On Affine Hypersurfaces with Parallel Nullity

Katsumi Nomizu
and
Barbara Opozda

Katsumi Nomizu
Department of Mathematics
Brown University
Providence, RI 02912
USA

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

Barbara Opozda
Instytut Matematyki, UJ
UL, Reymonta 4
30-059 Kraków
Poland

MPI / 91-74

On Affine Hypersurfaces with Parallel Nullity

Katsumi Nomizu (*) and Barbara Opozda (**)

Affine differential geometry for hypersurfaces in the classical sense of Blaschke is based on the hypothesis that the given hypersurface is nondegenerate (quote from [B, p.104]: Für parabolisch gekrümmte Flächen ("Torsen", $LN - M^2 = 0$) versagt die Grundform.). In relative geometry (for example, see [S]) and in the study of affine immersions [N-P1], [N-P2], the nondegeneracy condition is often important, although a few results (for example, Berwald'stheorem [N-P2], Radon's theorem [O]) have been established under a somewhat weaker assumption on the rank of the fundamental form h.

In this paper, we examine a general condition weaker than nondegeneracy under which geometry of a given hypersurface can be reduced to the classical situation. We start with an immersion $f : M^n \to R^{n+1}$. *For an arbitrary choice of a transversal vector field* ξ, *consider the condition that the kernel of* h *be parallel relative to the connection* ∇ *induced by* ξ. *It turns out that this condition is independent of a choice of* ξ. Under this condition of parallel nullity and under a completeness assumption which is also intrinsic, we shall show that f is globally a cylinder immersion of the form $M^n = M^r \times L, f = f_1 \times f_0$, where $f_1 : M^r \to R^{r+1}$ is a nondegenerate hypersurface, L is a leaf of T^0, and f_0 is a connection-preserving map of L of T^0 onto R^{n-r}, where R^{r+1} and R^{n-r} are affine subspaces in R^{n+1} which are mutually transversal. Such a representation is unique up to an equiaffine transformation. Thus the geometry of M^n is completely determined by that of a profile nondegenerate hypersurface M^r in R^{r+1} which is itself uniquely determined up to equiaffine equivalence. For later applications we include additional information on transversal vector fields.

(*) The work of the first author is supported by an Alexander von Humboldt research award at Technische Universität Berlin and Max-Planck- Institut für Mathematik, Bonn.

(**) The work of the second author is supported by an Alexander von Humboldt research fellowship at Universität zu Köln and Max-Planck-Institut für Mathematik, Bonn.

1. Preliminaries.

Let $f : M^n \to R^{n+1}$ be a connected hypersurface immersed in the affine space R^{n+1} provided with a fixed determinant function (volume element). Around each point of M^n let ξ be an arbitrarily chosen transversal vector field. As usual, we write

$$(I) \qquad D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi$$

and

$$(II) \qquad D_X \xi = -f_*(SX) + \tau(X)\xi,$$

where X, Y are vector fields on M^n, ∇ is the induced connection on M^n, h the affine fundamental form, S the shape operator, and τ the transvesal connection form, all depending on the chosen ξ. The following lemma is standard.

Lemma 1. *If we change ξ to another transversal vector field $\overline{\xi} = (f_*Z+\xi)/\lambda$, where Z is a certain vector field on M^n and λ a positive function, then the induced connection, the affine fundamental form, the transversal connection form, and the shape operator change as follows:*

$$(1) \qquad \overline{h} = \lambda h;$$

$$(2) \qquad \overline{\nabla}_X Y = \nabla_X Y - h(X,Y)Z;$$

$$(3) \qquad \overline{\tau} = \tau + \eta - d(\log\lambda),$$

where η is the 1-form such that $\eta(X) = h(X,Z)$ for all X;

$$(4) \qquad \overline{S}X = [SX - \nabla_X Z + \tau(X)Z + h(X,Z)Z]/\lambda.$$

By virtue of (1) we see that the rank of $\overline{h}$ at a point x is the same as that of h at x. We call it the rank of f at x. We also see that the null space $\{X : h(X,Y) = 0 \text{ for all } Y\}$ at x is the same as the null space of $\overline{h}$ at x. This null space of h is denoted by $T^0(x)$. We shall say that T^0 is parallel relative to ∇ if, for any curve from x to y, parallel translation along the curve maps $T^0(x)$ onto $T^0(y)$. In this case, the dimension of $T^0(x)$ remains constant on M^n. In general, it is known that a differentiable distribution, say T^0, is parallel if and only if for any vector field $Y \in T^0$ we have $\nabla_X Y \in T^0$ for every vector field X.

Lemma 2. *The condition that T^0 is parallel relative to ∇ is independent of a choice of transversal vector field.*

Proof. Suppose T^0 is parallel relative to ∇. For any curve $x_t, 0 \leq t \leq 1$, and for any ∇-parallel $Y_t \in T^0$, we have by (2)

$$\overline{\nabla}_t Y_t = \nabla_t Y_t - h(X_t, Y_t)Z = \nabla_t Y_t = 0,$$

where X_t is the tangent vector field of x_t. Thus Y_t is $\overline{\nabla}$-parallel. This means that T^0 is $\overline{\nabla}$-parallel.

From now on, we assume that our hypersurface satisfies the condition of parallel nullity (that is, T^0 is parallel relative to ∇). The distribution T^0 being parallel, it is integrable and totally geodesic. We say that T^0 is complete if each leaf L of T^0 is complete relative to ∇, that is, every ∇-geodesic in L extends infinitely for its affine parameter. In this regard we have

Lemma 3. *On each leaf L of T^0 the induced connection ∇ is the same for any choice of ξ. In particular, the property that T^0 is complete is independent of a choice of ξ.*

Proof. If X, Y are vector fields on L, then we have $\overline{\nabla}_X Y = \nabla_X Y - h(X, Y)Z = \nabla_X Y$. Thus two connections ∇ and $\overline{\nabla}$ coincide on L.

From (I), we easily get

Lemma 4. *For every leaf L of T^0, $f(L)$ is a totally geodesic submanifold in R^{n+1}. If T^0 is complete, then $f(L)$ is an entire affine subspace of dimension $s = \dim T^0$; f actually gives a connection-preserving diffeomorphism of L onto the affine subspace $f(L)$. Moreover, for two distinct leaves L_1 and L_2 of T^0, $f(L_1)$ and $f(L_2)$ are affine subspaces which are D-parallel in R^{n+1}.*

Remark 1. If the connection ∇ induced by some transversal vector field ξ is complete and if T^0 is parallel, then T^0 is complete.

Remark 2. If an affine hypersurface $f : M^n \to R^{n+1}$ has the property that $\nabla h = 0$ for some choice of transversal vector field, then it obviously satisfies the condition of parallel nullity.

Remark 3. For an affine hypersurface $f : M^n \to R^{n+1}$, the Gauss equation implies that for each point $x \in M^n$ we have

$$T^0(x) \subset \cap_{X,Y \in T_x(M^n)} \ker R(X, Y).$$

The two subspaces coincide if the rank of S is > 1 or if rank $h = 1$ and $R \neq 0$. If they coincide at every point and if $\nabla R = 0$, then T^0 is parallel.

We add the following facts for later use. Assume that two transversal vector fields ξ and $\overline{\xi}$ coincide mod T^0, that is, $\overline{\xi} = \xi + f_*(Z)$, where $Z \in T^0$. Then from Lemma 1 we see that

$$\overline{h} = h \text{ and } \overline{\tau} = \tau$$

$$\overline{S} = S \bmod T^0, \text{ and } \overline{\nabla} = \nabla \bmod T^0,$$

that is, $\overline{\nabla}_X Y - \nabla_X Y \in T^0$ for all vecotr fields X, Y. Now using these facts it is easy to establish the following.

Lemma 5. *Assume that $\overline{\xi} = \xi$ mod T^0. Then we have*

$$\overline{\nabla h} = \nabla h, \tag{5}$$

$$\overline{R} = R \bmod T^0, \tag{6}$$

that is, $\overline{R}(X,Y)W - R(X,Y)W \in T^0$ for all X, Y, W.

Moreover, if ξ satisfies $ST^0 \subset T^0$, then

$$\overline{\nabla S} = \nabla S \bmod T^0, \tag{7}$$

$$\overline{\nabla R} = \nabla R \bmod T^0. \tag{8}$$

2. Global cylinder representation of a hypersurface M^n

We now prove the following theorem.

Theorem. *Let $f : M^n \to R^{n+1}$ be a connected hypersurface such that its affine fundamental form h has parallel kernel T^0. Assume that T^0 is complete. Then we can express $f : M^n \to R^{n+1}$ as follows: $M^n = M^r \times L, f = f_1 \times f_0$, where $f_1 : M^r \to R^{r+1}$ is a connected nondegenerate hypersurface and f_0 is a connection-preserving map of a leaf L of T^0 onto R^{n-r}, and $R^{n+1} = R^{r+1} \times R^{n-r}$. Such a representation is unique up to an equiaffine transformation of R^{n+1} so that a nondegenerate profile hypersurface M^r is determined uniquely up to an equiaffine transformation of R^{r+1}.*

Proof. Let x_0 be an arbitrary but fixed point of M^n. For the leaf L_0 through x_0 of T_0, $f(L_0)$ is an entire affine subspace of dimension $s = n - r$ through $o = f(x_0)$ in R^{n+1}. Call it R^s. For any point $p \in R^{n+1}$ we denote by $R^s(p)$ the s-dimensional affine subspace through p which is parallel to

R^s. Again from Lemma 4 we know that if $x \in M^n$, then the image by f of the leaf L through x coincides with $R^s(f(x))$. Let us choose an affine subspace of dimension $r+1$, say, R^{r+1} through $f(x_0)$ which is transversal to R^s. Again, for any point p in R^{n+1} the $(r+1)$-dimensional affine subspace through p and parallel to R^{r+1} will be denoted by $R^{r+1}(p)$. The mapping $f : M^n \to R^{n+1}$ is then transversal to R^{r+1}. In fact, for any $x \in M^n$ such that $p = f(x) \in R^{r+1}$ we have $T_p(R^{n+1}) = T_p(R^s) + f_*(T_x(M^n))$, because $f_*(T_x(M^n))$ contains $R^s(p) = f(L)$, where L is the leaf of T^0 through x. By a well known theorem (for example, see [H], p.22]), it follows that $M^r = \{x \in M^n : f(x) \in R^{r+1}\}$ is an r-dimensional submanifold of M^n. We see that the restriction of $f : M^n \to R^{n+1}$ to M^r gives rise to a hypersurface $f_1 : M^r \to R^{r+1}$; we shall show that M^r is connected in a moment. In the case where the original immersion $f : M^n \to R^{n+1}$ is an imbedding, we may think of M^r as the intersection of M^n with R^{r+1}.

Now we define a one-to-one map $\Phi : M^n \to M^r \times L_0$ as follows. We consider $o = f(x_0)$ as the origin of R^{n+1}, R^s, and R^{r+1}, whenever we need a reference point in each of these affine spaces. Now for any $x \in M^n$, we define

$$\Phi(x) = (y, z) \in M^r \times L_0,$$

where y, z are determined as follows. Consider $p = f(x)$. For the leaf $L(x)$ of T^0 through x, $f(L(x))$ is the affine subspace $R^s(p)$, which meets R^{r+1} at a certain unique point, say, q. Since f maps one-to-one on $L(x)$, there is a unique point $y \in L(x) \subset M^n$ such that $f(y) = q$. This means $y \in M^r$. On the other hand, the vector from q to p is parallel to the vector from o to z, where z is a certain uniquely determined point of R^s. It is now easy to find the inverse map $M^r \times L \to M^n$ of Φ. Since Φ is differentiable, the existence of the projection $M^n \to M^r$ shows that M^r is connected. So we get a cylinder representation of M^n with a profile hypersurface M^r.

We have still to prove the uniqueness of such a representation. For this purpose we use the following lemma in analytic geometry which is easy to prove.

Lemma 5. *Let R^s be a fixed affine subspace of the affine space R^{n+1}. Suppose R^{r+1} and $\overline{R}^{r+1}$ are two affine subspaces that are transversal to map F of R^{r+1} onto $\overline{R}^{r+1}$ as follows: for each point $x \in M^{r+1}$, let $R^s(x)$ denote the affine subspace through x that is parallel to R^s. We let $\overline{x}$ be a uniquely determined point of intersection with $\overline{R}^{r+1}$ and set $F(x) = \overline{x}$. Then F is an affine transformation of R^{r+1} onto $\overline{R}^{r+1}$. Moreover, F is equiaffine (that is volume-preserving) if we fix a determinant function (parallel volume element) ω_{n+1} on R^{r+1} and a determinant function ω_s on R^s and define* *determinant functions ω_{r+1} and $\overline{\omega}_{r+1}$ on R^{r+1} and $\overline{R}^{r+1}$, respectively, such that $\omega_{n+1} = \omega_{r+1} \wedge \omega_s$ and $\omega_{n+1} = \overline{\omega}_{r+1} \wedge \omega_s$.*

Now suppose $\overline{\Phi} : M^{n+1} \to \overline{M}^r \times \overline{L}$ is another cylinder representation, where $\overline{f}_1 : \overline{M}^r \to \overline{R}^{r+1}$ is a nondegenerate hypersurface of $\overline{R}^{r+1}$ and $\overline{f}_0 : \overline{L} \to \overline{R}^s$ is a connection-preserving map of a leaf $\overline{L}$ of T^0 onto an affine subspace $\overline{R}^s$ transversal to $\overline{R}^{r+1}$. We may assume, without loss of generality, that $L = \overline{L}, R^s = \overline{R}^s$, and $f_0 = \overline{f}_0$. Then we get an equiaffine transformation $F_1 : R^{r+1} \to \overline{R}^{r+1}$ in the manner of Lemma 5. Combining this with the identity map: R^s we get an equiaffine transformation, denoted by F, of R^{n+1} onto itself. It is now clear that $F_1(M^r) = \overline{M}^r$ and $\overline{\Phi} = F \circ \Phi$. This completes the proof of the theorem.

Corollary. *Under the assumption of the theorem, we can find a unique transversal vector field ξ for M^n with the following properties:*

1) *ξ is D-parallel in the direction of T^0; the affine shape operator vanishes on T^0.*

2) *The restriction of ξ to a profile hypersurface M^r coincides with the affine normal of the nondegenerate hypersurface M^r.*

Such ξ is unique once a profile hypersurface is chosen.

Remark 4. If we do not assume the completeness for T^0, then for any point x_0 of M^n we can get a local cylinder decomposition of a neighborhood U of x_0 in the form $V \times W$, where V is a nondegenerate hypersurface in R^{r+1} and W is an open subset of R^s.

We add some more information on the relationship between the geometry of M^n and that of M^r. Continuing the notation in the proof of the theorem, we define a distribution T^1 by

$$T^1_x = f_* x^{-1}(R^{r+1}) \text{ for each } x \in M^n,$$

where R^{r+1} is now considered as the vector subspace instead of the affine space R^{r+1} through $f(x_0)$ This distribution is obviously integrable. We denote by π the projection of the vector space R^{n+1} onto R^{r+1} (parallel to the subspace R^s). We also denote by the same symbol the projection of TM onto T^1 parallel to T^0 so that $f_* \circ \pi = \pi \circ f_*$. Let ξ be a transversal vector field to f. We define $\overline{\xi} = \pi \circ \xi$. Then $\overline{\xi}$ is also transversal to f and equal to ξ mod T^0. By the formulas preceding Lemma 5 and those in Lemma 5 we have

Proposition.

$$\overline{h} = h, \qquad \overline{\tau} = \tau, \qquad \overline{S} = \pi \circ S, \qquad \overline{\nabla}_X Y = \pi(\nabla_X Y)$$

$$\overline{R}(X,Y)W = \pi(R(X,Y)W),$$

$$(\overline{\nabla}_X\overline{S})(Y) = \pi(\nabla_X S)(Y),$$

and

$$(\overline{\nabla}_W\overline{R})(X,Y)V = \pi(\nabla_W R)(X,Y)V);$$

for the last two identities we need to assume that ξ satisfies condition $ST^0 \subset T^0$ in Lemma 5. Moreover, the same relations hold if $\overline{\nabla}$ is considered the connection on M^r (that is, the restriction to M^r).

Remark 5. If ξ is assumed to be equiaffine, then certainly all the identities in Lemma 5 hold. Moreover, $\overline{\xi}$ is parallel relative to D along T^0.

Combining Remarks 3, 5 and the last identity in the proposition we obtain

Corollary. *Assume ξ is an equiaffine transversal vector field to a hypersurface $f : M^n \to R^{n+1}$ such that $\nabla R = 0$. If rank $S > 1$ everywhere, then M^n is locally a cylinder $M^r \times R^s$ and $\overline{\nabla}$ on M^r is locally symmetric.*

References

[B] Blaschke, W.: Vorlesungen über Differentialgeometrie II, Affine Differentialgeometrie, Springer, Berlin, 1923

[H] Hirsch, M. W.: Differential Topology, Springer, New York, Heidelberg, Berlin, 1976

[N-P1] Nomizu, K. and Pinkall, U.: *On the geometry of affine immersions* Math. Z. 195(1987), 165-178

[N-P2] Nomizu, K. and Pinkall, U.: *Cubic form theorem for affine immersions,* Results ini Math. 13(1988), 338-362

[O] Opozda, B.: *Some extensions of Radon's theorem*, to appear in Proceedings of the Conference on Global Analysis and DifferentialGeoemtry, Berlin, 1990.

[S] Simon, U.:*Zur Entwickelung der affinen Differentialgeometrie nach Blaschke,* Gesam. Werke Blaschke, Band 4, Thales, Essen, 1985)

Katsumi Nomizu
Department of Mathematics
Brown University
Providence, RI 02912
USA

Barbara Opozda
Instytut Matematyki, UJ
UL, Reymonta 4
30-059 Kraków
Poland