Algebraic Geometric Invariants for Homotopy Actions

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by

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To Bill Browder on his Sixtieth Birthday

0. Introduction

Let X be a paracompact topological space, and let $\mathcal{H}(X)$ denote the monoid of self-homotopy equivalences of X. The homotopy equivalences which are homotopy equivalent to the identity map of X form a submonoid $\mathcal{H}_1(X)$. The quotient $\mathcal{E}(X) \equiv \mathcal{H}(X)/\mathcal{H}_1(X)$ is a group, called *the group of self-equivalences of* X. Subgroups of $\mathcal{E}(X)$ represent "homotopy symmetries" of X, or symmetries of the homotopy type of X in the homotopy category. More generally, any group homomorphism $\rho : G \to \mathcal{E}(X)$ leads to a *homotopy action of* G *on* X by choosing representatives $\tilde{g} \in \mathcal{H}(X)$ for $\rho(g) \in \mathcal{E}(X)$ for each $g \in G$, and considering $\tilde{g} : X \to X$ acting on X. If $\varphi : G \times X \to X$ is a topological action, then φ induces a homotopy action. Conversely, given a homotopy action $\alpha : G \to \mathcal{E}(X)$, one is interested in finding a topological action $\psi : G \times X' \to X'$ and a homotopy equivalence $f : X' \to X$ which is equivariant up to homotopy, i.e. $f \circ g$ and $\tilde{g} \circ f$ are homotopic (using the

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above-mentioned notation). If such an (X', ψ) exists, then ψ is called a *topological replacement* (following George Cooke [18]) or a *topological realization* for α.

The concept of a homotopy-action and its first applications to algebraic topology appear in the 1978 paper of George Cooke [18]. Further generalization of Cooke's work and its applications to homotopy theory are due to Alex Zabrodsky [48] [49]. Zabrodsky had formulated an obstruction theoretic approach to finding a topological replacement for homotopy actions which was step-by-step and fairly difficult to compute (unpublished work). Bill Browder and the author were led to homotopy actions as a tool to study and construct finite group actions on simply-connected manifolds [7] [11]. These efforts motivated the author's present paper and an earlier paper [8]. There are also further research by Frank Quinn [36] [37], Justin Smith [42], Peter Kahn [28] [29] and Schwänzl-Vogt [39]. Mark Mahowald has kindly pointed out to the author the recent joint work of Mike Hopkins and Haynes Miller (to appear).

The purpose of this paper is to introduce new invariants for homotopy actions of finite groups via tools from abstract algebraic geometry. Roughly speaking, these invariants parametrize and encode replacements for familiar objects such as fixed-point sets and inertia subgroups from topological transformation groups. More details and applications of these ideas will appear elsewhere.

1. Homotopy Actions and Representations

In this section, we discuss some examples of homotopy actions and how they relate to other problems. While homotopy actions and topological actions of a group G on a space X are geometrically very different, they share some common algebraic features. Namely, if $\mathcal{F}$ is a homotopy functor from spaces to abelian groups, then $\mathcal{F}(X)$ affords a G-representation in both cases. When $\mathcal{F}(X)$ admits a richer algebraic structure, the G-representation $\mathcal{F}(X)$ carries more sophisticated information.

In the case of topological G-actions, the RG-modules $H^*(X;R)$ and $H_*(X;R)$ are called *homology representations* of X. Homology representations arise naturally in group theory. Cf. [1] [44] for a sample of examples in this context.

Example 1. Suppose X is a space homotopy equivalent to a bouquet of n copies of S^m. Then $\mathcal{E}(X) \cong \mathrm{Aut}(H_m(X)) \cong \mathrm{GL}(n, \mathbf{Z})$ by obstruction theory. Thus, any torsion-free $\mathbf{Z}G$-module of $\mathbf{Z}$-rank n gives rise to a homotopy G-action on X and vice versa.

Example 2. Let X be a closed-simply-connected topological 4-manifold, and let $\mu : H^2(X) \times H^2(X) \to \mathbf{Z}$ be its intersection form. Let $\mathrm{Aut}(H^2(X), \mu)$ denote the set of isometries of $H^2(X)$ with respect to the non-degenerate symmetric bilinear form μ. It is isomorphic to an arithmetic group of orthogonal type. According to Mike

Freedman [22] every such orthogonal form μ on $\mathbf{Z}^n$ corresponds to $\mathrm{Aut}(H^2(X), \mu)$ for an appropriate simply-connected closed 4-manifold X as above, and at most there are two such X up to homeomorphism for each isomorphism class of μ [22]. On the other hand, there is a surjection $\mathcal{E}(X) \to \mathrm{Aut}(H^2(X), \mu)$ with kernel isomorphic to $(\mathbf{Z}_2)^r$. Thus, a homotopy G-action $G \to \mathcal{E}(X)$ gives rise to a torsion-free orthogonal $\mathbf{Z}G$-module. Conversely, the results of Freedman-Quinn [23] imply that any homomorphism $G \to \mathrm{Aut}(H^2(X), \mu)$ lifts to $\mathcal{E}(X)$. Consequently, every torsion-free orthogonal $\mathbf{Z}G$-module corresponds to a homotopy G-action on a simply-connected closed 4-manifold (well-defined up to homotopy). In this case, the homotopy equivalences corresponding to action of $g \in G$ can be taken to be homeomorphisms. However, the group law in G is preserved only up to homotopy, so this homotopy action is still far from a topological G-action.

Example 3. In the late 1950's, Steenrod asked if every G-representation was obtained as the homology representation for a G-action on a finite Moore space [34]. The answer is negative in general. A reformulation of this problem, called *The Steenrod Problem*, has been studied by several authors [3] [4] [5] [8] [9] [13] [17] [28] [29] [42] [45] [46] [47]. It amounts to the problem of characterizing the homology representations of a group G acting on a Moore space.

Suppose M is a finitely generated RG-representation. When M is R-torsion-free, Example 1 shows that there exists a homotopy G-action (unique up to homotopy!) on a bouquet of spheres X such that $H_*(X; R) \cong M$. To replace this homotopy action by a topological action, one may appeal to G. Cooke's Theory [18]. According to [18], the representation $\alpha : G \to \mathcal{E}(X)$ gives rise to a map $B_\alpha : B_G \to B_{\mathcal{E}(X)}$ on the level of classifying spaces. The exact sequence $\mathcal{H}_1(X) \to \mathcal{H}(X) \to \mathcal{E}(X)$ gives rise to a fibration $B_{\mathcal{H}_1(X)} \to B_{\mathcal{H}(X)} \to B_{\mathcal{E}(X)}$.

Theorem (G. Cooke [18]). The homotopy action α is equivalent to a topological G-action if and only if there is a lift of B_α to $B_{\mathcal{H}(X)}$.

Suppose such a lift exists, say $f : B_\alpha \to B_{\mathcal{H}(X)}$. Then one pulls back the Stasheff universal fibration $X \to E_X \to B_{\mathcal{H}(X)}$ via f to the fibration $X \to W \xrightarrow{\pi} B_G$. Now consider the pull-back of the universal principal G-bundle $G \to E_G \to B_G$ via π to $G \to \tilde{W} \to W$. The free G-space $\tilde{W}$ is seen to be homotopy equivalent to X, and this G-action is equivalent to α. Cooke's point of view has been applied to the Steenrod Problem by Peter Kahn [28]. In this case, $\mathcal{E}(X) \cong \mathrm{GL}(n, \mathbf{Z})$, and through stabilization of n and Quillen's plus-construction the above-mentioned map $B_\alpha \to B_{\mathrm{GL}(n,\mathbf{Z})}$ is replaced by $\rho : B_G \to \mathrm{BGL}^+$, leading to algebraic K-theoretic invariants.

Example 4. When the $\mathbf{Z}G$-module M is not $\mathbf{Z}$-torsion free, the G-action on $\mathrm{Aut}(M)$ may not lift to a homotopy action on a Moore space X whose homology is isomorphic to the underlying abelian group of M. In fact, obstruction theory shows that the existence of such a homotopy action depends on the 2-torsion part of $H^*(G; M)$. There is a beautiful (unpublished) example of Pierre Vogel which realizes this obstruction. Let G be the Klein four-group $\mathbf{Z}_2 \times \mathbf{Z}_2$ regarded as the subgroup of $\mathrm{GL}(2, \mathsf{F}_4)$ via $\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathsf{F}_4 \right\}$. G acts naturally on $\mathsf{F}_4 \oplus \mathsf{F}_4 \cong (\mathbf{Z}_2)^4$ via left matrix multiplication on 2-vectors. This representation of $\mathbf{Z}_2 \times \mathbf{Z}_2$ on $M = (\mathbf{Z}_2)^4$ cannot be realized by a homotopy-action on a Moore space, and hence not by a topological G-action either.

Example 5. Another approach to the Steenrod Problem is due to Frank Quinn [36] [37] and Justin Smith [42] [43] in the general context of topological realization of chain complexes. The obstruction theories of Quinn and Smith are essentially different. There is an example of a $\mathbf{Z}$-torsion-free $\mathbf{Z}[\mathbf{Z}_2 \times \mathbf{Z}_2]$-representation due to G. Carlsson [17] which cannot arise as the homology representation of any $\mathbf{Z}_2 \times \mathbf{Z}_2$-action on a Moore space. Thus, there is a homotopy $\mathbf{Z}_2 \times \mathbf{Z}_2$-action α on a Moore space X whose associated $\mathbf{Z}_2 \times \mathbf{Z}_2$-representation is isomorphic to M. Peter Kahn's computation of the first non-vanishing obstruction via Cooke's Theory (Example 3) to replace α by an equivalent topological action and the analogous obstructions via Smith's theory applied to the appropriate $\mathbf{Z}[\mathbf{Z}_2 \times \mathbf{Z}_2]$-chain complex are closely related. This points out to the possibility of a deeper relationship between the works of Cooke, Carlsson, Quinn and Smith.

In the following, we describe algebraic-geometric invariants that can be associated to topological G-spaces, G-representations, G-chain complexes and homotopy G-actions by the same procedure. In the case of finite group actions on Moore spaces, these invariants coincide, and confirm the same answer by the previous methods.

2. Algebraic Geometric Invariants

Let $\mathcal{C}$ be any of the following categories:

(i) $\mathbf{T}$ = category of paracompact topological spaces and continuous maps.

(ii) $\mathbf{T}^h$ = the homotopy category corresponding to $\mathbf{T}$.

(iii) $\mathbf{M}$ = the category of R-modules (finitely generated) and R-homomorphisms.

(iv) $\mathbf{C}$ = the category of R-chain complexes and chain homomorphism.

Let G be a finite group. The G-equivariant category corresponding to $\mathcal{C}$ is denoted by $\mathcal{C}_G$. Thus objects of $\mathcal{C}_G$ are endowed with a G-action and the morphisms are G-equivariant (homotopy-actions and homotopy-equivariant maps for $\mathbf{T}^h$). Given objects X and Y in $\mathcal{C}_G$ and a G-morphism $f : X \to Y$, we wish to attach invariants for the G-actions on X and Y and establish a relationship between them via f. First, consider $\mathbf{T}_G$, where the fixed point set X^G, $X^H (H \subseteq G)$, the isotropy subgroups G_x for $x \in X$, the orbit space X/G, the Borel-construction $X_G \equiv E_G \times_G X$ ($X \to$

$E_G \times_G X \to BG$ is the associated fibre bundle to the universal principal G-bundle $G \to E_G \to BG$) and their many algebraic-topological invariants are defined as usual. When $\dim X < \infty$, X^G and its topological invariants are useful invariants of the G-action on X. The isotropy groups are rather weak invariants when one deals with homological properties of the G-action as opposed to more subtle geometric circumstances (such as G-manifolds). A better substitute is the set $\mathcal{A}(X) = \{A \subseteq G \mid H^*(X^A, X^{A'}; \mathbf{Z}_p) \neq 0$, where A and A' are elementary abelian p-subgroups and $A' \supsetneq A\}$ which we call *the set of essential stabilizers.* When only one prime $p \mid |G|$ must be considered, we use the notation $\mathcal{A}_p(X)$. Among the algebraic topological invariants in $\mathbf{T}_G$, the Borel equivariant cohomology $H^*_G(X; R) \equiv H^*(X_G; R)$ is one of the most useful and common invariants. Many of the invariants of G-actions on X and Y are obtained from the functor $E_G \times_G (-) : \mathbf{T}_G \to \mathbf{T}$ and the usual algebraic-topological invariants of $\mathbf{T}$. Note that $H^*_G(-; R)$ yields a graded module over $H^*(BG; R)$ and $f^*_G \equiv H^*_G(f; R)$ is $H^*(BG; R)$-linear.

In the following, we wish to construct analogues of the above-mentioned invariants of $\mathbf{T}_G$ for the remaining categories. The concepts corresponding to $E_G \times_G (-)$ and $H^*_G(-)$ for $\mathbf{M}$ and $\mathbf{C}$ were introduced and studied in homological algebra in the same period. The set-theoretic notion "G-fixed points" when applied to an RG-module M, with the same definition as X^G, leads to group cohomology with twisted coefficients $H^*(G; M)$. Let Q^* be an RG-free acyclic complex. $Q^* \otimes_G M$ in $\mathbf{M}_G$ is analogous to the Borel construction in $\mathbf{T}_G$. Thus, $H(Q^* \otimes_G M) \cong H^*(G; M)$ is a good candidate to imitate $H^*_G(X)$. This construction carries over to $\mathbf{C}_G$, and leads to the notion of group-cohomology with coefficients in a chain complex, originally called hypercohomology (cf. [15] [14] and [9] Section One.) For an RG-chain complex C^*, the cohomology of the total complex of the double complex $Q^* \otimes C^*$ is denoted by $\mathsf{H}^*(G; C^*)$. If C^* is the singular cochain complex of a paracompact G-space X, then $\mathsf{H}^*(G; C^*)$ agrees with $H^*_G(X; R)$, confirming the expected requirements. Moreover, if $C^0 = M$ and $C^i = 0$ for all $i > 0$, then $\mathsf{H}^*(G; C^*) = H^*(G; M)$. Thus, constructions in $\mathbf{C}_G$ yield generalizations for both $\mathbf{T}_G$ and $\mathbf{M}_G$. The case of homotopy actions $\mathbf{T}^h_G$ is not clear as readily, even if we ask for the analogue of the Borel equivariant cohomology $H^*_G(-)$.

It is worth mentioning that the direct generalization of fixed point set X^G to the algebraic set-up yields an algebraic object only. The question of finding a *geometric generalization* still remains to be explored. For example, consider a finite dimensional G-space X and its singular cochain complex $C^*(X; R)$. The derived functors of $(-)^G$ in $\mathbf{C}_G$ give back $H^*_G(X; R)$ and not $H^*(X^G; R)$. Thus, analogues of fixed point sets and essential stabilizer subgroups from a geometric point of view remain to be formulated for $\mathbf{M}_G$ and $\mathbf{C}_G$ besides $\mathbf{T}^h_G$.

In abstract algebraic geometry, one often needs appropriate finiteness conditions. In the sequel, we will assume that all RG-modules are finitely generated. This will include total homologies of G-spaces and G-chain complexes.

2.1 Proposition. Let R be a commutative ring and let G be a finite group. Then $H^*(G;R)$ is a (graded-commutative) noetherian R-algebra. For any finitely generated RG-module M, $H^*(G;M)$ is a finitely generated $H^*(G;R)$-module. Similarly, assume that $H^*(X;R)$ and $H^*(C^*;R)$ are finitely generated R-modules where X is a G-space and C^* is a G-complex. Then $H^*_G(X;R)$ and $\mathsf{H}^*(G;C^*)$ are finitely as $H^*(G;R)$-modules.

See Evens [21] for noetherian properties of $H^*(G;R)$ and related results.

Next, we shall restrict attention to the graded R-algebra H_G defined to be $\bigoplus_i H^{2i}(G;R)$ whenever R is not a ring of characteristic 2. There are two main reasons for this restriction. First, the ring $H^*(G;R)$ is not strictly commutative in general. One may consider the $\mathbb{Z}_2$-graded R-algebra corresponding to H_G and $\bigoplus_i H^{2i+1}(G;R)$ as a super-ring, and apply constructions from super-algebraic geometry. There are many important technical differences between the commutative and the $\mathbb{Z}_2$-graded theories. We shall study the super-algebraic geometric invariants analogous to the invariants of H_G elsewhere. The second reason is the more satisfactory geometric situation that arises for the case of elementary abelian p-groups when we consider the polynomial subalgebra of $H^*(G;R)$.

Consider the projective scheme $(S,\mathcal{O}_S)$ over Spec(R), where $S = \text{Proj}(H_G) =$ the projective scheme consisting of all proper homogeneous prime ideals of H_G. A finitely generated graded H_G-module F^* gives rise to a coherent algebraic sheaf $\mathcal{F}$ of modules over $(S,\mathcal{O}_S)$, also called an $\mathcal{O}_S$-module for short. Let us recall briefly this construction due to Serre and Grothendieck [24] [40]. A sub-basis for the Zariski topology on Proj(H_G) consists of open sets of the form Spec$((H_G)_{(f)})$ which are denoted by S_f. Here, $(H_G)_{(f)}$ is the homogeneous localization of H_G with respect to the homogeneous element $f \in H_G$ and it consists of all (equivalence classes of) elements $\frac{a}{f^n}$ with $a \in H_G$ homogeneous, $\deg(a) = \deg(f^n)$ and $n \geq 0$. Given F^*, let $\mathcal{F}(S_f) = \{\frac{b}{f^n} \mid b \in F^* \text{ is homogeneous}, \deg b = \deg f^n \text{ and } n \geq 0\}$ define the presheaf associated to $\mathcal{F}$.

Given a coherent $\mathcal{O}_S$-module $\mathcal{F}$, define the following exact sequence:

$$0 \to \mathcal{F}_\tau \to \mathcal{F} \to \mathcal{F}_\varphi \to 0.$$

The subsheaf $\mathcal{F}_\tau$ is defined via $\mathcal{F}_\tau(S_f) =$ torsion $(H_G)_{(f)}$-submodule of $\mathcal{F}(S_f)$, and $\mathcal{F}_\varphi$ is the quotient sheaf $\mathcal{F}/\mathcal{F}_\tau$. $\mathcal{F}_\tau$ is called the torsion subsheaf of $\mathcal{F}$ and $\mathcal{F}_\varphi$ is the largest torsion-free quotient of $\mathcal{F}$. The support of $\mathcal{F}_\tau$ is defined to be $\text{supp}(\mathcal{F}_\tau) = \{s \in S \mid \mathcal{F}_s \neq 0\}$. It defines a closed subscheme of $(S,\mathcal{O}_S)$ whose underlying topological space is $\text{supp}(\mathcal{F}_\tau)$.

Let W be any of the following: (i) a topological space with an action $G \times X \to X$; (ii) an R-free RG-chain complex C^*; (iii) an RG-module M.

Let F^* be the H_G-module in each case, respectively: (i) $H^*_G(X;R)$; (ii) $\mathsf{H}^*(G;C^*)$; (iii) $H^*(G;M)$.

Definition. Let $\mathcal{F}$ be the $\mathcal{O}_S$-module corresponding to F^*. $\mathcal{F}$ is called *the characteristic sheaf* of W. $\mathcal{F}_\tau$ and $\mathcal{F}_q$ are called respectively *the characteristic torsion* and the *characteristic torsion-free sheaves* of W.

Next, we relate the characteristic sheaves of a finite dimensional G-space to its more familiar geometric invariants. It turns out that the set of essential stabilizers $\mathcal{A}_p(X)$ and the corresponding $\dim_k H^*(X^A; k)$ for each $A \in \mathcal{A}_p(X)$ can be recovered from the characteristic sheaves associated to the G-action on X. Here, k is a field of characteristic p and $A \cong (\mathbb{Z}_p)^{n+1}$, $n \geq 0$. To see this, we need to look at the case $G \cong (\mathbb{Z}_p)^{n+1}$. For simplicity of notation, we shall consider the case $p = 2$. For $p =$ odd prime, one has the same statements by replacing H_G and $\mathrm{Proj}(H_G)$ in the following with the reduced k-algebra $H_G/\text{radical}$ and the corresponding reduced scheme. By considering the odd and even degrees separately, one can obtain stronger results, such as Euler characteristics of the fixed point sets. Cf. [9] for related results.

2.5 Theorem. Let $G = (\mathbb{Z}_2)^{n+1}$ and k be a field of characteristic 2. Let X be a finite-dimensional G-space with characteristic sheaf $\mathcal{F}$. Then:

(a) $\mathcal{F}_\tau$ determine $\mathcal{A}_2(X) =$ set of essential stabilizers of X, and conversely $\mathcal{A}_2(X)$ determines $\mathrm{supp}(\mathcal{F}_\tau)$.

(b) $\dim_k H^*(X^G; k) = \mathrm{rank}\, \mathcal{F}_q = \mathrm{rank}\, \mathcal{F}$.

Proof. First consider the case $X^G = \emptyset$. Then $H_G^*(X)$ is a torsion H_G-module (k-coefficients by Borel-Quillen-Hsiang localization theorem (cf. [35] [27], for example). Hence $\mathcal{F}_q = 0$ and $\mathcal{F}_\tau = \mathcal{F}$. Since $H_G \cong k[t_0, \ldots, t_n]$ is a polynomial ring, $\mathrm{Proj}(H_G) = \mathbb{P}^n(k) = \mathbb{P}^n(\mathbb{F}_2) \underset{\mathrm{Spec}\, \mathbb{F}_2}{\times} \mathrm{Spec}\, k$. The linear subspaces of $\mathbb{P}^n(\mathbb{F}_2)$ are easily seen to be in one-to-one correspondence with subgroups of G, while points of $\mathbb{P}^n(\mathbb{F}_2)$ correspond to cyclic subgroups of G. The defining ideal for the linear subspace corresponding to the subgroup $A \subsetneq G$ is given by the kernel of the induced homomorphism $H^*(G; \mathbb{F}_2) \to H^*(A; \mathbb{F}_2)$. The formulation of the localization theorem in equivariant cohomology by Hsiang [27] implies that $A \subsetneq G$ is an essential stabilizer if and only if there exists $u \in H_G(X; \mathbb{F}_2)$ such that $\mathrm{ann}(u) = \mathrm{Ker}(H^*(G; \mathbb{F}_2) \to H^*(A; \mathbb{F}_2))$. Hence, corresponding to each $A \in \mathcal{A}_2(X)$, there exists a subsheaf of $\mathcal{F}_\tau$ whose support is the $\mathbb{F}_2$-rational linear subspace of $\mathbb{P}^n(k)$ defined by the homogeneous prime ideal $I_A = \mathrm{Ker}(H^*(G; k) \to H^*(A; k))$. Further, I_A occurs as the annihilating ideal for a global section $u \in H^0(\mathbb{P}^n(k), \mathcal{F}_\tau(d))$ for a sufficiently large d. Therefore, $\mathcal{A}_2(X)$ determines $\mathrm{supp}(\mathcal{F}_\tau)$. Conversely, for any homogeneous element $u \in H_G^r(X; k)$, the annihilating ideal $\mathrm{ann}(u) \subseteq H_G$ is seen to be invariant under the action of the Steenrod algebra, i.e. invariant under the substitution $t_i \mapsto t_i + t_i^2$ for all polynomials generators of H_G. According to Serre [41], the variety defined by such an ideal is the union of the $\mathbb{F}_2$-rational linear subspaces of $\mathbb{P}^n(k)$. Consider an element $v \in H^0(\mathbb{P}^n(k); \mathcal{F}_\tau(d))$ (for a sufficiently high d) whose annihilating ideal is a homogeneous prime $I \subseteq H_G$. Accordingly, I determines a subgroup $A \subseteq G$ via, $I \cong I_A = \mathrm{Ker}(H^*(G;k) \to H^*(A;k))$. Therefore, there is a one-one correspondence between $\mathcal{A}_2(X)$ and the homogeneous prime ideals of H_G which occur as annihilating ideals for some $u \in H^0(\mathbf{P}^n(k), \mathcal{F}(d))$ for a sufficiently large d.

(b) rank $\mathcal{F}_q$ is equal to the dimension of the $(\mathcal{F}_q)_x$ at the generic point x of $\mathbf{P}^n(k)$ as a vector space over K (the quotient field of H_G). By the localization theorem, $H_G^*(X) \otimes_{H_G} K \cong H_G^*(X^G) \otimes_{H_G} K \cong H^*(X^G) \otimes_k K$. Hence $\dim_k H^*(X^G) = \mathrm{rank}\, \mathcal{F}_q = \mathrm{rank}\, \mathcal{F}$.

3. Invariants for Homotopy Actions.

In this section, we extend the construction of the previous section to homotopy actions. To do this, we need to introduce an appropriate stabilization procedure. Let M_i be RG-modules and P_i be projective RG-modules, $i = 1, 2$. M_i are called (projectively) stably equivalent if for suitable choices of P_i, $M_1 \oplus P_1 \cong M_2 \oplus P_2$. Define the operation ω on the stable equivalence classes of RG-modules via the short exact sequence applied to a representative of the class:

$$0 \to \omega(M) \to P \to M \to 0.$$

Here, P is RG-projective, and $\omega(M)$ is well-defined up to projective equivalence by Schanuel's Lemma [19]. Denote stable equivalence class of M by $\langle M \rangle$ and $\omega(\langle M \rangle)$ by $\langle \omega(M) \rangle$. We can define the operation ω^{-1} for R-free RG-modules via $0 \to M \to Q \to \omega^{-1}(M) \to 0$ where Q is RG-injective (equivalently RG-projective, since these two concepts agree for quasi-Frobenius rings such as RG) [19]. Equivalently, $\omega^{-1}(M) \equiv \mathrm{Hom}_R(\omega(\mathrm{Hom}_R(M, R)), R)$ could be used. Inductively, $\omega^{n+1}(M) \equiv \omega(\omega^n(M))$, $n \in \mathbf{Z}$. In general, $\omega^{-1}(M)$ can be defined as $\omega^{-2}(\omega(M))$, using the R-free RG-module $\omega(M)$. Two RG-modules M_1 and M_2 are called ω-stably equivalent if $\langle \omega^r(M_1) \rangle = \langle \omega^s(M_2) \rangle$ for some $r, s \in \mathbf{Z}$.

Next, we need to pass to the graded version of ω-stability. Given a graded finitely generated RG-module $M^\bullet = \bigoplus_{i \geq 0} M^i$, choose $N \in \mathbf{Z}$ sufficiently large so that $M^i = 0$ for $i \geq N$. Define a sequence $\overline{M}^i$ as follows: $\overline{M}^0 = \omega^N M^0$ and the sequence $0 \to \overline{M}^i \to \overline{M}^{i+1} \to \omega^{N-(i+1)}(M^{i+1}) \to 0$ is exact. Thus, up to ω-stability, $\overline{M}^N$ is a composite extension of composition factors $\omega^{N-i}(M^i)$.

3.1. Definition-Proposition. The ω-stable class of $\overline{M}^N$ is well-defined and it is called an ω-composite extension of $M^\bullet$. The ω-stable class of $\overline{M}^N$ is denoted by $\mu(M^\bullet)$.

Note that $\overline{M}^N$ depends on the choices of extensions. Hence, there are many possibilities for $\mu(M^\bullet)$ in general.

Notation. For a homotopy action $\alpha : G \to \mathcal{E}(X)$, define $\eta(X,\alpha)$ = the set of ω-stable classes of ω-composite extensions for $H^*(X;R)$. $\eta(X,\alpha)$ incorporates all possible ω-stable classes which may arise from all possible topologically equivalent G-actions.

To see the effect of ω-stability on characteristic sheaves, we need to recall the notion of Serre twist in algebraic geometry. Let Λ^* be a graded R-algebra, and let F^* be a graded Λ^*-module. Let $S = \mathrm{Proj}(\Lambda^*)$ and $\mathcal{F}$ be the coherent $\mathcal{O}_S$-module corresponding to F^*. The shift of grading by d yields the Λ^*-module $F^*(d)$ defined by $F^*(d)^i = F^{d+i}$. The corresponding sheaf is denoted by $\mathcal{F}(d)$. It follows that $\mathcal{F}(d) \cong \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_S(d)$. $\mathcal{O}_S(1)$ is called the Serre twisting sheaf, and $\mathcal{F}(d)$ is called a Serre twist of $\mathcal{F}$.

Definition. Two $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{F}'$ are called Serre-twist equivalent if $\mathcal{F}(d) \cong \mathcal{F}'(d)$ for some $d, d' \in \mathbb{Z}$.

3.2. Propostion. Let $\mathcal{F}$ be an $\mathcal{O}_S$-module, and let $\mathcal{F}_\tau$ and $\mathcal{F}_q$ be its torsion subsheaf and torsion-free quotients. Then:

(a) $\mathcal{F}(d)_\tau = \mathcal{F}_\tau(d)$. Hence $\mathrm{supp}(\mathcal{F})$ is invariant under Serre twists.

(b) $\mathcal{F}(d)_q = \mathcal{F}_q(d)$. Hence $\mathcal{F}_q$ is locally-free iff $\mathcal{F}_q(d)$ is locally free. Moreover, $\mathcal{F}_q$ and $\mathcal{F}_q(d)$ split into direct sums of sheaves of smaller rank in a similar manner.

The proof follows from tensoring $0 \to \mathcal{F}_\tau \to \mathcal{F} \to \mathcal{F}_q \to 0$ with $\mathcal{O}_S(d)$ over $\mathcal{O}_S$, and using the definitions.

Definition. Let $\alpha : G \to \mathcal{E}(X)$ be a homotopy action. The set $\mathcal{S}^\omega(X,\alpha) \equiv$ Serre-twist equivalence classes of characteristic sheaves $\mathcal{F}$ corresponding to one representative from each ω-stable class in $\eta(X,\alpha)$.

3.3. Proposition. $\mathcal{S}^\omega(X,\alpha)$ is well-defined. In particular, the set $\mathrm{supp}(X,\alpha) = \{\mathrm{supp}(\mathcal{F}_\tau) \mid \mathcal{F}_\tau$ is a characteristic sheaf corresponding to a representative from and element of $\eta(X,\alpha)\}$ and $\mathrm{Fix}(X,\alpha) = \{$Serre-twist equivalence classes of characteristic torsion-free sheaves corresponding to a representative from an element of $\eta(X,\alpha)\}$ are well-defined.

Proof. With a slight abuse of notation, apply cohomology $H^*(G,-)$ to the exact sequence $0 \to \omega(M) \to P \to M \to 0$. It follows that

$$H^i(G;M) \cong H^{i+1}(G;\omega(M)) = H^*(G;\omega(M))(1)^i. \qquad (1)$$

Hence, the characteristic sheaves of ω-stably equivalent RG-modules differ only by Serre twists. Note that two graded modules which are eventually isomorphic (i.e. isomorphic in all sufficiently large degrees) give rise to isomorphic sheaves upon Serre-Grothendieck construction. The conclusion follows from Proposition 3.2.

Remark. Supp(X, α) and Fix(X, α) incorporate the extensions in the ω-composite extension constructions on $H^*(X;R)$. They depend on R and the graded RG-representation $H^*(X;R)$. Otherwise, most other topological properties of X are weakened in the stability process.

Next, there are two closely-related notions of stability for G-spaces and G-complexes. Following [6], two G-spaces X_i, $(i = 1,2)$ are called freely equivalent, if there exists a G-space Y containing X_1 and X_2 as G-subspaces, and Y/X_i are compact G-spaces with free G-actions away from their base points corresponding to X_i. The corresponding notion for R-free RG-chain complexes are defined similarly: C_* and C'_* are freely equivalent if there is an RG-complex D_* containing both as RG-subcomplexes and such that D_*/C_* and D_*/C'_* are finitely generated and free over RG. Free equivalence is an equivalence relation.

3.4. Proposition. (a) Suppose X and Y are freely equivalent G-spaces. Then their characteristic sheaves are isomorphic.

(b) Assume further that $Y^G \neq \emptyset$ and Y is a Moore space with $H^n(Y;R) = M$ for some $n > 0$ and some RG-module M. Then the characteristic sheaves of X and M are Serre-twist equivalent. Similar statements hold for RG-complexes.

Proof. It suffices to consider the case where Y contains X as a G-subspace, and $Y - X$ is a finite dimensional free G-space. Thus, $H^*_G(Y, X)$ vanishes in all degrees greater than $\dim(Y - X)$. The inclusion $j : X \to Y$ induces an eventual isomorphism $j^* : H^*_G(Y) \to H^*_G(X)$, hence an isomorphism after Serre-Grothendieck construction.

To see (b), observe that $H^n(Y;R)$ is an ω-composite extension of $\oplus_i \overline{H}^i(X;R)$. Let $y_0 \in Y^G$. Then

$$\overline{H}^*_G(X) = \overline{H}^*_G(Y, y_0) \cong H^*(G;M)$$

in all sufficiently high degrees and after a possible degree shift. Hence the characteristic sheaves differ at most by a Serre-twist. Similar comments apply to RG-complexes.

Remark. The condition $Y^G \neq \emptyset$ is not necessary if one is willing to replace G-spaces by suitable RG-chain complexes (e.g. the cellular chain complex if X and Y are G-CW complexes). Up to free equivalence, any RG-chain complex is equivalent to one with precisely one cohomology group M, which is a ω-composite extension of $\oplus_i H^i(X;R)$. In this situation, the statement of 2.4 remains true again.

4. Obstructions for Topological Replacements.

Let (X, α) be a homotopy G-Action. As pointed out above, G. Cooke proved that (X, α) can be replaced by an equivalent topological G-action if and only if the map $B_\alpha : B_G \to B_{\mathcal{E}(X)}$ lifts to $B_{\mathcal{H}(X)}$ in the fibration $B_{\mathcal{H}(X)} \to B_{\mathcal{E}(X)}$. This lifting problem can be studied via obstructions theory as in [48] [49] [28]. In practice, only a few lifting problems can be successfully studied. Often, when the first (or at best the first few) obstructions vanish and a higher obstruction group is non-zero, it becomes exceedingly difficult to determine whether or not different choices for the earlier Stages of the lifts will avoid the higher obstructions. Moreover, the intermediate obstructions may not have a useful geometric or algebraic interpretation.

The strategy adopted in this paper is to find necessary conditions for existence of topological replacements for (X, α) via global invariants, namely characteristic sheaves associated to ω-composite extensions of $H^*(X)$. There are advantages in this point of view. First, these invariants are more readily computable. Through common algebraic tools. Secondly, these invariants are defined in terms of more basic features of (X, α), namely representations on homology. The stabilizations reduce further the dependence on more subtle invariants of the homotopy-type of X. Third, these invariants have geometric interpretations in terms of fixed point sets and stability subgroups when (X, α) is equivalent to a topological G-action. Thus, these invariants of homotopy actions are substitutes for their geometric counterparts in the case of topological actions. As such, they not only detect which ω-composite extension can possibly come from an equivalent topological action, but which geometric characteristics the potential G-action should have.

Throughout the rest of this section, we assume that $G = (\mathbf{Z}_p)^{n+1}$ and k is a field of characteristic p. Homology and cohomology groups have coefficients in k.

4.1. Theorem. Let (X, α) be a homotopy G-action on a connected CW complex X. Let $(S, \mathcal{O}_S)$ be The projective scheme $\mathrm{Proj}(H_G)$. Let $\eta(X, \alpha)$ be the set of ω-stable ω-composite extensions and $\mathcal{S}^\omega(X, \alpha) =$ the set of Serre-twist equivalence classes of characteristic sheaves obtained from Serre-Grothendieck construction on representatives of $\eta(X, \alpha)$. If (X, α) is equivalent to a 0 topological G-action, then there exists a $[\mathcal{E}] \in \mathcal{S}^\omega(X, \alpha)$ such that:

(a) ξ_φ is locally-free, and it splits into a direct sum of invertible $\mathcal{O}_S$-modules.

(b) $\mathrm{supp}[\xi_\tau]$ is a union of F_p-rational linear subspaces of S.

Outline of Proof: Suppose $\psi : G \times Y \to Y$ is a topological G-action replacing (X, α). Let $[\xi] \in \mathcal{S}^\omega(X, \alpha)$ be The Serre twist equivalence class of the characteristic sheaf of (Y, ψ) corresponding to $H_G^*(Y; k)$. As indicated before, we set $(S, \mathcal{O}_S)$ be $\mathrm{Proj}(H_G(\mathrm{Radical}))$, so that $S = \mathsf{P}^n(k)$ and ξ is an $\mathcal{O}_S$-module. If $\dim Y < \infty$, then Theorem 2.5 implies that support of ξ_τ consists of the essential stabilizers of the G-action on Y which are proper subgroups of G. Hence (b) follows in this case. Moreover, proof of 2.5 shows that $[\xi_\varphi]$ the Serre twist equivalence class of the sheaf obtained from the graded H_G-module $H_G^*(X^G; k)$. The splitting of $H_G^*(X^G) = H^*(G) \otimes H^*(X; k) = H^*(G) \otimes (\oplus_d H^d(X; k) \cong \oplus_d H^*(G)(d)$ proves (D) for this case. In the genral case, we note that $\mathrm{supp}[\xi_\tau]$ is determined by the components of the closed subspace of $\mathsf{P}^n(k)$ defined by $\mathrm{Ann}(u) \subset H_G$ for a suitable set of elements

$u \in H_G^*(Y)$. On the other hand, ann(u) is invariant under the action of Steenrod algebra $\mathcal{A}_p$ [30] [31] a result of Serre [40] [31] implies that these components are $\mathbf{F}_p$-rational linear subspaces of $\mathbf{P}^n(k)$, as required in (b). As for (a) in this case, we use the work of Lannes [32] [33]. Accordingly, the torsion-free quotient of $H_G^*(Y'; \mathbf{F}_p)$ is isomorphic in sufficiently high dimensions to $(H_G/\text{Radical}) \otimes \tilde{T}_G(H_G^*(Y'; \mathbf{F}_p))$, where $\tilde{T}_G$ is a suitable version of Lannes' functor T_G and $Y' = \text{Map}(E_G, Y)$ with the diagonal G-action. Since $H_G(Y'; \mathbf{F}_p)$ is finitely generated over $H_G/\text{Radical}$. $\tilde{T}_G(H_G^*(Y', \mathbf{F}_p))$ is a finite dimensional graded $\mathbf{F}_p$-vector space. On the other hand, $H_G^*(Y') = H_G^*(Y)$, so (a) follows just as in the provious case.

Remark. For a general finite group G, the Quillen stratification of Spec H_G [35] can be used to formulate a generalization of the above theorem. The proof above determines the behavior of $[\xi_\varphi]$ and $[\xi_\tau]$ on each stratum.

4.2. Corollary. Suppose (X, α) is a homotopy action and keep the notation above. Assume that for all $[\xi] \in \mathcal{S}^\omega(X, \alpha)$, either $[\xi_\varphi]$ is not represented by a sum of invertible $\mathcal{O}_S$-module or supp$[\xi_\tau]$ has components which are not $\mathbf{F}_p$-rational linear subspaces of $\mathbf{P}^n(k)$. Then (X, α) cannot be replaced by a topological action.

4.3. Example. Let K be a finite extension of $\mathbf{F}_p$ such that $[K : \mathbf{F}_p] > 1$. Let $P_i \in \mathbf{P}^n(k)$ be a set of points which are not $\mathbf{F}_p$-rational, $i = 1, \ldots, m$. Choose $(n+1)$-vectors $x_i = (x_{i0}, \ldots, x_{in}) \in K^n$, and form the elements

$$u_i = 1 + \sum_{j=0}^{n} x_{ij}(e_j - 1) \in KG,$$

where $\{e_0, \ldots, e_n\}$ is an $\mathbf{F}_p$-basis for G regarded as an $(n+1)$-dimensional $\mathbf{F}_p$-vector space. The subgroups $\langle u_i \rangle \subseteq KG$ are all isomorphic to $\mathbf{Z}_p$, and KG is a free $K\langle u_i \rangle$-module [16] [20]. Define the $\mathbf{Z}$-torsion free modules M_i via exact sequence $0 \to M_i \to (\mathbf{Z}G)^S \to KG \otimes_{K\langle u_i \rangle} K \to 0$. Computation of $H^*(G, M_i)$ shows that its characteristic sheaf is a sky-scraper sheaf over $\mathbf{P}^n(k)$ with support $\{P_i\}$. On the other hand, consider a space X homotopy equivalent to a bouquet of spheres of dimensions d_i, $i = 1, \ldots, m+1$. We assume that the d_i-th Betti number of X is the same as the $\mathbf{Z}$-rank of M_i for $1 \le i \le m$. As in Example 1.1, construct a homotopy action $\alpha : G \to \mathcal{E}(X)$ whose reduced homology representation is $\oplus_i M_i$. Here, we choose the trivial action on M_{m+1}. Suspending topological actions allows one to have a base point fixed by G. Such suspensions do not affect our arguments, so we shall consider reduced homology in the following.

We leave out M_0 for the moment, and consider the extensions

$$0 \to \overline{M}^i \to \overline{M}^{i+1} \to \omega^{N-(i+1)}(M^{i+1}) \to 0$$

as in the definition and notation of Section 3. Here $H^*(X; k) = M^*$ and $i \ge 1$ in these extensions. The extensions are determined by an appropriate $\text{Ext}_G^*(\overline{M}^i, M^{i+1})$

after dimension shifting, hence by a class in $H^*(G, \mathrm{Hom}(\overline{M}^i, M^{i+1}))$. On the other hand, a well-known result of Dade [20] (called Dade's Lemma, see also [12] [16]) and an argument similar to Mackey's formula can be used to show that $(KG \otimes_{K\langle u_i\rangle} K) \otimes (KG \otimes_{K\langle u_j\rangle} K)$ is a free KG-module if $i \neq j$. This type of argument and induction proves that the ω-composite extension corresponding to $M_1, \ldots, M_m$ is isomorphic to $\oplus_{i=1}^m \omega^{d_m - d_i}(M^i) = M'$. First, consider the case $M_{m+1} = 0$. Then, $\mathcal{S}^\omega(X, \alpha)$ has only one element $[\xi]$, and $\xi = \xi_\tau$. It follows that $\mathrm{supp}[\xi_\tau] = \{P_1, \ldots, P_m\} \subset \mathrm{P}^n(k)$ satisfies the hypothesis of Corollary 4.2. Hence (X, α) does not have a topological realization. Next, if $M_{m+1} \neq 0$, $\mathcal{S}^\omega(X, \alpha)$ can have more than one element. However, one can compute that for all $[\xi] \in \mathcal{S}^\omega(X, \alpha)$, $\mathrm{supp}[\xi_\tau] \subset \{P_1, \ldots, P_m\}$. If we take $m > \dim H^{m+1}(X; k)$, then $\mathrm{supp}[\xi_\tau]$ is seen to be non-empty. Hence (X, α) is not equivalent to a topological action in this case either.

4.4. Example. In this example, let P_1 be as in 4.3, and construct M_1 as before. Let $M_2 = \omega(M_1)$, and construct a homotopy action (X, α) as in 4.3 with reduced homology $H_d(X) \cong M_1$ and $H_{d+1}(X) = M_2$. Again, we may consider reduced homology. In this case, then $\mathcal{S}^\omega(X, \alpha)$ has only two classes. One corresponding to the case $\xi_\tau = 0$ and $\xi_\varphi = \mathcal{O}_S$ and another in which $\xi_\varphi = \mathcal{O}_S$ and ξ_τ is a sky-scraper sheaf with support $\{P_1\}$ and $H^i(S; \xi_\tau) \cong k^2$ for all sufficiently large i. In the first case, $[\xi]$ is represented by the characteristic sheaf of $\psi : G \times Y \to Y$, where Y is obtained as the mapping cone of $f : G_+ \wedge (S^d)^r \to G_+ \wedge (S^d)^r$. Such f_* in homology realizes the homomorphism ∂ in the sequence

$$0 \to \omega^2(M_1) \to (\mathbf{Z}G)^r \xrightarrow{\partial} (\mathbf{Z}G)^s \to M_1 \to 0.$$

Thus, this ω-composite extension corresponds to a semifree G-action on a finite dimensional space with a contractible fixed point set. In the second case, there is no topological action whose characteristic sheaf be equivalent to ξ. Hence this ξ does not correspond to a topological action.

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