Max-Planck-Institut für Mathematik
Bonn

Surjective separating maps on noncommutative L^p-spaces

by

Christian Le Merdy
Safoura Zadeh

Max-Planck-Institut für Mathematik
Preprint Series 2020 (49)

Date of submission: September 12, 2020

Surjective separating maps on noncommutative L^p-spaces

by

Christian Le Merdy
Safoura Zadeh

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Laboratoire de Mathématiques de Besançon
Université Bourgogne Franche-Comté
25030 Besançon Cedex
France

MPIM 20-49

SURJECTIVE SEPARATING MAPS ON NONCOMMUTATIVE L^p-SPACES

CHRISTIAN LE MERDY AND SAFOURA ZADEH

Abstract. Let $1 \leq p < \infty$ and let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded map between noncommutative L^p-spaces. If T is bijective and separating (i.e., for any $x, y \in L^p(\mathcal{M})$ such that $x^*y = xy^* = 0$, we have $T(x)^*T(y) = T(x)T(y)^* = 0$), we prove the existence of decompositions $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$, $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and maps $T_1\colon L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$, $T_2\colon L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$, such that $T = T_1 + T_2$, T_1 has a direct Yeadon type factorization and T_2 has an anti-direct Yeadon type factorization. We further show that T^{-1} is separating in this case. Next we prove that for any $1 \leq p < \infty$ (resp. any $1 \leq p \neq 2 < \infty$), a surjective separating map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded (resp. completely bounded) if and only if there exists a decomposition $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorization and $\mathcal{M}_2$ is subhomogeneous.

1. Introduction

This paper deals with separating maps between noncommutative L^p-spaces, $1 \leq p < \infty$. These operators were investigated recently in [1, 4, 5], to which we refer for background, motivation and historical facts. Recall that a bounded map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ between two noncommutative L^p-spaces is called separating if for any $x, y \in L^p(\mathcal{M})$, the condition $x^*y = xy^* = 0$ implies that $T(x)^*T(y) = T(x)T(y)^* = 0$. It was shown in [4, Proposition 3.11] and [1, Theorem 3.3 & Remark 3.4] that $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is separating if and only if there exist a w^*-continuous Jordan homomorphism $J\colon \mathcal{M} \to \mathcal{N}$, a positive operator B affiliated with $\mathcal{N}$ and commuting with the range of J, as well as a partial isometry $w \in \mathcal{N}$ such that $w^*w = s(B) = J(1)$ and

$$T(x) = wBJ(x), \qquad (x \in \mathcal{M} \cap L^p(\mathcal{M})).$$

Such a factorization (which is necessarily unique) is called a Yeadon type factorization in [4, 5]. We further say that T admits a direct Yeadon type factorization if the Jordan homomorphism J in this factorization is a $*$-homomorphism. It is proved in [5, Proposition 4.4] and [1, Theorem 3.6] that any separating map $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ with a direct Yeadon type factorization is necessarily completely bounded. It is also proved in [5, Proposition 4.5] that any such map is S^1-bounded (see Section 2 below for the definition). The main purpose of the present paper is to establish a form of converse of these results for surjective maps. More precisely, we prove the following characterizations.

Theorem. Let $1 \leq p < \infty$, let $\mathcal{M}, \mathcal{N}$ be semifinite von Neumann algebras and let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a surjective separating map. The following are equivalent :

(i) T is S^1-bounded;

Key words and phrases. von Neumann algebras, noncommutative L^p-spaces, separating maps, operator spaces.

(ii) There exists a direct sum decomposition $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ such that the restriction of T to $L^p(\mathcal{M}_1)$ has a direct Yeadon type factorization and $\mathcal{M}_2$ is subhomogeneous.

Moreover if $p \neq 2$, then (ii) is also equivalent to :

(iii) T is completely bounded.

These results will be proved in Section 4. We also provide an example showing that the surjectivity assumption cannot be dropped. In section 3, we establish a general decomposition result for bijective separating maps which plays a key role in the above characterization results. We prove in passing that the inverse of any bijective separating map is separating as well. Section 2 is preparatory.

2. Background

In this section we recall some necessary background on semifinite noncommutative L^p-spaces and subhomogeneous von Neumann algebras.

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal semifinite faithful trace $\tau_{\mathcal{M}}$. Assume that $\mathcal{M} \subset B(\mathcal{H})$ acts on some Hilbert space $\mathcal{H}$. Let $L^0(\mathcal{M})$ denote the $*$-algebra of all closed densely defined (possibly unbounded) operators on $\mathcal{H}$, which are $\tau_{\mathcal{M}}$-measurable. Then for any $1 \leq p < \infty$, the noncommutative L^p-space associated with $\mathcal{M}$ can be defined as

$$L^p(\mathcal{M}) := \big\{x \in L^0(\mathcal{M}) \,:\, \tau_{\mathcal{M}}(|x|^p) < \infty\big\}.$$

We set $\|x\|_p := \tau_{\mathcal{M}}(|x|^p)^{\frac{1}{p}}$ for any $x \in L^p(\mathcal{M})$. Then $L^p(\mathcal{M})$ equipped with $\|\cdot\|_p$ is a Banach space. The reader may consult [3,8,12] and the references therein for details and further properties.

We let S^p, $1 \leq p < \infty$, denote the noncommutative L^p-space built upon $B(\ell^2)$ with its usual trace; this is in fact the Schatten p-class of operators on ℓ^2. For any $m \geq 1$, we let S^p_m denote the Schatten p-class of $m \times m$ matrices. Whenever E is an operator space, we let $S^p_m[E]$ denote the E-valued Schatten space introduced in [6, Chapter1].

Recall that we may identify $L^p(\mathcal{M} \otimes M_m)$ with $L^p(\mathcal{M}) \otimes S^p_m$ in a natural way. Let $\mathcal{N}$ be, possibly, another semifinite von Neumann algebra. We say that an operator $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is completely bounded if there exists a constant $K \geq 0$ such that

$$\|T \otimes I_{S^p_m} : L^p(\mathcal{M} \otimes M_m) \to L^p(\mathcal{N} \otimes M_m)\| \leq K,$$

for any $m \geq 1$. In this case, the completely bounded norm of T is the smallest such uniform bound and is denoted by $\|T\|_{cb}$. We further say that T is a complete isometry if $T \otimes I_{S^p_m}$ is an isometry for any $m \geq 1$.

In [5, Section 3], we introduced S^1-valued noncommutative L^p-spaces, which naturally extend previous constructions from [2,6]. We recall this definition here.

For $1 \leq p < \infty$, the S^1-valued noncommutative L^p-space, $L^p(\mathcal{M}; S^1)$, is the space of all infinite matrices $[x_{ij}]_{i,j\geq 1}$ in $L^p(\mathcal{M})$ for which there exist families $(a_{ik})_{i,k\geq 1}$ and $(b_{kj})_{k,j\geq 1}$ in $L^{2p}(\mathcal{M})$ such that $\sum_{i,k} a_{ik}a_{ik}^*$ and $\sum_{k,j} b_{kj}^*b_{kj}$ converge in $L^p(\mathcal{M})$ and for all $i, j \geq 1$,

$$x_{ij} = \sum_{k=1}^{\infty} a_{ik}b_{kj}.$$

We equip $L^p(\mathcal{M};S^1)$ with the following norm

$$\|[x_{ij}]\|_{L^p(\mathcal{M};S^1)} = \inf\left\{ \Big\| \sum_{i,k=1}^{\infty} a_{ik}a_{ik}^* \Big\|_p^{\frac{1}{2}} \Big\| \sum_{k,j=1}^{\infty} b_{kj}^* b_{kj} \Big\|_p^{\frac{1}{2}} \right\}, \tag{1}$$

where the infimum is taken over all families $(a_{ik})_{i,k\geq 1}$ and $(b_{kj})_{k,j\geq 1}$ as above. The space $L^p(\mathcal{M};S^1)$ endowed with this norm is a Banach space.

For any integer $m \geq 1$, we let $L^p(\mathcal{M};S^1_m)$ be the subspace of $L^p(\mathcal{M};S^1)$ of matrices $[x_{ij}]_{i,j\geq 1}$ with support in $\{1,\ldots,m\}^2$.

Following [5, Definition 3.8], we say that a bounded operator $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded if there exists a constant $K \geq 0$ such that

$$\|T \otimes I_{S^1_m} : L^p(\mathcal{M};S^1_m) \longrightarrow L^p(\mathcal{N};S^1_m)\| \leq K,$$

for any $m \geq 1$. In this case, the S^1-bounded norm of T is the smallest such uniform bounded and is denoted by $\|T\|_{S^1}$. We further say that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is an S^1-isometry if for each $m \geq 1$, $T \otimes I_{S^1_m}$ is an isometry.

We proved in [5] that for any $n \geq 1$, $L^p(M_n;S^1_m) = S^p_n[S^1_m]$ isometrically. Further, if $\mathcal{M},\mathcal{N}$ are hyperfinite, then $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is S^1-bounded if and only if it is regular in the sense of [7].

We note that any direct sum $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ induces isometric identifications $L^p(\mathcal{M}) = L^p(\mathcal{M}_1) \overset{p}{\oplus} L^p(\mathcal{M}_2)$ and $L^p(\mathcal{M};S^1) = L^p(\mathcal{M}_1;S^1) \overset{p}{\oplus} L^p(\mathcal{M}_2;S^1)$ (see [5, Lemma 5.2] for the last identification).

Recall that a C^*-algebra $\mathcal{A}$ is called subhomogeneous of degree $\leq N$ if all irreducible representations of $\mathcal{A}$ are of maximum dimension N. If $\mathcal{A}$ is subhomogeneous of degree $\leq N$, for some N, we simply say that $\mathcal{A}$ is subhomogeneous. It is well-known (see for example [9, Theorem 7.1.1]) that $\mathcal{M}$ is a subhomogeneous von Neumann algebra of degree $\leq N$ if and only if there exist $r \geq 1$, integers $1 \leq n_1 \leq n_2 \leq \ldots \leq n_r \leq N$ and abelian von Neumann algebras $L^\infty(\Omega_1),\ldots,L^\infty(\Omega_r)$ such that

$$\mathcal{M} \simeq \overset{\infty}{\underset{1\leq j\leq r}{\oplus}} L^\infty(\Omega_j;M_{n_j}). \tag{2}$$

If a von Neumann algebra $\mathcal{M}$ is not subhomogeneous of degree $\leq N$, it is well-known that there is a non zero $*$-homomorphism $\gamma : M_{N+1} \to \mathcal{M}$. Lemma 2.1 below makes this more explicit in the semifinite case.

Lemma 2.1. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $N \geq 1$. If $\mathcal{M}$ is not subhomogeneous of degree $\leq N$, then there is a complete isometry from S^p_{N+1} into $L^p(\mathcal{M})$ that is also an S^1-isometry.*

Proof. Let $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ be the direct sum decomposition of $\mathcal{M}$ into a type I summand $\mathcal{M}_1$ and a type II summand $\mathcal{M}_2$ (see e.g. [11, Section 5]).

Assume that $\mathcal{M}_2 \neq \{0\}$. Following the same lines as in [5, Lemma 2.3], there is a projection e in $\mathcal{M}_2$, a trace preserving von Neumann algebra identification

$$\mathcal{M}_2 \simeq M_{N+1}\overline{\otimes}(e\mathcal{M}_2 e) \tag{3}$$

and a finite trace projection ε in $e\mathcal{M}_2 e$ such that the mapping

$$\gamma : M_{N+1} \to \mathcal{M}_2 \subset \mathcal{M}; \qquad \gamma(a) = a \otimes \varepsilon$$

is a non zero $*$-homomorphism taking values in $L^1(\mathcal{M})$, and therefore $L^p(\mathcal{M})$.

For every $[a_{ij}]_{1\le i,j\le m}$ in $S^p_{N+1} \otimes S^p_m$ we have that

$$\|[a_{ij} \otimes \varepsilon]\|_{L^p(\mathcal{M}_2 \otimes M_m)} = \|\varepsilon\|_p \|[a_{ij}]\|_{L^p(M_{N+1}\otimes M_m)},$$

and therefore $\|\varepsilon\|_p^{-1}\gamma$ is a complete isometry from S^p_{N+1} into $L^p(\mathcal{M})$. By [5, Lemma 5.1],

$$\|[a_{ij} \otimes \varepsilon]\|_{L^p(\mathcal{M}_2;\, S^1_m)} = \|\varepsilon\|_p \|[a_{ij}]\|_{S^p_{N+1}[S^1_m]},$$

and therefore $\|\varepsilon\|_p^{-1}\gamma$ is also an S^1-isometry from S^p_{N+1} into $L^p(\mathcal{M})$.

If $\mathcal{M}_2 = \{0\}$, then $\mathcal{M}$ is of type I. Since $\mathcal{M}$ is not subhomogeneous of degree $\le N$, it follows from [11, Theorem V.1.27] that there exist a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \ge N+1$ and an abelian von Neumann algebra W such that $\mathcal{M}$ contains $B(\mathcal{H})\overline{\otimes}W$ as a summand. Using this summand instead of (3) and arguing as above we obtain the result in this case as well. □

3. Bijective separating maps and their inverses

The goal of this section is to provide a decomposition for bijective separating maps that facilitates their study. We apply this decomposition to show that the inverse of a bijective separating map is separating as well.

First we recall some terminologies and results that we will use. A Jordan homomorphism between von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ is a linear map $J : \mathcal{M} \to \mathcal{N}$ such that

$$J(x^*) = J(x)^* \quad \text{and} \quad J(xy + yx) = J(x)J(y) + J(y)J(x)$$

for all x and y in $\mathcal{M}$. It is plain that $*$-homomorphisms and anti-$*$-homomorphisms are Jordan homomorphisms. In fact, every Jordan homomorphism is a sum of a $*$-homomorphism and an anti-$*$-homomorphism, as we recall here.

Let $J : \mathcal{M} \to \mathcal{N}$ be a Jordan homomorphism and let $\mathcal{D} \subset \mathcal{N}$ be the w^*-closed C^*-algebra generated by $J(\mathcal{M})$. Then $J(1)$ is the unit of $\mathcal{D}$. By e.g. [10, Theorem 3.3], there exist projections e and f in the center of $\mathcal{D}$ such that $e + f = J(1)$, $x \mapsto J(x)e$ is a $*$-homomorphism, and $x \mapsto J(x)f$ is an anti-$*$-homomorphism. Let $\mathcal{N}_1 = e\mathcal{N}e$ and $\mathcal{N}_2 = f\mathcal{N}f$. Define $\pi : \mathcal{M} \to \mathcal{N}_1$ and $\sigma : \mathcal{M} \to \mathcal{N}_2$ by $\pi(x) = J(x)e$ and $\sigma(x) = J(x)f$, for all $x \in \mathcal{M}$. Then J is valued in $\mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and $J(x) = \pi(x) + \sigma(x)$, for all $x \in \mathcal{M}$.

Assume that $\mathcal{M}$ and $\mathcal{N}$ are semifinite von Neumann algebras and let $1 \le p < \infty$. In [4], inspired by Yeadon's fundamental description of isometries between noncommutative L^p-spaces, we say that a bounded operator $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ has a Yeadon type factorization if there exist a w^*-continuous Jordan homomorphism $J\colon \mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a positive operator B affiliated with $\mathcal{N}$, which satisfy the following conditions:

(a) $w^*w = J(1) = s(B)$, the support projection of B;
(b) every spectral projection of B commutes with $J(x)$, for all $x \in \mathcal{M}$;
(c) $T(x) = wBJ(x)$ for all $x \in \mathcal{M} \cap L^p(\mathcal{M})$.

We call (w, B, J) the Yeadon triple associated with T. This triple is unique. Following [5], if J is a $*$-homomorphism (respectively, anti-$*$-homomorphism), we say that T has a direct (respectively, anti-direct) Yeadon type factorization.

Following [4], we say that a bounded operator $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is separating if for every $x, y \in L^p(\mathcal{M})$ such that $x^*y = xy^* = 0$, we have that $T(x)^*T(y) = T(x)T(y)^* = 0$. The following characterization has a fundamental role in the study of separating maps.

Theorem 3.1. *([1, Theorem 3.3], [4, Theorem 3.5]) A bounded operator* $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ *admits a Yeadon type factorization if and only if it is separating.*

It is easy to see that for a separating map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ with Yeadon triple (w, B, J), we have that

$$T(z^*) = wT(z)^*w \qquad (z \in L^p(\mathcal{M})). \tag{4}$$

Also, if T has a direct (respectively, anti-direct) Yeadon type factorization, we get that

$$T(zm) = T(z)J(m) \ \text{ (respectively, } T(mz) = T(z)J(m)\text{)}\,, \tag{5}$$

for every $z \in L^p(\mathcal{M})$ and $m \in \mathcal{M}$.

Remark 3.2. Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating map with Yeadon triple (w, B, J). We observe that if T is surjective, then w is a unitary. Indeed on the one hand, we see that T is valued in $wL^p(\mathcal{N})$. Since $ww^*w = w$, this implies that T is valued in $ww^*L^p(\mathcal{N})$. Hence, if T is surjective, we have $ww^*L^p(\mathcal{N}) = L^p(\mathcal{N})$, which implies that $ww^* = 1$. On the other hand, $T(x) = T(x)J(1)$, for any $x \in L^p(\mathcal{M})$. Hence, T is valued in $L^p(\mathcal{N})J(1)$. Hence, if T is surjective, we have $L^p(\mathcal{N})J(1) = L^p(\mathcal{N})$, which implies $w^*w = J(1) = 1$.

Proposition 3.3. *Let* $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ *be a separating map that is bijective. Then there exist direct sum decompositions*

$$\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2, \qquad \textit{and} \qquad \mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2,$$

and bounded bijective separating maps $T_1 : L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$ *with a direct Yeadon type factorization and* $T_2 : L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$ *with an anti-direct Yeadon type factorization such that* $T = T_1 + T_2$.

Proof. Assume that $w = 1$. Consider a decomposition for J, induced by central projections e and f, as recalled above. As detailed in [5, Remark 4.3], this induces a decomposition $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and separating maps

$$T_1\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{N}_1), \qquad T_1(x) = T(x)e,$$

with Yeadon triple (e, Be, π), and hence a direct Yeadon type factorization, and

$$T_2\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{N}_2), \qquad T_2(x) = T(x)f,$$

with Yeadon triple (f, Bf, σ), and hence an anti-direct Yeadon type factorization, such that $T = T_1 + T_2$.

Let $\mathcal{M}_1 := \ker(\sigma)$ and $\mathcal{M}_2 := \ker(\pi)$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are w^*-closed ideals of $\mathcal{M}$, there exist central projections $\alpha, \beta \in \mathcal{M}$ such that $\mathcal{M}_1 = \alpha\mathcal{M}$, and $\mathcal{M}_2 = \beta\mathcal{M}$. Set $\mathcal{M}_3 := (1-\alpha)(1-\beta)\mathcal{M}$. Note that $\alpha\beta \in \ker(\sigma) \cap \ker(\pi)$, and therefore $J(\alpha\beta) = 0$. Since T is one-to-one, by [4, Remark 3.14(a)], J is one-to-one and therefore we must have that $\alpha\beta = 0$. Hence,

$$1 = \alpha + \beta + (1-\alpha)(1-\beta).$$

Consequently, $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2 \overset{\infty}{\oplus} \mathcal{M}_3$, and so we have the decomposition

$$L^p(\mathcal{M}) = L^p(\mathcal{M}_1) \overset{p}{\oplus} L^p(\mathcal{M}_2) \overset{p}{\oplus} L^p(\mathcal{M}_3).$$

The result will follow if we can show that

$$L^p(\mathcal{M}_1) = \ker(T_2), \quad L^p(\mathcal{M}_2) = \ker(T_1) \quad \text{and} \quad \mathcal{M}_3 = \{0\}.$$

To see that $L^p(\mathcal{M}_1) \subseteq \ker(T_2)$, let $x \in \mathcal{M}_1 \cap L^p(\mathcal{M}_1)$, then

$$T_2(x) = B\sigma(x) = 0.$$

Hence, $\mathcal{M}_1 \cap L^p(\mathcal{M}_1) \subset \ker(T_2)$ and therefore $L^p(\mathcal{M}_1) \subset \ker(T_2)$. Now suppose that x belongs to $\ker(T_2)$. For any $n \geq 1$, let $p_n = \chi_{[-n,n]}(|x^*|)$, the projection associated with the indicator function of $[-n, n]$ in the Borel functional calculus of $|x^*|$, and $x_n := p_n x$. Then, using (5), we have that

$$T_2(x_n) = T_2(x)\sigma(p_n) = 0.$$

Hence, $B\sigma(x_n) = 0$. Since $s(B) = 1$, this implies that $\sigma(x_n) = 0$, that is x_n is in $\mathcal{M}_1$. Now because $x_n \to x$ in $L^p(\mathcal{M})$, we obtain that x belongs to $L^p(\mathcal{M}_1)$. Hence,

$$L^p(\mathcal{M}_1) = \ker(T_2).$$

Similarly, we can show that $L^p(\mathcal{M}_2) = \ker(T_2)$.

Finally, we show that $\mathcal{M}_3 = \{0\}$. Let $x \in L^p(\mathcal{M})$. By surjectivity of T, there is y in $L^p(\mathcal{M})$ such that $T(y) = T_1(x)$. Writing $T(y) = T_1(y) + T_2(y)$, we obtain that $T_1(x - y) = 0$ and $T_2(y) = 0$, that is $x - y$ belongs to $\ker(T_1) = L^p(\mathcal{M}_2)$ and y belongs to $\ker(T_2) = L^p(\mathcal{M}_1)$, thus x belongs to $L^p(\mathcal{M}_1) \overset{p}{\oplus} L^p(\mathcal{M}_2)$. Hence, $\mathcal{M}_3 = \{0\}$. This completes the proof in the case $w = 1$.

In the general case, consider the map $\widetilde{T} := w^*T(\cdot)$, which takes any $x \in \mathcal{M} \cap L^p(\mathcal{M})$ to $BJ(x)$. By Remark 3.2, $\widetilde{T}$ is also a bijective separating map. Its Yeadon triple is $(1, B, J)$. We may apply the above decomposition to the map $\widetilde{T}$ to obtain decompositions $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$, $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and bounded bijective separating maps $\widetilde{T_1} : L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$ with a direct Yeadon type factorization and $\widetilde{T_2} : L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$ with an anti-direct Yeadon type factorization such that $\widetilde{T} = \widetilde{T_1} + \widetilde{T_2}$. Since $w\widetilde{T} = T$, we obtain the result. □

Proposition 3.4. *Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is a bijective separating map, then*

(i) *$T^{-1} : L^p(\mathcal{N}) \to L^p(\mathcal{M})$ is separating.*
(ii) *If $J : \mathcal{M} \to \mathcal{N}$ is the Jordan homomorphism associated with T, then J is invertible and $J^{-1} : \mathcal{N} \to \mathcal{M}$ is the Jordan homomorphism associated with T^{-1}.*

Proof. Using the decomposition given in Proposition 3.3, it is enough to show parts (i) and (ii) for a bijective separating map with a direct Yeadon type factorization. So, throughout the proof we assume that this is the case. Note that by Remark 3.2, $J(1) = 1$.

(i) Suppose that $a, b \in L^p(\mathcal{N})$ such that $a^*b = ab^* = 0$. We show that $T^{-1}(a)^*T^{-1}(b) = T^{-1}(a)T^{-1}(b)^* = 0$. Let $x = T^{-1}(a)$ and $y = T^{-1}(b)$. Set $p_n := \chi_{[-n,n]}(|y|)$, for any

$n \geq 1$. We have that

$$\begin{aligned} T(x^*yp_n)B &= T(x^*)J(yp_n)B && \text{by (5)} \\ &= T(x^*)w^*T(yp_n) \\ &= wT(x)^*T(yp_n) && \text{by (4)} \\ &= wT(x)^*T(y)J(p_n) && \text{by (5)} \\ &= wa^*bJ(p_n) = 0. \end{aligned}$$

Since $s(B) = J(1) = 1$, we obtain $T(x^*yp_n) = 0$. Because T is one-to-one, we have that $x^*yp_n = 0$. Now, since $yp_n \to y$, we get that $x^*y = 0$. A similar argument using $ab^* = 0$ implies that $xy^* = 0$. Hence T^{-1} must be separating.

(*ii*) By part (*i*), T^{-1} is separating. We let J' denote the Jordan homomorphism of its Yeadon triple. Let $e \in \mathcal{N}$ be a projection with finite trace. For any $y \in e\mathcal{N}e$, we have that $T^{-1}(y) = T^{-1}(e)J'(y)$. Applying (5), we deduce that

$$y = TT^{-1}(y) = T\left(T^{-1}(e)J'(y)\right) = TT^{-1}(e)\ JJ'(y) = e\ JJ'(y).$$

Using the w^*-continuity of J and J', and the w^*-density of the union of the $e\mathcal{N}e$, for $\tau_{\mathcal{N}}(e) < \infty$, we deduce that $y = JJ'(y)$ for any $y \in \mathcal{N}$. By [4, Remark 3.14(a)], since T is one-to-one, J must be one-to-one. Hence, J is invertible with $J^{-1} = J'$. □

Remark 3.5. Part (*ii*) of Proposition 3.4 shows that a separating invertible map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ admits a direct Yeadon type factorization if and only if T^{-1} does.

4. A CHARACTERIZATION OF COMPLETELY/S^1-BOUNDED SURJECTIVE SEPARATING MAPS

In this section we show that a separating map can always be reduced to a one-to-one separating map and therefore we may confine ourself to the study of separating maps that are surjective rather than bijective. The goal of the section is to provide a characterization for surjective separating maps that are completely bounded (when $p \neq 2$) or S^1-bounded. We show that the surjectivity assumption is essential.

We require [5, Propositions 4.4 & 4.5] later on in our arguments in this section. We recall the statements for convenience.

Proposition 4.1. *Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded operator with a direct Yeadon type factorization. Then T is completely bounded and $\|T\|_{cb} = \|T\|$.*

Proposition 4.2. *Let $T\colon L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded operator with a direct Yeadon type factorization. Then T is S^1-bounded and $\|T\|_{S^1} = \|T\|$.*

Lemma 4.3. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating map. Then there exists a direct sum decomposition $\mathcal{M} = \mathcal{M}_0 \overset{\infty}{\oplus} \widetilde{\mathcal{M}}$ such that* $\ker(T) = L^p(\mathcal{M}_0)$.

Proof. Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a separating map and $J : \mathcal{M} \to \mathcal{N}$ be the Jordan homomorphism associated with T via its Yeadon type factorization. Let $\mathcal{M}_0 := \ker(J)$. Then $\mathcal{M}_0$ is an ideal. Since J is w^*-continuous, $\mathcal{M}_0$ is w^*-closed. Hence we have a direct sum decomposition

$$\mathcal{M} = \mathcal{M}_0 \overset{\infty}{\oplus} \widetilde{\mathcal{M}}.$$

It is clear that $L^p(\mathcal{M}_0) \subset \ker T$. Further $J|_{\widetilde{\mathcal{M}}}$ is one-to-one. By [4, Remark 3.14(a)] this implies that $T|_{L^p(\widetilde{\mathcal{M}})}$ is one-to-one. This yields the result. □

For any von Neumann algebra $\mathcal{M}$, we let $\mathcal{M}^{op}$ denote its opposite von Neumann algebra. Recall that the underlying dual Banach space structure and involution on $\mathcal{M}^{op}$ are the same as on $\mathcal{M}$ but the product of x and y is defined by yx rather than xy. Note that the Banach spaces $L^p(\mathcal{M})$ and $L^p(\mathcal{M}^{op})$ are the same. It is evident that, for von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, $J : \mathcal{M} \to \mathcal{N}$ is a $*$-homomorphism if and only if

$$J^{op} : \mathcal{M}^{op} \to \mathcal{N}; \quad x \mapsto J(x),$$

is an anti-$*$-homomorphism. Hence, a separating map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ has a direct Yeadon type factorization if and only if

$$T^{op} : L^p(\mathcal{M}^{op}) \to L^p(\mathcal{N}); \quad x \mapsto T(x),$$

has an anti-direct Yeadon type factorization.

Lemma 4.4 below is the principal ingredient of our characterization theorems. Its proof relies on the relation between the completely bounded norm or S^1-norm of the identity map

$$I^{op} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})$$

and the norms of the transformations

$$[x_{ij}]_{1\leq i,j\leq m} \mapsto [x_{ji}]_{1\leq i,j\leq m}$$

either on $L^p(\mathcal{M} \otimes M_m)$ or on $L^p(\mathcal{M}; S^1_m)$, in particular in the specific case when $\mathcal{M} = M_n$. We will use the fact that for any $n \geq 1$, we have $L^p(M_n \otimes M_m) \simeq S^p_m[S^p_n]$, isometrically, provided that S^p_n is equipped with the operator space structure given in [6].

Let t_m denote the transposition map on scalar $m \times m$ matrices. Assume that $\mathcal{M}$ is semifinite. The map

$$I_{\mathcal{M}^{op}} \otimes t_m : \mathcal{M}^{op} \otimes M_m \to \mathcal{M}^{op} \otimes M^{op}_m$$

is a trace preserving $*$-homomorphism, and so

$$I_{L^p(\mathcal{M}^{op})} \otimes t_m \colon L^p(\mathcal{M}^{op} \otimes M_m) \longrightarrow L^p(\mathcal{M}^{op} \otimes M^{op}_m)$$

is an isometry. Moreover $\mathcal{M}^{op} \otimes M^{op}_m = (\mathcal{M} \otimes M_m)^{op}$, hence $L^p(\mathcal{M}^{op} \otimes M^{op}_m) = L^p(\mathcal{M} \otimes M_m)$ isometrically. For any $[x_{ij}]_{1\leq i,j\leq m}$ in $L^p(\mathcal{M}) \otimes S^p_m$, since $I_{L^p(\mathcal{M}^{op})} \otimes t_m$ maps $[x_{ij}]$ to $[x_{ji}]$, we get that

$$\big\|[x_{ij}]\big\|_{L^p(\mathcal{M}^{op} \otimes M_m)} = \big\|[x_{ji}]\big\|_{L^p(\mathcal{M} \otimes M_m)}. \tag{6}$$

We now show that similarly, for any $[x_{ij}]_{1\leq i,j\leq m}$ in $L^p(\mathcal{M}) \otimes S^1_m$,

$$\big\|[x_{ij}]\big\|_{L^p(\mathcal{M}^{op};\, S^1_m)} = \big\|[x_{ji}]\big\|_{L^p(\mathcal{M};\, S^1_m)}. \tag{7}$$

To verify the identity (7), assume that $\|[x_{ij}]\|_{L^p(\mathcal{M}^{op};\, S^1_m)} < 1$. Taking into account the opposite product and (1), we can write

$$x_{ij} = \sum_k b_{kj} a_{ik}$$

for some a_{ik}, b_{kj} in $L^{2p}(\mathcal{M})$ such that $\sum_{i,k} a^*_{ik} a_{ik}$ and $\sum_{k,j} b_{kj} b^*_{kj}$ have norm < 1 in $L^p(\mathcal{M})$. This exacly means that $\|[x_{ji}]\|_{L^p(\mathcal{M};\, S^1_m)} < 1$. This shows the inequality $\geq$ in (7). Reversing the argument we find the other inequality.

Identities (6) and (7), respectively, imply

$$(8)\quad \big\|I^{op}\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb} = \sup_{m\geq 1}\big\|I_{L^p(\mathcal{M})}\otimes t_m\colon L^p(\mathcal{M}\otimes M_m) \longrightarrow L^p(\mathcal{M}\otimes M_m)\big\|,$$

and

$$(9)\quad \big\|I^{op}\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{S^1} = \sup_{m\geq 1}\big\|I_{L^p(\mathcal{M})}\otimes t_m\colon L^p(\mathcal{M};\ S^1_m) \longrightarrow L^p(\mathcal{M};S^1_m)\big\|.$$

When $\mathcal{M} = M_n$, the above identities can be more specific. In fact, as we show below, we have that for any $n \geq 1$,

$$(10)\qquad \big\|I^{op}\colon S^p_n \longrightarrow \{S^p_n\}^{op}\big\|_{cb} = \big\|t_n : S^p_n \to S^p_n\big\|_{cb}$$

and

$$(11)\qquad \big\|I^{op}\colon S^p_n \longrightarrow \{S^p_n\}^{op}\big\|_{S^1} = \big\|t_n : S^p_n \to S^p_n\big\|_{S^1}.$$

Using (6) applied to $\mathcal{M} = M_n$, to prove (10), it is enough to show that for any $[x_{ij}]_{1\leq i,j\leq m}$ in $S^p_n \otimes S^p_m$,

$$(12)\qquad \big\|[t_n(x_{ij})]\big\|_{S^p_m[S^p_n]} = \big\|[x_{ji}]\big\|_{S^p_m[S^p_n]}.$$

This follows from the fact that $t_m \otimes t_n = t_{nm}$ is an isometry on $S^p_m[S^p_n] \simeq S^p_{nm}$, and hence

$$\big\|(t_m\otimes t_n)[t_n(x_{ij})]\big\|_{S^p_m[S^p_n]} = \big\|[t_n(x_{ij})]\big\|_{S^p_m[S^p_n]}.$$

Since $(t_m \otimes t_n)[t_n(x_{ij})] = [x_{ji}]$, this yields (12).

Likewise, using (7) applied to $\mathcal{M} = M_n$, to prove (11), it is enough to show that for any $[x_{ij}]_{1\leq i,j\leq m}$ in $S^p_n \otimes S^1_m$,

$$(13)\qquad \big\|[t_n(x_{ij})]\big\|_{S^p_n[S^1_m]} = \big\|[x_{ji}]\big\|_{S^p_n[S^1_m]}.$$

Assume that $\|[t_n(x_{ij})]\|_{S^p_n[S^1_m]} < 1$. According to (1), we can write

$$t_n(x_{ij}) = \sum_k a_{ik}b_{kj}$$

for some a_{ik}, b_{kj} in S^{2p}_n such that $\sum_{i,k} a_{ik}a^*_{ik}$ and $\sum_{k,j} b^*_{kj}b_{kj}$ have norm < 1 in S^p_n. Then we have

$$x_{ij} = \sum_k t_n(a_{ik}b_{kj}) = \sum_k t_n(b_{kj})t_n(a_{ik}),$$

hence

$$x_{ji} = \sum_k t_n(b_{ki})t_n(a_{jk}).$$

Further

$$\sum_{k,j} t_n(a_{jk})^*t_n(a_{jk}) = t_n\Big(\sum_{j,k} a_{jk}a^*_{jk}\Big),$$

and t_n is an isomerty on S^p_n. Consequently, $\sum_{k,j} t_n(a_{kj})^*t_n(a_{jk})$ has norm < 1 in S^p_n. Similarly, $\sum_{i,k} t_n(b_{ki})t_n(b_{ki})^*$ has norm < 1 in S^p_n. This shows that $\|[x_{ji}]\|_{S^p_n[S^1_m]} < 1$. We have thus proved the inequality $\geq$ in (13). Reversing the argument we find the other inequality.

In the sequel, $E(x)$ denotes the integer part of x.

Lemma 4.4. *Suppose that $\mathcal{M}$ is a semifinite von Neumann algebra.*

(i) If $\mathcal{M}$ is subhomogeneous of degree $\leq N$ for some $N \geq 1$, then for all $[x_{ij}] \in M_m \otimes L^p(\mathcal{M})$, $m \geq 1$, we have that

$$\|[x_{ji}]\|_{L^p(\mathcal{M}\otimes M_m)} \leq N^{2|1/2-1/p|}\|[x_{ij}]\|_{L^p(\mathcal{M}\otimes M_m)},$$

and

$$\|[x_{ji}]\|_{L^p(\mathcal{M};S^1_m)} \leq N\|[x_{ij}]\|_{L^p(\mathcal{M};\, S^1_m)}.$$

(ii) Suppose that there exists $K \geq 1$ such that for all $[x_{ij}] \in L^p(\mathcal{M}) \otimes S^p_m$, $m \geq 1$,

$$\|[x_{ji}]\|_{L^p(\mathcal{M}\otimes M_m)} \leq K\|[x_{ij}]\|_{L^p(\mathcal{M}\otimes M_m)}. \tag{14}$$

Then if $p \neq 2$, $\mathcal{M}$ is subhomogeneous of degree $\leq N$ with $N = E\left(K^{\frac{1}{2|1/2-1/p|}}\right)$.

(iii) Suppose that there exists $K \geq 1$ such that for all $[x_{ij}] \in L^p(\mathcal{M}) \otimes S^p_m$, $m \geq 1$,

$$\|[x_{ji}]\|_{L^p(\mathcal{M};S^1_m)} \leq K\|[x_{ij}]\|_{L^p(\mathcal{M};\, S^1_m)}. \tag{15}$$

Then $\mathcal{M}$ is subhomogeneous of degree $\leq N$ with $N = E(K)$.

Proof. (*i*) Assume that $\mathcal{M} = L^\infty(\Omega; M_n)$. Let $m \geq 1$ be given. We have that

$$L^p(\mathcal{M} \otimes M_m) \simeq L^p(\Omega; S^p_m[S^p_n]).$$

By Pisier-Fubini Theorem [6, (3.6)],

$$L^p(\mathcal{M}; S^1_m) \simeq L^p(\Omega; S^p_n[S^1_m]).$$

Consequently,

$$\big\|I_{L^p(\mathcal{M})} \otimes t_m \colon L^p(\mathcal{M} \otimes M_m) \longrightarrow L^p(\mathcal{M} \otimes M_m)\big\| = \big\|t_m \otimes I_{S^p_n} \colon S^p_m[S^p_n] \longrightarrow S^p_m[S^p_n]\big\|. \tag{16}$$

and

$$\big\|I_{L^p(\mathcal{M})} \otimes t_m \colon L^p(\mathcal{M}; S^1_m) \longrightarrow L^p(\mathcal{M}; S^1_m)\big\| = \big\|I_{S^p_n} \otimes t_m \colon S^p_n[S^1_m] \longrightarrow S^p_n[S^1_m]\big\|. \tag{17}$$

Applying (8) to both sides of (16), we deduce

$$\big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb} = \big\|I^{op} \colon S^p_n \longrightarrow \{S^p_n\}^{op}\big\|_{cb},$$

and applying (9) to both sides of (17), we deduce that

$$\big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{S^1} = \big\|I^{op} \colon S^p_n \longrightarrow \{S^p_n\}^{op}\big\|_{S^1}.$$

By [5, Lemma 5.3],

$$\big\|t_n : S^p_n \to S^p_n\big\|_{cb} = n^{2|1/p-1/2|} \quad \text{and} \quad \big\|t_n : S^p_n \to S^p_n\big\|_{S^1} = n,$$

hence we obtain by (10) and (11) that

$$\big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb} = n^{2|1/p-1/2|} \quad \text{and} \quad \big\|I^{op} \colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{S^1} = n.$$

When $\mathcal{M}$ is subhomogeneous of degree $\leq N$, there exist $r \geq 1$, integers $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r \leq N$ and abelian von Neumann algebras $L^\infty(\Omega_1), \ldots, L^\infty(\Omega_r)$ such that (2) holds. Then for any $m \geq 1$, we have that

$$L^p(\mathcal{M} \otimes M_m) \simeq \mathop{\overset{p}{\oplus}}_{1\leq j\leq r} L^p(\Omega_j; S^p_m[S^p_{n_j}]) \quad \text{and} \quad L^p(\mathcal{M}; S^1_m) \simeq \mathop{\overset{p}{\oplus}}_{1\leq j\leq r} L^p(\Omega_j; S^p_{n_j}[S^1_m]).$$

Using our previous argument and direct sums we deduce that

$$\|I^{op}: L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})\|_{cb} \leq N^{2|\frac{1}{p}-\frac{1}{2}|} \quad \text{and} \quad \|I^{op}: L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})\|_{S^1} \leq N.$$

The result follows from (6) and (7).

(ii) Suppose that $\mathcal{M}$ is not subhomogeneous of degree $\leq N = E(K^{\frac{1}{2|1/2-1/p|}})$. By Lemma 2.1, there exists a complete isometry

$$S^p_{N+1} \hookrightarrow \mathcal{M}.$$

This embedding implies that for any $m \geq 1$,

$$\big\|t_m \otimes I_{S^p_{N+1}}\colon S^p_m[S^p_{N+1}] \longrightarrow S^p_m[S^p_{N+1}]\big\| \leq \big\|I_{L^p(\mathcal{M})} \otimes t_m\colon L^p(\mathcal{M}\otimes M_m) \longrightarrow L^p(\mathcal{M}\otimes M_m)\big\|.$$

According to (8) and (10), this implies that

$$\big\|t_{N+1}\colon S^p_{N+1} \longrightarrow S^p_{N+1}\big\|_{cb} \leq \big\|I^{op}\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb}.$$

Hence

$$\big\|I^{op}\colon L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}^{op})\big\|_{cb} \geq (N+1)^{2|\frac{1}{p}-\frac{1}{2}|}.$$

Comparing this with inequality (14) above and applying (6), we get a contradiction.

(iii) The proof is similar to the proof of part (ii). □

Proposition 4.5. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be separating. If $\mathcal{M}$ is subhomogeneous then T is completely bounded and S^1-bounded.*

Proof. Changing T to w^*T, we can assume that $w = J(1)$. By [5, Remark 4.3], we can write T as a sum $T = T_1 + T_2$ such that T_1 has a direct Yeadon type factorization and T_2 has an anti-direct Yeadon type factorization. By Propositions 4.1 and 4.2, T_1 is completely bounded and S^1-bounded. Hence it suffices to show that T_2 is completely bounded and S^1-bounded. Let $I^{op} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})$ be the identity map and set $T_2^{op} =: T_2 \circ I^{op-1}$. Since T_2 has an anti-direct Yeadon type factorization, T_2^{op} has a direct Yeadon type factorization. So, by Propositions 4.1 and 4.2, T_2^{op} is completely bounded and S^1-bounded. Since $\mathcal{M}$ is subhomogeneous, part (i) of Lemma 4.4 and its proof show that I^{op} is completely bounded and S^1-bounded. By composition, we obtain that $T_2 = T_2^{op} \circ I^{op}$ is completely bounded and S^1-bounded. □

Proposition 4.6. *Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is a bijective separating map with an anti-direct Yeadon type factorization.*

(i) *If $p \neq 2$ and T is completely bounded then $\mathcal{M}$ is subhomogeneous.*
(ii) *If T is S^1-bounded then $\mathcal{M}$ is subhomogeneous.*

Proof. (i) Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, is a bijective separating map with an anti-direct Yeadon type factorization. Assume that T is completely bounded. Let $I^{op} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op})$ be the identity map and set $T^{op} := T \circ I^{op-1}$. Since T is bijective with an anti-direct Yeadon type factorization, T^{op} is bijective with a direct Yeadon type factorization. By part (i) of Proposition 3.4 and Remark 3.5, T^{op-1} is also separating with a direct Yeadon type factorization. Therefore, by Proposition 4.1, T^{op-1} is completely bounded. Hence, $I^{op} := T^{op-1} \circ T$ is completely bounded. It now follows from part (ii) of Lemma 4.4 and (6) that $\mathcal{M}$ is subhomogeneous.

(ii) The same argument as in part (i) with S^1-bounded (norm) replacing completely bounded (norm), Proposition 4.2 replacing Proposition 4.1, part (iii) of Lemma 4.4 replacing its part (ii) and (7) replacing (6) yields the result. □

Remark 4.7. Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p < \infty$, is a surjective separating isometry with an anti-direct Yeadon type factorization. The proof of Proposition 4.6 shows that when T is completely bounded and $p \neq 2$, $\mathcal{M}$ is subhomogeneous of degree $\leq E(\|T\|_{cb}^{\frac{1}{2|1/2-1/p|}})$. When T is S^1-bounded, $\mathcal{M}$ is subhomogeneous of degree $\leq E(\|T\|_{S^1})$.

Theorem 4.8. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, be a bounded separating map that is surjective. Then the following are equivalent.*

(i) *T is completely bounded.*
(ii) *There exists a decomposition $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorization and $\mathcal{M}_2$ is subhomogeneous.*

Proof. $(i) \implies (ii)$ Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, is a surjective completely bounded separating map. In view of Lemma 4.3, we may assume T is bijective.

By Proposition 3.3, there exist decompositions $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ and $\mathcal{N} = \mathcal{N}_1 \overset{\infty}{\oplus} \mathcal{N}_2$ and surjective separating maps $T_1 : L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$ and $T_2 : L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$ such that T_1 has a direct Yeadon type factorization, T_2 has an anti-direct Yeadon type factorization and $T = T_1 + T_2$. Since T is completely bounded, T_2 is also completely bounded. By part (i) of Proposition 4.6, $\mathcal{M}_2$ must be subhomogeneous.

$(ii) \implies (i)$ This is a consequence of Propositions 4.1 and 4.5. □

Theorem 4.9. *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p < \infty$, be a separating map that is surjective. Then the following are equivalent.*

(i) *T is S^1-bounded.*
(ii) *There exists a decomposition $\mathcal{M} = \mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorization and $\mathcal{M}_2$ is subhomogeneous.*

Proof. The proof is similar to Theorem 4.8, replacing completely bounded with S^1-bounded, part (i) of Proposition 4.6 by its part (ii) and Proposition 4.1 by Proposition 4.2. □

The following example shows the surjectivity assumption in Theorems 4.8 and 4.9 is essential. In fact in this example, on a non-subhomogeneous semifinite von Neumann algebra $\mathcal{M}$ and for a given $\varepsilon > 0$, we construct a separating isometry $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ such that T is not surjective, $\|T\|_{cb} \leq 1 + \varepsilon$, $\|T\|_{S^1} \leq 1 + \varepsilon$ and part (ii) of Theorems 4.8 and 4.9 is not satisfied.

The isometry T in our example is set up between hyperfinite von Neumann algebras and so $\|T\|_{cb} \leq \|T\|_{S^1}$ (see [7, Proposition 2.2] and [5, Proposition 3.11]). Therefore, we only need to verify that for such T we have that $\|T\|_{S^1} \leq 1 + \varepsilon$.

Example 4.10. Let $1 < p < \infty$. Consider the von Neumann algebra

$$\mathcal{M} = \ell^\infty\{M_n\} = \big\{(x_n)_{n\geq 1} \,:\, \forall\, n \geq 1,\ x_n \in M_n \text{ and } \sup_{n\geq 1} \|x_n\|_\infty < \infty\big\},$$

the infinite direct sum of all M_n, $n \geq 1$. Let $\mathcal{N} := \mathcal{M} \overset{\infty}{\oplus} \mathcal{M}$, the direct sum of two copies of $\mathcal{M}$. The noncommutative L^p-space associated with $\mathcal{M}$ is

$$\ell^p\{S_n^p\} = \big\{(x_n)_{n\geq 1} \,:\, \forall\, n \geq 1,\ x_n \in S_n^p \text{ and } \sum_{n\geq 1} \|x_n\|_p^p < \infty\big\},$$

equipped with the norm

$$\big\|(x_n)_{n\geq 1}\big\|_p = \Big(\sum_{n=1}^{\infty} \|x_n\|_p^p\Big)^{\frac{1}{p}},$$

and so the noncommutative L^p-space associated with $\mathcal{N}$ is $\ell^p\{S_n^p\} \overset{p}{\oplus} \ell^p\{S_n^p\}$. Let $(\beta_n)_{n\geq 1}$ be a sequence in the interval $(0,1)$. We may define two operators

$$T_1 \colon \ell^p\{S_n^p\} \to \ell^p\{S_n^p\} \qquad \text{and} \qquad T_2 \colon \ell^p\{S_n^p\} \to \ell^p\{S_n^p\}$$

by setting

$$T_1\big((x_n)_{n\geq 1}\big) \;=\; \big((1-\beta_n)^{\frac{1}{p}} x_n\big)_{n\geq 1} \qquad \text{and} \qquad T_2\big((x_n)_{n\geq 1}\big) \;=\; \big(\beta_n^{\frac{1}{p}}\, t_n(x_n)\big)_{n\geq 1}$$

for any $x = (x_n)_{n\geq 1} \in \ell^p\{S_n^p\}$. Consider

$$T \colon \ell^p\{S_n^p\} \to \ell^p\{S_n^p\} \overset{p}{\oplus} \ell^p\{S_n^p\}, \qquad T(x) = (T_1(x), T_2(x)).$$

It is plain that T is an isometry. Indeed for any $x = (x_n)_{n\geq 1} \in \ell^p\{S_n^p\}$, we have

$$\begin{aligned}\|T(x)\|_p^p &= \|T_1(x)\|_p^p + \|T_2(x)\|_p^p \\ &= \sum_{n=1}^{\infty}(1-\beta_n)\|x_n\|_p^p + \sum_{n=1}^{\infty}\beta_n\|{}^t x_n\|_p^p = \sum_{n=1}^{\infty}\|x_n\|_p^p = \|x\|_p^p.\end{aligned}$$

Given $\varepsilon > 0$, consider the above construction with

$$\beta_n = \frac{(1+\varepsilon)^p - 1}{n^p - 1}.$$

We show that T is S^1-bounded with $\|T\|_{S^1} \leq 1+\varepsilon$. Indeed consider an integer $m \geq 1$. We have

$$\ell^p\{S_n^p\}\big[S_m^1\big] \;=\; \ell^p\{S_n^p[S_m^1]\},$$

and therefore, we also have that

$$\Big(\ell^p\{S_n^p\} \overset{p}{\oplus} \ell^p\{S_n^p\}\Big)\,\big[S_m^1\big] \;=\; \ell^p\{S_n^p[S_m^1]\} \overset{p}{\oplus} \ell^p\{S_n^p[S_m^1]\}.$$

Now let $x = (x_n)_{n\geq 1} \in \ell^p\{S_n^p[S_m^1]\}$ (here each x_n is an element of $S_n^p[S_m^1]$). Then

$$(I_{S_m^1} \otimes T)(x) = \Big(\big((1-\beta_n)^{\frac{1}{p}} x_n\big)_{n\geq 1}, \big(\beta_n^{\frac{1}{p}}\,(t_n \otimes I_{S_m^1})(x_n)\big)_{n\geq 1}\Big).$$

Consequently,

$$\begin{aligned}\big\|(I_{S_m^1} \otimes T)(x)\big\|_p^p &= \sum_{n=1}^{\infty}(1-\beta_n)\|x_n\|_{S_n^p[S_m^1]}^p + \sum_{n=1}^{\infty}\beta_n\|(t_n \otimes I_{S_m^1})(x_n)\|_{S_n^p[S_m^1]}^p \\ &\leq \sum_{n=1}^{\infty}(1-\beta_n)\|x_n\|_{S_n^p[S_m^1]}^p + n^p\beta_n\|x_n\|_{S_n^p[S_m^1]}^p \qquad \text{by [5, Lemma 5.3 (ii)]} \\ &\leq (1+\varepsilon)^p \sum_{n=1}^{\infty}\|x_n\|_{S_n^p[S_m^1]}^p = (1+\varepsilon)^p\|x\|_p^p.\end{aligned}$$

It is clear that T is separating and that the Jordan homomorphism $J \colon \mathcal{M} \to \mathcal{N}$ in its Yeadon triple is given by

$$J\big((x_n)_{n\geq 1}\big) = \big((x_n)_{n\geq 1}, (t_n(x_n))_{n\geq 1}\big).$$

It follows that whenever $\mathcal{M}_1$ is a non zero summand of $\mathcal{M}$, the Yeadon factorization of the restriction of T to $L^p(\mathcal{M}_1)$ is neither direct nor indirect. A fortiori, T does not satisfy the assertion (ii) of Theorem 4.9.

ACKNOWLEDGMENT

The second author would like to thank "Laboratoire de Mathématiques de Besançon" for hospitality and excellent working condition she received during her visit.

References

[1] G. Hong, S. K. Ray and S. Wang, *Maximal ergodic inequalities for positive operators on noncommutative L^p-spaces*, Preprint 2020, arXiv:1907.12967v5.
[2] M. Junge, *Doob's inequality for non-commutative martingales*, J. Reine Angew. Math. 549 (2002) 149-190.
[3] M. Junge and Q. Xu, *Noncommutative maximal ergodic theorems*, J. Amer. Math. Soc. 20 (2007), 385-439.
[4] C. Le Merdy and S. Zadeh, *ℓ^1-contractive maps on noncommutative L^p-spaces*, to appear in Journal of Operator Theory, arXiv:1907.03995.
[5] C. Le Merdy and S. Zadeh, *On factorization of separating maps on noncommutative L^p-spaces*, preprint 2020, arXiv:2007.04577.
[6] G. Pisier, *Non-commutative vector valued L^p-spaces and completely p-summing maps*, Astérisque, Vol. 247, 1998.
[7] G. Pisier, *Regular operators between non-commutative L_p-spaces*, Bull. Sci. Math. 119 (1995), 95-118.
[8] G. Pisier and Q. Xu, *Non-commutative L^p-spaces*, Handbook of the geometry of Banach spaces, Vol. 2, pp. 1459-1517, North-Holland, Amsterdam, 2003.
[9] V. Runde, *Amenable Banach algebras*, Springer-Verlag, New York, 2020.
[10] E. Størmer, *On the Jordan structure of C^*-algebras*, Trans. Amer. Math. Soc. 120 (1965), 438-447.
[11] M. Takesaki, *Theory of operator algebras. I*, Encyclopaedia of Mathematical Sciences, Vol. 124, Springer-Verlag, Berlin, 2002.
[12] M. Terp, *L_p-spaces associated with von Neumann algebras*, Notes, Math. Institute, Copenhagen Univ., 1981.
[13] F. Y. Yeadon, *Isometries of noncommutative L^p-spaces*, Math. Proc. Cambridge Philos. Soc. 90 (1981), 41-50.

Laboratoire de Mathématiques de Besançon, Universite Bourgogne Franche-Comté, France

Email address: christian.lemerdy@univ-fcomte.fr

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111, Bonn, Germany &
Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Germany.

Email address: jsafoora@gmail.com