Noetherian Symbolic blow up and examples in any dimension

by

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Introduction. - The following question was raised by Cowsik [Cw]: Consider P a prime ideal in a regular local ring (R,m), is the symbolic Rees's Ring

$$R^{(P)} := \bigoplus_{n\geq 0} P^{(n)}$$

a noetherian ring?
Where $P^{(n)}$ is the n-symbolic power of P (i.e. $P^{(n)} = P^n R_P \cap R$). This question appears in works of Rees [Re] and Nagata [Na].

P. Roberts gave a counterexample to the Cowsik's question, but very few examples were known of symbolic Rees's rings noetherian. Recently many works study the question of Cowsik, especially Huneke [Hu] and Schenzel [Sc].

The main interest of this paper is to give a practical criterium to decide when the ring $R^{(P)}$ is noetherian, also I use this criterium to show a reduction lemma who permits to find infinitely many examples in any embedding dimension from one example.

Also by using syzygies of monomial curves and one explicit description of the syzygy of the curve $k[t^a,t^b,t^c]$ in function of a,b,c . I can find a large class of monomial curves with ideal P in $k[X,Y,Z]$ s.t. $R^{(P)}$ is noetherian.

In a next paper [Mo] I develop the connection between the syzygies of a monomial curve and the "change money problem" of Frobenius. I give some combinatorial properties of the syzygies by using local cohomological methods.

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§ 1. Preliminaries

(R,m) will always denote a local commutative ring (noetherian) with residue field (R/m) infinite.

1.0 Definition - For an ideal I in R , $\bar{I}$ is the integral closure i.e. $\bar{I} = \{a \in R \,/\, a^n + b_1 a^{n-1} + \ldots + b_n = 0$ with $b_i \in I^i\}$.

1.1 The symbolic power of a prime ideal is noted by $P^{(n)}$ and we have two equivalent definitions

$$P^{(n)} = P^n R_P \cap R$$

$$P^{(n)} = \{x \in R \,/\, \exists y \notin P \text{ and } xy \in P^n\} .$$

1.2 Let (R,m) be a local noetherian ring, I a m-primary ideal, then for n large enough the Hilbert function $H(n) = \lg(R/I^n)$ coincides with a polynomial called the Hilbert-Samuel polynomial

$$H(n) = e(I)\,\frac{n^d}{d!} + \ldots\ldots$$

for $n \gg 0$.
Where d is the Krull dimension of R and $e(I)$ the multiplicity of the ideal I .

Let $x_1,\ldots,x_d$ be a sequence of parameters for E an R-module (i.e. $E/(x_1,\ldots,x_d)E$ is an Artinian module). We have the inequality

$$e(x_1,\dots,x_d) \le lg\left(E/_{(x_1,\dots,x_d)}\right)$$

and a ring is Cohen-Macaulay if and only if this is one equality for any system of parameters.

1.2.1 Localisation of the multiplicity

Let E be a noetherian R-module and q a system of parameters $q = (x_1,\dots,x_s)$ for E, then

$$e(qE) = \sum_P l_{R_P}(E_P)\, e_{R/P}\,(q\,R/P)$$

where the summation is over all prime ideals P minimal containing $ann(E)$, s.t. $\dim E = \dim(R/P)$.

1.2.2 Associative law for multiplicity

With the notations as above, let $1 \le i \le s$ then

$$e((x_1,\dots,x_s)E) = \sum_p e_{R_p}((x_1,\dots,x_i)E_p)e_{R/p}((x_{i+1},\dots,x_s)R/p) .$$

Here p ranges over all minimal prime ideals of $(x_1,\dots,x_i)$.

1.2.3 The relative multiplicity

Let I be an ideal in R, $x_1,\dots,x_r \in R$, we note by $\underline{x}$ the ideal generated by $x_1,\dots,x_r$, suppose that $\underline{x} + I$ is an m-primary ideal, then the function $e(\underline{x}\; R/_{I}n)$ takes the values of one polynomial of degree $\dim(A/I)$ for n large enough and the leading coefficient will be noted by $e(\underline{x},I)$. Also using the localisation of the multiplicity we have:

$$e(\underline{x},I) = \sum_{\substack{P\supset I \\ \dim(A/P)=\dim(A/I)}} e(\underline{x}\; R/P)e(I\; R_P) .$$

1.3 Let P be a prime ideal in R and $n \not\in P$ the multiplication by x

$$R/_P(n) \xrightarrow{x} R/_P(n)$$

is injective for all n .

In particular if $\dim(R/P) = 1$, $R/_P(n)$ is a Cohen-Macaulay ring for any n .

1.3.1 Suppose $\dim(R/P) = 1$. We can compute $e(x,P^{(k)})$ for any $x \in m-P$.

$$\begin{aligned} e(x,P^{(k)}) &= e(x\ R/P)\ e\ (P^{(k)}R_P) \\ &= lg(R/\underline{x}+P)\ e\ (P^{(k)}R_P) \\ &= lg(R/\underline{x}+P)\ e\ (PR_P)k^{d-1} \ . \end{aligned}$$

1.3.2 Suppose $\dim(R/P) = 1$, $x \not\in P$, fix a $k \in \mathbb{N}$ and put $S = R/x$, $J_n = P^{(nk)}S$, J_n is a $m/(x)$-primary ideal. Then

$$lg(S/J_n) := lg(R/\underline{x}+P^{(nk)}) = e(xR/P^{(nk)})$$

(because $R/P^{(nk)}$ is Cohen-Macaulay)

$$= e(x\ R/P)\ lg\ (R_P/P^{(nk)}R_P)$$

(by localisation of multiplicities)

$$= lg(R/_{\underline{x}+P})\ lg\ (R_P/P^{nk}R_P)$$

in particular $lg(S/J_n)$ coincides for $n >> 0$ with a polynomial having the leading term

$$e(x,P^{(k)})\ \frac{n^{d-1}}{(d-1)!}$$

(see 1.3.1).

1.4 Definition - Let $I \subset J$ two ideals, I is said a reduction of J if and only if $I\ J^n = J^{n+1}$ for large n, this is the same to say $\bar{I} = \bar{J}$, in particular $\sqrt{I} = \sqrt{J}$, I and J have the same minimal associated primes and so $h(I) = h(J)$ (h means heigth).

Suppose that R/m is infinite then the analytic spread $l(I)$ is the size of a minimal basis of a reduction of I.

1.4.1 A formally equidimensional ring (or quasi unmixed) R is a ring s.t. for any P minimal in the completion $\hat{R}$ of R we have $\dim(\hat{R}/P) = \dim \hat{R} = \dim R$.

In a formally equidimensional ring the altitude formula is true, i.e.

$$h(I) + \dim(R/I) = \dim(R)\ .$$

1.5 The following theorem is central in this paper:

Equimultiplicity criteria for $l(I) = h(I)$ (See [Li] pp. 118).

<u>Theorem</u> - (Dade)

Let (R,m) be a formally equidimensional local ring, with (R/m) infinite. For an ideal I the following conditions are equivalent:

(i) $l(I) = h(I)$

(ii) there exists a sequence $\underline{y} = (y_1,\dots,y_i)$ whose image in R/I is a system of parameters and such that

$$e(\underline{y},I) = e(\underline{y} + I) .$$

(iii) There exists a sequence $\underline{x} = (x_1,\dots,x_n)$ satisfying

(a) $\underline{x} + I$ is an m-primary ideal

(b) $\dim(R/\underline{x}R) = \dim R - r = \dim R - \dim(R/I)$

(in other words: x is part of a system of parameters in R and the image of $\underline{x}$ in R/I is a system of parameters)

and such that

- if $r < \dim R$ then $e(\underline{x},I) = e(I(R/\underline{x}))$
- if $r = \dim R$ then $e(\underline{x},I) = e(\underline{x})$.

§ 2. Theorem (The main theorem)

Let (R,m) be a d-dimensional formally equidimensional local ring, P a height $d-1$ prime ideal s.t. R_P is regular then the followings are equivalent

1) $R^{(P)}$ is a noetherian ring

2) $\exists k / P^{(kn)} = P^{(k)n} \ \exists n \geq 1$

3) $\exists k / l(P^{(k)}) = d-1 = h(P^{(k)})$

4) $\exists k, \exists f_1,\ldots,f_{d-1} \in P^{(k)}$ and for any $x \in m-P$

$$e(x,f_1,\ldots,f_{d-1}) = e(\underline{x},P^{(k)})$$

(see 1.3.1)

5) $\exists x \notin P, \exists k$ and $f_1,\ldots,f_{d-1} \in P^{(k)}$ such that

$$e(x,f_1,\ldots,f_{d-1}) = e(\underline{x},P^{(k)})$$

6) $\exists x \notin P, \exists k_1,\ldots,k_{d-1} \in \mathbb{N}$ and $f_i \in P^{(k_i)}$ such that

$$e(x,f_1,\ldots,f_{d-1}) = lg(R/(\underline{x}+P))k_1\ldots k_{d-1}$$

7) $\exists k/gr_{P^{(k)}}R$ is a Cohen-Macaulay ring.

<u>Remarks</u> 1) This theorem was proved by Huneke in the case $d = 3$.

2) The equivalence 1) $\Leftrightarrow$ 2) is due to Co sik [C] and Schenzel [Sc].

3) 2) $\Leftrightarrow$ 3) is due to Schenzel [Sc].

4) 5) $\Leftrightarrow$ 6) is purely formal and use the formula

$$e(\gamma_1^{n_1},\dots,\gamma_d^{n_d}) = n_1\dots n_d\, e(\gamma_1,\dots,\gamma_d)$$

(cf. [No] pp. 311).

5) The number k in the statements 2) to 7) is always the same.

Now we will prove 5) $\Rightarrow$ 3) $\Rightarrow$ 4).

<u>Proof</u>. - 5) $\Rightarrow$ 3) Put $S := R/(x)$, $I := P^{(k)}S$ is an m/x-primary ideal and we consider his Hilbert-Samuel function

$$H_I(n) = \lg(S/_{I}n) = \lg(R/_{\underline{x}+P}(k)n)) \ .$$

Now by hypothesis $f_1,\dots f_{d-1} \in P^{(k)}$ and $(x,f_1,\dots,f_{d-1})$ is a parameter system in R then $(f_1,\dots,f_{d-1})S$ is a parameter ideal included in I and

$$e(I) \leq e((f_1,\dots,f_{d-1})S) \leq e(x,f_1,\dots,f_{d-1}) = e(x,P^{(k)}) \ .$$

The second inequality is just the definition [No]

$$(e(x,f_1,\dots,f_{d-1}) = e(f_1,\dots,f_{d-1}(R/x)) - e((f_1,\dots,f_{d-1})(0_R : x))$$

and the third equality is by hypothesis.

Also $I^n = P^{(k)n} S \subset P^{(kn)}S = J_n$ (see 1.3.2) and $lg(S/_I n) \geq lg(S/J_n)$, and for large n this becomes an inequality between polynomials of degree $d-1$, in particular by taking leading terms (see 1.3.2) we obtain

$$e(I) \geq e(\underline{x},P^{(k)})$$

then $$e(I) = e(\underline{x},P^{(k)})$$

and by 1.5 $l(P^{(k)}) = d-1$.

Now we prove 3) → 4).

Let $x \in m-P$ and take $(f_1,\dots,f_{d-1})$ a reduction of $P^{(k)}$, $\underline{x} + P^{(k)}$ is contained in the integral closure of $(x,f_1,\dots,f_{d-1})$ and then $(x,f_1,\dots,f_{d-1})$ is a reduction of $\underline{x} + P^{(k)}$. Call $\underline{f} := f_1R +\dots+ f_{d-1} R$. Then we have

$$e(\underline{x},P^{(k)}) = e(x,\underline{f}) = e(x,f_1,\dots,f_{d-1}) .$$

Attention, here $e(x,\underline{f})$ is the relative multiplicity in the sense of 1.2.3.

The first equality follows from the general fact: If I is a reduction of J then $e(x,I) = e(x,J)$. The second equality is the associative law for multiplicity.

§ 3. Reduction theorem

3.1 Definition - Let C be a curve in k^d given parametrically by $X_1 = t^m$, $X_2 = \varphi_2(t),\ldots,X_d = \varphi_d(t)$.
Let $\alpha \in \mathbb{N}$, s.t. $(m,\alpha) = 1$.

Now let $\tilde{C}$ the curve in k^d given parametrically by

$$X_1 = t^m \ , \ X_2 = \varphi_2(t^\alpha),\ldots,X_d = \varphi_d(t^\alpha) \ .$$

Also for any $f(X_1,\ldots,X_d)$ in R ($R = k[X_1,\ldots,X_d]$ a, $k\{X_1,\ldots,X_d\}$, or $k\{[[X_1,\ldots,X_d]]\}$ we note by $\tilde{f}$ the element in R $\tilde{f}(X_1,\ldots,X_d) = f(X_1^\alpha,X_2,\ldots,X_d)$.

For the relations between C and $\tilde{C}$ see the appendix. Nevertheless we need the following

3.2 Lemma (See [Mo]. Let P (resp. $\tilde{P}$) the prime ideal of C (resp. of $\tilde{C}$) then for any f in R we have $f \in P \Longleftrightarrow \tilde{f} \in \tilde{P}$.
Also if $f_1,\ldots,f_s$ is a minimal set of generators of P then $\tilde{f}_1,\ldots,\tilde{f}_s$ is a minimal set of generators of $\tilde{P}$.

Remark - This is true even if d is not the embedding dimension of C.

3.3 Proposition (Reduction Lemma) - i) If $P^{(k)n} = P^{(kn)}$ for all $n \geq 1$ then $\tilde{P}^{(k)n} = \tilde{P}^{(kn)} \forall n \geq 1$.

ii) If C is a monomial curve of embedding dimension d then the converse is true.

<u>Proof</u> - R is as before the ring of polynomials, or $k\{X_1,\ldots,X_d\}$ or $k[[X_1,\ldots,X_d]]$.
Now applying the theorem 2.1 take $x = X_1$ and $f_1,\ldots,f_{d-1} \in P^{(k)}$ s.t.

$$\lg(R/_{(x,f_1,\ldots,f_{d-1})}) = \lg(R/_{\underline{x}+P})k^{d-1}$$

now by construction and lemma 3.2 $X_1R + P = X_1R + \tilde{P}$, on the other hand if $f \in P^{(k)}$ it is easy to see that $\tilde{f} \in \tilde{P}^{(k)}$, but we have also

$$(X_1,f_1,\ldots,f_{d-1}) = (X_1,\tilde{f}_1,\ldots,\tilde{f}_{d-1})$$

so we obtain

$$\lg(R/_{(x,\tilde{f}_1,\ldots,\tilde{f}_{d-1})}) = \lg(R/_{(\underline{x}+\tilde{P})})k^{d-1}$$

and the theorem 2.1 implies the Proposition i). In fact we have just that $P_m^{(kn)} = P_m^{(k)n}$ where m is the maximal ideal $(X_1,\ldots,X_d)$ in $k[X_1,\ldots,X_d]$ but the curves C and $\tilde{C}$ are smooth outside the origine, this implies that the sheafs $P^{(kn)}$ and $P^{(k)n}$ are equals outside the origin and this implies

$$P^{(kn)} = P^{(k)n} \text{ in } k[X_1,\ldots,X_d] .$$

Now we prove ii). First of all remark that $P^{(k)}$ is a graded ideal for all k , because P^k is a graded ideal and $P^{(k)}$ is the P-primary component of P^k . Now suppose that we have $x = X_1$ and $f_1,\dots,f_d \in \tilde{P}^{(k)}$ such that

$$lg(R/_{(x,f_1,\dots,f_{d-1})}) = lg(R/_{\underline{x}+\tilde{P}})k^{d-1}$$

we claim that any $f \in \tilde{P}^{(k)}$ can be written

$$(*) \qquad f = \tilde{f}^{(0)} + X_1\tilde{f}^{(1)} + \dots + X_1^{\alpha-1}\,\tilde{f}^{(\alpha-1)}$$

where $f^{(i)} \in P^{(k)}\ \forall\, i$. Using this and the fact that $(X_1,f_1,\dots,f_{d-1}) = (X_1,\tilde{f}_1^0,\dots,\tilde{f}_{d-1}^0)$ using 2.1 the part ii) of the proposition will follow. We can always write f as (*) we must prove that

$$f^{(i)} \in P^{(k)}\ \forall\, i$$

also $\deg X_i = \alpha\, e_i$ for $i \geq 2$

$\deg X_1 = e_1$ and $(\alpha,e_1) = 1$,

then $\deg \tilde{f}^{(0)} \in \alpha\,\mathbf{Z}$, $\deg \tilde{f}^{(i)} \in i\,e_1 + \alpha\,\mathbf{Z}$, in particular $X^i\tilde{f}^{(i)}$ all have different degrees and using that $\tilde{P}^{(k)}$ is graded we obtain $\tilde{f}^{(i)} \in P^{(k)}$ for any $i \in [0,\alpha-1]$. Now in order to finish we must prove that if $\tilde{f} \in \tilde{P}^{(k)}$ then $f \in P^{(k)}$. By definition there exists some $g \in \tilde{P}$, s.t.

$$g\,\tilde{f} \in \tilde{P}^k$$

write $g = \tilde{g}_0 + X_1\tilde{g}_1 + \ldots + X_1^{\alpha-1}g_{\alpha-1}$ with one of the g_i not $\tilde{P}$. Consider the product

$$g\tilde{f} = \tilde{f}(g_0 + X_1\tilde{g}_1 + \ldots + X_1^{\alpha-1}\tilde{g}_{\alpha-1}) \in \tilde{P}^k$$

and the degrees of $X^i\tilde{f}\tilde{g}_i$ are all different, in particular because $\tilde{P}^k$ is a graded ideal

$$X_1^{\overset{\circ}{i}}\tilde{f}\tilde{g}_i \in \tilde{P}^k$$

but there are some $\tilde{g}_{i0} \notin \tilde{P}^k$ and $X_1^{i_0}\tilde{g}_{i_0} \notin \tilde{P}^k$ and by multiplying $X_1^{\alpha-i_0}$, we see that there are some $\tilde{G} = X_1^{\alpha}\tilde{g}_{i_0}$ not in $\tilde{P}$ but $\tilde{G}\tilde{f} \in \tilde{P}^k$. The proof will be a consequence of the following lemma

<u>Lemma</u>. - $\tilde{G} \in \tilde{P}^k \Longleftrightarrow G \in P^k$.

<u>Proof</u>. - Let $\tilde{G} = p(\tilde{g}_1,\ldots,\tilde{g}_s)$ where $p \in R[Y_1,\ldots,Y_s]$ is a polynomial of order bigger than k . We write the coefficient h of p like

$$h = \tilde{h}_0 + X_1\tilde{h}_1 + \ldots + X_1^{\alpha-1}\tilde{h}_{\alpha-1}$$

this implies that

$$\tilde{G} = \tilde{p}_0(\tilde{g}_1,\ldots,\tilde{g}_s) + X_1\tilde{p}_1(\tilde{g}_1,\ldots,\tilde{g}_s) + \ldots + X_1^{\alpha-1}\tilde{p}_{\alpha-1}(\tilde{g}_1,\ldots,\tilde{g}_s)$$

with $\tilde{p}_i \in k[X_1^{\alpha}, X_2,\ldots,X_d][Y_1,\ldots,Y_s]$ and using the fact that the degree term are not in the same set we prove that

$$\tilde{G} = \tilde{p}_0(\tilde{g}_1,\ldots,\tilde{g}_s)$$

this implies that $G = p_0(g_1,\ldots,g_s) \in P^k$. The converse is trivial. We don't know if the property $R^{(p)}$ being noetherian depends only from R/P , in particular we don't know what happens if the embedding dimension changes. Nevertheless we have the following lemma.

Lemma. - Consider a curve C in k^d given parametrically by

$$X_1 = t^a$$
$$X_2 = \varphi_2(t)$$
$$\ldots\ldots\ldots$$
$$X_d = \varphi_d(t)$$

and add one new variable W and a new equation involving X_1 , (for example $W = X_1X_2$) then if we call P the ideal of C in $k[X_1,\ldots,X_d]$ and P_1 the corresponding ideal of $\tilde{C}$ in $k[X_1,\ldots,X_d, W]$ then if $P^{(k)n} = P^{(kn)}$ for $n \geq 1$ also $P_1^{(k)n} = P_1^{(kn)}$ for $n \geq 1$.

Proof. - We use the theorem 2.1. Take $x = X_1$, and $f_1,\ldots,f_{d-1} \in P^{(k)}$ s.t.

$$\lg\left(k[X_1,\ldots,X_d]/_{(X_1,f_1,\ldots,f_{d-1})}\right)_m = \lg\left(k[X_1,\ldots,X_d]/_{(X_1R+P)}\right)_m k^{d-1}$$

where $(\ )_m$ means localisation in m . Also $f \in P^{(k)}$ implies $f \in P_1^{(k)}$ and

$$(X_1,\ W-X_1X_2) = (X_1,\ W) \ .$$

And the ideal generated by X_1 and P_1 is the same as the ideal generated by X_1 , W and P , so we obtain

$$\lg\left(k[X_1,\ldots,X_d,\ W]/_{(X_1,\ f_1,\ldots,f_{d-1},\ W-X_1X_2)}\right)_m =$$

$$= \lg\left(k[X_1,\ldots,X_d,\ W]/_{(X_1,\ P_1)}\right)_m k^{d-1}$$

and the lemma follows from 2.1.

§ 4. Monomial curves in k^3

Let $a,b,c \in \mathbb{N}$ such that $(a,b,c) = 1$ and that $k[t^a,t^b,t^c]$ be a curve C of embedding dimension 3. Let $R = k[X,Y,Z]$ with grading given by weight $(X) = a$, weight $(Y) = b$, weight $(Z) = c$. After J. Herzog [He] [K] we know that if $k[t^a,t^b,t^c]$ is not a complete intersection, his syzygies are like follows

$$0 \longrightarrow R^2 \xrightarrow{M} R^3 \longrightarrow R \longrightarrow k[t^a,t^b,t^c] \longrightarrow 0$$

$$X \longmapsto t^a$$

$$Y \longmapsto t^b$$

$$Z \longmapsto t^c \quad .$$

Now we improve this result giving explicitely the matrix M . The equations of the curve C in k^3 being the 2×2 minors of the matrix M . The problem to find M is in fact equivalent to the solution of the Frobenius's change money problem in dimension 3 (cf. [Rö] and [Mo]):

<u>Problem</u>. - Find the biggest $g \in \mathbb{N}$ who can't write

$$g = \alpha a + \beta b + \gamma c \text{ , with } \alpha,\beta,\gamma \in \mathbb{N} \ (0 \in \mathbb{N}) \ .$$

See also the appendix for many comments in this problem. Now let s_0 the unique natural number such that

$$s_0 b \equiv c \bmod a \quad \text{and} \quad 0 < s_0 < a$$

put $s_{-1} := a$ and consider the continuous fraction

$$s_{-1} := a = q_1 s_0 - s_1 \qquad\qquad q_i \geq 2,\ s_i \geq 0$$

$$s_0 = q_2 s_1 - s_2$$

$$s_{m-1} = q_{m+1}\, s_m$$

and the sequences P_i, R_i defined by

$$P_{-1} = 0,\ P_0 = 1,\ P_{i+1} = P_i Q_i - P_{i-1},\ P_{m+1} = a$$

$$R_i = \frac{s_i b - P_i c}{a},\ R_{-1} = b,\ R_{m+1} = -c\ .$$

Then $\{s_i\}$ and $\{R_i\}$ are strictly decreasing sequences, P_i is a strictly increasing sequence.

Definition. - Let ν the unique integer number s.t.

$$R_{\nu+1} \leq 0 < R_\nu\ .$$

4.1 Theorem [Rö]

$$g = \max(bs_\nu - a R_{\nu+1},\ aR_\nu + c P_{\nu+1}) - a - b - c\ .$$

4.2 Theorem. - C is a complete intersection $\Longleftrightarrow$ $s_{\nu+1} = 0$ and the matrix syzygies is

$$M = \begin{pmatrix} X^{R_\nu} & Y^{s_\nu - s_{\nu+1}} \\ Y^{s_{\nu+1}} & Z^{P_\nu} \\ Z^{P_{\nu+1}-P_\nu} & X^{-R_{\nu+1}} \end{pmatrix} .$$

For the proof of this theorem see [Mo]. Now we apply this theorem to find a very large class of examples of monomial curves.

4.3 Theorem. - Let a,b,c,s_0 natural numbers such that

1) $s_0 c \equiv b \bmod a$, $0 < s_0 < a < b < c$

2) $a + 1 \in s_0 \mathbb{Z}$

3) $c\left(\frac{a+1}{s_0} + s_0 - 1\right) \geq b(s_0^2 - s_0 + 1)$.

Let P be the ideal in $k[X,Y,Z]$ of the curve $k[t^a,t^b,t^c]$ then

$$P^{(s_0)n} = P^{(s_0 n)} \quad \forall\, n \geq 1 .$$

Proof. - Using the notations before 4.3, the first two conditions implies that

$$s_{-1} = a,\ s_1 = 1,\ s_2 = 0,\ q := q_1 = \frac{a+1}{s_0},\ q_2 = s_0$$

$$p_{-1} = 0,\ p_0 = 1,\ p_1 = \frac{a+1}{s_0},\ p_2 = a$$

$$R_{-1} = b,\ R_0 = \frac{s_0 b - c}{a},\ R_1 = \frac{b - (\frac{a+1}{s_0}) c}{a},\ R_2 = -c$$

this implies that $\nu = 0$ or 1, but $\nu = 1$ is the complete intersection case, in this case $P^{(n)} = P^n$ for all n. We must study the case $\nu = 0$. The syzygy's matrix is

$$M^t = \begin{pmatrix} X^{\alpha} & Y & Z^{\frac{a+1}{s_0} - 1} \\ Y^{s_0 - 1} & Z & X^{\beta} \end{pmatrix}$$

where $\alpha = \frac{s_0 b - c}{a}$, $\beta = \frac{qc - b}{a}$.

In particular the generators of P are

$$F_1 = X^{\alpha+\beta} - Y^{s_0 - 1} Z^{q-1}$$

$$F_2 = Y^{s_0} - X^{\alpha} Z$$

$$F_3 = Z^q - YX^{\beta}$$

and the relations are

$$X^{\alpha} F_3 + YF_1 + Z^{q-1} F_2 = 0$$

$$X^{\beta} F_2 + Y^{s_0 - 1} F_3 + Z F_1 = 0$$

Now we will prove that there exist $d_{s_0} \in P^{(s_0)}$ s.t.

$$d_{s_0} = Z^a + XG_1 + Y^{s_0}G_2$$

and then we apply the theorem 2.1 with

$$x = X,\ f_1 = F_2,\ f_2 = d_{s_0}$$

and we compute

$$\lg\left(R/_{(X,F_2,d_{s_0})}\right) = \lg\left(R/_{(X,Y^{s_0},Z^a)}\right) = a\, s_0$$

$$= \lg\left(R/_{(XR+P)}\right) s_0$$

the theorem 2.1 then implies that $P^{(s_0)n} = P^{(s_0 n)}$ for all $n \geq 1$.
The element d_{s_0} will be find by an inductive process. Put $d_1 = F_1$ and using the relations between F_1, F_2, F_3 we get

$$\begin{aligned} Zd_1F_3 &= -F_3(X^\beta F_2 + Y^{s_0-1}F_3) \\ &= -X^\beta F_3F_2 - Y^{s_0-1}F_3^2 \\ &= X^{\beta-\alpha}F_2(YF_1 + Z^{q-1}F_2) - Y^{s_0-1}F_3^2 \end{aligned}$$

$$\Rightarrow Z(d_1F_3 - Z^{q-2}X^{\beta-\alpha}F_2^2) = Y(X^{\beta-\alpha}F_2F_1 - Y^{s_0-2}F_3^2)$$

this implies that there exists $d_2 \in P^{(2)}$ (by definition).

such that

$$Zd_2 = Y^{s_0-2}\ F_3^2 - X^{\beta-\alpha}F_2F_1 \ .$$

Now we compute $Z\ d_2\ F_3$ using the same method and we find that there exists some $d_3 \in P^{(3)}$ such that

$$Zd_3 = Y^{s_0-3}F_3^3 + X^{\beta-2\alpha}\ F_1^2\ F_2 \ .$$

The third condition on the theorem is equivalent to say that $\beta \geq (s_0-1)\alpha$ and we can continue the computations and find $d_{s_0} \in P^{(s_0)}$ such that

$$Z\ d_{s_0} = F_3^{s_0} - X^{\beta-(s_0-1)\alpha}\, F_1^{s_0-1}\, F_2$$

but $F_3^{s_0} = (Z^q - YX^\beta)^{s_0} = Z^{a+1} - X^\beta G_1$ because $qs_0 = a + 1$, then $d_{s_0} = Z^a + XG_1 + Y^{s_0}G_2$ and the theorem is proved.

4.4 <u>Corollary</u>. - Let $s_0 = 2$ and a,b,c satisfying the first condition on the theorem, if $c > b$ then the third condition is satisfied (because $a \geq 3$) and we have $P^{(2n)} = P^{(2)n}$ $\forall\ n \geq 1$.

<u>Proof</u>. - If a is odd the three conditions in 4.3 are satisfied. If a is even theorem 4.2 implies that P is a complete intersection.

4.5 <u>Corollary</u>. - Let P the ideal of a monomial curve of multiplicity 3 then $P^{(2n)} = P^{(2)n}$ $\forall\ n \geq 1$. In this case we can see that if P is not a complete intersection then $s_0 = 2$, and a,b,c satisfy the conditions in the theorem 4.3.

4.6 Corollary. - Let a and α two coprime numbers then the prime ideal P of the curve $k[t^{a}, t^{a+\alpha}, t^{a+2\alpha}]$ satisfy $P^{(2n)} = P^{(2)n}$ $\forall\, n \geq 1$.

Also I mention here that Huneke [Hu] and Schenzel [Sc] have many other examples.

For the moment using the reduction Lemma and the following example of Huneke I find a large class of examples in any dimension.

4.7 Example [Hu]. - Let $R = \mathbb{C}[X,Y,Z]$ and P the Kernel of the homomorphism of R into $\mathbb{C}[[t^{6}, t^{7} + t^{10}, t^{8}]]$ by sending X to t^{6} , Y to $t^{7} + t^{10}$, Z to t^{8} . Then $P^{(10)n} = P^{(10n)}$ for all $n \geq 1$. In particular P is a set theoretic complete intersection.

4.8 - Take the example in 4.7 and add a new variable W and a new equation

$$W = XY = t^{14} \quad \text{(for example)}$$

now take any $\alpha \in \mathbb{N}$, coprime with 14 than the curve $\tilde{P}$ in $\mathbb{C}[X,Y,Z,W]$ given parametrically by

$$X = t^{6\alpha} \ , \ Y = t^{7\alpha} + t^{10\alpha} \ , \ Z = t^{8\alpha} \ , \ W = t^{14}$$

is a curve of embedding dimension 4 and such that

$$\tilde{P}^{(10)n} = \tilde{P}^{(10n)} \quad \forall\, n \geq 1 \ .$$

Now add a new variable $V = XW = t^{14+6\alpha}$ and take any $\beta \in N$ coprime with $14 + 6\alpha$, then the curve $\tilde{\tilde{P}}$ in $[[X,Y,Z,W,V]]$ given parametrically by

$$X = t^{6\alpha\beta},\ Y = t^{7\alpha\beta} + t^{10\alpha\beta},\ Z = t^{8\alpha}$$

$$W = t^{14\beta},\ V = t^{14+6\alpha}$$

is a curve of embedding dimension 5 such that

$$\tilde{\tilde{P}}^{(10n)} = \tilde{P}^{(10)n} \quad \forall\, n \geq 1 .$$

You can continue this process and find examples in any dimension. In particular all these curves are set theoretic complete intersection!

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