GEOMETRIC FEATURES OF LATTICE POINT PROBLEMS

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Introduction.

The purpose of this survey article is to introduce the (non-expert) reader to some of the problems, techniques, and results in the study of "lattice point problems". This descriptive phrase appears in many different contexts and means different things to different subspecialties. In this article the common meaning attached to a "lattice point problem" consists of the following general counting problem.

Let $w > 0$ be a real or integral valued parameter. Let $\mathcal{R}_w$ be a family of bounded closed sets in $\mathbf{R}^n$ with positive Lebesgue measure. Assume $\mathcal{R} = \cup_w \mathcal{R}_w$ is unbounded. Let φ denote a rational function defined on $\mathcal{R}$. Define for each w

$$N(w,\varphi) = \sum_{m \in \mathbf{Z}^n \cap \mathcal{R}_w} \varphi(m), \qquad V(w,\varphi) = \int_{\mathcal{R}_w} \varphi \, dx_1 \cdots dx_n.$$

When $\varphi \equiv 1$, one often writes $N(w), V(w)$ for $N(w,1), V(w,1)$. Thus,

$$N(w) =_{\text{def}} card(\mathcal{R}_w \cap \mathbf{Z}^n), \qquad V(w) =_{\text{def}} vol(\mathcal{R}_w).$$

Problem. Describe the behavior of $N(w,\varphi)$ as $w \to \infty$ and relate the behavior to that of $V(w,\varphi)$ as $w \to \infty$.

A lattice point problem for a class $\mathcal{C}$ of rational functions is "geometric" if there exists a function $\psi(w,\varphi)$ such that $\psi(w,\varphi)/V(w,\varphi) \to 0$ as $w \to \infty$ and so that

$$N(w,\varphi) - V(w,\varphi) = O(\psi(w,\varphi)) \text{ as } w \to \infty \text{ for all } \varphi \in \mathcal{C}.$$

Geometric lattice point problems are of interest in particular in numerical integration because they give classes of expanding domains and rational integrands for which integration and summation agree up to lower order error terms in the parameter of expansion.

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In general for multidimensional problems, standard numerical integration methods do not appear to work as well as they do in one variable. In part this is due to a lack of geometric machinery available to develop alternative integration schemes. GLPs offer one class of such problems for which considerable geometric and analytic machinery is available for comparing integration and summation with some precision.

This article discusses a few ideas and results that exhibit interesting uses of singularity theory to the analysis of GLPs. The discussion falls nicely into two parts. Part 1 treats the case in which the domains $\mathcal{R}_w$ are determined by homogeneous and positive definite polynomials on $\mathbf{R}^n$. Sections 1,2 treat the circle problem and a generalization to quadratic forms. This is a classical GLP originating within analytic number theory. The main problem, at least when $\varphi \equiv 1$ is not to show that one is dealing with a GLP. That can be verified in an elementary way. Instead, the emphasis is placed on obtaining good estimates for the smallest rate of growth of the error term

$$E(w) = N(w) - V(w).$$

Even for such a concrete problem, the determination of the precise order in $E(w)$ is a difficult problem that remains unsolved. The purpose of the discussion is to give a flavor of certain underlying insights and techniques that have led to nontrivial information about the growth of $E(w)$.

Section 3 then discusses more recent work of Randol and Colin de Verdière that analyses the growth of $E(w)$ in terms of singularity theory when one views $\mathcal{R}_w$ as the "homothetic" expansion of a bounded domain

$$\{P \leq 1\}$$

with boundary $\{P = 1\}$. Here, if P is a positive definite form of degree d, then $\mathcal{R}_w = w^{1/d}\mathcal{R}_1$. For this class of domains, the most natural class of weight function are rational functions with homogenous numerator and denominator, although more general functions are certainly useable. This is discussed at the end of the section.

Part 2 discusses certain ideas needed if one wants to analyse $N(w, \varphi)$ for a family of domains that expands in a manner different than the homothetic one of Part 1. In particular, suppose $P \in \mathbf{R}[x_1, \ldots, x_n]$ has degree d, is not homogeneous, but still determines a proper mapping on $\mathbf{R}^n$. Then it appears to be more natural to define

$$\mathcal{R}_w = \{P \leq w\}$$

rather than

$$\mathcal{R}'_w = w^{1/d}\mathcal{R}_1,$$

since $V(\mathcal{R}'_w) = vol(\{P \leq 1\}\, w^{n/d}$ but $N(\mathcal{R}_w)$ need not have this asymptotic.

Two cases are treated, using functional methods generalizing those of section 2. Section 4 describes the work of Mahler, who treated the case of elliptic polynomials on $[0, \infty)^n$, rather than $\mathbf{R}^n$, and analyzed $N(w, \varphi), V(w, \varphi)$ using the lattice $\mathbf{N}^n$ rather than $\mathbf{Z}^n$. The results are similar to those in section 3. They are obtained however by the analysis of a Dirichlet series that does not possess a reflection type functional equation but which can still be analytically continued to the complex plane as a reasonably well behaved function. An estimate for

$$E(w,\varphi) = N(w,\varphi) - V(w,\varphi)$$

is then derived via purely analytic methods. The main result, Theorem 4.10, says that $E(w,\varphi)$ is of strictly smaller order whenever φ is the quotient of elliptic polynomials on $[0,\infty)^n$. However, no explicit estimate for the error can be derived using Mahler's methods.

A natural desire, emanating from section 4, is to loosen the restrictive condition of ellipticity. Recently, the author of these notes has discovered a method to do this for hypoelliptic polynomials. Hypoellipticity is a growth condition at infinity that is considerably weaker than ellipticity in that it allows, as one example, an *arbitrary* form of top degree, but requires compensation in the growth with lower order terms that do grow at infinity. Section 5 treats the salient features of the analysis containing results on GLPs determined by hypoelliptic polynomials on $[1,\infty)^n$. The main result, Theorem 5.10, is an extension of Mahler's work, but exploits resolution of singularities at "infinity", and applies to all weights φ which are quotients of hypoelliptic polynomials. Again, one can show that $E(w,\varphi)$ is of strictly smaller order in w, but no explicit estimate seems to follow easily from the proof of 5.10. The reason for such difficulties essentially arises from the contributions to $E(w,\varphi)$ that come from the behavior of φ/P^s near the boundary of the chain of integration $[1,\infty)^n$.

In order to obtain simple and general estimates for $E(w,\varphi)$, it is therefore considerably easier to work with the entire lattice $\mathbf{Z}^n$. Here, one can incorporate one of Bochner's ideas, applied originally to elliptic polynomials in [Bo], but whose method evidently extends to hypoelliptic polynomials on $\mathbf{R}^n$. As a result of this technique, one can give a simple and explicit estimate for $E(w,\varphi)$. This is sketched at the end of section 5. The main result, stated in Theorem 5.33, gives a *general* estimate, applicable to the situations in which the methods of section 3 do not yet apply. The natural goal, therefore, should be to improve these estimates by combining the ideas of section 3 and section 5.

Three appendices to the article discuss certain technical points which the reader may find helpful.

The polynomials $W(x) = x_1^k + \ldots + x_n^k$ typically analyzed in the context of Waring's problem certainly fall into the class studied in section 4, while those of other Pham-Brieskorn type polynomials (that is, additive and weighted homogeneous) belong to the class studied in section 5. However, the methods used in the analysis of Waring's problem are considerably more technical, due to the desire to study the asymptotic of $card(\{x \in [1,\infty)^n : W(x) = \ell\} \cap \mathbf{N}^n$. To keep the length of these notes reasonably modest, nothing further will be said about Waring's problem or others that arise within diophantine approximation theory. The primary reason is that considerations primarily of a subtle arithmetic, not geometric, character dominate the analysis. Because the typical reader here is expected to be more geometric in orientation, this seems to be a reasonable constraint.

Part 1. Positive definite forms

Section 1. *Landau's analysis of the classical circle problem*

In this section P will denote the polynomial $x_1^2 + x_2^2$. Define the counting function

$$N(w) = card\{(m_1, m_2) \in \mathbb{Z}^2 : m_1^2 + m_2^2 \leq w\}.$$

Evidently, the area of this region is πw. The first result is elementary.

THEOREM 1.1.

$$N(w) = \pi w + O(w^{1/2}).$$

One of the reasons why there is interest in the asymptotic of $N(w)$ is-

Remark. The ring of integers in the field $\mathbf{Q}[i]$ equals $\mathbb{Z}[i]$. The norm of a Gaussian integer $m_1 + im_2$ equals $P(m_1, m_2)$. So, asymptotic information about the circle problem gives the asymptotic distribution of Gaussian integers. ∎

PROOF:

$$\begin{aligned}
N(w) &= \sum_{m \in \mathbb{Z}^2 : m_1^2 + m_2^2 \leq w} 1 = \sum_{|m_1| \leq w^{1/2}} \sum_{|m_2| \leq \sqrt{w - m_1^2}} 1 \\
&= \sum_{|m_1| \leq w^{1/2}} \left(1 + 2[\sqrt{w - m_1^2}]\right) \\
&= 2 \sum_{|m_1| \leq w^{1/2}} \left(\sqrt{w - m_1^2} + O(1)\right) + O(w^{1/2}) = 2 \sum_{|m_1| \leq w^{1/2}} \sqrt{w - m_1^2} + O(w^{1/2}) \qquad (1.2) \\
&= 4 \sum_{m_1 = 0}^{[w^{1/2}]} \sqrt{w - m_1^2} + O(w^{1/2}) \\
&= 4 \int_0^{\sqrt{w}} \sqrt{w - u^2}\, du + O(w^{1/2}) \quad (\text{ via monotonicity of } u \to \sqrt{w - u^2}, u \in [0, \sqrt{w}]) \\
&= \pi w + O(w^{1/2}). \quad ∎
\end{aligned}$$

Define $E(w) = N(w) - \pi w$.

One has introduced the $O(w^{1/2})$ error initially in (1.2). Remarkably, one can reduce the *best* exponent for $E(w)$ to a number in $[1/4, 1/3]$. It is not yet known, however, what is the best exponent. Here, an argument of Landau, adapted from that given in [Kr], will be given that shows

$$E(w) = O_\epsilon(w^{1/3+\epsilon}) \text{ for any } \epsilon > 0.$$

This notation means that for each $\epsilon > 0$ there exist C_ϵ, B_ϵ such that

$$|E(w)| \leq C_\epsilon w^{1/3+\epsilon} \quad \text{if } w \geq B_\epsilon.$$

The proof uses an expansion of the continuous function $w \to \int_0^w N(u)du$ into a series of Bessel functions and then a Tauberian argument of Landau. Although this argument is based upon the rather special nature of the polynomial P, it seems worthwhile to present the details because it can be modified, in principle, to suit other polynomials. Moreover, such arguments tend to be known to experts and scattered throughout the literature. So, it is perhaps useful to include a few examples in this article.

Recall the definitions of the first two Bessel functions.

$$J_1(y) = \frac{2}{\pi^{1/2}\,\Gamma(3/2)}\left(\frac{y}{2}\right)\int_0^1 (1-u)^{1/2} cos(yu)du$$

$$J_2(y) = \frac{2}{\pi^{1/2}\,\Gamma(5/2)}\left(\frac{y}{2}\right)^2\int_0^1 (1-u)^{3/2} cos(yu)du\,.$$

The following facts will be needed below.

(1.3.1) There exists $c > 0$ such that for $i = 1, 2$,

$$|J_i(y)| < \frac{c}{\sqrt{y}} \quad \text{for } y > 0\,.$$

(1.3.2) For i = 1,2

$$J_i(y) = \sqrt{\frac{2}{\pi y}}\, cos\left(y - \frac{\pi j}{2} - \frac{\pi}{4}\right) + O(y^{-3/2})\,.$$

(1.3.3) For each positive integer n

$$\frac{d}{dy}\left(\frac{y}{\pi n}\, J_2(2\pi\sqrt{ny})\right) = \sqrt{\frac{y}{n}}\, J_1(2\pi\sqrt{ny})\,.$$

Remark. The higher exponent Bessel functions will be used in Section 2. For this section, these two will suffice. ∎

For two integrable functions f, g on **R** define

$$f(w) * g(w) = \int_0^w f(u)g(w-u)du\,.$$

In addition, introduce the function

$$\psi(u) = u - [\![u]\!] - 1/2,$$

whose significance for the analysis of lattice point problems has been understood since the classical works of Hardy-Littlewood, Landau, and van der Corput. The main properties of this function are summarized in Appendix A.

LEMMA 1.4.

$$1 * N(w) = \frac{\pi}{2}w^2 - 8w^{1/2}\psi(w^{1/2}) + 4\psi(w^{1/2}) * \psi(w^{1/2}).$$

PROOF: Introduce the notations

$\sum_{a<m\leq b}{}' f(m)$ resp. $\sum_{a\leq m\leq b}{}'' f(m)$ to indicate that $f(b)$ resp. $f(a), f(b)$ are weighted by $1/2$.

Then,

$$\begin{aligned}
1 * N(w) &= \int_0^w N(u)du = 4\int_0^w \left(\sum_{\substack{m_1^2+m_2^2\leq u \\ m_1,m_2\geq 0}}{}'' 1 \right) du \\
&= 4 \sum_{\substack{m_1^2+m_2^2\leq w \\ m_1,m_2\geq 0}}{}'' (w - m_1^2 - m_2^2) \\
&= 4\int_0^w \sum_{0\leq m_1^2\leq u}{}' 1 \cdot \sum_{0\leq m_2^2\leq w-u}{}' 1du \\
&= 4\left([\![w^{1/2}]\!] + \frac{1}{2}\right) * \left([\![w^{1/2}]\!] + \frac{1}{2}\right) \\
&= 4\left(w^{1/2} - \psi(w^{1/2})\right) * \left(w^{1/2} - \psi(w^{1/2})\right).
\end{aligned}$$

From this, the above formula is clear. ∎

Introduce the notation

$$c(n) = card\{m \in \mathbb{Z}^2 : P(m) = n\}.$$

THEOREM 1.5.

$$1 * N(w) = \frac{\pi}{2}w^2 + \frac{w}{\pi}\sum_{n=1}^{\infty}\frac{c(n)}{n}J_2(2\pi\sqrt{nw}).$$

PROOF: Use Lemma 1.4 and the series expansion (cf. (A-1)) for ψ. As pointed out in (A-6), one can interchange summation and integration in the integrals for $w^{1/2}\psi(w^{1/2})$ and $\psi(w^{1/2}) * \psi(w^{1/2})$. One calculates and finds

$$\begin{aligned} w^{1/2}\psi(w^{1/2}) &= \frac{-1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}w^{1/2} * sin(2\pi n w^{1/2}) \\ &= \frac{-w}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}J_2(2\pi n w^{1/2}), \end{aligned}$$

and

$$\begin{aligned} \psi(w^{1/2}) * \psi(w^{1/2}) &= \frac{-1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\psi(w^{1/2}) * sin(2\pi n w^{1/2}) \\ &= \frac{1}{\pi^2}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{k\ell}sin(2\pi k w^{1/2}) * sin(2\pi \ell w^{1/2}) \\ &= \frac{w}{\pi}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{J_2(2\pi\sqrt{(k^2+\ell^2)w})}{k^2+\ell^2}. \end{aligned}$$

By the growth estimate (1.3.1), one sees that the series converges absolutely. By grouping together the terms with indices (k,ℓ) for which $P(k,\ell) = n$, one completes the proof of the Theorem. ∎

One can proceed to prove

THEOREM 1.6. *For any* $\epsilon > 0$

$$E(w) = O_\epsilon(w^{1/3+\epsilon}).$$

PROOF (LANDAU): Introduce a parameter h and consider the differences

$$\int_w^{w+h} E(u)\,du \text{ and } \int_{w-h}^{w} E(u)\,du.$$

One notes that the monotonicity of $N(u)$ implies

$$N(u) - \pi u\big|_{u\in[w,w+h]} \ge N(w) - \pi(w+h) \quad \text{and} \quad N(u) - \pi u\big|_{u\in[w-h,w]} \le N(w) - \pi(w-h).$$

Thus,

$$\frac{1}{h}\int_{w-h}^{w} E(u)du - \pi h \le E(w) \le \frac{1}{h}\int_{w}^{w+h} E(u)du + \pi h. \tag{1.7}$$

By (1.5),

$$\begin{aligned}\int_w^{w+h} E(u)du &= \int_0^{w+h} N(u)\,du - \int_0^w N(u)\,du - \pi\left[\frac{(w+h)^2}{2} - \frac{w^2}{2}\right] \\ &= \sum_{n=1}^{\infty} c(n)\left[\frac{w+h}{\pi n}\cdot J_2(2\pi\sqrt{n(w+h)}) - \frac{w}{\pi n}\cdot J_2(2\pi\sqrt{nw})\right] \\ &= \sum_{n\le z} c(n)\left[\frac{w+h}{\pi n}\cdot J_2(2\pi\sqrt{n(w+h)}) - \frac{w}{\pi n}\cdot J_2(2\pi\sqrt{nw})\right] \\ &\quad + \sum_{n>z}\frac{c(n)}{\pi n}\left[(w+h)J_2(2\pi\sqrt{n(w+h)}) - tJ_2(2\pi\sqrt{nw})\right],\end{aligned}$$

where the parameter z will be chosen below (cf. (1.10)) to be an appropriate function of w, h.

Use (1.3.3) in the sum over $n \le z$. This implies

$$\frac{w+h}{\pi n}\cdot J_2(2\pi\sqrt{n(w+h)}) - \frac{w}{\pi n}\cdot J_2(2\pi\sqrt{nw}) = \int_w^{w+h}\sqrt{\frac{u}{n}}\,J_1(2\pi\sqrt{nu})\,du\,.$$

Thus,

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{c(n)}{\pi n}\left[(w+h)J_2(2\pi\sqrt{n(w+h)}) - wJ_2(2\pi\sqrt{nw})\right] \qquad (1.8) \\ &= \sum_{n\le z}\frac{c(n)}{\sqrt{n}}\int_w^{w+h}\sqrt{u}J_1(2\pi\sqrt{nu})du + \sum_{n>z}\frac{c(n)}{\pi n}\left[(w+h)J_2(2\pi\sqrt{n(w+h)}) - wJ_2(2\pi\sqrt{nw})\right]\end{aligned}$$

Estimate the two terms in (1.8) as follows. First, if $n < z$ then (1.3.1) implies

$$\begin{aligned}\int_w^{w+h}\sqrt{u}J_1(2\pi\sqrt{nu})du &< \frac{c}{n^{1/4}}\left[(w+h)^{5/4} - w^{5/4}\right] \\ &= \frac{cw^{5/4}}{n^{1/4}}\left[(1+\frac{h}{w})^{5/4} - 1\right].\end{aligned}$$

If $h < w$ then

$$(1+\frac{h}{w})^{5/4} = 1 + \frac{5}{4}\cdot\frac{h}{w} + O((\frac{h}{w})^2)\,.$$

Thus, the prior estimate is

$$\frac{chw^{1/4}}{n^{1/4}} + \frac{1}{n^{1/4}}\cdot O(h^2w^{-3/4})\,.$$

This implies

$$\sum_{n\le z}\frac{c(n)}{\pi n}\left[(w+h)J_2(2\pi\sqrt{n(w+h)})-wJ_2(2\pi\sqrt{nw})\right]\ll\left[hw^{1/4}+O(h^2w^{-3/4})\right]\sum_{n\le z}\frac{c(n)}{n^{3/4}}.$$

The sum over $n>z$ can be estimated by using (1.3.1) on the two J_2 terms and treating the difference as a sum. This gives

$$\sum_{n>z}\frac{c(n)}{\pi n}\left[(w+h)J_2(2\pi\sqrt{n(w+h)})-wJ_2(2\pi\sqrt{nw})\right]\ll\sum_{n>z}\frac{c(n)}{\pi n^{5/4}}\left((w+h)^{3/4}+w^{3/4}\right).$$

The term in the parentheses is $w^{3/4}O(1)$ when $h<w$. Thus, combining these two estimates, one concludes

$$\int_w^{w+h}E(u)du\ll hw^{1/4}\sum_{n\le z}\frac{c(n)}{n^{3/4}}+w^{3/4}\sum_{n>z}\frac{c(n)}{n^{5/4}}+O(h^2w^{-3/4})\sum_{n\le z}\frac{c(n)}{n^{3/4}}.$$

It is easy to see that

$$\sum_{n\le z}\frac{c(n)}{n^{\frac{3}{4}}}\ll_\epsilon\sum_{n\le z}\frac{1}{n^{\frac{3}{4}-\epsilon}}\ll_\epsilon z^{\frac{1}{4}+\epsilon}$$
$$\sum_{n>z}\frac{c(n)}{n^{\frac{5}{4}}}\ll_\epsilon\sum_{n>z}\frac{1}{n^{\frac{5}{4}-\epsilon}}\ll z^{\frac{-1}{4}+\epsilon}.$$

Thus,

$$\int_w^{w+h}E(u)du\ll_\epsilon hw^{\frac{1}{4}}z^{\frac{1}{4}+\epsilon}+w^{\frac{3}{4}}z^{\frac{-1}{4}+\epsilon}+O\left(\frac{h^2z^{\frac{1}{4}+\epsilon}}{w^{\frac{3}{4}}}\right). \tag{1.9}$$

One forces h,w to satisfy the inequality $h^2<w^{3/4}$. Then, set $z=h^aw^b$ and find a,b so that the first two terms in (1.9) are the same. One finds that $a=-2,b=1$. Set

$$z=h^{-2}w. \tag{1.10}$$

One now wants to impose an additional constraint, if necessary, upon h,w that insures

$$O(\frac{h^{3/2-2\epsilon}}{w^{1/2-\epsilon}})=O(1)\quad\text{as } w\to\infty.$$

This will hold if $h<w^{1/3+\epsilon}$. Thus, one sets $z=h^{-2}w$ subject to the constraint $h<w^{1/3+\epsilon}$. This implies

$$\int_w^{w+h}E(u)du\ll_\epsilon(wh)^{1/2}w^\epsilon h^{-2\epsilon}.$$

Thus, (1.7) implies

$$E(w) \ll_\epsilon \frac{w^{1/2+\epsilon}}{h^{1/2+2\epsilon}} + \pi h\,.$$

One now chooses c so that if $h = w^c$ then these two terms have the same order in w for each ϵ. One solves for c and finds

$$c = \frac{1}{3} + \epsilon\,.$$

This implies

$$E(w) = O_\epsilon(w^{1/3+\epsilon}) \quad \text{for each } \epsilon > 0\,,$$

as claimed. ∎

Section 2. *A functional method for analyzing a generalized circle problem*

In this section $Q = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ denotes a positive definite quadratic form defined over $\mathbf{R}$. Further, define

$$N(w) = card\{m \in \mathbf{Z}^n : Q(m) \le w\}\,.$$

Set

$$D = det\,(a_{ij})\,,$$
$$A = vol(\{Q \le 1\}) = \frac{\pi^{n/2}}{\sqrt{D}\,\Gamma(\frac{n}{2}+1)}\,.$$

So $V(w) =_{\text{def}} vol(\{Q \le w\}) = Aw^{n/2}$. This section will sketch Landau's argument, given in [La-1], that proves

THEOREM 2.1.

$$N(w) = V(w) + O_\epsilon(w^{\frac{n-1}{2}\cdot\frac{n}{n+1}+\epsilon})\,.$$

To appreciate the significance of (2.1), one should first observe what is the "trivial" estimate for the error $E(w) = N(w) - V(w)$. This is

PROPOSITION 2.2.

$$N(w) = V(w) + O(w^{\frac{n-1}{2}})\,.$$

PROOF: To each lattice point m contained in the set $\{Q \le w\}$ assign the open box

$$C(m) = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : |x_i - m_i| < \frac{1}{2}, \text{ for each } i\}\,.$$

Clearly, if $m \ne m'$ then $C(m) \cap C(m') = \emptyset$. Moreover, since each box has volume equal to 1, it is clear that $vol(\cup_m C(m)) = N(w)$.

One now observes the existence of functions $c(w), c'(w)$ such that $c(w), c'(w) = O(w^{1/2})$ and

$$\{Q \leq w - c'(w)\} \subset \cup_m C(m) \subset \{Q \leq w + c(w)\}.$$

Thus, by the homogeneity of Q one obtains

$$vol(\{Q \leq w - c'(w)\}) = A(w - c'(w))^{n/2} \leq N(w) \leq vol(\{Q \leq w + c(w)\}) = A(w + c(w))^{n/2}.$$

Hence, $N(w) - Aw^{n/2} = O(w^{\frac{n-1}{2}})$. ∎

PROOF OF (2.1): The key functional tool is the quadratic theta function associated to Q and its functional equation of reflection type, discovered by Epstein [Ep]. For $y \in \mathbf{R}, h, z \in \mathbf{R}^n$, set

$$\Theta_Q(z, h, y) = \sum_{m \neq 0 \in \mathbf{Z}^n} e^{-\pi y Q(m+z) + 2\pi i (h \cdot m)}.$$

Next, set $\tilde{Q}$ to denote the quadratic form associated to the inverse matrix of (a_{ij}). That is,

$$\tilde{Q}(x_1, \ldots, x_n) = \frac{1}{D} \sum_{ij} \frac{\partial D}{\partial a_{ij}} x_i x_j.$$

A fundamental property of Θ_Q is the identity [Ep]

$$\Theta_Q(z, h, y) = \frac{e^{-2\pi i (h \cdot z)}}{y^{n/2}\sqrt{D}} \Theta_{\tilde{Q}}(h, -z, 1/y).$$

Set $\lambda_1(z) \leq \lambda_2(z) \leq \cdots$ resp. $\eta_1(h) \leq \eta_2(h) \leq \cdots$ to denote the elements of $\{Q(m + z)\}_{m \neq 0}$ resp. $\{\tilde{Q}(m + h)\}_{m \neq 0}$, arranged in order and with multiplicities of membership possibly larger than 1.

Define the following Dirichlet series, which converge absolutely if $Re(s) > n/2$:

$$D(s, z, h) = \sum_{m \neq -z \in \mathbf{Z}^n} \frac{e^{2\pi i (m \cdot h)}}{Q(m + z)^s} = \sum_{k=1}^{\infty} a_k(z, h)\, e^{-s \log \lambda_k(z)}$$

$$\tilde{D}(s, z, h) = \sum_{m \neq -h \in \mathbf{Z}^n} \frac{e^{-2\pi i (m \cdot z)}}{Q(m + h)^s} = \sum_{k=1}^{\infty} b_k(z, h)\, e^{-s \log \eta_k(h)},$$

Note. For simplicity, $a_k(z, h), b_k(z, h)$ are denoted a_k, b_k in the following. Similarly, $\lambda_k(z), \eta_k(h)$ are denoted λ_k, η_k. By the notational convention used in the prior paragraph, one notes that $|a_k| = |b_k| = 1$ for each k. In particular, each coefficient equals 1 when $z, h \in \mathbf{Z}^n$. ∎

Now set

$$\gamma = \begin{cases} 1 & \text{if } h \in \mathbf{Z}^n \\ 0 & \text{if } h \notin \mathbf{Z}^n \end{cases}$$

$$\delta = \begin{cases} 1 & \text{if } z \in \mathbf{Z}^n \\ 0 & \text{if } z \notin \mathbf{Z}^n \end{cases}.$$

Epstein showed the following pair of identities that determine analytic continuations of these two series to the entire s plane as meromorphic functions.

$$\begin{aligned} \pi^{-s}\,\Gamma(s)\,D(s,z,h) = {} & \frac{\gamma D^{-1/2}}{s-\frac{n}{2}} - \frac{\delta e^{-2\pi i(z\cdot h)}}{s} + \int_1^\infty y^{s-1}\left[\Theta_Q(z,h,y)-1\right]dy \\ & + \frac{e^{-2\pi i(z\cdot h)}}{D^{1/2}}\int_1^\infty y^{\frac{n}{2}-s-1}\left[\Theta_{\tilde{Q}}(h,-z,y)-1\right]dy\,. \end{aligned} \tag{2.3}$$

$$\begin{aligned} \pi^{-s}\,\Gamma(s)\,\tilde{D}(s,z,h) = {} & \frac{\delta D^{1/2}}{s-\frac{n}{2}} - \frac{\gamma e^{2\pi i(z\cdot h)}}{s} + \int_1^\infty y^{s-1}\left[\Theta_{\tilde{Q}}(h,-z,y)-1\right]dy \\ & + D^{1/2}\,e^{2\pi i(z\cdot h)}\int_1^\infty y^{\frac{n}{2}-s-1}\left[\Theta_Q(-z,-h,y)-1\right]dy\,. \end{aligned}$$

Setting

$$c = D^{-1/2}e^{-2\pi i(z\cdot h)}\pi^{-n/2}\,,$$

one deduces the following functional equation:

$$D(s,z,h) = c\,\pi^{2s}\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s)}\tilde{D}(\frac{n}{2}-s,z,h)\,. \tag{2.4}$$

In particular, if $\sigma < 0$ then $\frac{n}{2}-\sigma > 0$. So, one obtains the following expression for $D(s,z,h)$ if $\sigma < 0$:

$$D(s,z,h) = c\cdot\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s)}\cdot\sum_{k=1}^{\infty}\frac{b_k}{\eta_k^{n/2}}(\eta_k\pi^2)^s\,.$$

Moreover, the series on the right converges absolutely. One concludes that for fixed h, z

$$\begin{aligned} \sigma > n/2 \text{ implies} \quad & |D(\sigma+it,z,h)| = O(1)\,, \\ \sigma < 0 \text{ implies} \quad & |D(\sigma+it,z,h)| = \left|\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s)}\right|. \end{aligned}$$

Stirling's asymptotic (2.5) for $\Gamma(s)$ implies that in any vertical strip $\sigma_1 \le \sigma \le \sigma_2$

$$\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s)} = O(|t|^{\frac{n}{2}-2\sigma})\,.$$

Thus, for any $\epsilon > 0$ and z, h,

$$|D(-\epsilon + it, z, h)| = O(|t|^{\frac{n}{2}+2\epsilon}).$$

Furthermore, from the fact that $1/\Gamma(s) = O(e^{2|t|})$, (2.3) implies, that for $\sigma \in [-\epsilon, \frac{n}{2} + \epsilon]$

$$D(s, z, h) = O(e^{2|t|}).$$

One concludes from the Phragman-Lindelöf theorem:

$$D(s, z, h) = O(|t|^{\frac{n}{2}+2\epsilon}) \quad \text{uniformly for } \sigma \in [-\epsilon, \frac{n}{2} + \epsilon]. \tag{2.5}$$

One now recalls three properties of the Bessel function of any exponent. First, there is the series expansion. Given $v \in \mathbf{R}$ set

$$J_v(y) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\Gamma(v + i + 1)} \left(\frac{y}{2}\right)^{v+2i}. \tag{2.6.1}$$

Secondly, there is the estimate for large y

$$|J_v(y)| = O(\frac{1}{\sqrt{y}}). \tag{2.6.2}$$

Thirdly, one has for each positive integer r

$$\frac{d^r}{dy^r} \left(y^{\frac{v+r}{2}} J_{v+r}(2\sqrt{y})\right) = y^{\frac{v}{2}} J_v(2\sqrt{y}) \tag{2.6.3}$$

Using (2.6.1), an elementary exercise of residue calculus shows

LEMMA 2.7. *If $\epsilon \in (0, 1/4), y > 0, r \in \mathbf{N}$ then*

$$\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\Gamma(\frac{n}{2} - s)}{\Gamma(s)} y^s \, ds = \frac{2\pi i}{y^{\frac{r}{2}-\frac{n}{4}}} \cdot J_{\frac{n}{2}+r}(2\sqrt{y}).$$

Corresponding to the finite sums of weighted coefficients, produced by the "weighted Perron's formula" (cf. Appendix B), define the following. (Note that the dependence upon z, h is not indicated in the notation for simplicity.)

$$\begin{aligned}
A(w) &= \sum_{\lambda_n \le w} a_n \\
A_1(w) &= \sum_{\lambda_n \le w} a_n(w - \lambda) \\
A_2(w) &= \frac{1}{2!} \sum_{\lambda_n \le w} a_n(w - \lambda_n)^2 \\
&\vdots \qquad \vdots \\
A_k(w) &= \frac{1}{k!} \sum_{\lambda_n \le w} a_n(w - \lambda_n)^k \; k = 3, 4, \dots.
\end{aligned}$$

A tedious calculation with step functions shows, setting $A_0(w) = A(w)$,

$$\int_0^w A_k(u)du = A_{k+1}(w)\,,$$

$$\text{and} \qquad A_k(w) = \int_0^w dw_1 \int_0^{w_1} dw_2 \cdots \int_0^{w_{k-1}} A(w_k)\,dw_k\,.$$

A second important property concerning these functions is the relation between the $A_k(w)$ and the k^{th} difference operators. Define the operator and its iterates

$$\Delta_z f(w) = f(w+z) - f(w),$$
$$\Delta_z^{(v)} f(w) = \Delta_z \left(\Delta_z^{(v-1)} f\right)(w) \quad v \geq 2\,.$$

Then

$$\Delta_z^{(v)} f(w) = \sum_{j=0}^{v} (-1)^{v-j} \binom{v}{j} f(w+jz)\,.$$

Moreover, a straightforward calculation, left to the reader, shows

$$\begin{aligned} \Delta_z A_1(w) &= \int_w^{w+z} A(w_1)\,dw_1\,, \\ \Delta_z^{(2)} A_2(w) &= \int_w^{w+z} dw_1 \int_{w_1}^{w_1+z} A(w_2)\,dw_2\,, \\ &\vdots \qquad \vdots \\ \Delta_z^{(v)} A_V(w) &= \int_w^{w+z} dw_1 \int_{w_1}^{w_1+z} dw_2 \ldots \int_{w_{v-1}}^{w_{v-1}+z} A(w_v)\,dw_v\,. \end{aligned} \tag{2.8}$$

One starts with the formula below, in which $w > 0, a > n/2, r \in \mathbf{N}$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} D(s,z,h)w^s \frac{ds}{[s]_r} = \frac{1}{r!}\sum_{\lambda_k \leq t} a_k\left(1-\frac{\lambda_k}{w}\right)^r = \frac{A_r(w)}{w^r}\,.$$

Now move the vertical line to $\sigma = -\epsilon$ where, for convenience, one assumes $\epsilon \in (0, 1/4)$. Residue calculus, combined with the growth estimate (2.5) and analytic continuation formula (2.3), implies that for r sufficiently large

$$\begin{aligned} \frac{A_r(w)}{w^r} &= \frac{\pi^{\frac{n}{2}}\gamma}{D^{\frac{1}{2}}\,\Gamma(\frac{n}{2})\cdot\frac{n}{2}\cdot(\frac{n}{2}+1)\cdots(\frac{n}{2}+r)} \cdot w^{\frac{n}{2}} + \frac{D(0,z,h)}{r!} + \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} D(s,z,h)w^s \frac{ds}{[s]_r} \\ &= \frac{\pi^{\frac{n}{2}}\gamma}{D^{\frac{1}{2}}\,\Gamma(\frac{n}{2}+r+1)} \cdot w^{\frac{n}{2}} + \frac{D(0,z,h)}{r!} \\ &\quad + \frac{c}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \left(\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s)} \sum_{k=1}^{\infty} \frac{b_k}{\eta_k^{\frac{n}{2}}}(\pi^2\eta_k)^s\right) \cdot w^s \frac{ds}{[s]_r}\,. \end{aligned}$$

The Bessel functions arise naturally once one interchanges summation and integration, which one may because the series of functions

$$\sum_{k=1}^{\infty} \frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s)} \cdot \frac{b_k}{\eta_k^{\frac{n}{2}}} (\pi^2 \eta_k)^s$$

converges absolutely and uniformly over the line $\sigma = -\epsilon$. Then, Lemma 2.7 says that the integral over this line equals

$$\frac{w^{\frac{n}{4}-\frac{r}{2}}}{\pi^{r-\frac{n}{2}}} \sum_{k=1}^{\infty} \frac{b_k}{\eta_k^{\frac{n}{4}+\frac{r}{2}}} \cdot J_{\frac{n}{2}+r}(2\pi\sqrt{\eta_k\, w}).$$

One concludes

$$\begin{aligned} A_r(w) = {} & \frac{\pi^{\frac{n}{2}}\, \gamma}{D^{\frac{1}{2}}\, \Gamma(\frac{n}{2}+r+1)} \cdot w^{\frac{n}{2}+r} + \frac{D(0,z,h)}{r!} \\ & + \frac{c}{\pi^{r-\frac{n}{2}}} \cdot w^{\frac{n}{4}+\frac{r}{2}} \sum_{k=1}^{\infty} \frac{b_k}{\eta_k^{\frac{n}{4}+\frac{r}{2}}} \cdot J_{\frac{n}{2}+r}(2\pi\sqrt{\eta_k\, w}). \end{aligned} \tag{2.9}$$

Note. For the rest of the discussion one imposes the condition that $h, z \in \mathbf{Z}^n$. With this condition, it is clear that $A(w) = N(w)$, defined at the beginning of the section. The reader will hopefully not find this confusing. ∎

Since each coefficient a_k in the definition of $A(w)$ is nonnegative, one can relate $A_r(w)$ to $A(w)$ as follows. First, the chain of intervals, corresponding to the domain of integration of $\Delta_z^{(r)} A_r(w)$,

$$[w_{r-1}, w_{r-1}+z] \subset [w_{r-2}, w_{r-2}+2z] \ldots \subset [w, w+rz]$$

implies this sequence of inequalities for any $w_r \in [w_{r-1}, w_{r-1}+z]$:

$$A(w) \leq A(w_{r-1} \leq A(w_r) \leq A(w_{r-1}+z) \leq A(w+rz).$$

Thus,

$$z^r A(w) \leq \Delta_z^{(r)} A_r(w) \leq z^r A(w+rz). \tag{2.10}$$

One now proves the following important estimate.

LEMMA 2.11. *Let $w, y > 0$ and $z \in (0, w]$. Set*

$$f(w,y) = w^{\frac{n}{4}+\frac{r}{2}} J_{\frac{n}{2}+r}(2\pi\sqrt{wy}).$$

Then, there exists $E = E(n)$ such that

$$\Delta_z^{(r)} f(w,y) < \frac{E w^{\frac{n-1}{4}}}{y^{\frac{1}{4}}} \left(min\{w, yz^2\}\right)^{\frac{r}{2}},$$

where the difference operator is taken with respect to w.

PROOF: The proof uses three properties established above. These are

(1) The expression for $\Delta_z^{(r)} f$ in terms of the $f(w+vz)$ (2.8);
(2) The estimate for $|J_v|$ (2.6.2);
(3) The differential equation (2.6.3).

Details are left to the reader. ∎

One uses this Lemma to estimate $\Delta_z^{(r)} A_r(w)$. The resulting estimate will be combined with (2.10) to estimate $A(w)$.

First, since $\gamma = \delta = 1$, one writes, with f denoting the function defined in (2.11),

$$\Delta_z^{(r)} A_r(w) = \frac{\pi^{\frac{n}{2}}}{D^{\frac{1}{2}}\,\Gamma(\frac{n}{2}+r+1)} \cdot \Delta_z^{(r)}(w^{\frac{n}{2}+r}) + \frac{D(0,z,h)}{r!}\Delta_z^{(r)}(w^r)$$
$$+\frac{c}{\pi^{r-\frac{n}{2}}}\sum_{k=1}^{\infty}\frac{b_k}{\eta_k^{\frac{n}{4}+\frac{r}{2}}}\,\Delta_z^{(r)} f(w,\eta_k)\,.$$

One finds by a simple calculation

$$\Delta_z^{(r)}(w^r) = \int_w^{w+z} dw_1 \int_{w_1}^{w_1+z} dw_2 \cdots \int_{w_{r-1}}^{w_{r-1}+z} r!\,dw_r = r!\,z^r\,, \tag{2.12}$$
$$\Delta_z^{(r)}(w^{\frac{n}{2}+r}) = (\frac{n}{2}+r)\cdots(\frac{n}{2}+1)\left[z^r w^{\frac{n}{2}} + O\left(w^{\frac{n}{2}-1}z^{r+1}\right)\right]\,.$$

The last item is to estimate

$$\Big|\sum_{k=1}^{\infty}\frac{b_k}{\eta_k^{\frac{n}{4}+\frac{r}{2}}}\,\Delta_z^{(r)} f(w,\eta_k)\Big|\,.$$

Since each $b_k = 1$, one obtains from Lemma 2.11,

$$\Big|\sum_{k=1}^{\infty}\frac{b_k}{\eta_k^{\frac{n}{4}+\frac{r}{2}}}\,\Delta_z^{(r)} f(w,\eta_k)\Big| < E w^{\frac{n-1}{4}}\sum_{k=1}^{\infty}\frac{1}{\eta_k^{\frac{n+1}{4}+\frac{r}{2}}}\left(min\{w, z^2\eta_k\}\right)^{\frac{r}{2}}\,.$$

Now split up the sum according to

$$z^2\,\eta_k \le w \quad \text{or} \quad z^2\,\eta_k > w\,.$$

Then

$$E w^{\frac{n-1}{4}}\sum_{k=1}^{\infty}\frac{1}{\eta_k^{\frac{n+1}{4}+\frac{r}{2}}}\left(min\{w,z^2\eta_k\}\right)^{\frac{r}{2}} = E w^{\frac{n-1}{4}} z^r \sum_{\eta_k\le w/z^2}\frac{1}{\eta_k^{\frac{n+1}{4}}}$$
$$+\,E w^{\frac{n-1}{4}+\frac{r}{2}}\sum_{\eta_k> w/z^2}\frac{1}{\eta_k^{\frac{n+1}{4}+\frac{r}{2}}}\,. \tag{2.13}$$

One therefore needs an estimate of the following sort.

LEMMA 2.14.

(1) $\sum_{\{k:\eta_k \le R\}} 1/\eta_k^\kappa = O(R^{\frac{n}{2}-\kappa})$ *if* $\kappa < \frac{n}{2}$.
(2) $\sum_{\{\eta_k > R\}} 1/\eta_k^\kappa = O(R^{\frac{n}{2}-\kappa})$ *if* $\kappa > \frac{n}{2}$.

PROOF: Set

$$G(R) = \sum_{\{k:\eta_k \le R\}} 1 = \sum_{\eta_k=1}^{R} \sum_{\{m\in\mathbf{Z}^n:\tilde{Q}(m)=\eta_k\}} 1 .$$

It is not difficult to see that $G(R) = O(R^{\frac{n}{2}})$. Indeed, there exists $C > 0$ such that $\tilde{Q}(x) \ge C\|x\|^2$. Hence,

$$\tilde{Q}(m) = \eta_k \le R \quad \text{implies} \quad \|m\|^2 \le C^{-1}R .$$

Since there exist $O(R^{\frac{n}{2}})$ lattice points in the R sphere of $\mathbf{R}^n$, one concludes $G(R) = O(R^{\frac{n}{2}})$.

Introduce the notation $k(y)$ to denote the largest index for which $\eta_{k(y)} \le y$ for any real number y.

Case 1. $\kappa < \frac{n}{2}$.

Partial summation states

$$\begin{aligned}
\sum_{\{k:\eta_k \le R\}} 1/\eta^\kappa &= \frac{G(\eta_1)}{\eta_1^\kappa} + \sum_{k=2}^{k(R)} \frac{G(\eta_k) - G(\eta_{k-1})}{\eta_k^\kappa} \\
&= \sum_{k=1}^{k(R)-1} G(\eta_k)\left(\frac{1}{\eta_k^\kappa} - \frac{1}{\eta_{k+1}^\kappa}\right) + \frac{G(R)}{\eta_{k(R)}^\kappa} \\
&= \kappa \sum_{k=1}^{G(R)-1} G(\eta_k) \int_{\eta_k}^{\eta_{k+1}} \frac{du}{u^{\kappa+1}} + \frac{G(R)}{\eta_{k(R)}^\kappa} \\
&= (\text{since } G(u) \text{ is constant in } [\eta_k, \eta_{k+1})\ )\ \ \kappa \cdot \int_{\eta_1}^{\eta_{k(R)}} \frac{G(u)}{u^{\kappa+1}}\, du + \frac{G(R)}{\eta_{k(R)}^\kappa} \\
&= O\left(\int_{\eta_1}^{\eta_{k(R)}} u^{\frac{n}{2}-\kappa-1}\, du\right) + O(\eta_{k(R)}^{\frac{n}{2}-\kappa}) \\
&= O(R^{\frac{n}{2}-\kappa}) .
\end{aligned}$$

Case 2. $\kappa > \frac{n}{2}$.

$$
\begin{aligned}
\sum_{k:\eta_k>R} 1/\eta_k^\kappa &= \sum_{k=k(R)+1}^{\infty} \frac{G(\eta_k)-G(\eta_{k-1})}{\eta_k^\kappa} \\
&= \sum_{k=k(R)+1}^{\infty} G(\eta_k)\left(\frac{1}{\eta_k^\kappa}-\frac{1}{\eta_{k+1}^\kappa}\right)-\frac{G(R)}{\eta_{k(R)+1}^\kappa} \\
&< \kappa\cdot \sum_{k=k(R)+1}^{\infty} G(\eta_k)\int_{\eta_k}^{\eta_{k+1}} \frac{du}{u^{\kappa+1}} = O\left(\int_{\eta_{k(R)+1}}^{\infty} u^{\frac{n}{2}-\kappa-1}\,du\right) \\
&= O(R^{\kappa-\frac{n}{2}}). \quad \blacksquare
\end{aligned}
$$

Using Lemma 2.14, a simple calculation shows that the estimates in (2.13) agree and equal

$$
O\left(z^{r+\frac{1-n}{2}}\cdot w^{\frac{n-1}{2}}\right). \tag{2.15}
$$

Now set $z=w^\gamma$ and choose γ so that the order in w of (2.12) and (2.15) agree. A simple calculation shows that

$$
\gamma=\frac{1}{n+1}.
$$

Then, replacing z by this power of w inside the $O(\cdot)$ terms, and recalling the definition of A from the beginning of the section, one sees that

$$
\Delta_z^{(r)} A_r(w) = z^r\left[Aw^{\frac{n}{2}}+O\left(w^{\frac{n-1}{2}\cdot\frac{n}{n+1}}\right)\right].
$$

Hence,

$$
A(w)\le Aw^{\frac{n}{2}}+O\left(w^{\frac{n-1}{2}\cdot\frac{n}{n+1}}\right).
$$

Moreover,

$$
A\left(w+rw^{\frac{1}{n+1}}\right)\ge Aw^{\frac{n}{2}}+O\left(w^{\frac{n-1}{2}\cdot\frac{n}{n+1}}\right).
$$

Setting $y=w+rw^{\frac{1}{n+1}}$ one notes that $w=y+O(y^{\frac{1}{n+1}})$. Thus,

$$
A(y)\ge Ay^{\frac{n}{2}}+O\left(y^{\frac{n-1}{2}\cdot\frac{n}{n+1}}\right).
$$

One concludes that for all large w

$$
A(w)=Aw^{\frac{n}{2}}+O\left(w^{\frac{n-1}{2}\cdot\frac{n}{n+1}}\right).
$$

As already noted, $A(w)=N(w)$, when $h,z\in\mathbf{Z}^n$. This completes the proof of Theorem 2.1. $\blacksquare$

Section 3. *Results obtained using singularity theory*

In this section $\mathcal{R}_1$ will denote a compact region

$$\{P \leq 1\} \subset \mathbf{R}^n,$$

where P is a positive definite homogeneous polynomial of degree d. The appropriate homothetic expansions will then be

$$\mathcal{R}_w = \{P \leq w\} = w^{1/d}\,\mathcal{R}_1.$$

Randol and Colin de Verdière have given estimates for the growth of the error

$$E(w) = N(w) - V(w)$$

in terms of the geometry of the hypersurface $\partial\mathcal{R}_1$. To carry out the analysis, they have used Poisson summation to express a smooth approximation to $N(w)$ as a sum of fourier transforms. Then geometric methods are used to estimate the absolute values of the transforms. The discussion here will sketch that given in [CdV] which exploits the local analysis of singularities to give asymptotics of certain oscillatory integrals arising naturally in the problem. However, many details can not be included in so limited a discussion as that given here.

The first point is to express $N(w)$ in a naive way. Set

$$\begin{aligned} \chi = {} & \text{ characteristic function of } \mathcal{R}_1, \\ \text{and} \quad \chi_w(\cdot) = {} & \chi(\cdot / w^{1/d}). \end{aligned}$$

It is clear that χ_w is the characteristic function of $\mathcal{R}_w$. Thus,

$$N(w) = \sum_{m \in \mathbf{Z}^n} \chi_w(m).$$

To apply Poisson summation, one needs a C^∞ function, so one smooths χ_w as follows. Let $\rho : \mathbf{R}^n \to [0,1]$ be a C^∞ function with support in the ball of radius 1 and satisfying

$$\int_{\mathbf{R}^n} \rho\, dx = 1.$$

For $\epsilon > 0$, set

$$\rho_\epsilon(\cdot) = \epsilon^{-n} \cdot \rho(\cdot/\epsilon).$$

Thus, ρ_ϵ satisfies the same property as ρ but has support in the ball of radius ϵ. Consider now the convolution

$$\chi_w * \rho_\epsilon(x) = \int_{\mathbf{R}^n} \chi_w(y)\, \rho_\epsilon(x-y)\, dy\,,$$

and define the series

$$N_\epsilon(w) = \sum_{m \in \mathbf{Z}^n} \chi_w * \rho_\epsilon(m).$$

One has the

LEMMA 3.1. *For each $w, \epsilon > 0$, $\chi_w * \rho_\epsilon$ is C^∞ with compact support.*

Defining, for $f \in C_c^\infty(\mathbf{R}^n)$,

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\, e^{-2\pi i(\xi \cdot x)}\, dx,$$

a standard fact is that

$$\widehat{f * g} = \hat{f} * \hat{g} \quad \text{for any } g \in C_c^\infty(\mathbf{R}^n).$$

Thus, Poisson summation shows

PROPOSITION 3.2.

$$N_\epsilon(w) = \sum_{m \in \mathbf{Z}^n} \hat{\chi}_w \cdot \hat{\rho}_\epsilon(m).$$

Consider the $m = 0$ term. It is easy to see via the homogeneity of P and definition of ρ_ϵ that

PROPOSITION 3.3.

$$\hat{\chi}_w(0) \cdot \hat{\rho}_\epsilon(0) = vol(\mathcal{R}_1)\, w^{n/d}.$$

Separating this term from the rest of the series in (3.2), define the "error term"

$$E_\epsilon(w) = \sum_{m \neq 0 \in \mathbf{Z}^n} \hat{\chi}_w \cdot \hat{\rho}_\epsilon(m).$$

As $w \to \infty$ one must estimate the growth of $E_\epsilon(w)$. To do this, one first rewrites the $\hat{\chi}_w(m)$ factor in each summand of $E_\epsilon(w)$.

For any $z \neq 0 \in \mathbf{R}^n$, set $\nu_z = z/\|z\|$ and $w(z) = w^{1/d}\|z\|$.

Then,

$$\begin{aligned}
\hat{\chi}_w(m) &= \int_{\mathbf{R}^n} \chi(x/w^{1/d})\, e^{-2\pi i(m,x)}\, dx \\
&= w^{n/d} \int_{\mathbf{R}^n} \chi(y)\, e^{-2\pi i w^{1/d}\|m\|(\nu_m, y)}\, dy \\
&= w^{n/d} \int_{\mathcal{R}_1} e^{-2\pi i w(m)(\nu_m, y)}\, dy\,.
\end{aligned}$$

Thus,

$$\hat{\chi}_w(m) = w^{n/d}\, \hat{\chi}(w(m)\nu_m).$$

Moreover,

$$\hat{\rho}_\epsilon(m) = \hat{\rho}(\epsilon m).$$

One concludes

$$E_\epsilon(w) = N_\epsilon(w) - vol(\mathcal{R}_1)\, w^{n/d} = w^{n/d} \sum_{m\neq 0} \hat{\chi}(w(m)\nu_m)\hat{\rho}(\epsilon m). \tag{3.4}$$

To estimate $E_\epsilon(w)$ it is therefore necessary to determine useful estimates in w, z of $\hat{\chi}(w(z)\nu_z)$, as $w \to \infty$ that are summable in z when z assumes values in $\mathbf{Z}^n - \{0\}$. Moreover, when ϵ becomes a function of w as done below, it will then also be necessary to do the same with $\hat{\rho}(\epsilon m)$.

Assume this is done, for the moment. This means that one can prove the existence of $\alpha < 0$ and for each ϵ a constant $C(\epsilon)$ which is bounded as $\epsilon \to 0$ such that

$$|E_\epsilon(w)| < w^{\frac{n+\alpha}{d}}\, C(\epsilon) \quad \text{for all } w \text{ sufficiently large.} \tag{3.5}$$

One then needs to understand in what way $N_\epsilon(w)$ converges to $N(w)$ for large w as $\epsilon \to 0$. To do this, one first notes that if the lattice point m is contained in the ϵ neighborhood of $\mathcal{R}_w$ but not in $\mathcal{R}_w$ then $\chi_w * \rho_\epsilon(m)$ need not equal zero. Extending this, one finds easily,

LEMMA 3.6.

(1) *$N_\epsilon(w)$ is uniformly bounded for small $\epsilon > 0$ and w contained in bounded intervals.*

(2) *Let $\mathcal{I} = \{w > 0 : \partial\mathcal{R}_w \cap \mathbf{Z}^n = \emptyset\}$. Then*

$$\lim_{\epsilon\to 0} N_\epsilon(w) = N(w) \quad \text{if } w \in \mathcal{I}.$$

Thus, if one allows w to increase without bound but remain in $\mathcal{I}$ then combining (3.4), (3.5), and this lemma would give an asymptotic for $N(w)$ whenever w stays in $\mathcal{I}$. Since $\mathcal{I}^c$ is at most countable and has no finite cluster point, this is not a very restrictive condition measure theoretically. However, aesthetically it is not too pleasing since the exact nature of $\mathcal{I}^c$ is very unclear. To circumvent this difficulty, Colin de Verdière developed a modification of the above discussion, due to Randol [R]. The rest of this section will sketch his ideas.

Given $\beta \geq 0$ define

$$\mathcal{R}_1(\beta) = \{x \in \mathbf{R}^n : d(x, \mathcal{R}_1) \leq \beta\},$$

and for $\beta < 0$ set

$$\mathcal{R}_1(\beta) = \mathbf{R}^n - \{x : d(x, \mathbf{R}^n - \mathcal{R}_1) \leq -\beta\}.$$

For arbitrary β and positive w set

$$\mathcal{R}_w(\beta) = w^{1/d}\, \mathcal{R}_1(\beta)$$

$$\text{and} \quad \chi_w(\beta) = \text{ characteristic function of } \mathcal{R}_w(\beta).$$

Note that $\mathcal{R}_w(\beta)$ is not necessarily the β neighborhood of $\mathcal{R}_w$, although $\mathcal{R}_w(0) = \mathcal{R}_w$.

In the following, χ_β will denote the characteristic function of $\mathcal{R}_1(\beta)$.

Now define

$$N(w,\beta) = \sum_{m\in\mathbb{Z}^n} \chi_w(\beta)(m)$$
$$N_\epsilon(w,\beta) = \sum_{m\in\mathbb{Z}^n} \chi_w(\beta) * \rho_\epsilon(m)$$

and

$$E_\epsilon(w,\beta) = \sum_{m\neq 0} \chi_w(\beta) * \rho_\epsilon(m).$$

As in (3.4), one notes that

$$E_\epsilon(w,\beta) = w^{n/d} \sum_{m\neq 0} \hat{\chi}_\beta(w(m)\nu_m)\,\hat{\rho}(\epsilon m).$$

It is easy to check that

$$N_\epsilon(w, -\epsilon/w^{1/d}) \leq N(w) \leq N_\epsilon(w, \epsilon/w^{1/d})$$

and

$$vol(\mathcal{R}_w(\beta)) = vol(\mathcal{R}_w) + O(w^{n/d}\,\beta^{n-1}). \tag{3.7}$$

Thus, by setting

$$\begin{aligned} \epsilon &= w^{\frac{1}{d}\cdot\frac{n-1}{n+1}} \\ \beta &= \epsilon^{\frac{2n}{(n-1)^2}}/w^{\frac{1}{d}}, \end{aligned} \tag{3.8}$$

one finds that

$$|E(w)| \leq \max\{|E_\epsilon(w,-\beta)|, |E_\epsilon(w,\beta)|\} + O\left(w^{\frac{n-2}{d}+\frac{2}{d(n+1)}}\right). \tag{3.9}$$

So, one is forced to obtain estimates for the $E_\epsilon(w,\beta)$ that are uniform in β when β is chosen according to (3.8).

Remark 3.10. The necessity of choosing the exponent of ϵ as in (3.8) will be clarified at the end of the proof of Theorem 3.26. The $O\left(w^{\frac{n-2}{d}+\frac{2}{d(n+1)}}\right)$ error term should be understood as an "elementary estimate" that arises solely from estimate (3.7). The entire effort will be directed to showing that the local classification of singularities implies, under the assumptions of Theorem 3.26, that one can never do worse than this elementary estimate. That is, the two other terms appearing in (3.9) can never grow faster than this power of w. It should also be pointed out that when $n = 2$ Landau has shown that one cannot hope to do better than $O(w^{2/3})$ in general. This appears to have been the motivation for insisting upon proving a theorem with the error term estimated by the above power of w. ∎

In the discussion below, one will first insist upon studying the magnitudes of all $\hat{\chi}_\beta(w(z)\nu_z)$ for $|\beta| \ll 1$. It will not suffice to do this only if $\beta = 0$.

Now, since each $\mathcal{R}_w(\beta)$ has a smooth oriented boundary, one can apply Stokes' theorem. To state the identity, introduce the following convenient notation. Set

$$\begin{aligned}
\boldsymbol{n}_\beta(x) &= \text{ outward unit normal vector to } \partial\mathcal{R}_1(\beta) \text{ at } x \\
d\sigma(\beta) &= \text{ area differential over } \partial\mathcal{R}_1(\beta).
\end{aligned}$$

One then has

PROPOSITION 3.11.

$$\begin{aligned}
\hat{\chi}_\beta(w(z)\nu_z) &=_{def} \int_{\mathbf{R}^n} \chi_\beta(x)\, e^{-2\pi i w(z)(\nu_z,x)}\, dx \\
&= \int_{\mathcal{R}_1(\beta)} e^{-2\pi i w(z)(\nu_z,x)}\, dx \\
&= \frac{-i}{2\pi w(z)} \int_{\partial\mathcal{R}_1(\beta)} e^{-2\pi i w(z)(\nu_z,x)}\, (\nu_z, \boldsymbol{n}_\beta(x))\, d\sigma(\beta).
\end{aligned}$$

For $\xi \in S^{n-1}$, and any β, define $\eta = (\xi, \beta)$ and

$$\begin{aligned}
\tilde{\varphi}(\eta, x) &= (\xi, x)\big|_{\partial\mathcal{R}_1(\beta)}, \\
a(\eta, x) &= (\xi, \boldsymbol{n}_\beta(x))\big|_{\partial\mathcal{R}_1(\beta)}, \\
I_{w(z)}(\eta) &= \int_{\partial\mathcal{R}_1(\beta)} e^{-2\pi i w(z)\tilde{\varphi}(\eta,x)}\, a(\eta, x)\, d\sigma(\beta).
\end{aligned}$$

$I_{w(z)}(\eta)$ is an example of an "oscillatory integral" with phase $\tilde{\varphi}(\eta, x)$ and amplitude $a(\eta, x)$. For each fixed η, the asymptotic in $w(z)$ is determined, up to terms that decrease exponentially, by the singular set of the map $x \to \tilde{\varphi}(\eta, x)$. This follows from the simple to check property that if x' is a regular point of $\tilde{\varphi}(\eta, x)$ then $I_{w(z)}(\eta)$ *is* exponentially decreasing in $w(z)$.

The critical points of φ are easily seen to consist of the following set of points.

PROPOSITION 3.12. *The critical set of $\tilde{\varphi}$ equals*

$$\{(\xi, x) : \xi \text{ is normal to } \partial\mathcal{R}_1(\beta) \text{ at } x\}.$$

To find useful estimates for $E_\epsilon(w, \beta)$ that are uniform in β, one should first eliminate the dependance in β of the domain of integration. To do this precisely requires certain ideas from the theory of Fourier integral operators. Due to a desire to keep this paper within a reasonable length, the proof of the following result will not be given.

Set

$$\varphi(\xi,x)=\tilde{\varphi}(\xi,0,x)=(\xi,x)\big|_{\partial R_1} \qquad d\sigma=d\sigma(0). \tag{3.13}$$

THEOREM 3.14. *There exists a C^∞ function $A(\eta,x)$ such that for all β whose absolute value is sufficiently small, one has*

(1) $A(\xi,0,x)=a(\xi,0,x)$.
(2) *Define*

$$J_{w(z)}(\eta)=\int_{\partial\mathcal{R}_1} e^{-2\pi i w(z)\,\varphi(\xi,x)}\,A(\eta,x)\,d\sigma.$$

Then,

$$I_{w(z)}(\eta)=e^{-2\pi i w(z)\beta}\cdot J_{w(z)}(\eta)\,.$$

Impose now the following "finiteness" condition on the type of singularities possessed by φ. This condition will be assumed to hold for the rest of this section.

For each $\xi\in S^{n-1}$, the function $x\to\varphi(\xi,x)$ has only isolated singular points. (H)

Using (H), one can "localize" the estimate for the growth of $J_{w(z)}(\eta)$ in a standard way to one of an oscillatory integral with phase given by the universal unfolding of a singularity. This will now be described in a general setting.

Let $x\to\psi(y,x)$ denote a family of functions depending smoothly upon the parameter y. Assume that x resp. y is contained in an open set $\mathcal{X}\subset\mathbf{R}^k$ resp. $\mathcal{Y}\subset\mathbf{R}^m$. Assume further:

for each $y,\psi_y(x)$ admits only finitely many isolated singular points in $\mathcal{X}$. (3.15)

Let $b\in C^\infty(\mathcal{X})$. Define

$$\mathcal{I}_\tau(y,b)=\int_{\mathcal{X}} e^{i\tau\psi(y,x)}\,b(x)\,dx.$$

The behavior in τ, as $\tau\to\infty$, of $\mathcal{I}_\tau(y,b)$ can be determined by localizing the domain of integration to one of a sufficiently small ball centered at each singular point of ψ_y contained in $\mathcal{X}$, and then patching together all locally determined estimates by a C^∞ partition of unity. A typical result needed to do this follows from the Malgrange preparation theorem.

PROPOSITION 3.16. *Assume (3.15) holds. Then there exists $M\in\mathbf{N}$ such that for each $y\in\mathcal{Y}$, $\psi_y(x)$ possesses at most M singular points in $\mathcal{X}$.*

Let $\Sigma(\psi_y)=$ singular set in $\mathcal{X}$ of ψ_y.

PROPOSITION 3.17. *For each $p \in \Sigma(\psi_y)$ there exists an open neighborhood $\mathcal{V}_p \subset \mathcal{V}_p$ and $\gamma(y,p) \in \mathbf{R}$ such that*

(1) $\mathcal{V}_p \cap \Sigma(\psi_y) = \{p\}$;
(2) *For any $b \in C_c^\infty(\mathcal{V}_p)$,*

$$\int_{\mathcal{V}_p} e^{i\tau\psi(y,x)}\, b(x)\, dx = O(\tau^{\gamma(y,p)}), \quad \tau \to \infty. \tag{*}$$

For example, if ψ_y has a nondegenerate critical point at p, then $\gamma(y,p) = -k/2$.

Define the index

$$\sigma(y,p) = inf\,\{\gamma(y,p) : (*) \text{ holds }\}. \tag{3.18}$$

If x' is a regular point of ψ_y then one sets $\sigma(y,x') = -\infty$ since in this case there exists a neighborhood $\mathcal{V}_{x'}$ so that the oscillatory integral in (*) exponentially decays.

Although the above can be defined for any parametrised family of singularities, for purposes of estimating the growth of oscillatory integrals when one is not given very precise (or any) information about the singularities, it is useful to assume that locally one uses the universal unfolding of a singularity for the phase function.

Recall the

Definition. If $f \in \mathbf{R}\{x_1,\ldots,x_k\}$ has a singular point at 0, then the universal unfolding of f is the function

$$F(\zeta,x) = f(x) + \sum_{i=1}^{m} \zeta_i f_i(x),$$

when:

(i)

$$m =_{def} dim\,(x_1,\ldots,x_n)\mathbf{R}\{x_1,\ldots,x_n\}/(\partial f/\partial x_1,\ldots,\partial f/\partial x_n) \text{ is finite;}$$

(ii) The $\{f_i\}$ form a basis of $\mathcal{J}(f)$. ∎

The vector space in (i) will be denoted by $\mathcal{J}(f)$. The codimension of the singularity equals m.

Given any $\psi(y,x)$ and $p \in \Sigma(\psi_y)$ for which the germ of ψ_y at p is $f(x)$, and such that conditions (i), (ii) hold, the universality of F implies the

PROPOSITION 3.19. *There exist:*

(1) *A point $\tilde{\zeta}$,*
(2) *Neighborhoods Z of $\tilde{\zeta}$ in $\mathbf{R}^m$, Y of y in $\mathcal{Y}$, and $\mathcal{B}$ of the origin in $\mathbf{R}^k$,*
(3) *C^∞ functions*

$$h : (Y,y) \to (Z,\tilde{\zeta}) \qquad \text{and} \quad H : (Y \times \mathcal{V}_p, (y,p)) \to (\mathcal{B}, 0),$$

such that

a) *For each* $y \in Y, H(y, \cdot)$ *defines a* C^∞ *diffeomorphism;*
b)

$$\psi(y, x) = F(h(y), H(y, x)) \text{ for all } (y, x) \in Y \times \mathcal{V}_p. \tag{\#}$$

Thus, an estimate for the index function $\gamma(\zeta, 0)$ of the oscillatory integral

$$\int_B e^{i\tau F(\zeta, x)}\, b(x)\, dx$$

that is uniform in ζ, b would automatically give an estimate for the integral defined in Proposition (3.17).

From hereon, one will assume that $Y, \mathcal{V}_p$ have been chosen so that (a), (b) of Proposition 3.19 hold when the phase function ψ *is* the function $\varphi(\xi, x)$ defined in (3.13).

Since $S^{n-1} \times \partial\mathcal{R}_1$ is a compact space, the open cover $\{Y \times \mathcal{V}_p\}_{p \in \Sigma(\varphi_y)}$, has a finite subcover $\{Y_i \times \mathcal{V}_{P_i}\}_{i=1}^{L}$. To each $\mathcal{U}_i =_{def} Y_i \times \mathcal{V}_{p_i}$ there is the function, denoted $f_i(x)$, with isolated singular point at p_i and universal unfolding

$$F_i(\zeta, x) = f_i(x) + \sum_{j=1}^{m_i} \zeta_j\, \mu_j(x),$$

defined in a bounded open set $\mathcal{Z}_i \times \mathcal{B}_i$, such that equation (#) holds, after specializing ψ to be φ.

Suppose now that $(y, p) \in \mathcal{U}_i$. Thus, the germ of φ_y at p determines an element of the family of functions appearing in F_i. Set $\mathcal{C}_i$ to equal the set of smooth functions h on $\mathcal{B}_i$ such that for each $\zeta \in \mathcal{Z}_i, h(\zeta, \cdot)$ has compact support. Given $b \in \mathcal{C}_i$ define the oscillatory integral

$$I_i(\tau, \zeta, b) = \int_{\mathcal{B}_i} e^{i\tau\, F_i(\zeta, x)}\, b(x)\, dx,$$

and the indices σ_i, σ :

$$\begin{aligned} \sigma_i(\zeta) &= inf\,\{\gamma(\zeta, 0) : I_i(\tau, \zeta, b) = O(\tau^{\gamma(\zeta,0)}) \text{ for all } b \in \mathcal{C}_i\}, \\ \sigma_i &= sup_{\zeta \in \mathcal{Z}_i}\, \sigma_i(\zeta), \\ \sigma &= max_i\, \{\sigma_i\}. \end{aligned} \tag{3.20}$$

By the definition of $J_{w(z)}(\eta)$, given in Theorem 3.14, and the prior discussion, it follows that

PROPOSITION 3.21.

$$J_{w(z)}(\eta) = O(w(z)^\sigma) \text{ as } w \to \infty,$$

and this estimate is uniform in η.

Clearly, this is the first result to try and exploit in order to obtain an estimate for $E_\epsilon(w,\beta)$. The naive way to proceed would be first to understand how big can σ be. Since the Dirichlet series

$$\sum_{0\neq m\in\mathbf{Z}^n} \frac{1}{\|m\|^s}$$

has its first pole at $s = n$ it is clear that if $1-\sigma > n$ then (3.21) would suffice to give an estimate for $E_\epsilon(w,\beta)$ satisfying (3.5). However, to force $\sigma < 1-n$ requires σ to be much too small to be of any use. This is the case because a nondegenerate singularity on $\mathbf{R}^{n-1}$ must appear in any neighborhood of a deformation of f_i. This then forces $\sigma_i \geq -(n-1)/2$.

Thus, to convert (3.21) into a *useable* estimate of the $E_\epsilon(w,\beta)$, one must do some additional analysis of each $I_i(\tau,\zeta,b)$. Colin de Verdière worked this out for certain cases. However, much more work needs to be done to extend his results. The discussion here will treat the possibility that the singularity $f_i(x)$ is a "simple singularity class" in Arnold's classification of [A].

Remark. By the phrase "singularity class" of an element f of $\mathbf{R}\{x_1,\ldots,x_k\}$ is meant the set of all germs of smooth functions right equivalent to f under germs of smooth diffeomorphisms fixing the origin. ∎

These classes are types A_k, D_k, E_6, E_7, E_8. For purposes of the analysis here, a crucial property they satisfy is that there exists a weighted homogeneous polynomial that can be used to represent each of them. These are written down in the following list, in which the notation Q will refer to a nondegenerate quadratic form in the variables not appearing explicitly in the canonical form polynomial of each class. Of course, the total number of variables in each polynomial must equal the dimension of $\partial\mathcal{R}_1$.

$$\begin{aligned}
A_{n+1}:&\quad x_1^{n+2}+Q,\\
D_{n+1}:&\quad x_1^2x_2 \pm x_2^{n+2}+Q,\\
E_6:&\quad x_1^3 \pm x_2^4+Q,\\
E_7:&\quad x_1^3 + x_1x_2^3+Q,\\
E_8:&\quad x_1^3 + x_2^5+Q\,.
\end{aligned}$$

Recall that one singularity class σ' is said to be *adjacent* to the singularity class σ, in which case one writes this as

$$\sigma' \longleftarrow \sigma,$$

if a function representing the class σ' appears in the universal unfolding of a function representing σ.

The adjacency diagram of these 5 classes of singularities is given in the following diagram:

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] & A_6 \arrow[l] & A_7 \arrow[l] & \cdots \arrow[l] \\
 & & D_4 \arrow[u] & D_5 \arrow[l] \arrow[u] & D_6 \arrow[l] \arrow[u] & D_7 \arrow[l] \arrow[u] & D_8 \arrow[l] \arrow[u] & \cdots \arrow[l] \\
 & & & E_6 \arrow[u] \arrow[uur] & E_7 \arrow[l] \arrow[u] \arrow[uur] & E_8 \arrow[l] \arrow[u] \arrow[uur] & &
\end{tikzcd}$$

Figure 1

To each of the functions $f_j, j = 1, \dots, L$, defined above by the covering of $S^{n-1} \times \partial\mathcal{R}_1$, one has the parameter space $\mathcal{Z}_j$ of dimension $m_j = dim\,\mathcal{J}(f_j)$. One now breaks up each $\mathcal{Z}_j$ as follows. For each $t = 0, 1, \dots, m_j$ set

$$\mathcal{W}_t(j) = \{\zeta \in \mathcal{Z}_j : dim\,\mathcal{J}(F_j(\zeta, \cdot)) = t \quad \text{at some point in } \mathcal{B}_j\}.$$

To state the main result of the analysis, one must first define certain indices which determine the growth or decay of certain integrals arising in the analysis. The definitions of the indices is in terms of the weights with respect to which $f_j(x_1, \dots, x_{n-1})$ is homogeneous.

Let $r_1(j), \dots, r_{n-1}(j)$ be the weights. Thus, for any $t > 0$,

$$f_j(t^{r_1(j)}x_1, \dots, t^{r_{n-1}(j)}x_{n-1}) = t f_j(x_1, \dots, x_{n-1}).$$

One can choose monomials for the basis of $\mathcal{J}(f_j)$ and determine their weights, where

$$\text{weight of } x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} = \sum r_u k_u.$$

Order the weights by magnitude and denote them by $0 < s_1(j) \leq s_2(j) \leq \dots \leq s_{m_j}(j)$. Since f_j is simple, $s_{m_j}(j) < 1$. Define

$$\mu_u(j) = \frac{1}{1 - s_u(j)}.$$

Secondly, define for each $j = 1, \dots, L$

$$\epsilon(j) = -\sum_{u=1}^{n-1} r_u(j).$$

The significance of this number is given by

PROPOSITION 3.22.

$$\epsilon(j) = inf\left\{\gamma : \int_{\mathcal{B}_j} e^{i\tau f_j(x)}\, b(x)\, dx = O(\tau^\gamma) \text{ for all } b \in C_c^\infty(\mathcal{B}_j)\right\}.$$

PROOF: This is proved in [A]. ▮

Similarly, define, using notation from (3.),

$$\epsilon_t(j) = sup\{\sigma_j(\zeta) : \zeta \in \mathcal{W}_t(j)\}. \tag{3.23.1}$$

Thirdly, define for each t and j,

$$\begin{aligned} \nu_t(j) &= sup_u\, \mu_u(j)\,[\epsilon(j) - \epsilon_t(j)], \\ \alpha'_t(j) &= (inf_u\, \mu_u)(m_j - \sum_u s_u(j)) - \nu_t(j) \end{aligned} \tag{3.23.2}$$

Finally, define by induction on the codimension the numbers $\alpha_t(j)$, for $t = 0, 1, \ldots, m_j$ as follows:

$$\alpha_{m_j}(j) = m_j \text{ for each } j.$$

Next, set

$\mathcal{A}_j = \{$ singularity classes adjacent to f_j whose codimension at some point of $\mathcal{B}_j = m_j - 1\}$.

Set for any $t < m_j$

$$\tilde{\alpha}_t(j) = inf\,\{\alpha_t(h) : h \in \mathcal{A}_j, h \neq f_j\}.$$

Then define

$$\alpha_t(j) = inf\,\{\alpha'_t(j), \tilde{\alpha}_t(j)\}.$$

Evidently, if $\mathcal{A}_j = \emptyset$, then $\alpha_t(j) = \alpha'_t(j)$ for all t.

The first table below gives the indices of the five singularity types defined above. The second table gives expressions of $\alpha'_t(j)$ in terms of $\epsilon_t(j)$ whenever f_j is one of these five classes. A useful exercise for the reader would be to verify as many of them as possible.

	m(class)	ϵ(class)	$sup_u\{\mu_u\}$	$inf_u\{\mu_u\}$	$m - \sum_u s_u$
A_{n+1}	n	$\frac{1}{2} - \frac{1}{n+2}$	$\frac{n+2}{2}$	$\frac{n+2}{n+1}$	$\frac{n(n+3)}{2(n+2)}$
D_{n+1}	n	$\frac{1}{2} - \frac{1}{2n}$	n	$\frac{n}{n-1}$	$\frac{n^2+1}{2n}$
E_6	5	$\frac{5}{12}$	6	$\frac{4}{3}$	$\frac{5}{2}$
E_7	6	$\frac{4}{9}$	9	$\frac{9}{7}$	$\frac{26}{9}$
E_8	8	$\frac{7}{15}$	15	$\frac{5}{4}$	$\frac{49}{15}$

In the following table, the notation ϵ_t, α'_t is now intended to be read as the value of the indices, defined in (3.23), when the singularity class of f_j is indicated on the left.

$$\begin{aligned}
A_{n+1}: &\quad \alpha'_t = \frac{n+2}{2}\epsilon_t + \frac{n(n+5)}{4(n+1)}, \\
D_{n+1}: &\quad \alpha'_t = n\epsilon_t + \frac{n}{n-1}, \\
E_6: &\quad \alpha'_t = 6\epsilon_t + \frac{5}{6}, \\
E_7: &\quad \alpha'_t = 9\epsilon_t - \frac{2}{7}, \\
E_8: &\quad \alpha'_t = 15\epsilon_t - \frac{35}{12},
\end{aligned}$$

Looking closely at these values and working out all the values of α_t, one will observe the following relation,

LEMMA 3.24. *For each t and j*

$$\epsilon_t(j) \leq \frac{m_j}{2(m_j+1)}\,\alpha_t(j).$$

where m_j is the codimension of f_j (the first column of table 1).

The point of this peculiar observation will be made clear at the end of the proof of Theorem 3.26.

The main analytic result in [CdV] consists of the following estimates. The proof will not be given here. Suffice it to say however that the assumption of weighted homogeneity of the defining equations f_j appears to be needed for certain delicate parts of the analysis.

THEOREM 3.25.

(1) *If f_j is one of the five classes written above, then there exists $C > 0$ such that for each $t = 0, 1, \ldots, m_j$ and each $b \in C_c^\infty(\mathcal{B}_j)$*

$$|I_j(\tau,\zeta,b)|_{\mathcal{W}_t(j)}| \leq C\tau^{-\frac{n-1}{2}+\epsilon_t(j)}\,\Theta_{t,j}(\zeta) \quad \text{for } \tau \text{ sufficiently large},$$

where the domain of $\Theta_{t,j}$ is $\mathcal{W}_t(j)$ and

$$\Theta_{t,j}(\zeta) = \prod_{u=1}^{m_j-t} d(\zeta, \mathcal{W}_{t+u}(j))^{-\nu_u(j)} \quad \text{for some positive } \nu_u(j).$$

(2) *If $t \neq 0$ when f_j is of type E_7, or t is arbitrary in all other cases, then*

$$\int_{\{\zeta: d(\zeta,\mathcal{W}_t(j))\leq h\}\cap supp(b)} |\Theta_{t,j}(\zeta)|\,d\zeta = O(h^{\alpha_t(j)}) \quad \text{as } h \to 0.$$

(3) *If $t = 0$ when f_j is type E_7 then one can choose for $\Theta_{0,j}(\zeta)$ a function of form*

$$\prod_{u=1}^{5} d(\zeta, \mathcal{W}_u(j))^{-\nu_u(j)}(\sum_i \zeta_i^{\mu_i(j)})^{-1/6} \qquad \textit{for some positive } \nu_u(j)\,.$$

Using Theorem 3.25, one can proceed to a sketch of the proof of one of the main results proved in [ibid] (in a different form than is given there). This theorem essentially says that the main source of error in $E_\epsilon(w)$ is the "elementary" estimate (3.7) when any possible singularity exhibited by φ is simple.

THEOREM 3.26. *Let P be a positive definite form on $\mathbf{R}^n$. Assume that the germ of the function*

$$(\xi, x) \in S^{n-1} \times \partial\mathcal{R}_1 \;\rightarrow\; \varphi(\xi, x) = (\xi, x)\big|_{\partial\mathcal{R}_1}$$

is C^∞ equivalent to the germ of a universal unfolding of a simple singularity at any singular point of $\varphi(\xi, \cdot)$. Then

$$E(w) = O(w^{\frac{n-2}{d} + \frac{2}{d(n+1)}}) \qquad w \rightarrow \infty.$$

PROOF: Denote the five simple singularity classes by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$. Suppose that the classes $\mathcal{C}_1$ resp. $\mathcal{C}_2$ contain all A_k resp. D_k. For any ξ and each point $x \in \Sigma(\varphi_\xi)$, the germ of $\varphi(\xi, \cdot)$ at x therefore belongs to exactly one of these classes.

For each $j = 1, \ldots, 5$ and $t = 0, 1, \ldots,$ set

$$\begin{aligned} W_t(j) = \{\xi \in S^{n-1} :\ & \text{the germ of } \varphi(\xi, \cdot) \text{ at some singular point has codimension } t \\ & \text{and this germ is adjacent to a singularity in class } \mathcal{C}_j\}. \end{aligned}$$

For a given $m \neq 0 \in \mathbf{Z}^n$, the vector ν_m can belong to at most 5 classes.

The discussion below will assume t is arbitrary for $\mathcal{C}_j \neq E_7$ and $t \neq 0$ if $\mathcal{C}_j = E_7$. The modifications needed in the only other case are left to the reader to discover.

Consider the cone over $W_t(j)$, denoted $\mathbf{R}^+\, W_t(j)$. Let

$$\begin{aligned} \mathcal{B}_t(j) &= \{y \in \mathbf{R}^n : d(y, \mathbf{R}^+\, W_t(j)) \leq 1\}\,, \\ \mathcal{B}'_t(j) &= \{y \in \mathbf{R}^n : d(y, \mathbf{R}^+\, W_t(j)) \leq 3/2\}\,, \end{aligned}$$

and set

$$\begin{aligned} \mathcal{A}_t(j) &= \mathcal{B}_t(j) - \cup_{q>t}\mathcal{B}_q(j) \\ &= \{y \in \mathbf{R}^n : d(y, \mathbf{R}^+\, W_t(j)) \leq 1,\ d(y, \mathbf{R}^+\, W_{t+i}(j)) > 1,\ i > 0\}\,. \end{aligned}$$

Combining this with (3.25) one gets

$$|\hat{\chi}(w(m)\nu_m)| \leq w(m)^{-\frac{(n+1)}{2} + \epsilon_t(j)}\, \Theta_t(\nu_m) \quad \text{if } \nu_m \in W_t(j)\,.$$

One next shows the following two estimates, whose proofs are left to the reader.

LEMMA 3.27. *For each t there exists $C_t > 0$ such that for any $\xi \in \mathcal{A}_t(j)$, and each $\xi' \in \mathcal{A}_t(j)$ for which $d(\xi, \xi') \le 1/2$, one has*

$$\Theta_{t,j}(\nu_\xi) \le C_t\, \Theta_{t,j}(\nu_{\xi'})\,.$$

LEMMA 3.28. *Assume that $\epsilon = w^{-\frac{q}{d}}$. Then for all $N \gg 1$*

$$\hat{\rho}(\epsilon\, z) = O\left([1 + w^{-\frac{q}{d}}\|z\|]^{-N}\right)\,.$$

Combining these two lemmas implies that if $\nu_m \in W_t(j)$, then

$$|\hat{\chi}(w(m)\nu_m)\,\hat{\rho}(\epsilon m)| \le C_t \int_{B_{\frac{1}{2}}(m)} w(z)^{-\frac{(n+1)}{2}+\epsilon_t(j)}\,\Theta_{t,j}(\nu_z)\,(1 + w^{-\frac{q}{d}}\|z\|)^{-N}\,dz\,.$$

Since $B_{\frac{1}{2}}(m) \subset \mathcal{B}'_t \cap \{\|z\| \ge 1/2\}$ if $m \in \mathcal{A}_t(j)$, one concludes

$$\begin{aligned}\sum_{m\in\mathcal{A}_t(j)} |\hat{\chi}(w(m)\nu_m)\,\hat{\rho}(\epsilon m)| &\le C_t \int_{\mathcal{B}'_t(j)\cap\{\|z\|\ge 1/2\}} w(z)^{-\frac{(n+1)}{2}+\epsilon_t(j)}\,\Theta_{t,j}(\nu_z)\,(1 + w^{-\frac{q}{d}}\|z\|)^{-N}\,dz \qquad (3.29)\\ &= C_t w^{-\frac{(n+1)}{2d}+\epsilon_t(j)} \int_{\mathcal{B}'_t(j)\cap\{\|z\|\ge 1/2\}} \|z\|^{-\frac{(n+1)}{2}+\epsilon_t(j)}\,(1 + w^{-\frac{q}{d}}\|z\|)^{-N}\,\Theta_{t,j}(\nu_z)\,dz\,.\end{aligned}$$

Now convert to polar coordinates (r, θ). The second integral in (3.29) now becomes bounded from above by the product of a constant, possibly different than C_t, and

$$\int_{1/2}^{\infty} r^{\frac{n-3}{2}+\epsilon_t(j)} \left(\int_{\{\xi\in S^{n-1}: d(\xi, W_t(j))\le 3/2r\}} \Theta_{t,j}(\theta)\,d\theta\right)(1 + w^{-\frac{q}{d}}r)^{-N}\,dr$$

which is, by parts 2,3 of (3.25),

$$\le C' \int_{1/2}^{\infty} r^{\frac{n-3}{2}+\epsilon_t(j)-\alpha_t(j)}\,(1 + w^{-\frac{q}{d}}r)^{-N}\,dr\,.$$

An estimate in w of this latter integral is given by

LEMMA 3.30. *For any positive c*

$$\int_c^\infty r^\alpha(1 + w^{-\beta}r)^{-N}\,dr = O(w^{\beta(\alpha+1)}) \quad \text{as } w \to \infty,$$

when $N \gg 1, \alpha > -1, \beta > 0$.

PROOF: Split the integral into one over the interval $[c, w^\beta]$ and the second over $[w^\beta, \infty)$. Over each interval, one gets the $O(w^{\beta(\alpha+1)})$ by an elementary calculation. ∎

Thus, there exists C'_t such that

$$w^{n/d} \sum_{m \in \mathcal{A}_t(j)} |\hat{\chi}(w(m)\nu_m)\hat{\rho}(\epsilon m)| \leq C'_t w^{\frac{n-1}{2d} + \epsilon_t(j) + \frac{\alpha}{d}[\frac{n-1}{2} + \epsilon_t(j) - \alpha_t(j)]} . \tag{3.31}$$

Using the value for α indicated by (3.8), one now checks that the exponent of w in (3.31) will be at most the "elementary estimate", stated in the theorem, iff

$$\epsilon_t(j) \leq \frac{n-1}{2n} \alpha_t(j).$$

Since the function $v \to (v-1)/2v$ is increasing, and $m_j \leq n$, Lemma 3.24 shows that this inequality holds for any simple singularity. Finally, since each possible m is included in at most 5 different $\mathcal{A}_t(j)$, the series defining $E_\epsilon(w, \beta), \beta$ defined by (3.8), is bounded from above by the sum

$$w^{n/d} \sum_{j=1}^{5} \sum_t \sum_{m \in \mathcal{A}_t(j)} |\hat{\chi}(w(m)\nu_m)\hat{\rho}(\epsilon m)|.$$

So, the theorem follows. ∎

Additional Remarks.

(1) The same proof with obvious modifications will work if P is weighted homogeneous and positive definite on $\mathbf{R}^n$ (like $x^4 + y^6$).

(2) If one wants to introduce weights, determined by a rational function φ, defined everywhere on $\mathbf{R}^n$, and thereby compare $N_P(w, \varphi)$ and $V_P(w, \varphi)$, the first problem encountered is determining the order in w of $V_P(w, \varphi)$. This of course is clear if $\varphi \equiv 1$. For φ not constant, in general it is not so clear how one can find a "main term" in the asymptotic of $V_P(w, \varphi)$. Doing this is essential before one even begins to look for an error term.

Suppose one can decompose φ by degree, $\varphi = \varphi_r + \varphi_{r-1} + \dots$ where

$$\varphi_i(tx_1, \dots, tx_n) = t^i \varphi(x_1, \dots, x_n).$$

Then it is clear that

$$V_P(w, \varphi) = \sum_{i \leq r} \left(\int_{\mathcal{R}_1} \varphi_i \, dx \right) w^{\frac{n+i}{d}}.$$

Thus, if

$$\int_{\mathcal{R}_1} \varphi_r \, dx \neq 0 \tag{3.32}$$

then the order one wants the error term to grow less than should evidently be $(n+r)/d$. Using the same arguments and notations as in this section, the error term can be written

$$w^{\frac{n+r}{d}} \sum_{m \neq 0} \widehat{\chi \cdot \varphi}(w(m)\nu_m) \, \hat{\rho}(\epsilon m).$$

Since the exponents used in this section are by definition uniform in the amplitudes, the effect of introducing φ is *not* detected in the estimation of this series. As a result, whenever (3.32) holds, one is assured of a lower order in w for $E(w,\varphi)$, that is, the lattice point problem with weight determined by φ is geometric. This may also be the case if (3.32) fails. However, the smaller the largest degree is of that homogeneous term of φ for which (3.32) holds, the bigger is the estimate for the error relative to $V_P(w,\varphi)$. In particular, it becomes increasingly less clear that the error does not begin to overwhelm the main term from $V_P(w,\varphi)$. This is the main danger of not having (3.32) satisfied for φ_r.

(3) Whenever (3.32) does hold, the asymptotic

$$N_P(w,\varphi) - \sum_{i\leq r}\left(\int_{\mathcal{R}_1} \varphi_i\,dx\right) w^{\frac{n+i}{d}} = O(w^{\frac{n+r}{d}-\theta})$$

implies that

$$\int_{\mathcal{R}_1} \varphi_r\,dx - w^{-\frac{n+r}{d}} N_P(w,\varphi) = O(w^{-\theta'}).$$

Thus, an integral over $\mathcal{R}_1$ is approximated by a finite sum, depending upon a parameter, and the difference goes to zero as the parameter increases without bound. The finite sum is over points in $\mathcal{R}_1$ of the form $m/w, m \in \mathbf{Z}^n$. The rate with which the error decays has been approximated by local analysis using singularity theory. This technique should be of interest in numerical integration.

(4) The conditions needed to be verified in order that Theorem 3.26 holds are not at all easy to verify, simply starting from P, unless $n = 2$. Here, one can only encounter an A_k singularity and the value of k equals the order of vanishing of the curvature along the curve $\partial\mathcal{R}_1$ at the singular points.

From this point of view, it is preferable to analyze the lattice point problem from a different point of view, one in which the error term can always be shown to be of lower order, although an explicit estimate may be harder to give. Because such a procedure would be more general, one should expect the size of the error to be bigger. The analytical methods of Part 2 do this. An important goal in the analysis of these types of problems is to combine (and considerably extend) the geometric ideas of Section 3 with the analytical ones from Part 2, with the hope of understanding much better the error, for classes of polynomials P and weights φ.

Part 2. Functional methods for some GLPs

In this part, $x = (x_1, \ldots, x_n)$ denotes a variable point of $[0,\infty)^n$ for section 4 and $[1,\infty)^n$ for subsequent sections. One sets $\|x\| = max\,\{x_i\}$, and $dx = dx_1 \ldots dx_n$. All polynomials will be assumed to be defined over $\mathbf{R}$.

Section 4. *Elliptic polynomials*

Given $P(x)$ of degree d, write

$$P(x) = P_d(x) + P_{d-1}(x) + \ldots + P_1(x) + P_0, \qquad \deg P_j = j \text{ for each } j\,.$$

Definition 4.1. P is elliptic on $[0,\infty)^n$ if for all x

$$|P_d(x)| > 0 \quad \text{and} \quad |P(x)| > 0\,. \quad \blacksquare$$

The condition on $|P|$ is not essential but does simplify the discussion below. The reader will easily be able to modify the discussion if $|P(x)|$ is allowed to vanish in $[0,\infty)^n$.

Remark. An interesting example of an elliptic polynomial arises in number theory. Given a number field K, a "ray class" C of K is an equivalence class of ideals defined by the relation $\mathcal{A} \sim \mathcal{B}$ iff $\mathcal{A} = (\alpha)\mathcal{B}$, where $\alpha \in K$ is "totally positive", that is, each real embedding of K maps α to a positive number. Shintani showed [Sh] that each ray class zeta function can be expressed as a finite sum of Dirichlet series $D_P(s,\varphi)$, of the type defined prior to Proposition 4.9, where each polynomial P is elliptic on $[0,\infty)^n$. (Although he did not explicitly state this, it is clear from his construction, that the "norm form" polynomials appearing in his analysis, are in fact elliptic and defined over $\mathbf{R}$.) A consequence of Theorem 4.10, proved in this section, is then a completely "elementary" analytic continuation of any ray class zeta function. $\blacksquare$

Let $\varphi = Q/T$ where Q, T are also elliptic on $[0,\infty)^n$. Set

$$e = deg\,Q, f = deg\,T, \delta = e - f.$$

For $w > 0$ set

$$N(w,\varphi) = \sum_{\{m \in \mathbf{N}^n : |P(m)| \le w\}} \varphi(m) \qquad \text{and} \quad V(w,\varphi) = \int_{\{|P| \le w\} \cap [1,\infty)^n} \varphi\, dx.$$

This section will show

THEOREM 4.2. *(Mahler) There exists $\theta > 0$ such that*

$$N(w,\varphi) - V(w,\varphi) = O\left(V(w,\varphi)/w^{\theta}\right) .$$

That is, the lattice point problem with $\mathcal{R}_w = \{x \in [1,\infty)^n : |P(x)| \le w\}$ is a GLP whenever P, Q, T are elliptic on $[0,\infty)^n$.

Note. The discussion below will assume that $P_d, T_f, Q_e > 0$ and $P > 0$ on $[0,\infty)^n$. The trivial changes needed if some of the signs are negative are left to the reader. ∎

Simple growth estimates become available immediately from the ellipticity condition. First, ellipticity implies the existence of $c_1 > 0$ such that

$$inf\, \{P_d(x), Q_e(x), T_f(x)\} > c_1 \quad \text{for all } x \in S^{n-1} \cap [0,\infty)^n.$$

Thus,

PROPOSITION 4.3. *For all $x \in [0,\infty)^n$*

$$\begin{aligned} P_d(x) &\ge c_1 \|x\|^d \\ Q_e(x) &\ge c_1 \|x\|^e \\ T_f(x) &\ge c_1 \|x\|^f . \quad ∎ \end{aligned}$$

Secondly, writing

$$\begin{aligned} p(x) &= \sum_{i=0}^{d-1} \frac{P_i(x)}{P_d(x)} \\ q(x) &= \sum_{i=0}^{e-1} \frac{Q_i(x)}{Q_e(x)} \\ t(x) &= \sum_{i=0}^{f-1} \frac{T_i(x)}{T_f(x)} , \end{aligned}$$

one sees that for each x

$$\begin{aligned} P(x) &= P_d(x)\,[1 + p(x)] , \\ Q(x) &= Q_e(x)\,[1 + q(x)] , \\ T(x) &= T_f(x)\,[1 + t(x)] . \end{aligned}$$

It is clear that

$$p(x), q(x), t(x) = O\left(\frac{1}{\|x\|}\right) . \tag{4.4}$$

Thus,

PROPOSITION 4.5. *There exists $C > 0$ such that for all $x \in [0,\infty)^n$*

$$P(x) \geq C\|x\|^d$$
$$Q(x) \geq C\|x\|^e$$
$$T(x) \geq C\|x\|^f\,.$$

PROOF: The proof will be given only for P and the similar details for the other polynomials are left to the reader.

There exists $R > 0$ such that $\|x\| \geq R$ implies $|p(x)| \leq 1/2$. Thus,

$$\left|\,\frac{P(x)}{P_d(x)} - 1\,\right| < \frac{1}{2}\,.$$

So, $\|x\| \geq R$ implies

$$P(x) \geq P_d(x)/2 \geq \frac{c_1}{2}\|x\|^d\,.$$

On the compact set $[0,\infty)^n \cap \{\|x\| \leq R\}$, there exists $c_2 > 0$ such that $P(x) \geq c_2$. Thus, there exists $c_3 > 0$ such that $c_2 \geq c_3\|x\|^d$ if $\|x\| \leq R$. Now set $C = min\{c_3, c_1/2\}$ to prove the proposition. ∎

Remark. Of course one also has upper bounds of exactly the same form with a possibly different constant. ∎

Thirdly, if $x \in [1,\infty)^n$ and ω denotes any point in $S^{n-1} \cap [0,\infty)^n$ then there exist smooth functions $u_j(\omega), j = 1,\ldots,n$ such that

$$x_j = 1 + ru_j(\omega) \quad j = 1,\ldots,n,$$

defines a coordinate change whose jacobean equals $r^{n-1}dr d\omega$. Set $\overline{1} = (1,\ldots,1)$. The point $u(\omega) = (u_1(\omega),\ldots,u_n(\omega))$ is denoted by ω unless it is helpful to emphasize otherwise. In these coordinates one writes

$$\begin{aligned} P(x) &= P(\overline{1} + r\omega) = P_d(\overline{1} + r\omega)\left(1 + p(\overline{1} + r\omega)\right) \\ &= r^d P_d(\omega + \overline{1}/r)\left(1 + p(\overline{1} + r\omega)\right) \\ Q(x) &= r^e Q_e(\omega + \overline{1}/r)\left(1 + q(\overline{1} + r\omega)\right)\,, \\ T(x) &= r^f T_f(\omega + \overline{1}/r)\left(1 + t(\overline{1} + r\omega)\right)\,. \end{aligned} \tag{4.6}$$

The functions

$$Q_e(\omega + \overline{1}/r)\left(1 + q(\overline{1} + r\omega)\right) \quad T_f(\omega + \overline{1}/r)\left(1 + t(\overline{1} + r\omega)\right)$$

are necessarily smooth functions on $[1,\infty) \times S^{n-1}$ with values in some interval *not* containing the origin.

Note. Introduce the notation. Given the multiindex $J = (j_1, \ldots, j_n)$,

$$|J| = j_1 + \ldots + j_n, \qquad J! = j_1! \cdots j_n!,$$
$$P_d^{(J)}(\omega) = D_{x_1}^{j_1} \cdots D_{x_n}^{j_n} P_d(\omega). \quad \blacksquare$$

One now expands out the P_d factor using Taylor's formula (in the original x coordinates)

$$\begin{aligned} P_d(\omega + \bar{1}/r) &= P_d(\omega) + \sum_{1 \le |J| \le d} \frac{P_d^{(J)}(\omega)}{J!} \cdot \left(\frac{1}{r}\right)^{|J|} \\ &= P_d(\omega) \left[1 + \sum_{1 \le |J| \le d} \frac{P_d^{(J)}(\omega)}{J! P_d(\omega)} \cdot \left(\frac{1}{r}\right)^{|J|} \right]. \end{aligned}$$

Ellipticity of P implies $P_d(\omega) > 0$ for each ω. Now define

$$\mathcal{P}(r, \omega) = \sum_{1 \le |J| \le d} \frac{P_d^{(J)}(\omega)}{J! P_d(\omega)} \cdot \left(\frac{1}{r}\right)^{|J|}.$$

It is clear there exists $B > 0$ such that for each J

$$\left| \frac{P_d^{(J)}(\omega)}{J! P_d(\omega)} \right| < B \qquad \text{for all } \omega \in S^{n-1} \cap [0, \infty)^n.$$

Thus, since each $|J| \ge 1$, one concludes

$$\mathcal{P}(r, \omega) = O\left(\frac{1}{r}\right). \tag{4.7}$$

Moreover, combining (4.4) and (4.7), one observes the existence of $R > 0$ such that $r \ge R$ implies for all $\omega \in S^{n-1} \cap [0, \infty)^n$

$$|p(r, \omega)| \le 1/2 \quad \text{and} \quad |\mathcal{P}(r, \omega)| \le 1/2. \tag{4.8.1}$$

Thus, the definitions of $p, \mathcal{P}$ imply

$$\begin{aligned} p(\bar{1} + r\omega) &= \frac{1}{r} \cdot \sum_{i=0}^{\infty} \frac{A_i(\omega)}{r^i} \\ \mathcal{P}(\bar{1} + r\omega) &= \frac{1}{r} \cdot \sum_{i=0}^{d-1} \frac{B_i(\omega)}{r^i}, \end{aligned} \tag{4.8.2}$$

where each $A_i(\omega), B_i(\omega)$ is a rational function of $u(\omega)$ that is bounded over S^{n-1}, and the series converges absolutely and uniformly over $[R, \infty) \times S^{n-1}$.

Theorem 4.2 follows from the more precise result Theorem 4.10. To formulate this, one introduces the two functions

$$D_P(s,\varphi) = \sum_{m\in\mathbf{N}^n} \frac{\varphi(m)}{P(m)^s},$$
$$I_P(s,\varphi) = \int_{[1,\infty)^n} \frac{\varphi(x)}{P(x)^s}\,dx\,.$$

One first observes

PROPOSITION 4.9. *$D_P(s,\varphi), I_P(s,\varphi)$ are analytic if $\sigma > (n+\delta)/d$.*

PROOF: Restricted to $[1,\infty)^n$, the growth of P, Q, T given in (4.5) implies that for any $x \in [1,\infty)^n$

$$P(x) > C(x_1\cdots x_n)^{d/n},$$
$$\varphi(x) < C(x_1\cdots x_n)^{\delta/n}.$$

Thus,

$$\left|\int_{[1,\infty)^n} \frac{\varphi}{P^s}\,dx\right| \le \int_{[1,\infty)^n} \frac{\varphi}{P^\sigma}\,dx$$
$$< C^{-\sigma}\int_{[1,\infty)^n} (x_1\cdots x_n)^{\frac{-d\sigma+\delta}{n}}\,dx\,.$$

Clearly, if $(-d\sigma+\delta)/n < -1$, then the integral converges. This implies $I_P(s,\varphi)$ is analytic if $\sigma > (n+\delta)/d$, as claimed. The proof for absolute convergence of $D_P(s,\varphi)$ in this halfplane is similarly easy and left to the reader. ∎

THEOREM 4.10.

(1) *$I_P(s,\varphi)$ admits an analytic continuation as a meromorphic function on $\mathbf{C}$.*
(2) *The first pole of $I_P(s,\varphi)$ is simple and occurs at $s = (n+\delta)/d$.*
(3) *There exists $M > 0$ such that for any $\sigma_1 < \sigma_2 \le (n+\delta)/d$ and $\epsilon > 0$ then there exists a constant $C = C(\epsilon,\sigma_1,\sigma_2)$ so that*

$$|I_P(s,\varphi)| < C\left(1 + |t|^{M(\frac{n+\delta}{d}-\sigma)+\epsilon}\right),$$

for all $\sigma \in [\sigma_1,\sigma_2]$ and $|t| \ge 1$.

(4) *$D_P(s,\varphi)$ admits an analytic continuation as a meromorphic function on $\mathbf{C}$.*
(5) *The first pole of $D_P(s,\varphi)$ is simple and occurs at $s = (n+\delta)/d$.*
(6) *There exists $\mu > 0$ such that if $\sigma_1 < \sigma_2 \le (n+\delta)/d$ and $\epsilon > 0$ then there exists $C' = C'(\epsilon,\sigma_1,\sigma_2)$ so that*

$$|D_P(s,\varphi)| < C'\left(1 + |t|^{\mu(\frac{n+\delta}{d}-\sigma)+\epsilon}\right),$$

for all $\sigma \in [\sigma_1,\sigma_2]$ and $|t| \ge 1$.

(7) *$D_P(s,\varphi) - I_P(s,\varphi)$ is analytic at $s = (n+\delta)/d$.*

Remark. The estimate in part (3) is much weaker than what can be proved using more detailed arguments. The argument used to establish (6) shows that one can take $\mu = nd$. The proof of (4.10) will follow Mahler's with some (evident) modifications, needed to deal with $I_P(s, \varphi)$, rather than the integral taken over all of $[0, \infty)^n$, which is what Mahler used. ∎

PROOF OF "THEOREM 4.10 IMPLIES THEOREM 4.2": The needed observation comes from the pair of equations, valid for any $c > n/d$:

$$N_P(w, \varphi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} D_P(s, \varphi) w^s \frac{ds}{s}, \tag{4.11.1}$$

$$V_P(w, \varphi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} I_P(s, \varphi) w^s \frac{ds}{s}. \tag{4.11.2}$$

(4.11.1) is Perron's formula, cf. (B-1). The proof of the second equation is an exercise with Mellin inversion and uses the existence of $\gamma > 0, k \in \mathbf{Z}_+, B \neq 0$, such that

$$V_P(w, \varphi) \sim B w^\gamma log^k w,$$

as well as the fact that $V_P(w, \varphi) = 0$ for $y \ll 1$, a corollary of (4.5).

Set

$$\nu_P(w, \varphi) = \text{ right side of (4.11.2).}$$

Then Mellin inversion implies that

$$I_P(s, \varphi) = s \int_0^\infty w^{-s} \nu_P(w, \varphi) \frac{dw}{w}.$$

A well known identity states that for $w > 0$ a regular value of P,

$$V_P'(w, \varphi) = \int_{\{P=w\} \cap [1,\infty)^n} \frac{\varphi\, dx}{dP},$$

where $\varphi\, dx/dP$ denotes the Leray residue of $\varphi\, dx/(P-w)$ along $\{P = w\}$. Thus, for $\sigma \gg 1$,

$$\begin{aligned} I_P(s, \varphi) = \int_0^\infty w^{-s} V_P'(w, \varphi) dw &= w^{-s} V_P(w, \varphi)\Big|_0^\infty + s \int_0^\infty w^{-s} V_P(w, \varphi) \frac{dw}{w} \\ &= s \int_0^\infty w^{-s} V_P(w, \varphi) \frac{dw}{w}. \end{aligned}$$

So, $\nu_P(w, \varphi)$ and $V_P(w, \varphi)$ are both the Mellin inversion of $I_P(s, \varphi)/s$ and must therefore be equal.

Now (4.11.1), (4.11.2), combined with (4.10) and the standard residue argument used in Part 1, evidently shows (4.2). ∎

The rest of Section 4 will be devoted to proving Theorem 4.10 along the lines of Mahler's original argument. It is hopefully instructive for the reader to compare this "elementary" argument with the general argument of section 5.

PROOF OF PARTS (1)-(3): Choose R so that (4.8.1) is satisfied. Then

$$I_P(s,\varphi) = I_1(s) + I_2(s),$$

where

$$I_1(s) = \int_{S^{n-1}\cap[0,\infty)^n} \int_R^\infty \frac{r^{n-1}\varphi(r,\omega)}{P(r,\omega)^s}\, dr\, d\omega$$

$$I_2(s) = \int_{S^{n-1}\cap[0,\infty)^n} \int_0^R \frac{r^{n-1}\varphi(r,\omega)}{P(r,\omega)^s}\, dr\, d\omega\,.$$

By (4.1) and (4.5), P is never zero over the compact subset of $\mathbf{R}^n$ that is the domain of integration for $I_2(s)$. Thus, it is clear that $I_2(s)$ is entire. Moreover, the growth estimate asserted in part (3) of (4.10), is easily verified for $I_2(s)$. So, it suffices to argue for $I_1(s)$.

To do this, one writes P as in (4.6) and expands out the factors $(1+p)^{-s}$, and $(1+\mathcal{P})^{-s}$ via Taylor's formula with remainder. For each integer $h \geq 1$,

$$(1+p)^{-s} = \sum_{i=0}^{h} \binom{-s}{i} p^i + h\binom{-s}{h} p^h \cdot \int_0^1 (1-v)^{h-1}(1+vp)^{-s-h}\, dv\,,$$

$$(1+\mathcal{P})^{-s} = \sum_{i=0}^{h} \binom{-s}{i} \mathcal{P}^i + h\binom{-s}{h} \mathcal{P}^h \cdot \int_0^1 (1-v)^{h-1}(1+v\mathcal{P})^{-s-h}\, dv.$$

Using (4.8.2), one derives an explicit expression of each power of $\mathcal{P}$ resp. p as a polynomial in $1/r$ resp. an absolutely convergent series in $1/r$ that converges uniformly over S^{n-1}. Moreover, one has the estimates which are uniform in ω

$$|p^h| \ll r^{-h} \quad \text{and} \quad |\mathcal{P}^h| \ll r^{-h}\,.$$

A simple calculation also shows that the only term without a factor of $1/r$ in $(1+\mathcal{P})^{-s}(1+p)^{-s}$ is the constant 1. Thus,

$$P(\overline{1}+r\omega)^{-s} = r^{-ds} P_d(\omega)^{-s} \left[1 + \sum_{i=1}^{h-1} C_i(s,\omega) r^{-i} + O_{s,\omega}(r^{-h})\right] \tag{4.13}$$

where

(1) Each $C_i(s,\omega)$ is polynomial in s, rational in $u(\omega)$, and bounded over S^{n-1}.
(2) $O_{s,\omega}(r^{-h})$ denotes a function that is polynomial in s, rational in $u(\omega)$, bounded over S^{n-1}, and which satisfies the estimate:
there exists $C > 0$ such that for $|s| \geq 1, r \geq R$,

$$|O_{s,\omega}(r^{-h})| \leq C|s|^h r^{-h} \tag{4.14}$$

uniformly in ω.

It follows from a simple argument that uses the polar coordinate expressions for Q, T that (4.13) gives an analytic continuation of $I_1(s)$ into the region

$$\sigma > \frac{n+\delta-h}{d}$$

as a meromorphic function with a simple pole at $s = (n+\delta)/d$ with residue equal to

$$\int_{S^{n-1}\cap[0,\infty)^n} \frac{Q_e(\omega)}{T_f(\omega)} \cdot P_d^{-n/d}(\omega)\, d\omega\,,$$

and with any other pole possibly occurring at $s = \frac{n+\delta-i}{d}, i = 1, \ldots, h-1$. The growth estimate of part (3) follows easily from (4.14) by choosing $M = d$, and using a simple argument of convexity, based upon Phragman-Lindelöf. An argument of this type is detailed in [Sa-2, 6.1]. This completes the proof of parts (1)-(3) of Theorem 4.10.

Needed for the rest of the proof of (4.10) is the following simple extension of (1)-(3).

LEMMA 4.15. *For any differential monomial $D_x^J, |J| \geq 1$, and $F = P, Q, T$,*

$$\lim_{\|x\|\to\infty} \frac{D_x^J F(x)}{F(x)} = 0\left(\frac{1}{\|x\|^{|J|}}\right).$$

PROOF: Since $D_x^J F$ is a sum of polynomials of degree at most $deg\, F - |J|$, the asserted limit follows immediately from (4.5). ∎

Now, for multiindices $B_1, \ldots, B_{\ell_1}, C_1, \ldots, C_{\ell_2} \neq 0$, set $\boldsymbol{B} = (B_1, \ldots, B_{\ell_1})$, and $\boldsymbol{C} = (C_1, \ldots, C_{\ell_2})$. Define $|\boldsymbol{B}| = |B_1| + \ldots + |B_{\ell_1}|$ with the same definition for $|\boldsymbol{C}|$. Now define

$$I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi) = \int_{[1,\infty)^n} \prod_{i=1}^{\ell_1} \left[\frac{D_x^{B_i}\varphi}{\varphi}\right] \cdot \prod_{i=1}^{\ell_2} \left[\frac{D_x^{C_i} P}{P}\right] \cdot \frac{\varphi}{P^s}\, dx\,.$$

The proof of parts (1), (3) of (4.10) extend straightforwardly to each $I_{\boldsymbol{B},\boldsymbol{C}}(s)$. This is left to the reader to establish. Of particular interest here however is the approximation of the first pole.

PROPOSITION 4.16. *The first possible pole of any $I_{\boldsymbol{B},\boldsymbol{C}}(s)$ must be smaller than $(n+\delta)/d$. In particular, $I_{\boldsymbol{B},\boldsymbol{C}}(s)$ is analytic if $\sigma > \frac{n+\delta-|\boldsymbol{B}|-|\boldsymbol{C}|}{d}$.*

PROOF: It suffices to show that $I_{\boldsymbol{B},\boldsymbol{C}}(s)$ is analytic in a halfplane containing $\sigma > (n+\delta)/d$. Combining Propositions 4.9 and (4.15), it is clear that there exists $C > 0$ such that for all $x \in [1,\infty)^n$

$$\left|\prod_{i=1}^{\ell_1} \left[\frac{D_x^{B_i}\varphi}{\varphi}\right] \cdot \prod_{i=1}^{\ell_2} \left[\frac{D_x^{C_i} P}{P}\right] \cdot \frac{\varphi}{P^s}\right| \leq C\|x\|^{-|\boldsymbol{B}|-|\boldsymbol{C}|-d\sigma+\delta} \leq C(x_1\cdots x_n)^{\frac{-|\boldsymbol{B}|-|\boldsymbol{C}|-d\sigma+\delta}{n}}\,.$$

Thus, $I_{B,C}(s)$ is analytic if $\sigma > \frac{n+\delta-|B|-|C|}{d}$. ∎

PROOF OF PARTS (4)-(7): These will follow easily from (1)-(3) and the Euler-Maclaurin formula extended to n variables. Thus, one must insure that the decay condition (C-4) is satisfied by $D_x^I(\varphi/P^s)$ for any index I. An elementary exercise, left to the reader, shows that this function can be written as a linear combination of all possible functions of the form

$$\prod_{i=1}^{\ell_1}\left[\frac{D_x^{B_i}\varphi}{\varphi}\right]\cdot\prod_{i=1}^{\ell_2}\left[\frac{D_x^{C_i}P}{P}\right]\cdot\frac{\varphi}{P^s} \tag{4.17}$$

where

$$B_1+\ldots+B_{\ell_1}+C_1+\ldots+C_{\ell_2}=I$$

and the coefficients are polynomials in s with integral coefficients. Thus, (4-16) implies that (C-4) is satisfied over the interval $[1,\infty)$ in each coordinate plane.

As discussed in Appendix C, the Euler-Maclaurin summation formula constructs for each $k=1,2,\ldots$, numbers $c_\ell(k)$, $i=0,1,\ldots,k-1$, and a periodic C^∞ function $\sigma_k(u)$, where u denotes a coordinate on $\mathbf{R}$, so that if $f(u+iv)$ is any holomorphic function satisfying (C-4), then

$$\sum_{\nu=1}^{\infty} f(\nu)=\int_1^\infty f(u)du+\sum_{i=0}^{k-1}c_i(k)f^{(i)}(1)+\int_1^\infty \sigma_k(u)f^{(k)}(u)\,du\,.$$

The precise values of the $c_i(k)$ and expressions of σ_k are given in the appendix.

Set, for each $k=1,2,\ldots$ and $i=0,1,\ldots,k-1$

$$h_i^{(k)}(u)=c_i(k)$$
$$h_k^{(k)}(u)=\sigma_k(u)\,.$$

One now sets $k=1$ and iterates the Euler-Maclaurin summation formula n-times to show

PROPOSITION 4.18. *If $\sigma>(n+\delta)/d$, and $I=(i_1,\ldots,i_n)$, then*

$$\begin{aligned} D_P(s,\varphi) &= \sum_{i_1=0}^{1}\cdots\sum_{i_n=0}^{1}\int_{[1,\infty)^n} h_{i_1}^{(1)}(x_1)\cdots h_{i_n}^{(n)}(x_n)D_x^I(\frac{\varphi}{P^s})\,dx \\ &= I_P(s,\varphi)+\sum_{\substack{i_1,\ldots,i_n=0\\ I\neq 0}}^{1}\int_{[1,\infty)^n} h_{i_1}^{(1)}(x_1)\cdots h_{i_n}^{(1)}(x_n)D_x^I(\frac{\varphi}{P^s})\,dx. \end{aligned}$$ ∎

Moreover, one observes that Proposition 4.16 implies that for each $I\neq(0,\ldots,0)$,

$$\int_{[1,\infty)^n} h_{i_1}^{(1)}(x_1)\cdots h_{i_n}^{(1)}(x_n)D_x^I(\frac{\varphi}{P^s})\,dx \quad \textit{is analytic if}\quad \sigma>\frac{n+\delta-1}{d}\,. \tag{4.19}$$

One can then repeat this procedure $k>1$ times. In this way one proves

PROPOSITION 4.20. *If $\sigma > (n+\delta)/d$ then*

$$D_P(s,\varphi) = \sum_{i_1=0}^{k} \cdots \sum_{i_n=0}^{k} \int_{[1,\infty)^n} h_{i_1}^{(k)}(x_1)\cdots h_{i_n}^{(k)}(x_n) D_x^I\left(\frac{\varphi}{P^s}\right) dx\,. \tag{1}$$

Thus, there exist constants $c(I)$ for each $I \neq (0,\ldots,0) \in \mathcal{I}_k'$ so that

$$\begin{aligned} D_P(s,\varphi) = I_P(s,\varphi) + \sum_{I\neq(0,\ldots,0)\in\mathcal{I}_k'} c(I) \int_{[1,\infty)^n} D_x^I\left(\frac{\varphi}{P^s}\right) dx \\ + \sum_{I\in\mathcal{I}_k''} \int_{[1,\infty)^n} h_{i_1}^{(k)}(x_1)\cdots h_{i_n}^{(k)}(x_n) D_x^I\left(\frac{\varphi}{P^s}\right) dx\,. \end{aligned} \tag{2}$$

As in (4.19) one observes that for any $I \in \mathcal{I}_k''$

$$\int_{[1,\infty)^n} h_{i_1}^{(k)}(x_1)\cdots h_{i_n}^{(k)}(x_n) D_x^I\left(\frac{\varphi}{P^s}\right) dx \quad \textit{is analytic if} \quad \sigma > \frac{n+\delta-k}{d}\,. \tag{4.21}$$

Thus, formula (1), combined with the analytic continuability of each $I_{\mathbf{B},\mathbf{C}}(s)$, determines an analytic continuation of $D_P(s,\varphi)$ into the region $\sigma > \frac{n+\delta-k}{d}$ as a meromorphic function.

By Proposition 4.16, it is then clear that $D_P(s,\varphi)$ must have a pole at $s = (n+\delta)/d$ and $D_P(s,\varphi) - I_P(s,\varphi)$ must be analytic at $s = (n+\delta)/d$. This proves (4), (5), (7) of the theorem. In any vertical strip of finite width, contained in $|t| \geq 1$, the proof of part (3), applied to each $I_{\mathbf{B},\mathbf{C}}(s)$ implies immediately the polynomial growth of $D_P(s,\varphi)$ in any such strip. This proves (6) and completes the proof of Theorem 4.10. ∎

Additional Remarks.

(1) The extension of Mahler's argument to elliptic polynomials on $\mathbf{R}^n$ is straightforward by adapting Euler-Maclaurin so as to give a summatory formula for a series taken over $\mathbf{Z}^n$ rather than $\mathbf{N}^n$. A different (but essentially related) method of studying such a lattice point problem, using analytic methods, has been carried out by Bochner [Bo], using Poisson summation (not surprisingly). Indeed, the approach taken by Randol and Colin de Verdière, cf. Section 3, while directed at the contribution that geometry makes to the error term, starts with the analytic point of view taken by Bochner.

(2) Instead of a series, like $D_P(s,\varphi)$, defined over $\mathbf{N}^n$, one often wants, in problems of a diophantine approximation nature, to sum over lattice points inside some semialgebraic subset of $\mathbf{R}^n$. It appears to be very difficult to give an analytic continuation of the Dirichlet series, so obtained, using the methods in [Ma-1]. In [Ma-2], Mahler was able to analyze the functional properties of a Dirichlet series when summed over the lattice points of a cone in $[0,\infty)^2$ with the special property that the slope of a boundary ray of the cone was a quadratic irrationality γ. The polynomial P continued to be elliptic on $[0,\infty)^2$. By a very pretty and clever argument, Mahler used the periodicity of the continued fraction for γ to

determine an analytic continuation of the series to the entire s plane as a meromorphic function. Remarkably, the nature of the polar locus of this function is considerably different than that obtained in Theorem 4.10. Mahler discovered that there could be countably many poles on *vertical* lines, not just rational poles. Moreover, the imaginary parts of the nonrational poles depended upon the fundamental unit of the field generated by γ. One could not hope to obtain such a property using geometric considerations only. This indicates the subtle and different nature of this type of problem, for which a successful analysis requires a combination of arithmetic and geometric reasoning. ∎

Section 5. *Hypoelliptic polynomials*

A natural extension of the class of elliptic polynomials on $[0,\infty)^n$ is the class of hypoelliptic polynomials on $[1,\infty)^n$. Unlike elliptics, the top order term is in some sense not significant for a hypoelliptic polynomial. Only the relative behavior at infinity of the polynomial with its partial derivatives is important. As a result, the geometry at infinity for a hypoelliptic polynomial can be more varied than an elliptic polynomial.

Definition 5.1. P is hypoelliptic on $[1,\infty)^n$ if for each differential monomial D_x^A

$$\lim_{\substack{\|x\|\to\infty \\ x\in[1,\infty)^n}} \frac{D_x^A P(x)}{P(x)} = 0\,.$$

In this section P will always denote a hypoelliptic polynomial on $[1,\infty)^n$ that is defined over $\mathbf{R}$.

Hörmander [Hö, ch. 11] found a class of hypoelliptics exhibiting the relative insignificance of the top order term.

Example. Let Q be any polynomial of degree δ and R any elliptic polynomial on $[0,\infty)^n$ of degree $r \in [2k\delta - 2(k-1), 2k\delta)$. Set $P = Q^{2k} + R$. Then, P is hypoelliptic on $[1,\infty)^n$. ∎

The following property was proved by Hörmander [ibid].

PROPOSITION 5.2. *There exist $c, C, D > 0$ such that*

$$\left| \frac{D_x^A P(x)}{P(x)} \right| \le C\|x\|^{-c|A|} \quad \textit{if } \|x\| \ge D\,.$$

Implicit in (5.2) is the possibility that $P(x)$ can equal 0. However, this can only occur on a compact subset of $[1,\infty)^n$.

Recall that $d =$ degree P. One concludes

COROLLARY 5.3. *There exist $\alpha, C, D > 0$ with $\alpha \ge cd$ such that*

$$|P(x)| \ge C\|x\|^{\alpha} \quad \textit{if } \|x\| \ge D.$$

Definition 5.4. The Lojasiewicz exponent of P at infinity is the largest α for which for some $D, C > 0$ one has

$$|P(x)| \geq C\|x\|^{\alpha} \quad \text{if } \|x\| \geq D.$$

Denote this exponent by α_P. Set c_P to be the largest c for which the inequality in (5.2) holds. Call this the hypoellipticity exponent of P. ∎

Remark 5.5.
(1) It is clear that $\alpha_P \leq d$. If P is elliptic, then $\alpha_P = d$.
(2) The above inequalities imply that outside a compact subset of $[1, \infty)^n$, the sign of P is constant. For simplicity, one assumes in the following that the sign is positive. ∎

Now let $\varphi = Q/T$ where Q, T are both hypoelliptic on $[1, \infty)^n$. Again, for simplicity, one assumes that φ is positive outside a compact subset of $[1, \infty)^n$. One can then choose positive constants α, D so that (5.3) holds for P, Q, T. In addition, an elementary calculation shows that a positive constant c^* exists so that (5.2) also holds for φ.

Define $N_P(w, \varphi), V_P(w, \varphi)$ as in Section 4. The main result of this section is

THEOREM 5.6. *There exists $\theta > 0$ such that*

$$N_P(w, \varphi) - V_P(w, \varphi) = O(V_P(w, \varphi)/w^{\theta}) \quad \text{as } w \to \infty.$$

A sketch of the proof that hypoelliptics determine a GLP follows.

Choose $D_1 > D_2 > 0$ so that (5.3) holds if $\|x\| \geq D_i, i = 1, 2$. Define

$$D_P(s, \varphi) = \sum_{\substack{m \in \mathbf{N}^n \\ T \cdot P(m) \neq 0}} \frac{\varphi(m)}{P(m)^s}.$$

Now let χ denote a C^{∞} function with values in $[0, 1]$ and satisfying the property

$$\chi(x) = \begin{cases} 1 & \text{if } \|x\| \geq D_1 \\ 0 & \text{if } \|x\| \leq D_2 . \end{cases}$$

Thus, for any differential monomial D_x^A of positive order,

$$supp\ (D_x^A(\chi)) \subset \{x : \|x\| \in [D_1, D_2]\}. \tag{5.7}$$

Set

$$D_P(s, \varphi, \chi) = \sum_{m \in \mathbf{N}^n} \frac{\chi(m)\varphi(m)}{P(m)^s}.$$

Next, let $K_1 \Subset K_2$ be compact subsets of $[1, \infty)^n$ such that $\{P = 0\} \cap [1, \infty)^n \subset K_1$. Further, using the notations of Section 4, let ℓ_1, ℓ_2 be any nonnegative integers, L be any nonzero multiindex, and set $\boldsymbol{B}$ resp. $\boldsymbol{C}$ to denote ℓ_1 resp. ℓ_2 tuples of multiindices whose sum is L. The notation $\boldsymbol{B}+\boldsymbol{C}$ is also used to denote this sum. Define

$$\begin{aligned}
I_P(s,\varphi) &= \int_{[1,\infty)^n - K_2} \frac{\varphi}{P^s}\, dx\,, && (*)\\
I_P(s,\varphi,\chi) &= \int_{[1,\infty)^n} \frac{\chi\varphi}{P^s}\, dx\,,\\
I(L,s,\varphi,\chi) &= \int_{[1,\infty)^n} D_x^L\left(\frac{\chi\varphi}{P^s}\right) dx\,,\\
I_L(s,\varphi,\chi) &= \int_{[1,\infty)^n} \chi D_x^L\left(\frac{\varphi}{P^s}\right) dx\,,\\
I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi,\chi) &= \int_{[1,\infty)^n} \chi \prod_{i=1}^{\ell_1}\left[\frac{D_x^{B_i}\varphi}{\varphi}\right]\cdot\prod_{i=1}^{\ell_2}\left[\frac{D_x^{C_i}P}{P}\right]\cdot\frac{\varphi}{P^s}\, dx\,.
\end{aligned}$$

A simple estimation argument, similar to (4.9), uses (5.2)-(5.4) to show

PROPOSITION 5.8. *$D_P(s,\varphi,\chi), I_P(s,\varphi,\chi)$, and each function defined in (*) are analytic and absolutely convergent if $\sigma > n/\alpha_P$.*

Following the argument given in the proof of parts 4-7 of Theorem 4.10, one concludes that the integral representations given in (4.20) of $D_P(s,\varphi,\chi)$ are valid whenever $\sigma > n/\alpha_P$. Although the $I(L,s,\varphi,\chi)$ are the functions that actually appear in these representations, the important analytic continuation properties are found in the $I_L(s,\varphi,\chi)$. This is the point of

LEMMA 5.9. *The following differences of the functions defined above are all entire functions of s.*

(1) $D_P(s,\varphi) - D_P(s,\varphi,\chi)$.
(2) $I_P(s,\varphi) - I_P(s,\varphi,\chi)$.
(3) $I(L,s,\varphi,\chi) - I_L(s,\varphi,\chi)$.

PROOF: (1) is clear since the difference is a finite sum of functions of the form $(1-\chi(m))/P(m)^s$. (2) is clear since the difference is an integral over a compact subset on which P is never 0. Similarly, (5.7) implies that (3) is a finite sum of integrals, each of which is supported on a compact subset over which P is never 0. ∎

Thus, it suffices to study the functional properties of $I_P(s,\varphi,\chi)$ and each $I_L(s,\varphi,\chi)$. One notes from the discussion in Section 4 that each $I_L(s,\varphi,\chi)$ is a linear combination (with coefficients in $\mathbf{Z}[s]$ of degree $|L|-1$ in s) of the $I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi,\chi)$, with $\boldsymbol{B}+\boldsymbol{C}=L$. Thus, it suffices to determine the functional properties of $I_P(s,\varphi,\chi)$, and any $I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi,\chi)$.

Following the "tauberian program" of Section 4, Theorem 5.6 will then follow from

THEOREM 5.10.

(1) $D_P(s,\varphi,\chi), I_P(s,\varphi,\chi)$ *and each* $I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi,\chi)$ *can be continued to the entire* s *plane as meromorphic functions with rational poles.*

Let $\rho_1(\varphi)$ *resp.* $\lambda_1(\varphi)$ *denote the first pole of* $D_P(s,\varphi,\chi)$ *resp.* $I_P(s,\varphi,\chi)$.

(2) $\rho_1(\varphi) = \lambda_1(\varphi)$.

Let $\rho(\varphi)$ *denote the common value of* $\rho_1(\varphi), \lambda_1(\varphi)$.

(3) $D_P(s,\varphi,\chi) - I_P(s,\varphi,\chi)$ *is analytic at* $\rho(\varphi)$.

(4) *There exist* $\mu, M > 0$ *such that for any* $\sigma_1 < \sigma_2 \leq \rho$ *and* $\epsilon > 0$ *one can find* $C = C(\epsilon, \sigma_1, \sigma_2)$ *so that for any* $\sigma \in [\sigma_1, \sigma_2]$, $,|t| \geq 1$ *and* $\mathbf{C}$

$$|D_P(s,\varphi,\chi)| < C\left(1 + |t|^{\mu(\rho(\varphi)-\sigma_1)+\epsilon}\right) \tag{\#}$$

$$|I_P(s,\varphi,\chi)| < CM^{\sigma}, \tag{\#\#}$$

$$|I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi,\chi)| < CM^{\sigma}.$$

Remarks.

(1) As in the proof of Theorem 4.10, one can choose $\mu = nd$ for the constant in part (4). Different values for μ have been given in [Li-2,3].

(2) An appealing, but probably naive, conjecture, in light of (5.5), is that

$$\rho(\varphi) = \frac{n}{\alpha_P}. \tag{5.11}$$

For this would give a characterization of the first pole of $D_P(s,\varphi), I_P(s,\varphi)$ via the geometry of φ, P at infinity. (5.11) would extend to hypoelliptics the relation, proved in Theorem 4.10, between the first pole of a Dirichlet series and the Lojasiewicz exponent determined by elliptic polynomials. The analogous assertion for the local isolated hypersurface singularity case would claim, for example, that the largest root of the local b-function was controlled solely by the multiplicity of the defining function, which is known not to be the case, cf. [Ig]. In general, a perhaps more reasonable question to consider is a geometric characterization for an estimate of $\frac{n}{\alpha_P} - \rho$. ∎

SKETCH OF PROOF OF THEOREM 5.10: One sees, using the expression for $D_P(s,\varphi,\chi)$, given by (4.20), and by the above reductions, that the proof splits into two parts:

Claim 1. For any pair of tuples of multiindices $\boldsymbol{B},\boldsymbol{C}$, the functions $I_P(s,\varphi,\chi)$ and $I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi,\chi)$, admit analytic continuations to the s plane as meromorphic functions with rational poles and satisfy the growth estimate (##) of (4).

Claim 2. The first pole of $I_P(s,\varphi,\chi)$ is strictly larger than the first pole of any $I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi,\chi)$ if $\boldsymbol{B}$ or $\boldsymbol{C}$ do not both consist of zero vectors.

Remark 5.12. Claim 1 has been established in [Li-3] while Claim 2 is proved in [Li-4]. An alternative proof can be based upon an argument of Bochner and is sketched at the end of this section. These assertions can be viewed as analogues of the well known properties established for the distributions $P_{\pm}^{s}$ on $\mathcal{S}(\mathbf{R}^n)$, defined by

$$\varphi \rightarrow \int_{P\neq 0} \varphi\, P_{\pm}^{s}\, dx\,.$$

The general significance of these distributions was first emphasized by Gelfand. Two proofs of the analytic continuation of these distributions are known. The first uses resolution of singularities, cf. [Ati], [Be-Ge], while the second uses the functional equation and b-function [Be].

In [Li-1,2,3] the functional equation at infinity was used to prove parts (1), (4) of Theorem 5.10. In particular, the "moderate growth" of the series outside the domain of analyticity was proved using the functional equation, just like the reflection type functional equation (2.4) yielded the growth estimate (2.5) for the Dirichlet series determined by a positive definite quadratic form. However, it seems to be difficult, so far, to use the general algebraic functional equation to establish the relatively precise analytic properties (2), (3). For purposes of a unified discussion, the proof of Claims 1,2, given here, will therefore be based upon resolution of singularities at infinity. ▮

Since the analysis needed to prove both claims is carried out at infinity, it is first necessary to define the following objects.

Definitions/Notations.

1) The chart at infinity in $(P^1\mathbf{R})^n$ will be denoted $(\mathbf{R}^n, (w_1, \ldots, w_n))$. The hyperplane at infinity $\{w_1 \cdots w_n = 0\}$ is denoted H_∞. The notations $1/w$ resp. dw are used to denote the point $(1/w_1, \ldots, 1/w_n)$ resp. the differential $dw_1 \cdots dw_n$.

2) Define the rational functions

$$R(w) =_{def} \frac{1}{P(1/w)}\,,$$
$$\Phi(w) =_{def} \varphi(1/w)\,.$$

The letter χ will also be used to denote the function $\chi(1/w)$.

3) Set

$$\eta_{\boldsymbol{B},\boldsymbol{C}}(w) = \chi \cdot \prod_{i=1}^{\ell_1} \left[\frac{D_x^{B_i}\varphi}{\varphi}\right] \cdot \prod_{i=1}^{\ell_2} \left[\frac{D_x^{C_i} P}{P}\right](1/w).$$

It is clear that $\eta_{\boldsymbol{B},\boldsymbol{C}}$ is defined over the set $(0,1]^n$ in the chart at infinity. Moreover, by (5.2), one sees the existence of $c', C' > 0$ such that for each $\boldsymbol{B}, \boldsymbol{C}$

$$|\eta_{\boldsymbol{B},\boldsymbol{C}}(w)| < C' |w_1 \cdots w_n|^{c'}\,.$$

Thus, for each $p \in \partial[0,1]^n \cap H_\infty$

$$\lim_{\substack{w \to p \\ w \in (0,1]^n}} \eta_{\boldsymbol{B},\boldsymbol{C}}(w) = 0\,. \tag{5.13}$$

Note. In the following one fixes a particular $\boldsymbol{B},\boldsymbol{C}$ and then drops the subscript from η whenever there is no possibility of confusion. ∎

The proof of Claims (1), (2) is based upon analyzing the integrands in $I_P(s,\varphi,\chi)$, $I_{\boldsymbol{B},\boldsymbol{C}}(s,\varphi,\chi)$, using a resolution of singularities.

There exist a nonsingular real algebraic manifold Y and projective morphism $\pi : Y \to (\mathbf{R}^n, (w_1, \ldots, w_n))$ such that the following properties are satisfied.

(5.14)

i) There exists a divisor $\mathcal{D} \subset Y$ so that $\pi : Y - \mathcal{D} \to \mathbf{R}^n$ is an isomorphism onto its image;

ii) $\mathcal{D}$ is a normally crossing divisor. That is, $\mathcal{D} = \cup \mathcal{D}_\alpha$ where each $\mathcal{D}_\alpha$ is smooth and at each point $p \in \mathcal{D}$ the set of divisors containing p are mutually transverse;

iii) The divisor determined by

$$[\prod_{i=1}^{n}(w_i - 1) \cdot \prod_{i=1}^{n} w_i \cdot R \cdot \Phi \cdot \chi \cdot \eta] \circ \pi$$

has support in $\mathcal{D}$ (so that it too is locally normal crossing);

iv) $(0,1)^n \cap \pi(\mathcal{D}) = \emptyset$.

Thus, $(0,1)^n$ is disjoint from the locus of blowing up determined by π.

Next, one takes an open polydisc U containing $[0,1]^n$ in the chart at infinity and sets

$$\begin{aligned} X &= \pi^{-1}(U) \\ D &= \mathcal{D} \cap X \\ B &= \overline{\pi^{-1}(0,1)^n} \cap X \,. \end{aligned}$$

An elementary observation is the

LEMMA 5.15.

i) $\partial B \subset D$.

ii) $B \cap D = \partial B$.

PROOF: (i) follows from (5.14)(iii). To verify (ii), one notes that (5.14)(i,iv) imply

$$\pi^{-1}(0,1)^n \cap D = \emptyset.$$

Moreover, since π is continuous, $\pi^{-1}(0,1)^n$ is open in X and equals $int(B)$. Thus, $B \cap D = \partial B \cap D = \partial B$ by (i). ∎

A second elementary result will also be needed below. For each point $q \in \partial B$ there exists an open neighborhood $\mathcal{U}_q$ and coordinates $(z_1, \ldots, z_n)$, defined in $\mathcal{U}_q$ and centered at q, such that

$$\mathcal{U}_q \cap D \subset \cup_{i=1}^{n} \{z_i = 0\}. \tag{5.16}$$

A "sign distribution" is a function

$$\epsilon : \{1, \ldots, n\} \to \{+, -\}.$$

To each sign distribution one defines an open subset of any $\mathcal{U}_q$ by setting

$$\mathcal{O}_\epsilon = \{z \in \mathcal{U}_q : \epsilon(i) z_i > 0, \text{ for each } i = 1, \ldots, n\}.$$

One notes that the only geometric property of interest possessed by these sets is their disjointness from D.

LEMMA 5.17. *For each $q \in \partial B$ there exists a set $\mathcal{E}_q$ of sign distributions such that*

$$\cup_{\epsilon \in \mathcal{E}_q} \mathcal{O}_\epsilon = int(B) \cap \mathcal{U}_q .$$

PROOF: By (5.14)(i) and (5.15)(i), it is clear that

$$int(B) \cap \mathcal{U}_q \subset \cup_\epsilon \mathcal{O}_\epsilon .$$

Suppose for some ϵ_0 that $int(B) \cap \mathcal{U}_q \cap \mathcal{O}_{\epsilon_0} \neq \emptyset$. Further, suppose that $\mathcal{O}_{\epsilon_0} \not\subset int(B) \cap \mathcal{U}_q$. Then, Lemma (5.15) and (5.16) imply that

$$\mathcal{O}_{\epsilon_0} \cap (int(B) \cap \mathcal{U}_q) \neq \emptyset \quad \text{and} \quad \mathcal{O}_{\epsilon_0} \cap (B^c \cap \mathcal{U}_q) \neq \emptyset$$
$$\text{but} \qquad \mathcal{O}_{\epsilon_0} \cap (\partial B \cap \mathcal{U}_q) = \emptyset.$$

Since $\mathcal{O}_{\epsilon_0}$ is connected this decomposition of $\mathcal{O}_{\epsilon_0}$ into two disjoint open subsets cannot occur. Thus, $\mathcal{O}_{\epsilon_0} \subset int(B) \cap \mathcal{U}_q$. This implies Lemma (5.17). ∎

To each irreducible component D_α of D one defines the following orders.

$$\begin{aligned} M_\alpha &= \mathrm{ord}_{D_\alpha} R \circ \pi , \\ m_\alpha &= \mathrm{ord}_{D_\alpha} \Phi \circ \pi , \\ \kappa_\alpha &= \mathrm{ord}_{D_\alpha} \eta \circ \pi , \\ \gamma_\alpha &= \mathrm{ord}_{D_\alpha} Jac(\pi) - \mathrm{ord}_{D_\alpha} (w_1^2 \cdots w_n^2) \circ \pi , \end{aligned} \tag{5.18}$$

where $Jac(\pi)$ denotes the jacobian of π.

To each D_α for which $M_\alpha \neq 0$ and $e \in \mathbf{N}$, define the ratios

$$\rho(e, D_\alpha) = \frac{-(e + \gamma_\alpha + m_\alpha)}{M_\alpha}$$
$$\beta(e, D_\alpha) = \frac{-(e + \kappa_\alpha + \gamma_\alpha + m_\alpha)}{M_\alpha} .$$

If $M_\alpha = 0$ one sets $\rho(e, D_\alpha) = \beta(e, D_\alpha) = -\infty$, for each e.

Define

$$\rho(\pi) = sup_\alpha\{\rho(1, D_\alpha)\}, \qquad \beta(\pi) = sup_\alpha\{\beta(1, D_\alpha)\}. \tag{5.19}$$

Note. As will be explained below, the regularization procedure, described in [G-S, chs. 1,3], can be applied very naturally to determine the analytic continuation of $I_P(s, \varphi, \chi)$, $I_{\boldsymbol{B},\boldsymbol{C}}(s, \varphi, \chi)$. In light of this, it becomes clear that $\rho(1, D_\alpha)$ resp. $\beta(1, D_\alpha)$ are possible values for the *first* pole of $I_P(s, \varphi, \chi)$ resp. $I_{\boldsymbol{B},\boldsymbol{C}}(s, \varphi, \chi)$. Thus, any pole of $I_P(s, \varphi, \chi)$ is at most $\rho(\pi)$, and any pole of $I_{\boldsymbol{B},\boldsymbol{C}}(s, \varphi, \chi)$ is at most $\beta(\pi)$. ∎

A key step in the proof of Theorem 5.10 is therefore the proof of the inequality

$$\rho(\pi) > \beta(\pi). \tag{5.20}$$

This will follow immediately from

LEMMA 5.21. *Suppose q is a point in ∂B such that $\pi(q) \in H_\infty$. Let D_α be any component of D containing q. Then $\kappa_\alpha > 0$.*

PROOF: Assume there exists a point $q \in \partial B$ with $\pi(q) \in H_\infty$ for which $\kappa_{\alpha'} \le 0$ for some divisor $D_{\alpha'}$ containing q. Let $\mathcal{U}_q$ denote a neighborhood of the point so that (5.16) holds. Assume that coordinates are chosen so that the divisor $D_{\alpha'}$ satisfies the property $D_{\alpha'} \cap \mathcal{U}_q = \{z_1 = 0\}$. There exists at least one sign distribution ϵ so that $\mathcal{O}_\epsilon \subset int(B) \cap \mathcal{U}_q$. Given any point $p = (p_1, p') \in \mathcal{O}_\epsilon$ the path $\nu(t) = (1-t)p + t(0, p'), t \in [0, 1)$ is entirely contained in $\mathcal{O}_\epsilon$. By definition, one has that

$$ord_t\,(\eta \circ \pi \circ \nu) = \kappa_{\alpha'}\,.$$

Thus, $\kappa_{\alpha'} \le 0$ implies

$$\lim_{t \to 0} \eta \circ \pi \circ \nu(t) \ne 0\,.$$

On the other hand, $\mathcal{O}_\epsilon \subset int(B) \cap \mathcal{U}_q$ implies that for all $t > 0$, $\pi \circ \nu(t) \in (0, 1)^n$. Moreover, as $t \to 0$, $\pi \circ \nu(t)$ approaches a point in H_∞. Thus, by (5.13) the limit of η along the path $\pi \circ \nu(t)$ must equal 0. So, the point q with the above properties must not exist. This proves the Lemma. ∎

An entirely similar argument that uses (5.2), as expressed in the $(w_1, \ldots, w_n)$ coordinates, shows the important

LEMMA 5.22. *Suppose q is a point of ∂B such that $\pi(q) \in H_\infty$. Let D_α be any component of D containing q. Then $M_\alpha > 0$. Moreover, if $q \in \partial B$ is such that $\pi(q) \notin H_\infty$ then $M_\alpha = 0$ for any component D_α containing q.*

Remark 5.23. Geometrically, Lemma 5.22 says that the strict transform of the denominator of $R(w)$ is a component of D that is *disjoint* from B in X. That is, the polar divisor of $R \circ \pi\big|_X$ cannot intersect B. This property is very useful because it implies that the regularization procedure of Gelfand-Shapiro-Shilov can be applied in essentially the same manner as has been done to find the first pole of the distributions $P^s_{\pm}$, cf. [Ig, Li-5, Va]. ∎

Furnished with these preliminary observations, one can now proceed to the

PROOF OF CLAIMS (1), (2): In light of (5.20), it evidently suffices to show,

(1′) $I_P(s,\varphi,\chi)$, resp. $I_{\mathbf{B},\mathbf{C}}(s,\varphi,\chi)$ can be analytically continued as meromorphic functions with *possible* poles contained in the set $\{\rho(e,D_\alpha)\}_{e,\alpha}$ resp. in the set $\{\beta(e,D_\alpha)\}_{e,\alpha}$.

(1″) The analytically continued functions satisfy the growth estimate (##) of part (4) in the statement of Theorem 5.10.

(2′) $\rho(\pi)$ *is* the first pole of $I_P(s,\varphi,\chi)$.

By (5.20), it will suffice to prove (1′), (1″), (2′) for $I_P(s,\varphi,\chi)$ only. The proof of (1′), (1″) for $I_{\mathbf{B},\mathbf{C}}(s,\varphi,\chi)$ is similar and left to the reader. (5.20) is then invoked to establish that the first pole of $D_P(s,\varphi,\chi)$ can only equal the first pole of $I_P(s,\varphi,\chi)$.

One has for $\sigma > n/\alpha_P$,

$$\begin{aligned} I_P(s,\varphi,\chi) &= \int_{[0,1]^n} R^s\varphi\chi\,\frac{dw}{w_1^2\cdots w_n^2} \\ &=_{def} \lim_{\epsilon\to 0}\int_{[\epsilon,1]^n} R^s\varphi\chi\,\frac{dw}{w_1^2\cdots w_n^2} \\ &= \int_B (R\circ\pi)^s(\varphi\cdot\chi\circ\pi)|\pi^*(\frac{dw}{w_1^2\cdots w_n^2})|, \end{aligned}$$

where $|\pi^*(dw/w_1^2\cdots w_n^2)|$ denotes a density on X.

Since π is proper and B is a closed subset of the compact set $\pi^{-1}[0,1]^n$, B is also compact. For each $q\in B$ there exists an open neigborhood $\mathcal{U}_q$ so that (5.16) holds iff $q\in\partial B$. The open cover $\{\mathcal{U}_q\}$ of B admits a finite open subcover $\{\mathcal{U}_i\}_{i=1}^N$, where $\mathcal{U}_i$ is centered at q_i. One now takes a finite partition of unity $\{v_c\}$ subordinate to the cover $\{\mathcal{U}_i\}$. Thus, for $\sigma > n/\alpha_P$

$$\int_B (R\circ\pi)^s\,(\varphi\cdot\chi\circ\pi)\,|\pi^*(\frac{dw}{w_1^2\cdots w_n^2})| = \sum_c\sum_i\int_{\mathcal{U}_i\cap B}(R\circ\pi)^s\,(\varphi\cdot\chi\circ\pi)\,v_c\,|\pi^*(\frac{dw}{w_1^2\cdots w_n^2})|. \tag{5.25}$$

One next fixes an arbitrary $\mathcal{U}_i$. One chooses the coordinates centered at q_i so that

$$\mathcal{U}_i\cap D = \cup_{j=1}^r\{z_j=0\}.$$

Assume that $\{\epsilon_1,\ldots,\epsilon_{R(i)}\}$ are the sign distributions so that $\mathcal{O}_{\epsilon_k}\subset int(B)\cap\mathcal{U}_i, k = 1,\ldots,R(i)$. Define for each $j=1,\ldots,r$

$$\begin{aligned} M_j(i) &= \mathrm{ord}_{D_j}\,(R\circ\pi) \\ m_j(i) &= \mathrm{ord}_{D_j}\,(\varphi\cdot\chi\circ\pi) \\ \gamma_j(i) &= \mathrm{ord}_{D_j}\,|\pi^*(dw/w_1^2\cdots w_n^2)|. \end{aligned}$$

By keeping $\mathcal{U}_i$ sufficiently small, one observes that Lemma 5.22 implies that $M_j(i) \geq 0$ for each i, j.

Next, define for each $i = 1, \ldots, N$

$$\nu(i, \rho(\pi)) = \#\{j : \frac{-(1 + \gamma_j(i) + m_j(i)}{M_j(i)} = \rho(\pi)\},$$

and set

$$\mathcal{J}(\rho(\pi)) = \{i : \nu(i, \rho(\pi)) \geq 1\}.$$

By definition, $\mathcal{J}(\rho(\pi)) \neq \emptyset$. In this regard, one should also note that $r = 0$ is possible. This occurs iff $q_i \in int(B)$. In this case, each $M_j(i) = 0$ and $i \notin \mathcal{J}(\rho(\pi))$.

It now follows that the Gelfand-Shapiro-Shilov regularization method [G-S] applies to the integral over *each* open set $\mathcal{O}_{\epsilon_k}, k = 1, \ldots, R(i)$ and $i = 1, \ldots, N$. One thereby obtains an analytic continuation of each summand on the right side of (5.25). This proves ($1'$). The growth estimate of (##) is an easy consequence of the explicit expressions that determine the regularization. This is left to the reader to verify.

Thus, it suffices to prove ($2'$). This is done as follows. Assume that $i \in \mathcal{J}(\rho(\pi))$, so that the principal part at $s = \rho(\pi)$ of

$$\int_{\mathcal{U}_i \cap B} (R \circ \pi)^s (\varphi \cdot \chi \circ \pi) v_c \, |\pi^*(\frac{dw}{w_1^2 \cdots w_n^2})|$$

consists of at most $\nu(i, \rho(\pi))$ nonzero terms. One then shows that the term of order equal to $\nu(i, \rho(\pi))$ must be *positive.*

Note. When one i is fixed, $i, \rho(\pi)$ are subsequently dropped as the argument for ν. ∎

After reindexing, if necessary, one may assume that

$$\{j : \frac{-(1 + \gamma_j(i) + m_j(i))}{M_j(i)} = \rho(\pi)\} = \{1, 2, \ldots, \nu\}.$$

One sets $z' = (z_{\nu+1}, \ldots, z_n)$.

Then the contribution from $\mathcal{U}_i$ to the term of order ν in the principal part has the form

$$\sum_c \sum_{k=1}^{R(i)} \int_{\mathcal{U}_i \cap D_1 \ldots \cap D_\nu} (z_{\nu+1})^{\zeta_{\nu+1}}_{\epsilon_k(\nu+1)} \cdots (z_n)^{\zeta_n}_{\epsilon_k(n)} \, g_1(z')^{\rho(\pi)} \, g_2(z') \, v_c(z') \, g_3(z') \, dz' \tag{5.26}$$

where the following properties are satisfied:

(1) $\zeta_{\nu+1}, \ldots, \zeta_n > -1$, (cf. [Va] where this property was first used for a related problem);

(2) $g_1(z')$ is the restriction to $\cap_{j=1}^{\nu} D_j$ of the strict transform of $R \circ \pi$ in $\mathcal{U}_i$;

(3) $g_2(z')$ is the restriction to $\cap_{j=1}^{\nu} D_j$ of the product of the strict transform of $\Phi \circ \pi$ and (a continuous extension of) $\chi\Big|_{(0,1]^n} \circ \pi$ in $\mathcal{U}_i$;

(4) $g_3(z')$ is the restriction to $\cap_{j=1}^{\nu} D_j$ of the strict transform of the quotient of $|Jac(\pi)|$ with $w_1^2 \cdots w_n^2$ in $\mathcal{U}_i$.

From Lemma 5.22 and the positivity of $P \cdot \varphi \cdot \chi$ over all but a compact subset of $[1, \infty)^n$ (cf. (5.5)), one concludes that $g_1(z'), g_2(z'), g_3(z')$ are *finite and positive* over the domain of integration in (5.26). Moreover, since $\{v_c\}$ forms a partition of unity, one concludes that the double sum in (5.26) must be positive.

This implies that $\rho(\pi)$ *must* be a pole of $I_P(s, \varphi, \chi)$. Furthermore, any rational number larger than $\rho(\pi)$ could *not* be a pole of $I_P(s, \varphi, \chi)$ since it would be larger than any candidate pole $\rho(1, D_\alpha)$, used to define $\rho(\pi)$. This proves (2′) and therefore Claim (2). By the above remarks, the proof of Theorem 5.10 follows. ∎

Concluding Remarks.

(1) The reader might wonder if there exist any polynomials determining a proper polynomial P on $[1, \infty)^n$ for which the lattice point problem with $\mathcal{R}_w = \{P \le w\} \cap [1, \infty)^n$ is not geometric. Such examples are not difficult to find. Choose $n = 2$ and $a < b$ positive integers. Set $P(x, y) = x^a y^b$. Explicit calculations will verify that $N(w)$ and $V(w)$ both have growth $w^{1/a}$ but so does $E(w)$.

(2) The allowable choice of $\mu = nd$ in Theorems 4.10, 5.10 implies, by the discussion in Appendix B, one of two interesting conclusions. Either there is no pole of $D_P(s, \varphi)$ in the interval $(\rho(\varphi) - \frac{1}{nd}, \rho(\varphi))$, in which case there exists a nonzero polynomial $A(u)$ such that

$$N_P(w, \varphi) = w^{\rho(\varphi)} A(\log w) + O_\epsilon(w^{\rho(\varphi) - \frac{1}{nd} + \epsilon}), \tag{5.27}$$

or there is at least one other pole in this interval and one has therefore found a second and lower order term in the asymptotic of $N_P(w, \varphi)$, which experience shows to be quite difficult. However, from the analysis of Appendix B, it does not yet follow that (5.27) implies

$$E(w, \varphi) = O_\epsilon(w^{\rho(\varphi) - \frac{1}{nd} + \epsilon}).$$

This is because the asymptotic of $V_P(w, \varphi)$ is only controlled by the analytic properties of $I_P(s, \varphi)$ whereas the asymptotics of $N_P(w, \varphi)$ are controlled, a priori, by the analytic properties both of $I_P(s, \varphi)$ as well as the other integrals appearing in the integral representation (4.20) (appropriately modified for hypoelliptic polynomials via the smoothing term χ). It is at least possible that a pole of one of these functions of s might sneak into the above interval and determine a second pole of the Dirichlet series. This is the reason why an explicit estimate for $E(w, \varphi)$ cannot easily be given using the integral representation of $D_P(s, \varphi)$ based upon the iterated Euler-Maclaurin formula.

In order to circumvent this difficulty, an alternative integral representation of $D_P(s, \varphi)$ is useful, when one works with the entire lattice $\mathbf{Z}^n$. For here one can exploit Poisson summation and ignore all problems arising from the constributions near the boundary of $[1,\infty)^n$. In [Bo], Bochner worked out the details if the polynomials P, Q, T are elliptic over $\mathbf{R}^n$. However, using the existence of hypoellipticity constants (see (5.4)), it is easy to see that the argument extends to hypoelliptic polynomials. This will now be sketched.

Assuming hypoellipticity of P, T on $\mathbf{R}^n$ it follows that there exists $D > 0$ such that

$$\{P \cdot T = 0\} \subset \{x : \|x\| \leq D\}.$$

Choose $D' > D$ and a smooth function $\chi : \mathbf{R}^n \to [0,1]$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } \|x\| \geq D' \\ 0 & \text{if } \|x\| \leq D. \end{cases}$$

Define, as above, but now over $\mathbf{R}^n$, (no confusion should result by using the same notation)

$$D_P(s,\varphi) = \sum_{\substack{m \in \mathbf{Z}^n \\ T \cdot P(m) \neq 0}} \frac{\varphi(m)}{P(m)^s}$$

$$I_P(s,\varphi) = \int_{\mathbf{R}_n - \{\|x\| \leq D\}} \frac{\varphi}{P^s}\, dx_1 \cdots dx_n\,,$$

$$D_P(s,\varphi,\chi) = \sum_{m \in \mathbf{Z}^n} \frac{\chi \cdot \varphi(m)}{P(m)^s}\,,$$

$$I_P(s,\varphi,\chi) = \int_{\mathbf{R}^n} \frac{\chi \cdot \varphi}{P^s}\, dx_1 \cdots dx_n\,.$$

The analogue of Lemma 5.9 evidently still holds, that is,

LEMMA 5.28. *The following differences of the functions defined above are entire functions of s.*

(1) $D_P(s,\varphi) - D_P(s,\varphi,\chi)$.
(2) $I_P(s,\varphi) - I_P(s,\varphi,\chi)$.

Let $\epsilon > 0$ and define the decay function

$$E_\epsilon(x_1,\ldots,x_n) = e^{-(\epsilon\|x\|)^2}\,.$$

Define

$$D(s,\epsilon) = \sum_{m \in \mathbf{Z}^n} \frac{E_\epsilon \cdot \chi \cdot \varphi(m)}{P(m)^s}\,,$$

$$\hat{D}(s,\epsilon,\xi) = \int_{\mathbf{R}^n} e^{-2\pi i (x,\xi)}\, \frac{E_\epsilon \cdot \chi \cdot \varphi}{P^s}\, dx_1 \cdots dx_n\,.$$

An application of Poisson summation then shows

PROPOSITION 5.29. *For each $\epsilon > 0$ and $\sigma \gg 1$*

$$D(s,\epsilon) = \sum_m \hat{D}(s,\epsilon,m).$$

Clearly, one has, for $\sigma \gg 1$

$$\begin{aligned} \lim_{\epsilon\to 0} D(s,\epsilon) &= D_P(s,\varphi,\chi), \\ \lim_{\epsilon\to 0} \hat{D}(s,\epsilon,0) &= I_P(s,\varphi,\chi). \end{aligned} \tag{5.30}$$

Now, applying the existence of positive hypoellipticity constants for φ, P, one can show, via the argument in [Bo, pgs. 33-36], sketched below for the reader's convenience,

THEOREM 5.31. *The function defined by*

$$\lim_{\epsilon\to 0} \sum_{m\neq 0} \hat{D}(s,\epsilon,m)$$

is an entire function of s.

Thus, one concludes from (5.28),(5.31)

$$D_P(s,\varphi) - I_P(s,\varphi) \text{ is an entire function of } s. \tag{5.32}$$

The main conclusion that one can deduce from (5.32) concerns an explicit and easy to state estimate for $E(w,\varphi)$, when one uses the sets $\mathcal{R}_w = \{x \in \mathbf{R}^n : |P(x)| \leq w\}$.

THEOREM 5.33. *If P, Q, T are hypoelliptic on $\mathbf{R}^n$, $\deg P = d$, and $\varphi = Q/T$, then*

$$E(w,\varphi) = O_\epsilon(w^{\rho(\varphi) - \frac{1}{nd} + \epsilon}) \quad \text{as } w \to \infty.$$

PROOF: For each of the 2^n possible sign functions $\tau : \{1,\ldots,n\} \to \{+,-\}$ define the quadrant

$$Q_\tau = \{x : \tau(i)x_i \geq 0\},$$

series

$$D_\tau(s,\varphi) = \sum_{\substack{m\in Q_\tau\cap\mathbf{Z}^n \\ T\cdot P(m)\neq 0}} \frac{\varphi(m)}{P(m)^s},$$

and integral

$$I_\tau(s,\varphi) = \int_{Q_\tau - \{\|x\|\leq D\}} \frac{\varphi}{P^s}\, dx_1 \cdots dx_n.$$

Thus,

$$D(s,\varphi) = \sum_{\tau} D_\tau(s,\varphi) - \mathcal{D}(s,\varphi)$$

where $\mathcal{D}(s,\varphi)$ is a finite linear combination of Dirichlet series over lattice points contained in the intersection of at least two quadrants. The growth of each such series in vertical bands of the s-plane can be analyzed via the same reasoning used to prove (5.10). This is left to the reader.

Moreover, one has that for $\sigma \gg 1$

$$D_P(s,\varphi) - I_P(s,\varphi) = \sum_{\tau} [D_\tau(s,\varphi) - I_\tau(s,\varphi)] - \mathcal{D}(s,\varphi).$$

By (5.32), the left side is entire. By evident applications of part 3 of (5.10), applied to the sum over the τ, as well as each of the series appearing in $\mathcal{D}(s,\varphi)$, one concludes that for each $\theta > 0$ there exists a constant $C_\theta > 0$ such that for any $\sigma_2 < \sigma_1 \leq \rho(\varphi)$

$$|D_P(\sigma+it,\varphi) - I_P(\sigma+it,\varphi)| < C_\theta |t|^{\frac{1}{nd}(\rho(\varphi)-\sigma+\theta)} \qquad \text{if } \sigma \in [\sigma_2, \sigma_1].$$

Then, since this difference has no poles, it follows that when $c \gg 1$

$$E(w,\varphi) =_{\text{def}} \frac{1}{2\pi i} \int_{\sigma=c} [D_P(s,\varphi) - I_P(s,\varphi)] w^s \frac{ds}{s} = O_\epsilon(w^{\rho(\varphi)-\frac{1}{nd}+\epsilon}),$$

by the argument of Landau given in appendix B. ∎

SKETCH OF PROOF OF THEOREM 5.31: The idea is to show two properties.

(1) Given a compact subset K of the s-plane, there exists an integer $b = b_K \gg 1$ such that if $m \neq 0$ then

$$\lambda(s,m) =_{\text{def}} \lim_{\epsilon\to 0} \|m\|^{2b} \hat{D}(s,\epsilon,m) \text{ exists and is finite},$$

where the convergence is uniform for $s \in K$;

(2) The series

$$\sum_{m\neq 0} \frac{\lambda(s,m)}{\|m\|^{2b}} \text{ converges absolutely and uniformly if } s \in K.$$

It is clear that (1) and (2) imply the assertion of the theorem.

Define the operator

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

An application of integration by parts and Stokes' theorem, applied $4b$ times, implies that for $\sigma \gg 1$ and any $m \neq 0$

$$(-2\pi i)^{2b}\|m\|^{2b}\hat{D}(s,\epsilon,m) = \int_{\mathbf{R}^n} e^{-2\pi i(x,m)} \cdot \Delta^b \left(E_\epsilon \cdot \chi \cdot \varphi/P^s\right) dx_1 \cdots dx_n \, .$$

Now use the existence of positive numbers c_P resp. c^* so that the inequality in (5.2) holds for P resp. φ. Set $c = min\{c_P, c^*\}$.

By expanding out $\Delta^b (E_\epsilon \cdot \chi \cdot \varphi/P^s)$, using Leibniz' rule, one sees the following by a tedious calculation, best left to the industrious reader. Given a compact set K, set $\sigma_K = inf\,\{\sigma|_K\}$. Using the notation from (5.2)ff, set

$$b = 1 + [\![\frac{(\deg Q - \alpha_T) - \sigma_K \alpha_P + n}{2c}]\!] \, .$$

Then there exists a constant C_K such that for any $\epsilon > 0$ and $s \in K$

$$\int_{\mathbf{R}^n} |E_\epsilon||\Delta^b (\chi\varphi/P^s)\,|\, dx_1 \cdots dx_n < C_K \, . \tag{5.34}$$

Moreover, if $I \neq 0$ and J are indices with $|I + J| = 2b$ then the exponential decay of E_ϵ at infinity implies

$$\lim_{\epsilon \to 0} \int_{\mathbf{R}^n} |D_x^I(E_\epsilon) D_x^J(\chi\varphi/P^s)|\, dx_1 \cdots dx_n = 0 \, , \tag{5.35}$$

where the convergence is uniform in K. Properties (1), (2) now follow from (5.34), (5.35), by choosing b to be the maximum of the expression above and $\frac{n}{2} + 1$. This completes the proof of (5.31). ∎

The reader therefore sees an instructive difference in the methods described in sections 3-5. The first technique, when it can be applied, focusses more directly upon $E(w, \varphi)$ and so, obtains more precise results. The second method shows that under fairly general hypotheses on P, φ (over $\mathbf{R}^n$) one has a GLP, for which a general, but no doubt improvable estimate for $E(w, \varphi)$ can be derived. Clearly, it would be very interesting and worthwhile to know how to combine both methods and improve the error estimates.

(3) Whereas (2) has treated the case in which the series is defined over $\mathbf{Z}^n$, the reader of these notes might also be interested to know of an elementary argument that estimates $E(w)$ (that is, where $\varphi \equiv 1$) when P is hypoelliptic over $[1, \infty)^n$. This will give an "explicit" estimate for $E(w)$, when the sum is taken over the lattice points in $\mathbf{N}^n$. As the reader will note, the level of explicitness is considerably less than that achieved in (2). This appears to be typical of the type of problem encountered when working over $[1, \infty)^n$.

To do this, one needs to start with two theorems that give the asymptotic form of $N(w), V(w)$. As pointed out in Remark 5.12, parts (1), (4) of (5.10) have been proved by different methods than those used here. Combining these parts with the tauberian argument, due to Landau (see appendix B), one knows the following more precise form of (5.27).

THEOREM 5.36. *Let $\rho_1 > \rho_2 > \cdots > \rho_\ell > \rho_1 - \frac{1}{nd} \geq \rho_{\ell+1} \cdots$ be the first $\ell + 1$ poles of $D_P(s, 1)$. Then there exist nonzero polynomials $A_1(u), \ldots, A_\ell(u) \in \mathbf{R}[u]$ such that*

$$N(w) = \sum_{i=1}^{\ell} w^{\rho_i} A_i(\log w) + O_\epsilon(w^{\rho_1 - \frac{1}{nd} + \epsilon}) \quad \text{as } w \to \infty.$$

A complete asymptotic expansion for $V_P(t)$ is also known to exist.

THEOREM 5.37. *Let $\lambda_1 > \lambda_2 > \cdots$ be the poles of $I_P(s, 1)$. Then there exist nonzero polynomials $B_1(u), B_2(u), \ldots, \in \mathbf{R}[u]$ such that*

$$V(w) = \sum_{i=1}^{\infty} w^{\lambda_i} B_i(\log w) \quad \text{as } w \to \infty.$$

One can now show

THEOREM 5.38. *The lattice point problem with $\varphi = 1$ is geometric and an explicit estimate can be written down in terms of the hypoellipticity exponent for $P\big|_{[1,\infty)^n}$ and the first two exponents appearing in (5.37).*

PROOF: Let $\epsilon > 0$. Define the sets

$$\begin{aligned}
\mathcal{U}_\epsilon(w) &= \{x \in [1, \infty)^n : P(x) \leq w + w^{1-\epsilon c_P}\}, \\
\mathcal{L}_\epsilon(w) &= \{x \in [1, \infty)^n : \|x\| \geq w^\epsilon - \frac{1}{2} \text{ and } P(x) \leq w - w^{1-\epsilon c_P}\}, \\
\ell_\epsilon(w) &= \{x \in [1, \infty)^n : \|x\| \leq w^\epsilon - \frac{1}{2} \text{ and } P(x) \leq w - w^{1-\epsilon c_P}\}, \\
\mathcal{C}_\epsilon(w) &= \bigcup_{\substack{m \in \mathbf{N}^n \\ \|m\| \geq w^\epsilon \\ P(m) \leq w}} C(m),
\end{aligned}$$

where

$$C(m) = \{x : |x_i - m_i| < 1/2, \text{ for each i}\}.$$

The following is easily verified.

$$vol(\mathcal{L}_\epsilon(w)) + vol(\ell_\epsilon(w)) = vol\left(\{x \in [1, \infty)^n : P(x) \leq w - w^{1-\epsilon c_P}\}\right), \tag{5.39}$$

$$vol(\ell_\epsilon(w)) \leq vol\left(\{x \in [1, \infty)^n : \|x\| \leq w^\epsilon - 1/2\}\right) \leq C(w^\epsilon - 1/2)^n.$$

for some $C > 0$. Now set $\beta = \lambda_1 - \lambda_2$, one has

$$\begin{aligned}
vol(\mathcal{U}_\epsilon(w)) &= \sum_{i=1}^{\infty} (w + w^{1-\epsilon c_P})^{\lambda_i} B_i\left(\log(w + w^{1-\epsilon c_P})\right) \\
&= (w + w^{1-\epsilon c_P})^{\lambda_1} B_1\left(\log(w + w^{1-\epsilon c_P})\right) + O_\kappa((w + w^{1-\epsilon c_P})^{\lambda_1 - \beta + \kappa}) \\
&= w^{\lambda_1} B_1(\log w) + O_\kappa(w^{\lambda_1 - \epsilon c_P + \kappa}) + O_\kappa(w^{\lambda_1 - \beta + \kappa})
\end{aligned} \tag{5.40.1}$$

and similarly

$$vol(\mathcal{L}_\epsilon(w)) = w^{\lambda_1} B_1(\log w) + O_\kappa(w^{\lambda_1 - \epsilon c_P + \kappa}) + O_\kappa(w^{\lambda_1 - \beta + \kappa}). \tag{5.40.2}$$

Proposition (5.2) implies, by means of the Taylor expansion of P around each point m, used in the definition of $\mathcal{C}_\epsilon(w)$, that for all w sufficiently large,

$$\mathcal{L}_\epsilon(w) \subset \mathcal{C}_\epsilon(w) \subset \mathcal{U}_\epsilon(w). \tag{5.41}$$

Moreover,

$$vol(\mathcal{C}_\epsilon(w)) = N(w) - \nu_\epsilon(w),$$

where

$$\nu_\epsilon(w) = \#\{m \in [1,\infty)^n \cap \mathbf{N}^n : \|m\| \leq w^\epsilon - \frac{1}{2}, \text{ and } P(m) \leq w\}.$$

Clearly,

$$\nu_\epsilon(w) = O(w^{\epsilon n}).$$

Thus, combining this estimate with (5.39)-(5.41) implies

$$N(w) = V(w) + O(w^{\epsilon n}) + O_\kappa(w^{\lambda_1 - \epsilon c_P + \kappa}) + O_\kappa(w^{\lambda_1 - \beta + \kappa}).$$

One can choose ϵ, κ so that two of the three exponents are equal, in which case the order is the larger of the two values. In either case, one will arrive at an "explicit" description of the order of $E(w)$. For example, suppose one chooses ϵ so that

$$\lambda_1 - \epsilon c_P + \kappa = \lambda_1 - \beta + \kappa,$$

that is,

$$\frac{\lambda_1 - \lambda_2}{c_P},$$

and discovers that this is useful in the sense that $\epsilon n < \lambda_1$. This occurs iff

$$\lambda_1 < \frac{\lambda_2}{1 - \frac{c_P}{n}}.$$

One concludes that if, in addition, $\lambda_1 < (1 + c_P)\lambda_2$ then an error estimate is $O_\kappa(w^{\lambda_2 + \kappa})$. However, if $\lambda_1 > (1 + c_P)\lambda_2$ then an error estimate is $O(w^{n(\lambda_1 - \lambda_2)/c_P})$. One then sees that "good" estimates for $\lambda_1 - \lambda_2$ and c_P, if such exist, could be used to give reasonable estimates for $E(w)$. However, so far, no work to the author's knowledge has treated this problem.

Appendix A

The purpose of this appendix is to give a proof of the uniform boundedness of the partial sums $\sum_{n=1}^{N} sin(2\pi nu)/n$, which is consistently needed in arguments involving estimates of trigonometric sums. The discussion is adapted from Lindelöf's book [L].

It is clear that $\psi(u)$, defined in Section 1, is periodic on $\mathbf{R}$ with period 1. It therefore admits a Fourier series, which can easily be calculated to equal

$$\psi(u) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{sin(2\pi nu)}{n}, \quad \text{with equality only for } u \notin \mathbf{Z}. \tag{A-1}$$

The series converges uniformly to $\psi(u)$ on any closed interval inside $\mathbf{R} - \mathbf{Z}$.

PROPOSITION A-2. *The partial sums $\sum_{n=1}^{N} sin(2\pi nu)/n$ are uniformly (in u, N) bounded* .

This will follow as an immediate corollary from Proposition A-4 proved below.

Let $u \in (0,1)$. Choose circles C_j of radii $r_1 < r_2 < r_3 < \dots$, centered at the origin in the complex z plane subject to the following condition for each j :

$$r_j \in (2\pi j, 2\pi(j+1)), \text{ and there exists } c > 0 \text{ such that } inf_{j,k}|r_j - 2\pi k| > c. \tag{A-3}$$

The tail of the Fourier series of $\psi(u)$ can be represented as

PROPOSITION A-4. *For each m*

$$\sum_{n=m+1}^{\infty}\frac{sin(2\pi un)}{\pi n} = \frac{-1}{2\pi i}\int_{C_m}\frac{e^{uz}}{e^z - 1}\frac{dz}{z}.$$

PROOF: Let $\epsilon > 0$ and form the set

$$D_\epsilon = \bigcup_{k\in\mathbf{Z}}\{z : |z - 2\pi ik| > \epsilon\}.$$

Thus, it is easy to see the existence of M_ϵ such that

$$\left|\frac{1}{e^z - 1}\right| < M_\epsilon \quad \text{if } z \in D_\epsilon.$$

Consider the function

$$\frac{e^{uz}}{e^z - 1}.$$

It is not difficult to show the existence of M'_ϵ such that

$$\left|\frac{e^{uz}}{e^z - 1}\right| < M'_\epsilon \quad \text{if } z \in D_\epsilon. \tag{A-5}$$

Choose $\epsilon = c$, defined in (A-3). Then there exists M_c' such that for each $m = 1, 2, \ldots$,

$$\left|\frac{e^{uz}}{e^z - 1}\right|_{C_m} < M_c' \quad \text{if } z \in C_m .$$

One now checks that

$$Res_{z=\pm 2\pi ik}\left(\frac{e^{uz}}{e^z-1}\cdot\frac{dz}{z}\right) = \frac{e^{\pm 2\pi iku}}{\pm 2\pi ik}.$$

Thus,

$$Res_{z=2\pi ik}\left(\frac{e^{uz}}{e^z-1}\cdot\frac{dz}{z}\right) + Res_{z=-2\pi ik}\left(\frac{e^{uz}}{e^z-1}\cdot\frac{dz}{z}\right) = \frac{sin(2\pi uk)}{\pi k}.$$

Using the easily verified property, valid for each $u \in (0,1)$, and ϵ sufficiently small:

$$\lim_{|z|\to\infty}\frac{e^{uz}}{e^z-1} = 0 \text{ uniformly in } Arg\, z \text{ for } Arg\, z \in (-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}+\epsilon) \cup (\frac{\pi}{2}+\epsilon, \frac{3\pi}{2}-\epsilon),$$

one shows that

$$\lim_{R\to\infty}\int_{|z|=R}\frac{e^{uz}}{e^z-1}\frac{dz}{z} = 0.$$

Thus, one concludes

$$\frac{-1}{2\pi i}\int_{C_m}\frac{e^{uz}}{e^z-1}\frac{dz}{z} = \sum_{k\geq m+1}\left[Res_{z=2\pi ik}\left(\frac{e^{uz}}{e^z-1}\cdot\frac{dz}{z}\right) + Res_{z=-2\pi ik}\left(\frac{e^{uz}}{e^z-1}\cdot\frac{dz}{z}\right)\right],$$

completing the proof. ∎

A consequence of use in Section 1 is the

PROPOSITION A-6. *If $g(u)$ is an integrable function on $[a,b]$ then*

$$\int_a^b g(u)\sum_{n=1}^{\infty}\frac{sin(2\pi nu)}{n}du = \sum_{n=1}^{\infty}\int_a^b g(u)\frac{sin(2\pi nu)}{n}du .$$

Appendix B

An integral formula of considerable use in the functional approach to lattice point problems, as well as many other problem requiring asymptotic information on averages of coefficients of Dirichlet series, is "Perron's formula". This gives an integral representation for the function

$$\kappa(w) = \begin{cases} 1 & \text{if } w > 1 \\ \frac{1}{2} & \text{if } w = 1 \\ 0 & \text{if } w < 1, \end{cases}$$

as follows, in which c is any positive number:

$$\kappa(w) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} w^s \, \frac{ds}{s} \,. \tag{B-1}$$

The integral is understood to be a principal value when $w = 1$. The proof of this is a standard exercise with residue calculus and left to the reader (cf. [Ti-2]).

In order to be useful however, as exemplified by the discussion in Section 2, one often needs a version of (B-1) with weights. Thus, let r be a positive integer. Define

$$[s]_r = s(s+1)\cdots(s+r).$$

A second use of residue calculus shows that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} w^s \, \frac{ds}{[s]_r} = \begin{cases} \frac{1}{r!}(1-\frac{1}{w})^r & \text{if } w \geq 1 \\ 0 & \text{if } w < 1. \end{cases} \tag{B-2}$$

Suppose one has a Dirichlet series

$$D(s) = \sum_k \frac{a_k}{\lambda_k^s}$$

for which one knows the existence of a (right) halfplane of absolute convergence and an analytic continuation as a meromorphic function outside this halfplane that satisfies two properties:

(1) In any vertical strip of bounded width there are only finitely many poles.
(2) There exists "moderate growth" at infinity in any vertical strip, that is, for any interval $[a, b]$ there exists $M(a, b)$ such that

$$|D(\sigma + it)| \ll |t|^{M(a,b)} \quad \text{for all } \sigma \in [a, b]\,,$$

then one can use (B-2) to give an asymptotic for a weighted sum of the a_k

$$N_P(w, r) =_{\text{def}} \frac{1}{r!} \sum_{\lambda_k \leq w} a_k \left(1 - \frac{\lambda_k}{w}\right)^r .$$

The most rudimentary version of the procedure is essentially the following. Assume that the series converges absolutely if $\sigma \geq c$. Suppose one wants to move the contour $\sigma = c$ to the line $\sigma = c'$, where the band $\{\sigma + it : \sigma \in (c', c)\}$ contains poles of the analytically continued series. Set $[nM(c', c)] + 2 = r$. Let $R(c', T)$ denote the rectangle, symmetric with respect to the σ axis, of height $2T$ and width $c - c'$, and whose vertical sides are contained on the two lines $\sigma = c, c'$. Let $\mathcal{P}$ denote the set of poles in this band. The choice of r now insures that

$$\lim_{T\to\infty} \int_{\{t=\pm T\}\cap R(c',T)} D(s)\, w^s \frac{ds}{[s]_r} = 0\,.$$

Thus, residue calculus implies

$$N_{\mathcal{P}}(w,r) = \sum_{\rho\in\mathcal{P}} Res_{s=\rho}\left(D(s)\, w^s \frac{ds}{[s]_r}\right) + \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} D(s)\, w^s \frac{ds}{[s]_r}\,.$$

Since the integral over the leftmost vertical line converges absolutely, the integral is $O(w^{c'})$. On the other hand, if the sum of residue terms is not zero then this sum is of order strictly larger than $w^{c'}$, so that one has extracted an asymptotic for $N(w, r)$.

The argument of Landau in section 2 is a considerable refinement of this procedure when the analytically continued series possesses a "reflection type" functional equation, of a kind often encountered in analytic number theory problems. On the other hand, for (geometric) lattice point problems not determined by quadratic polynomials, one almost never will have a reflection type functional equation. In this case, it is useful to know a modification of Landau's tauberian argument, presented in its most complete form in [La-2].

Suppose that the series $D(s)$, given above, has positive coefficients but does not satisfy a reflection type functional equation of the form (2.4). Instead, suppose only that it possesses an analytic continuation to the complex plane as a meromorphic function with moderate growth in any vertical strip of bounded width. A sketch of this modification will be given here. The author would like to thank P. Sargos for communicating the argument below.

Suppose that the first pole of $D(s)$ equals ρ_0. By a theorem of Landau, one knows that ρ_0 is real. Assume that each pole is real. Further, assume there exists $\mu > 0$ such that for any $\epsilon > 0, \sigma_1 < \sigma_2 \leq \rho$ there exists $C = C(\epsilon, \sigma_1, \sigma_2) > 0$ such that

$$|D(\sigma + it)| < C(1 + |t|^{\mu(\rho-\sigma)+\epsilon}) \tag{B-3}$$

for $\sigma \in [\sigma_1, \sigma_2]$ and $|t| \geq 1$. An application of the Phragman-Lindelöf theorem will show that the moderate growth satisfied by $D(s)$ implies (B-3).

One proves

THEOREM B-4. *Let $\rho_0 > \rho_1 > \ldots > \rho_k > \rho - \frac{1}{\mu} \geq \rho_{k+1} > \ldots$ denote the poles of $D(s)$. Then for any $\epsilon > 0$,*

$$N(w) = \sum_{i=0}^{k} Res_{s=\rho_i}\left(D(s)\, w^s \frac{ds}{s}\right) + O_\epsilon(w^{\rho_0 - \frac{1}{\mu}+\epsilon})\,.$$

SKETCH OF PROOF: Start with the identity, valid if $c > \rho_0$ by (B-1),

$$N(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} D(s)w^s \frac{ds}{s}.$$

Define the primitives $N_1(w), N_2(w), \ldots,$ as done in (2.8). Then the difference operator Δ_z applied to $N_1(w)$ can be realized as an integral operator $w \to \int_w^{w+z} N(w_1)dw_1$. Applying this operator to both sides of the above equation and then interchanging the two integrations gives

$$\Delta_z N_1(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_w^{w+z} D(s)\, w_1^s dw_1 \frac{ds}{s} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} D(s)\Delta_z (w_1^s)\frac{ds}{s}$$

where of course,

$$\Delta_z(w_1^s) = \int_w^{w+z} w_1^s dw_1 .$$

Iterating this v times and using (2.8), gives the formula

$$\Delta_z^{(v)} N_V(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} D(s)\Delta_z^{(v)} (w^s)\frac{ds}{s},$$

where

$$\begin{aligned}\Delta_z^{(v)} (w^s) &=_{\text{def}} \int_w^{w+z} dw_1 \int_{w_1}^{w_1+z} dw_2 \cdots \int_{w_{v-1}}^{w_{v-1}+z} w_v^s\, dw_v \\ &= \left(\prod_{i=1}^{v}\frac{1}{s+i}\right)\sum_{j=0}^{v}(-1)^j\binom{v}{j}(w+(v-j)z)^{s+v} .\end{aligned} \tag{B-5}$$

One now estimates the kernel $\Delta_z^{(v)}(w^s)$. When z is small relative to w one has

$$\Delta_z^{(v)} (w^s) - z^v w^s = \int_w^{w+z} dw_1 \int_{w_1}^{w_1+z} dw_2 \cdots \int_{w_{v-1}}^{w_{v-1}+z} (w_v^s - w^s)dw_v .$$

Since $w_v \in [w, w+vz]$ one can then estimate the integrand from above by

$$(w+vz)^s - w^s = w^s(1 + O(z/w)),$$

where the $O(\cdot)$ depends upon s and is bounded when s is confined to bounded subsets. Thus, for z small relative to w and s bounded

$$\Delta_z^{(v)} (w^s) = z^v w^s (1 + O(z/w)) . \tag{B-6}$$

Using (B-5) one can transport the line $\sigma = c$ to the left. In particular, for given v one moves to $\sigma = \rho - v/\mu + \epsilon$, where ϵ denotes an arbitrarily small parameter below. Let $\mathcal{P}_v$ denote the poles of $D(s)\Delta_z^{(v)}(w^s)$ encountered between $\sigma = c$ and $\sigma = \rho - v/\mu + \epsilon$. The estimate on $|D(s)|$ implies, by (B-5), (B-6),

$$\begin{aligned}\Delta_z^{(v)} N_v(w) &= \sum_{\rho \in \mathcal{P}_v} Res_{s=\rho}\left(D(s)\Delta_z^{(v)}(w^s)\right) + \frac{1}{2\pi i}\int_{\rho - v/\mu+\epsilon-i\infty}^{\rho - v/\mu+\epsilon+i\infty} D(s)\Delta_z^{(v)}(w^s)\,\frac{ds}{s} \\ &= \sum_{\rho \in \mathcal{P}_v} z^v\, Res_{s=\rho}\left(D(s)w^s\,\frac{ds}{s}\right) + O_\epsilon\left(z^{v+1}\, w^{\rho-1+\epsilon}\right) + O\left(w^{\rho+v-v/\mu+\epsilon}\right). \qquad \text{(B-7)}\end{aligned}$$

One obtains a similar expression with $\Delta_z^{(v)} N_v(w - vz)$. Guided by (2.10), one now divides by z^v and wants to find $z = w^c$ that insures the two error terms in (B-7), with $\rho = \rho_0$, have the same order in w. One finds that $z = w^{1-v/\mu(v+1)}$. This forces the two error terms to agree up to an arbitrary ϵ. The order in w of this error term is easily checked to be

$$O\left(w^{\rho_0 - \frac{v}{\mu(v+1)}+\epsilon}\right).$$

The same result is found with $\Delta_z^{(v)} N_v(w - vz)$. Since ϵ is arbitrarily small and v can be taken arbitrarily large, this implies that $N(w)$ itself differs from the sum of residues by $O\left(w^{\rho_0 - 1/\mu+\epsilon}\right)$ and establishes Theorem B-4. ∎

For many series of interest, it is usually not possible to prove the moderate growth condition in as strong a form as stated above. For such series, it is useful to have a "truncated Perron's formula" (without weights!) at one's disposal. The following formulation is taken from Titchmarsh's book [Ti-1, 3.12].

LEMMA B-8. *Let*

$$D(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}$$

be a Dirichlet series that has a single pole at $s = 1$ of order r. Suppose that $a_k = O(\phi(k))$ where $k \to \phi(k)$ is monotonically increasing. Then for any $c > 1$, and **integral** *valued parameter w*

$$\sum_{k \le w-1} a_k + \frac{a_w}{2} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(s)\, w^s \frac{ds}{s} + O\Big(\frac{w^c}{T(c-1)^r}\Big) + O\Big(\frac{\phi(2w)\, w \log w}{T}\Big) + O\Big(\frac{\phi(w)}{T}\Big).$$

Appendix C

Let $f, g : [0, \infty) \to \mathbf{R}$ be two functions. Set

$$G(x) = \begin{cases} \sum_{n \le x} g(n) & \text{if } x \ge 1 \\ 0 & \text{if } x < 1 . \end{cases}$$

PROPOSITION C-1. *(Partial Summation) For any $0 \le a < b$,*

$$\sum_{a<n\le b} f(n)g(n) = f([b])G([b]) - f([a]+1)G([a]) + \sum_{a<n\le b-1} (f(n) - f(n+1))\, G(n).$$

If $f(u) \in C^1[a,b]$, then

$$\sum_{a<n\le b} f(n)g(n) = f(b)G(b) - f(a)G(a) - \int_a^b f'(u)G(u)\, du.$$

PROOF: The first formula is simple. Substitute $g(n) = G(n) - G(n-1)$ in the sum. This gives

$$\begin{aligned}
\sum_{a<n\le b} f(n)g(n) &= \sum_{n=[a]+1}^{[b]} f(n)G(n) - \sum_{n=[a]}^{[b]-1} f(n+1)G(n) \\
&= f([b])G([b]) - f([a]+1)G([a]) + \sum_{n=[a]+1}^{[b]-1} (f(n) - f(n+1))\, G(n) \\
&= f([b])G(b) - f([a]+1)G(a) + \sum_{a<n\le b-1} (f(n) - f(n+1))\, G(n) . \quad \blacksquare
\end{aligned}$$

Now assume $f \in C^1[a,b]$. Then $f(n) - f(n+1) = -\int_n^{n+1} f'(u)du$. Since $G(x) = G(n)$ is constant for $x \in [n, n+1)$, one can write

$$G(n)\,(f(n) - f(n+1)) = -\int_n^{n+1} G(u)f'(u)du.$$

Thus,

$$\sum_{a<n\le b} f(n)g(n) = f([b])G(b) - f([a]+1)G(a) - \int_{[a]+1}^{[b]} G(u)f'(u)du. \qquad \text{(C-2)}$$

Furthermore,

$$\int_a^b G(u)f'(u)du = \int_a^{[a]+1} G(u)f'(u)du + \int_{[a]+1}^{[b]} G(u)f'(u)du + \int_{[b]}^b G(u)f'(u)du.$$

For the first resp. third integral, one has $G(u) = G([\![a]\!]) = G(a)$ resp. $G(u) = G([\![b]\!]) = G(b)$. Thus,

$$\int_{[\![a]\!]+1}^{[\![b]\!]} G(u)f'(u)du = \int_a^b G(u)f'(u)du - G(a)\left(f([\![a]\!]+1) - f(a)\right) - G(b)\left(f(b) - f([\![b]\!])\right).$$

Combining this equation with (C-2) gives the formula to be proved. ∎

Setting $g = 1$ implies $G(u) = [\![u]\!]$. Thus,

$$\sum_{a<n\leq b} f(n) = [\![b]\!]f(b) - [\![a]\!]f(a) - \int_a^b [\![u]\!]f'(u)du.$$

Replacing $[\![b]\!], [\![a]\!], [\![u]\!]$ by $b - \psi(b) - \frac{1}{2}, a - \psi(a) - \frac{1}{2}, u - \psi(u) - \frac{1}{2}$ and using the formula

$$\int_a^b (u - \frac{1}{2})f'(u)du = f(b)(b - \frac{1}{2}) - f(a)(a - \frac{1}{2}) - \int_a^b f(u)du$$

proves one version of the Euler-Maclaurin summation formula:

$$\sum_{a<n\leq b} f(n) = \int_a^b f(u)du - \psi(b)f(b) + \psi(a)f(a) + \int_a^b \psi(u)f'(u)du. \tag{C-3}$$

In order to apply integration by parts $k \geq 1$ times to this formula, a technique that can be used for analytic continuation of certain Dirichlet series, one needs a sequence of functions $\phi_1, \ldots, \phi_k$ such that $\phi_j' = \phi_{j-1}$ and $\phi_1' = \psi$. In addition, one must obviously assume $f \in C^{k+1}[a, b]$. For purposes of these notes, one imposes the following conditions upon f :

$$f(u+iv) \text{ is holomorphic if } u \geq 0 \quad \text{and} \quad \lim_{u\to\infty} f^{(k)}(u) = 0 \text{ for each } \ k \geq 1. \tag{C-4}$$

To construct the antiderivatives starting with ψ, one defines the Bernoulli polynomials as follows.

Define the functions $\phi_i(u)$ by expanding in a Laurent series at 0 the function

$$\frac{e^{uz}}{e^z - 1} = \frac{1}{z} + \sum_{i=1}^{\infty}{}' \frac{\phi_i(u)}{i!} z^{i-1}, \tag{C-5}$$

so that

$$\frac{\phi_i(u)}{i!} = Res_{z=0}\left(z^{-i} \cdot \frac{e^{uz}}{e^z - 1}\right).$$

In addition, define the numbers $B_1, B_2, \ldots$, by setting for $|z| \in (0, 2\pi)$,

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} \cdot z^{2i-1}\,.$$

Define

$$\begin{aligned} \sigma_{2i}(u) &= \phi_{2i}(u) + (-1)^i B_i \quad & i = 1, 2, \ldots, \\ \sigma_{2i+1}(u) &= \phi_{2i+1}(u) & i = 0, 1, \ldots\,. \end{aligned}$$

The B_i resp. σ_i are the Bernoulli numbers resp. polynomials.

An explicit description of the $\phi_i(u)$ can be determined by multiplying the Laurent series in (C-5) by the series expansion for e^{uz} and extracting the coefficient of $1/z$. One finds

$$\phi_i(u) = u^i - \frac{i}{2} u^{i-1} + \binom{i}{2} B_1 u^{i-2} - \binom{i}{4} B_2 u^{i-4} + \ldots + (-1)^{k-1} \binom{i}{2k} B_k u^{i-2k} + \ldots\,.$$

Now expand out these polynomials in trigonometric series when u is restricted to $[0, 1]$.

PROPOSITION C-6. *For* $u \in [0, 1]$ *and* $k = 1, 2, \ldots$,

$$(1) \quad \phi_{2k}(u) = (-1)^{k+1} 2 \cdot (2k)! \sum_{j=1}^{\infty} \frac{cos(2j\pi u)}{(2j\pi)^{2k}}\,.$$

$$(2) \quad \phi_{2k+1}(u) = (-1)^{k+1} 2 \cdot (2k+1)! \sum_{j=1}^{\infty} \frac{sin(2j\pi u)}{(2j\pi)^{2k+1}}\,.$$

PROOF: As in the proof of Proposition A-4, one has

$$\lim_{m \to \infty} \frac{1}{2\pi i} \int_{C_m} \frac{e^{uz}}{e^z - 1} \frac{dz}{z^n} = \sum_{k=-\infty}^{\infty} Res_{z = 2\pi i k} \left(\frac{e^{uz}}{e^z - 1} \frac{dz}{z^n} \right)\,. \tag{C-7}$$

For $n \geq 1$

$$\lim_{m \to \infty} \left. \frac{e^{uz}}{e^z - 1} \cdot \frac{1}{z^n} \right|_{C_m} = 0 \quad \text{uniformly in } z\,.$$

Thus, the left side of (C-7) equals 0. By definition, $\phi_n(u)/n!$ equals the residue at $z = 0$. A simple calculation now proves (1), (2). ∎

Remark. One denotes by $\overline{\phi}_i(u), i = 2, 3, \ldots$ the periodic function defined by the series in (1), (2). Similarly, one sets $\overline{\sigma}_i(u), i \geq 2$, to be the periodic functions defined, using the same formulae as in the definitions of $\sigma_i(u)$, but using the $\overline{\phi}_i$ instead. These functions must agree with the corresponding $\phi_i(u)$ for $u \in [0, 1]$. Also, one sets $\overline{\phi}_1(u)$ to equal the series corresponding to (2) with $k = 0$. This just gives a new notation for $\psi(u)$. One then notes that $\overline{\phi}_1(u) = u - \frac{1}{2}$ only if $u \in (0, 1)$. ∎

The following properties are easily verified.

PROPOSITION C-8.

(1) For $i = 1, 2, \ldots$, and $u \in [0,1]$, $\overline{\phi}'_{i+1}(u) = (i+1)\overline{\phi}_i(u)$;

(2) $\lim_{u\to m^-} \overline{\phi}_1(u) = 1/2$, $\lim_{u\to m^+} \overline{\phi}_1(u) = -1/2$.

Given $a < b$, set $M = [\![a]\!] + 1, N = [\![b]\!]$. Then, using the notation $\overline{\phi}_1$ in place of ψ in (C-4), one has

$$\sum_M^N f(n) = \int_a^b f(u)du + \overline{\phi}_1(a)f(a) - \overline{\phi}_1(b)f(b) + \int_a^b \overline{\phi}_1(u)f'(u)du\,, \tag{C-9}$$

where a resp. b is now understood to be any number in $(M-1, M)$ resp. $(N, N+1)$.

Now assume that f satisfies the decay at infinity condition (C-4). Then (C-9) can be extended as follows. First, one can let $b \to \infty$. The resulting summatory formula is:

$$\sum_M^\infty f(n) = \int_M^\infty f(u)du + \overline{\phi}_1(a)f(a) + \int_a^\infty f'(u)\overline{\phi}_1(u)du\,.$$

Now let $a \nearrow M$. By (C-8)(2), $\overline{\phi}_1(a) \to 1/2$. Thus,

$$\sum_M^\infty f(n) = \int_M^\infty f(u)du + \frac{f(M)}{2} + \int_M^\infty f'(u)\overline{\phi}_1(u)du\,.$$

For any $b > a$ one uses (C-8)(1) to write

$$\int_a^b f'(u)\overline{\phi}_1(u)du = \frac{f'(u)\overline{\phi}_2(u)}{2}\Bigg|_a^b - \frac{1}{2}\int_a^b f''(u)\overline{\phi}_2(u)du\,.$$

Again, (C-4) implies that one can take the limit as $b \to \infty$ and obtain, after simplification,

$$\sum_M^\infty f(n) = \int_M^\infty f(u)du + \frac{f(M)}{2} - \frac{f'(M)\overline{\phi}_2(M)}{2} - \frac{1}{2}\int_M^\infty f''(u)\overline{\phi}_2(u)du\,.$$

Continuing this k times yields

$$\begin{aligned}\sum_M^\infty f(n) = \int_M^\infty f(u)du + \frac{f(M)}{2} - \frac{f'(M)\overline{\phi}_2(M)}{2} + \cdots (-1)^{k-1}\frac{f^{(k-1)}(M)\overline{\phi}_k(M)}{k!}\\ + \frac{(-1)^k}{k!}\int_M^\infty f^{(k)}(u)\overline{\phi}_k(u)du\,.\end{aligned} \tag{C-10}$$

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