z)\phi^\eta(w)+\varphi^\eta_\alpha(z+w)\phi^\eta(z)\varphi^\eta_\gamma(w)= \tag{C.9}$$

$$\phi^\eta(z+w)\varphi^\eta_\alpha(z)\varphi^\eta_\beta(w)+\varphi^\eta_\gamma(z+w)\varphi^\eta_\beta(z)\varphi^\eta_\alpha(w)$$

$$\varphi^\eta_\beta(z-w)\varphi^\eta_\gamma(z)\phi^\eta(w)-\varphi^\eta_\alpha(z-w)\phi^\eta(z)\varphi^\eta_\gamma(w)= \tag{C.10}$$

$$-\phi^\eta(z-w)\varphi^\eta_\alpha(z)\varphi^\eta_\beta(w)+\varphi^\eta_\gamma(z-w)\varphi^\eta_\beta(z)\varphi^\eta_\alpha(w)$$

$$\begin{gathered}(E_1(\eta+\beta)+E_1(\eta-\beta)-E_1(\eta+\alpha)-E_1(\eta-\alpha))\times\\ (\varphi^\eta_\gamma(z+w)\varphi^\eta_\gamma(z)\phi^\eta(w)-\phi^\eta(z+w)\phi^\eta(z)\varphi^\eta_\gamma(w))=\\ (E_1(\eta+\gamma)+E_1(\eta-\gamma)-2E_1(\eta))\times\\ (-\varphi^\eta_\alpha(z+w)\varphi^\eta_\alpha(z)\varphi^\eta_\beta(w)+\varphi^\eta_\beta(z+w)\varphi^\eta_\beta(z)\varphi^\eta_\alpha(w))\end{gathered} \tag{C.11}$$

$$\begin{gathered}\phi^\eta(w)(-\varphi^\eta_\alpha(z-\omega_\alpha)\varphi^\eta_\beta(w-\omega_\beta)\varphi^\eta_\alpha(z-w)+\varphi^\eta_\beta(z-\omega_\beta)\varphi^\eta_\alpha(w-\omega_\alpha)\varphi^\eta_\beta(z-w))=\\ -\phi^{-\eta}(w)(\varphi^\eta_\alpha(z-\omega_\alpha)\varphi^\eta_\beta(w-\omega_\beta)\varphi^\eta_\alpha(z+w)-\varphi^\eta_\beta(z-\omega_\beta)\varphi^\eta_\alpha(w-\omega_\alpha)\varphi^\eta_\beta(z+w))\end{gathered} \tag{C.12}$$

$$\begin{gathered}\phi^\eta(w+\omega_\alpha)(\varphi^\eta_\gamma(z-\omega_\gamma)\phi^\eta(w)\varphi^\eta_\beta(z-w)-\phi^\eta(z)\varphi^\eta_\gamma(w-\omega_\gamma)\varphi^\eta_\alpha(z-w))=\\ \phi^\eta(-w+\omega_\alpha)(\varphi^\eta_\gamma(z-\omega_\gamma)\phi^\eta(w)\varphi^\eta_\beta(z+w)+\phi^\eta(z)\varphi^\eta_\gamma(w-\omega_\gamma)\varphi^\eta_\alpha(z+w))\end{gathered} \tag{C.13}$$

9.4 Appendix D. Comments to Proof of Reflection Equation

Here we give some comments on the proof of the Proposition 3.2.

A direct substitution of (8.1-8.2) into (8.3) yields three types expressions proportional to $\sigma\otimes 1$, $1\otimes\sigma$ and $\sigma\otimes\sigma$. Consider, for example, $1\otimes\sigma_\gamma$ which contains the additional constants.

By the usage of (C.5-C.8) it simplifies to

$$
\begin{aligned}
&[\hat{S}_\gamma,\hat{S}_0]\,\big(2\phi^w(\hbar)(\varphi^\hbar_\gamma(z)\phi^\hbar(w)\varphi^\hbar_\gamma(z-w)-\phi^\hbar(z)\varphi^\hbar_\gamma(w)\phi^\hbar(z-w))+\\
&2\phi^{-w}(\hbar)(\varphi^\hbar_\gamma(z)\phi^\hbar(w)\varphi^\hbar_\gamma(z+w)-\phi^\hbar(z)\varphi^\hbar_\gamma(w)\phi^\hbar(z+w))\big)+\\
&[\tilde{\nu}_\gamma,\hat{S}_0]\,\big(2\phi^w(\hbar)(\varphi^\hbar_\gamma(z-\omega_\gamma)\phi^\hbar(w)\varphi^\hbar_\gamma(z-w)-\phi^\hbar(z)\varphi^\hbar_\gamma(w-\omega_\gamma)\phi^\hbar(z-w))+\\
&2\phi^{-w}(\hbar)(\varphi^\hbar_\gamma(z-\omega_\gamma)\phi^\hbar(w)\varphi^\hbar_\gamma(z+w)-\phi^\hbar(z)\varphi^\hbar_\gamma(w-\omega_\gamma)\phi^\hbar(z+w))\big)+\\
&4i\tilde{\nu}_\alpha\hat{S}_\beta\left(\varphi^\hbar_\alpha(z-\omega_\alpha)\varphi^\hbar_\beta(w)(\varphi^\hbar_\alpha(z-w)\phi(\hbar,w)+\varphi^\hbar_\alpha(z+w)\phi(\hbar,-w))-\right.\\
&\left.\varphi^\hbar_\beta(z)\varphi^\hbar_\alpha(w-\omega_\alpha)(\varphi^\hbar_\beta(z-w)\phi(\hbar,w)+\varphi^\hbar_\beta(z+w)\phi(\hbar,-w))\right)+\\
&4i\tilde{\nu}_\beta\hat{S}_\alpha\left(\varphi^\hbar_\alpha(z)\varphi^\hbar_\beta(w-\omega_\beta)(\varphi^\hbar_\alpha(z-w)\phi(\hbar,w)+\varphi^\hbar_\alpha(z+w)\phi(\hbar,-w))-\right.\\
&\left.\varphi^\hbar_\beta(z-\omega_\beta)\varphi^\hbar_\alpha(w)(\varphi^\hbar_\beta(z-w)\phi(\hbar,w)+\varphi^\hbar_\beta(z+w)\phi(\hbar,-w))\right)=\\
&i[\hat{S}_\alpha,\hat{S}_\beta]_+\left(2\phi^w(\hbar)(-\varphi^\hbar_\alpha(z)\varphi^\hbar_\beta(w)\varphi^\hbar_\alpha(z-w)+\varphi^\hbar_\beta(z)\varphi^\hbar_\alpha(w)\varphi^\hbar_\beta(z-w))+\right.\\
&\left.2\phi^{-w}(\hbar)(-\varphi^\hbar_\alpha(z)\varphi^\hbar_\beta(w)\varphi^\hbar_\alpha(z+w)+\varphi^\hbar_\beta(z)\varphi^\hbar_\alpha(w)\varphi^\hbar_\beta(z+w))\right)+\\
&i[\tilde{\nu}_\alpha,\tilde{\nu}_\beta]_+\left(2\phi^w(\hbar)(-\varphi^\hbar_\alpha(z-\omega_\alpha)\varphi^\hbar_\beta(w-\omega_\beta)\varphi^\hbar_\alpha(z-w)+\right.\\
&\varphi^\hbar_\beta(z-\omega_\beta)\varphi^\hbar_\alpha(w-\omega_\alpha)\varphi^\hbar_\beta(z-w))+\\
&\left.2\phi^{-w}(\hbar)(-\varphi^\hbar_\alpha(z-\omega_\alpha)\varphi^\hbar_\beta(w-\omega_\beta)\varphi^\hbar_\alpha(z+w)+\varphi^\hbar_\beta(z-\omega_\beta)\varphi^\hbar_\alpha(w-\omega_\alpha)\varphi^\hbar_\beta(z+w))\right),
\end{aligned}
\tag{D.1}
$$

where (α,β,γ) is equivalent to $(1,2,3)$ under cyclic permutations. The expression behind $[\tilde{\nu}_\alpha,\tilde{\nu}_\beta]_+$ vanishes due to (C.12). At the same time the expression behind $[\tilde{\nu}_\gamma,\hat{S}_0]$ has a pole at $w=\omega_\gamma$ different from those of behind expressions $[\hat{S}_\gamma,\hat{S}_0]$ and $[\hat{S}_\alpha,\hat{S}_\beta]_+$. Thus $[\tilde{\nu}_\gamma,\hat{S}_0]=0$. At the moment we have

$$
\begin{aligned}
&[\hat{S}_\gamma,\hat{S}_0]\,\big(2\varphi^\hbar_\gamma(z)\phi^\hbar(w)(\phi^w(\hbar)\varphi^\hbar_\gamma(z-w)+\phi^{-w}(\hbar)\varphi^\hbar_\gamma(z+w))-\\
&2\phi^\hbar(z)\varphi^\hbar_\gamma(w)(\phi^w(\hbar)\phi^\hbar(z-w)+\phi^{-w}(\hbar)\phi^\hbar(z+w))\big)\\
&4i\tilde{\nu}_\alpha\hat{S}_\beta\left(\varphi^\hbar_\alpha(z-\omega_\alpha)\varphi^\hbar_\beta(w)(\varphi^\hbar_\alpha(z-w)\phi(\hbar,w)+\varphi^\hbar_\alpha(z+w)\phi(\hbar,-w))-\right.\\
&\left.\varphi^\hbar_\beta(z)\varphi^\hbar_\alpha(w-\omega_\alpha)(\varphi^\hbar_\beta(z-w)\phi(\hbar,w)+\varphi^\hbar_\beta(z+w)\phi(\hbar,-w))\right)+\\
&4i\tilde{\nu}_\beta\hat{S}_\alpha\left(\varphi^\hbar_\alpha(z)\varphi^\hbar_\beta(w-\omega_\beta)(\varphi^\hbar_\alpha(z-w)\phi(\hbar,w)+\varphi^\hbar_\alpha(z+w)\phi(\hbar,-w))-\right.\\
&\left.\varphi^\hbar_\beta(z-\omega_\beta)\varphi^\hbar_\alpha(w)(\varphi^\hbar_\beta(z-w)\phi(\hbar,w)+\varphi^\hbar_\beta(z+w)\phi(\hbar,-w))\right)=\\
&i[\hat{S}_\alpha,\hat{S}_\beta]_+\left(-2\varphi^\hbar_\alpha(z)\varphi^\hbar_\beta(w)(\phi^w(\hbar)\varphi^\hbar_\alpha(z-w)+\phi^{-w}(\hbar)\varphi^\hbar_\alpha(z+w))\right.\\
&\left.2\varphi^\hbar_\beta(z)\varphi^\hbar_\alpha(w)(\phi^w(\hbar)\varphi^\hbar_\beta(z-w)+\phi^{-w}(\hbar)\varphi^\hbar_\beta(z+w))\right)
\end{aligned}
\tag{D.2}
$$

Using then (C.4) and cancelling $2\phi(z,\hbar)$ we have

$$
\begin{aligned}
&[\hat{S}_\gamma,\hat{S}_0]\,\big(\varphi^\hbar_\gamma(z)\phi^\hbar(w)(\varphi_\gamma(z-w)+\varphi_\gamma(z+w))-\\
&\varphi^\hbar_\gamma(w)(\phi^\hbar(w)\phi^\hbar(z-w)+\phi^\hbar(-w)\phi^\hbar(z+w))\big)+\\
&2i\tilde{\nu}_\alpha\hat{S}_\beta\left(\varphi^\hbar_\alpha(z-\omega_\alpha)\varphi^\hbar_\beta(w)(\varphi_\alpha(z-w)+\varphi_\alpha(z+w))-\right.\\
&\left.\varphi^\hbar_\beta(z)\varphi^\hbar_\alpha(w-\omega_\alpha)(\varphi_\beta(z-w)+\varphi_\beta(z+w))\right)+\\
&2i\tilde{\nu}_\beta\hat{S}_\alpha\left(\varphi^\hbar_\alpha(z)\varphi^\hbar_\beta(w-\omega_\beta)(\varphi_\alpha(z-w)+\varphi_\alpha(z+w))-\right.\\
&\left.\varphi^\hbar_\beta(z-\omega_\beta)\varphi^\hbar_\alpha(w)(\varphi_\beta(z-w)+\varphi_\beta(z+w))\right)=
\end{aligned}
\tag{D.3}
$$

$$
=i[\hat{S}_\alpha,\hat{S}_\beta]_+\left(-\varphi^\hbar_\alpha(z)\varphi^\hbar_\beta(w)(\varphi_\alpha(z-w)+\varphi_\alpha(z+w))+\ \varphi^\hbar_\beta(z)\varphi^\hbar_\alpha(w)(\varphi_\beta(z-w)+\varphi_\beta(z+w))\right)
$$

To get the final result one should compare the structure of poles ($w=0$ and $z=-w$). (C.9-C.13). Other types of expressions can be simplified in the same way through the use of (C.9-C.13).

References

[1] D.Arinkin, S.Lysenko Isomorphisms between moduli spaces of SL(2)-bundles with connections on $\mathbb{P}^1/\{x_1,\ldots,x_4\}$, *Math. Res. Lett.*, **4**, (1997), 181–190,
On the moduli spaces of SL(2)-bundles with connections on $\mathbb{P}^1/\{x_1,\ldots,x_4\}$, *Internat. Math. Res. Notices*, no. 19, (1997), 983–999.

[2] R.J.Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain, *I.Ann.Phys.*, **76**, (1973), 48–71.

[3] A. Belavin and V. Drinfeld, Solutions of the classical Yang-Baxter equation for simple Lie algebras, *Funct. Anal and Applic.*, **16**, (1982), no. 3, 1–29.

[4] A.V.Borisov, I.S.Mamaev, *Modern Methods of the Theory of Integrable Systems*, Moscow - Izhevsk: Institute of Computer Science, (2003).

[5] F.Calogero, Exactly solvable one dimensional many-body problem, *Lettere al Nuovo Cimento*, **13**, (1975), 411-416.

[6] L.Faddeev, L.Takhtajan, *Hamiltonian approach to solitons theory*, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, (1987)

[7] B. Feigin and A. Odesski, Sklyanin's elliptic algebras, *Funct. Anal. and Applic.*, **23**, (1989), no. 3, 207–214.

[8] Yu.N.Fedorov, Lax Representation with Spectral Parameter on the Coverings of Hyperelliptic Curves, *Math. notes*, **54**, (1993), 94–109.

[9] B. Gambier, Sur les équations differentielles du second ordre et du premier degré dont l'integral générale a ses points critiques fixes, *Acta Math. Ann*, **33**, (1910), 1-55.

[10] J.Gibbons and T.Hermsen, A generalization of Calogero-Moser system, *Physica D*, **11D**, (1984), 337–348.

[11] K.Hasegawa, Ruijsenaars' commuting difference operators as commuting transfer matrices, *Comm. Math. Phys.*, **187**, (1997), 289–325.

[12] N. Hitchin, Stable bundles and Integrable Systems, *Duke Math. Journ.*, **54**, (1987), 91-114.

[13] V.Inozemtsev, Lax Representation with spectral parameter on a torus for integrable particle systems, *Lett. Math. Phys.*, **17**, (1989), 11-17.

[14] T.Inami, H.Konno, Integrable XYZ spin chain with boundaries, *J.Phys.A: Math.Gen.*, **27**, (1994), L913-L918.

[15] K.Iwasaki, Fuchsian moduli on a Riemann surface - its Poisson structure and Poincaré-Lefschetz duality, *Pacific J. Math.*, **155**, (1992), 319-340

[16] B.Khesin, A.Levin, M.Olshanetsky, Bihamiltonian structures and quadratic algebras in Hydrodynamics and on non-commutative torus, *Comm. Math. Phys.*, **270**, (2004), 581–612.

[17] I.M.Krichever, Elliptic solutions of Kadomtsev-Petviasvili equation and integrable system of particles, *Funct. Anal. Appl.* **14**, (1980), 282-290.

[18] V.B. Kuznetsov, M.F. Jorgensen, P.L. Christiansen, New boundary conditions for integrable lattices, *J.Phys. A* **28**, (1995), 4639-4654.

[19] A.Levin, M.Olshanetsky, Hierarchies of isomonodromic deformations and Hitchin systems. Moscow Seminar in Mathematical Physics, *Amer. Math. Soc. Transl. Ser. 2*, **191**, Amer. Math. Soc., Providence, RI, (1999), 223–262.

[20] A.Levin, M.Olshanetsky, Non-autonomous Hamiltonian systems related to higher Hitchin integrals, (Russian) *Teoret. Mat. Fiz.* , **123**, no. 2, (2000), 237–263; translation in *Theoret. and Math. Phys.*, **123** (2000), 609–632.

[21] A.Levin, M.Olshanetsky, A.Zotov, Hitchin systems– symplectic Hecke correspondence and two-dimensional version, *Comm. Math. Phys.*, **236**, (2003), 93–133.

[22] Yu.I.Manin, Sixth Painlevé equation, universal elliptic curve, and mirror of P^2, *Amer. Math. Soc. Transl. (2)*, **186**, (1998) 131-151.

[23] J.Moser, Three integrable Hamiltonian systems connected with isospectral deformations, *Advances in Math.* **16**, (1975), 197–220.

[24] D.Mumford, *Tata Lectures on Theta I, II,* Birkhäuser Boston, 1983, 1984.

[25] N. Nekrasov, Holomorphic bundles and many-body systems, *Comm. Math. Phys.*, **180** (1996) 587-604.

[26] S.Oblezin, Isomonodromic deformations and the Hecke correspondence, *Moscow Math. Journ.*, **5**, (2005), 415-441.

[27] M. Olshanetsky, The large N limits of integrable models, *Mosc. Math. J.*, **3**, (2003), 1307–1331.

[28] P.Painlevé, Sur les équations différentielles du second odre à points critics fixes, *CRAS*, **143**, (1906), 1111-1117.

[29] A.Reyman and M.Semenov-Tian-Schansky, Lie algebras and Lax equations with spectral parameter on elliptic curve, (Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **150** (1986), Voprosy Kvant. Teor. Polya i Statist. Fiz. 6, 104–118, 221; translation in *J. Soviet Math.*, **46**, no. 1, (1989), 1631–1640.

[30] S.T.Simpson, Harmonic bundles on non-compact curves, *Journ. of AMS* **3** (1990), 713–770.

[31] E.Sklyanin, Boundary conditions for integrable equations, *Func. Anal. Appl.*, Vol.21, 86-87, (1987);

[32] E.Sklyanin, Some algebraic structures connected with the Yang-Baxter equation, *Funct. Anal. and Applic.*, **16** (1982), no. 4, 27–34.

[33] E.Sklyanin, Boundary conditions for integrable quantum systems, *J. Phys. A: Math. Gen*, **21**, (1988) 2375–2389.

[34] E.Sklyanin, T.Takebe, Algebraic Bethe ansatz for the XYZ Gaudin model. *Phys. Lett. A*, **219** (1996), 217–225.

[35] E.Sklyanin, T.Takebe, Separation of Variables in the Elliptic Gaudin Model, *Comm. Math. Phys.*, **204** no. 1, (1999), 17–38.

[36] K.Takasaki, Spectral Curves and Whitham Equations in the Isomonodromic Problems of Schlesinger Type, *Asian J. Math.*, **2** no. 4, (1998), 1049–1078.

[37] V.Vakulenko, Note on the Ruijsenaars-Schneider model, math.QA/9909079

[38] V.Volterra, Sur la théorie des variations des latitudes, *Acta Math.* **22** (1899), 201-357.

[39] S.Wojciechowski, An integrable marriage of the Euler equations with the Calogero-Moser system, *Phys. Lett.*, **A111** (1985) 101

[40] Workshop on Painlevé Transcedents, Their asymptotics and Physical Applications, *NATO ASI Ser. B: Physics* Vol. 278 (1990) Sainte Adele, Quebec, ed. by D. Levi and P. Winternitz

[41] N.E.Zhukovsky, *Jour. Phys. Chem. Soc.*, **17** (1885) 81-113, 145-199, 231-280.

[42] A.Zotov, Elliptic Linear Problem for Calogero-Inozemtsev Model and Painlevé VI Equation, *Lett.Math.Phys.*, **67** (2004) 153-165.