Operations in some graded Hopf categories

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1. Introduction

Crane and Frenkel proposed a notion of a Hopf category in [CF94]. It was motivated by Lusztig's approach to quantum groups – his theory of canonical bases. In particular, Lusztig obtains braided deformations $U_q\mathfrak{n}_+$ of universal enveloping algebras $U\mathfrak{n}_+$ for some nilpotent Lie algebras $\mathfrak{n}_+$ together with canonical bases of these braided Hopf algebras [Lus90, Lus91, Lus93]. The elements of the canonical basis are identified with isomorphism classes of simple perverse sheaves – certain objects of equivariant derived categories. They are contained in a semisimple abelian category of semisimple complexes. Conjectural properties of this category were collected into a system of axioms of a Hopf category, equipped with functors of multiplication and comultiplication, isomorphisms of associativity, coassociativity and coherence which satisfy four equations [CF94]. Crane and Frenkel gave an example of a Hopf category resembling the semisimple category encountered in Lusztig's theory corresponding to one-dimensional Lie algebra $\mathfrak{n}_+$ – nilpotent subalgebra of $\mathfrak{sl}(2)$. The mathematical framework and some further examples of Hopf categories were provided by Neuchl [Neu].

We discuss an example of a related notion – triangulated Hopf category – the whole equivariant derived category equipped with multiplication and comultiplication functors and structure isomorphisms. In particular, it is a monoidal category. The additive relations between operations proposed in [CF94] are replaced with distinguished triangles. The preliminary study of the subject can be found in [Lyu99a, Lyu99b].

1.1. Operations in a graded Hopf algebra

Let Q_+ be a commutative monoid additively generated by elements of a finite set I. Denote $R = \mathbb{Z}[q, q^{-1}]$. Let $\mathcal{H}$ be a Q_+-graded braided Hopf R-algebra, for instance, the algebra $U_q\mathfrak{n}_+$ of Lusztig [Lus93]. The comultiplication in $\mathcal{H} = \oplus_{v\in Q_+}\mathcal{H}_v$ can be written as

$$\Delta = \sum_{u,v\in Q_+} \Delta_{u,v}, \qquad \Delta_{u,v} : \mathcal{H}_{u+v} \to \mathcal{H}_u \otimes_R \mathcal{H}_v.$$

Similarly for iterated comultiplication $\Delta^{(l)} = (\Delta^{(l-1)} \otimes 1) \circ \Delta : \mathcal{H} \to \mathcal{H}^{\otimes l}$,

$$\Delta^{(l)} = \sum_{v_r\in Q_+} \Delta^{(l)}_{v_1,\dots,v_l}, \qquad \Delta^{(l)}_{v_1,\dots,v_l} : \mathcal{H}_{v_1+\dots+v_l} \to \mathcal{H}_{v_1} \otimes_R \cdots \otimes_R \mathcal{H}_{v_l}.$$

The associativity, the coassociativity and the bialgebra axiom imply the equation

$$\Delta^{(l)}(x^1)\cdot\ldots\cdot\Delta^{(l)}(x^k)=\Delta^{(l)}(x^1\cdot\ldots\cdot x^k) \tag{1.1.1}$$

for arbitrary elements $x^m\in\mathcal{H}$. Note that multiplication in the left hand side uses the braiding. Apply equation (1.1.1) to homogeneous elements x^m of degree v^m and write down its homogeneous component of multidegree $(v_1,\ldots,v_l)\in Q_+^l$:

$$\sum_{\substack{\sum_r v_r^m=v^m\\ \sum_m v_r^m=v_r}}\Delta^{(l)}_{v_1^1,\ldots,v_l^1}(x^1)\cdot\cdots\cdot\Delta^{(l)}_{v_1^k,\ldots,v_l^k}(x^k)=\Delta^{(l)}_{v_1,\ldots,v_l}(x^1\cdot\ldots\cdot x^k). \tag{1.1.2}$$

Each summand in the left hand side can be viewed as an operation with k inputs and l outputs. These operations are not distinguished in algebra setup. However, in graded Hopf categories their explicit use seems advantageous.

2. The main ingredients

2.1. The basic data

Categories will be equivariant derived categories $\frac{X}{G}:=D_G^{b,c}(X)$, where X is a complex algebraic variety, equipped with the action of a complex algebraic group G, as defined by Bernstein and Lunts [BL94].

The functors will be compositions of functors of the three types (see [BL94]). Let $\phi:G\to H$ be a group homomorphism, let X be a G-space, let Y be an H-space, and let $f:X\to Y$ be a ϕ-equivariant map. Then there are
— the inverse image functor $\frac{f}{\phi}^*:\frac{Y}{H}\to\frac{X}{G}$,
— if $\phi:G\to H$ is surjective, $K=\mathrm{Ker}(\phi)$, X is K-free, and $Y=K\backslash X$, the direct image functor (in this case it is an equivalence) $\frac{f}{\phi}_*:\frac{X}{G}\to\frac{Y}{H}$.
— if $\phi=1:G=H$ is the identity, the direct image functor with proper supports $\frac{f}{1}_!:\frac{X}{G}\to\frac{Y}{G}$.

Quiver. Let (H,I) be a finite oriented graph with the set of vertices I, the set of edges H, the structure map $H\to I\times I$, $h\mapsto(h',h'')$, where $h'\in I$ is the source of $h\in H$, and $h''\in I$ is the target of $h\in H$, such that $h'\neq h''$.

Let V be a finite dimensional I-graded $\mathbb{C}$-vector space, (a function $V:I\to\mathrm{Ob}\,\mathbb{C}\text{-vect}$, $i\mapsto V(i)$). Its automorphism group is

$$G_V=\mathrm{Aut}_{I\text{-grad-vect}}V=\prod_{i\in I}GL(V(i)).$$

Define a linear space

$$E_V=\oplus_{h\in H}\mathrm{Hom}_{\mathbb{C}}(V(h'),V(h'')).$$

The union of all E_V is the class of representations of the quiver. The group G_V acts on E_V by $(g.x)_h=g_{h''}x_hg_{h'}^{-1}$. The union of all $G_V\backslash E_V$ is the set of isomorphism classes of representations of the quiver. We consider the collection of equivariant derived categories $\frac{E_V}{G_V}$ as our Hopf category.

Filtrations. To introduce operations we need to consider decompositions of V

$$\mathcal{V}: \qquad V^1 \oplus V^2 \oplus \cdots \oplus V^k = V$$

into I-graded subspaces. Associate with it a filtration of V

$$0 = V^{(0)} \subset V^{(1)} \subset \cdots \subset V^{(k)} = V, \qquad V^{(m)} = V^1 \oplus \cdots \oplus V^m. \tag{2.1.1}$$

Associate with it the parabolic group $P_{\mathcal{V}}$

$$P_{\mathcal{V}} = \{g \in G_V \mid \forall m \;\; g(V^{(m)}) \subset V^{(m)}\}.$$

The unipotent radical of $P_{\mathcal{V}}$ is denoted $U_{\mathcal{V}}$. The group

$$L_{\mathcal{V}} = \{g \in G_V \mid \forall m \;\; g(V^m) \subset V^m\} = \prod_{m=1}^{k} G_{V^m} \simeq P_{\mathcal{V}}/U_{\mathcal{V}}$$

is a Levi subgroup of $P_{\mathcal{V}}$.

Let $F_{\mathcal{V}}$ be the linear subspace of E_V respecting the filtration:

$$F_{\mathcal{V}} = \{x \in E_V \mid \forall m, h \;\; x_h(V^{(m)}(h')) \subset V^{(m)}(h'')\}.$$

The group $P_{\mathcal{V}}$ acts in $F_{\mathcal{V}}$ as well as in its subspace $f_{\mathcal{V}}$, consisting of maps lowering the filtration:

$$f_{\mathcal{V}} = \{x \in E_V \mid \forall m, h \;\; x_h(V^{(m)}(h')) \subset V^{(m-1)}(h'')\}.$$

Operations. Let two decompositions of V into a direct sum be given:

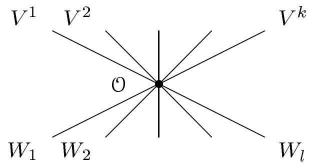
$$\mathcal{V}: \qquad V^1 \oplus V^2 \oplus \cdots \oplus V^k \xrightarrow{\sim} V, \tag{2.1.2}$$

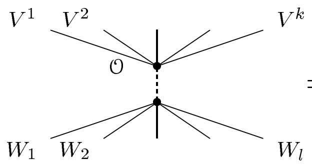
$$\mathcal{W}: \qquad W_1 \oplus W_2 \oplus \cdots \oplus W_l \xrightarrow{\sim} V. \tag{2.1.3}$$

We denote by $\mathcal{V}$ and $\mathcal{W}$ also the associated filtrations (2.1.1) and

$$0 = W_{(0)} \subset W_{(1)} \subset \cdots \subset W_{(l)} = V, \qquad W_{(r)} = W_1 \oplus \cdots \oplus W_r.$$

Let $\mathcal{O} \subset G_V$ be a left $P_{\mathcal{W}}$-invariant and right $P_{\mathcal{V}}$-invariant locally closed subset. We associate with it an operation

$\mathcal{K}^{\mathcal{O};\mathcal{V}}_{\mathcal{W}} =$ 

$=$ 

$= \pitchfork^{\mathcal{V}}_{\mathcal{W}} \circ \Psi^{\mathcal{O};\mathcal{V}}_{\mathcal{W}}$.

The components of it are generalized multiplication and comultiplication functors defined below.

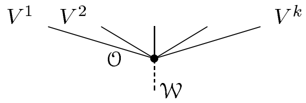
Graphical conventions. In graphical notations a straight line $|X$ labeled by an I-graded space X denotes category $\frac{E_X}{G_X}$. A dashed line $\mathcal{X}$ labeled by a filtration $\mathcal{X}$ of an I-graded space X denotes category $\frac{E_X}{P_{\mathcal{X}}}$. If a horizontal line crosses a diagram intersecting transversally with several (solid or dashed) open intervals labeled by spaces Y_1, Y_2, … and filtrations $\mathcal{X}_1$, $\mathcal{X}_2$ of spaces X_1, X_2, … , this line represents the category

$$\frac{E_{Y_1} \times E_{Y_2} \times \cdots \times E_{X_1} \times E_{X_2} \times \cdots}{G_{Y_1} \times G_{Y_2} \times \cdots \times P_{\mathcal{X}_1} \times P_{\mathcal{X}_2} \times \cdots}.$$

The order of factors repeats the order of intersection points on the horizontal line.

A vertex (or a horizontal row of vertices) represents a functor from the category, corresponding to horizontal line just above the vertex, to the category represented by a line just below the vertex.

Multiplication half. The multiplication half operation is



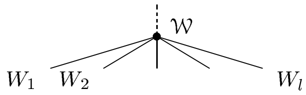
$$= \Psi^{\mathcal{O};\mathcal{V}}_{\mathcal{W}} = \left(\frac{\prod_{i=1}^k E_{V^i}}{\prod_{i=1}^k G_{V^i}} \xrightarrow{\phi^*} \frac{\mathcal{O} \times F_{\mathcal{V}}}{P_{\mathcal{W}} \times P_{\mathcal{V}}} \xrightarrow{\pi_*} \frac{\mathcal{O} \times_{P_{\mathcal{V}}} F_{\mathcal{V}}}{P_{\mathcal{W}}} \xrightarrow{\alpha_!} \frac{E_V}{P_{\mathcal{W}}} \right).$$

The scheme of multiplication is similar to that of Lusztig [Lus90, Lus91, Lus93]:

$$\prod_{i=1}^k E_{V^i} \xleftarrow{\phi} \mathcal{O} \times F_{\mathcal{V}} \xrightarrow{\pi} \mathcal{O} \times_{P_{\mathcal{V}}} F_{\mathcal{V}} \xrightarrow{\alpha} E_V,$$

where $\phi(o,f) = \kappa(f)$ is the forgetful map, $\kappa : F_{\mathcal{V}} \to \prod_{i=1}^k E_{V^i}$ is the natural projection, π is the canonical projection, $\alpha(o,f) = o.\iota(f)$ is induced from the action map, and $\iota : F_{\mathcal{V}} \to E_V$ is the natural embedding.

Comultiplication half. The comultiplication half operation functor is



$$= \pitchfork^{\mathcal{V}}_{\mathcal{W}} = \left(\frac{E_V}{P_{\mathcal{W}}} \xrightarrow{\iota^*} \frac{F_{\mathcal{W}}}{\prod_{j=1}^l G_{W_j}} \xrightarrow{\kappa_!} \frac{\prod_{j=1}^l E_{W_j}}{\prod_{j=1}^l G_{W_j}} \xrightarrow{\tau} \frac{\prod_{j=1}^l E_{W_j}}{\prod_{j=1}^l G_{W_j}} \right),$$

where τ is the shift

$$\tau L = L[2 \sum_{r>s;h} \dim W_r(h') \cdot \dim W_s(h'')].$$

The scheme of comultiplication is made of the natural embedding ι and the natural projection κ (as in Lusztig [Lus90, Lus91, Lus93]):

$$E_V \xleftarrow{\iota} F_{\mathcal{W}} \xrightarrow{\kappa} \prod_{j=1}^l E_{W_j}.$$

Braiding. Let $(A_m)_{m=1}^k$, $(B_n)_{n=1}^l$ be sequences of I-graded vector spaces. We define the braiding functor as

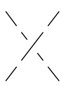
$$c = $$


$$= \Big(\frac{\prod E_{A_m} \times \prod E_{B_n}}{\prod G_{A_m} \times \prod G_{B_n}} \xrightarrow{\tau} \frac{\prod E_{A_m} \times \prod E_{B_n}}{\prod G_{A_m} \times \prod G_{B_n}} \xrightarrow{\sigma^*} \frac{\prod E_{B_n} \times \prod E_{A_m}}{\prod G_{B_n} \times \prod G_{A_m}} \Big),$$

where σ is the permutation isomorphism of groups and modules and the functor τ is the shift

$$\tau(L) = L\Big[-2 \sum_{\substack{m,n \\ i\in I}} \dim A_m(i) \dim B_n(i) + 2 \sum_{\substack{m,n \\ h \in H}} \dim A_m(h') \dim B_n(h'') \Big].$$

When $\mathcal{A}_m$, $\mathcal{B}_n$ are filtrations of I-graded vector spaces A_m, B_n we use the same τ and similar σ to define the braiding functor

$$c = $$


$$= \Big(\frac{\prod E_{A_m} \times \prod E_{B_n}}{\prod P_{\mathcal{A}_m} \times \prod P_{\mathcal{B}_n}} \xrightarrow{\tau} \frac{\prod E_{A_m} \times \prod E_{B_n}}{\prod P_{\mathcal{A}_m} \times \prod P_{\mathcal{B}_n}} \xrightarrow{\sigma^*} \frac{\prod E_{B_n} \times \prod E_{A_m}}{\prod P_{\mathcal{B}_n} \times \prod P_{\mathcal{A}_m}} \Big).$$

2.1.1. Minimal parametrizing subsets. Minimal bi-invariant subsets $\mathcal{O} \subset G_V$ are orbits of the action of $P_{\mathcal{W}} \times P_{\mathcal{V}}$ in G_V, $(p, r).g = pgr^{-1}$. They are in natural bijection with orbits of left action of G_V in the space of pairs of filtrations $P_{\mathcal{W}} \backslash G_V \times G_V / P_{\mathcal{V}} \simeq G_V / P_{\mathcal{W}} \times G_V / P_{\mathcal{V}}$, $(P_{\mathcal{W}} g, hP_{\mathcal{V}}) \leftrightarrow (g^{-1} P_{\mathcal{W}}, hP_{\mathcal{V}})$. Indeed, multiplication

$$P_{\mathcal{W}} \backslash G_V \times_{G_V} G_V / P_{\mathcal{V}} \to P_{\mathcal{W}} \backslash G_V / P_{\mathcal{V}}, \qquad (P_{\mathcal{W}} g, hP_{\mathcal{V}}) \mapsto P_{\mathcal{W}} ghP_{\mathcal{V}}$$

is a bijection. By [BLM90] these orbits are in bijection with $k \times l$-matrices (v_r^m), $1 \leqslant m \leqslant k$, $1 \leqslant r \leqslant l$ with elements in $Q_+ = \mathbb{Z}_+^I$, such that $\sum_r v_r^m = v^m$ is the dimension of V^m and $\sum_m v_r^m = v_r$ is the dimension of W_r. Indeed, to filtration $g^{-1}\mathcal{W}$ of length l and filtration $h\mathcal{V}$ of length k Beilinson, Lusztig and MacPherson associate a filtration $\mathcal{U}$ of the same space of length kl

$$0 = X_0^1 \subset X_1^1 \subset \dots \subset X_l^1 = X_0^2 \subset \dots \subset X_l^2 = X_0^3 \subset \dots \subset X_{l-1}^k \subset X_l^k = V,$$

where

$$X_r^m = hV^{(m-1)} + (g^{-1} W_{(r)} \cap hV^{(m)}) = (hV^{(m-1)} + g^{-1} W_{(r)}) \cap hV^{(m)}. \tag{2.1.4}$$

The matrix (v_r^m) consists of relative dimensions $v_r^m = \dim_{\mathbb{C}} X_r^m / X_{r-1}^m$. For our purposes it suffices to take $g = 1$. We conclude that the orbits of $P_{\mathcal{W}} \times P_{\mathcal{V}}$ in G_V are in bijection with the summands in the left hand side of equation (1.1.2).

Distinguished triangles. The $P_{\mathcal{W}} \times P_{\mathcal{V}}$-invariant subsets of G_V are unions of orbits, thereby, they correspond to sums of several summands in (1.1.2). The additive relation (1.1.2) in algebra is replaced for our Hopf category by a system of functorial distinguished triangles

$$\mathcal{K}^{\mathcal{O}_U} \to \mathcal{K}^{\mathcal{O}_X} \to \mathcal{K}^{\mathcal{O}_F} \to$$

given for any bi-invariant subset $\mathcal{O}_X \subset G_V$ and a bi-invariant closed subset $\mathcal{O}_F \subset \mathcal{O}_X$ with $\mathcal{O}_U = \mathcal{O}_X - \mathcal{O}_F$. The following diagram made with given distinguished triangles is an octahedron

$$\begin{tikzcd}
\mathcal{K}^{\mathcal{O}_R} \arrow[rr] & {} & \mathcal{K}^{\mathcal{O}_W} \arrow[dl] \arrow[dd] \\
{} & \mathcal{K}^{\mathcal{O}_Z} \arrow[ul, "1"'] \arrow[dr] \arrow[u, phantom, "d"] \arrow[d, phantom, "d"] \arrow[l, phantom, "="] \arrow[r, phantom, "="] & {} \\
\mathcal{K}^{\mathcal{O}_Q} \arrow[uu, "1"] \arrow[ur] & {} & \mathcal{K}^{\mathcal{O}_F} \arrow[ll, "1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}^{\mathcal{O}_R} \arrow[rr] \arrow[dr] & {} & \mathcal{K}^{\mathcal{O}_W} \arrow[dd] \\
{} & \mathcal{K}^{\mathcal{O}_U} \arrow[ur] \arrow[dl] \arrow[u, phantom, "="] \arrow[d, phantom, "="] \arrow[l, phantom, "d"] \arrow[r, phantom, "d"] & {} \\
\mathcal{K}^{\mathcal{O}_Q} \arrow[uu, "1"] & {} & \mathcal{K}^{\mathcal{O}_F} \arrow[ll, "1"] \arrow[ul, "1"']
\end{tikzcd}$$

for any pair of closed embeddings $\mathcal{O}_F \subset \mathcal{O}_Z \subset \mathcal{O}_W$, where $\mathcal{O}_U = \mathcal{O}_W - \mathcal{O}_F$, $\mathcal{O}_Q = \mathcal{O}_Z - \mathcal{O}_F$, $\mathcal{O}_R = \mathcal{O}_W - \mathcal{O}_Z$. This means commutativity of two squares formed by diagonal maps and of the four triangles marked "=". This is proven in [Lyu, Proposition 3.11.1].

2.2. Coherence isomorphism

Assume given a vector space decomposition

$$V = \oplus_{m=1}^k \oplus_{r=1}^l V_r^m.$$

We shall denote it $V_-^= = \mathcal{U}$ if the summands are ordered as follows

$$V_1^1, V_2^1, \dots, V_l^1, V_1^2, V_2^2, \dots, V_l^2, \dots, V_1^k, V_2^k, \dots, V_l^k. \tag{2.2.1}$$

The same decomposition with the order of summands

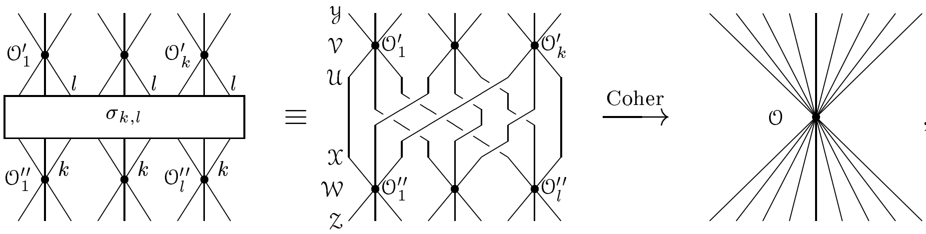
$$V_1^1, V_1^2, \dots, V_1^k, V_2^1, V_2^2, \dots, V_2^k, \dots, V_l^1, V_l^2, \dots, V_l^k$$

will be denoted $V_=^- = \mathcal{X}$. We also have decompositions

$$V_-^m = \mathcal{V}^m : \quad V^m = \oplus_{r=1}^l V_r^m, \qquad V^- = \mathcal{V} : \quad V = \oplus_{m=1}^k V^m,$$
$$V_r^- = \mathcal{W}_r : \quad W_r = \oplus_{m=1}^k V_r^m, \qquad V_- = \mathcal{W} : \quad V = \oplus_{r=1}^l W_r,$$

which produce filtrations $V_{(r)}^m$ of V^m, $V^{(m)}$ of V, $V_r^{(m)}$ of W_r and $W_{(r)}$ of V. We associate with these filtrations the parabolic groups $P_{\mathcal{V}^m} \subset G_{V^m}$, $P_{\mathcal{V}} \subset G_V$, $P_{\mathcal{W}_r} \subset G_{W_r}$ and $P_{\mathcal{W}} \subset G_{\mathcal{W}}$.

We claim that for any collection of indices and for any collection of bi-invariant locally closed subsets $(\mathcal{O}'_1, \dots, \mathcal{O}'_k, \mathcal{O}''_1, \dots, \mathcal{O}''_l)$, which may occur in the following diagram, there exists a bi-invariant locally closed subset $\mathcal{O}$ and a coherence isomorphism



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explicitly described below. Here $\sigma_{k,l} = (s_{k,l})^\sim_+$ is the braid, corresponding to the permutation $s_{k,l}$ of the set $\{1, 2, \dots, kl\}$,

$$s_{k,l}(1 + t + nl) = 1 + n + tk \quad \text{for} \quad 0 \leqslant t < l, 0 \leqslant n < k, \tag{2.2.2}$$

under the standard splitting $S_{kl} \to B_{kl}$, which maps the elementary transpositions to the generators of the braid group. The subset $\mathcal{O}$ is computed as follows.

Bi-invariance of initial parametrising subsets means that

$$G_{V^m} \supset \mathcal{O}'_m \in P_{V_-^m}\text{-sets-}P_{\mathcal{Y}^m},$$
$$G_{W_r} \supset \mathcal{O}''_r \in P_{\mathcal{Z}_r}\text{-sets-}P_{V_r^-},$$

where $\mathcal{Y}^m$ is the incoming filtration of V^m, and $\mathcal{Z}_r$ is the outgoing filtration of W_r. The subsets

$$P_{\mathcal{V}} \supset \overline{\mathcal{O}} \overset{\text{def}}{=} U_{\mathcal{V}} \cdot \prod_m \mathcal{O}'_m = \prod_m \mathcal{O}'_m \cdot U_{\mathcal{V}} \in P_{V_=^-}\text{-sets-}P_{\mathcal{Y}},$$
$$P_{\mathcal{V}} \supset \underline{\mathcal{O}} \overset{\text{def}}{=} U_{\mathcal{W}} \cdot \prod_r \mathcal{O}''_r = \prod_r \mathcal{O}''_r \cdot U_{\mathcal{W}} \in P_{\mathcal{Z}}\text{-sets-}P_{V_=^-}$$

are locally closed. The subset

$$G_V \supset \mathcal{O} \overset{\text{def}}{=} \underline{\mathcal{O}} \cdot \overline{\mathcal{O}} \in P_{\mathcal{Z}}\text{-sets-}P_{\mathcal{Y}}, \quad \mathcal{O} = \underline{\mathcal{O}} P_{\mathcal{U}} \times_{P_{\mathcal{U}}} \overline{\mathcal{O}} = \underline{\mathcal{O}} \times_{P_{\mathcal{W}} \cap P_{\mathcal{V}}} \overline{\mathcal{O}}$$

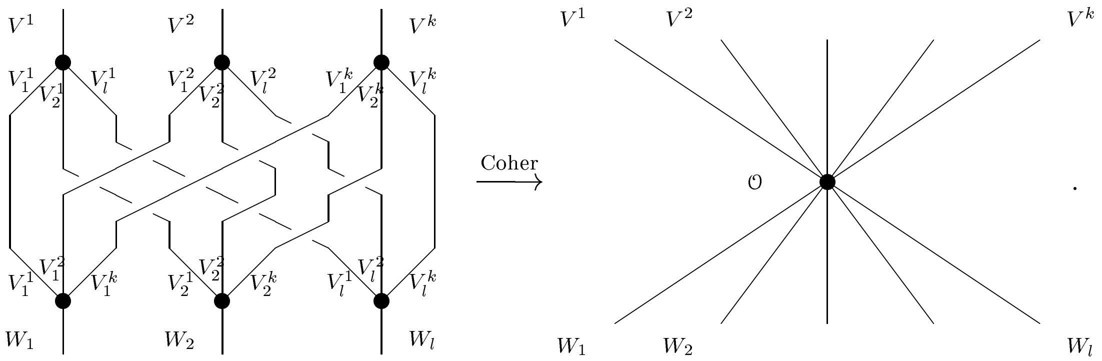
is also locally closed. Indeed, $\underline{\mathcal{O}} \times \overline{\mathcal{O}}$ is locally closed in $P_{\mathcal{W}} \times P_{\mathcal{V}}$, hence, $\underline{\mathcal{O}} \times_{P_{\mathcal{W}} \cap P_{\mathcal{V}}} \overline{\mathcal{O}}$ is locally closed in $P_{\mathcal{W}} \times_{P_{\mathcal{W}} \cap P_{\mathcal{V}}} P_{\mathcal{V}} \simeq P_{\mathcal{W}} \cdot P_{\mathcal{V}}$. The subspace $C = P_{\mathcal{W}} \cdot P_{\mathcal{V}}$ of G_V is locally closed, since it is an orbit of $P_{\mathcal{W}} \times P_{\mathcal{V}}$ in G_V, and the number of such orbits is finite by Section 2.1.1. Indeed, C is embedded in its closure as

$$C = \overline{C} - \bigcup \{ \overline{\mathcal{O}_a} \mid \mathcal{O}_a \subset \overline{C} \text{ is an orbit of } P_{\mathcal{W}} \times P_{\mathcal{V}}, \ \mathcal{O}_a \neq C \}$$

due to dimension considerations.

Both associativity isomorphism and coassociativity isomorphism of [Lyu99b] are particular cases of the general coherence isomorphism.

In the particular case, when the upper row of operations consists of comultiplications (operations with single input), and the lower row of operations consists of multiplications (operations with single output), we have necessarily $\mathcal{O}'_m = G_{V^m}$ and $\mathcal{O}''_r = G_{W_r}$. In such cases we omit the parametrizing set, since it is unique. Graphical notation here is the following:



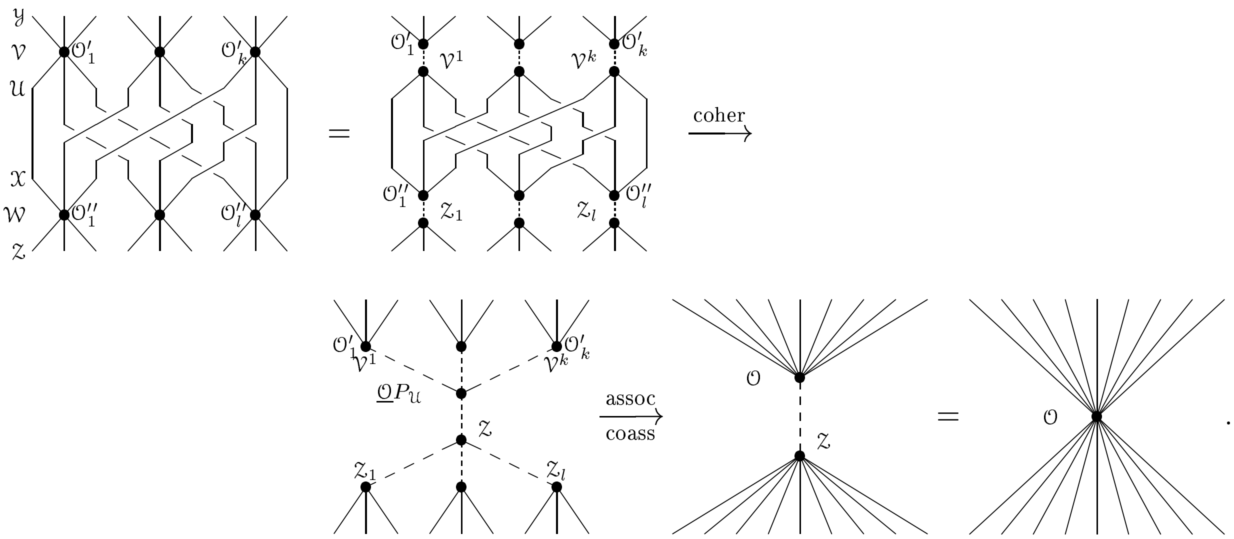
By general formulas we find here $\overline{\mathcal{O}} = P_{\mathcal{V}}$, $\underline{\mathcal{O}} = P_{\mathcal{W}}$, hence, $\mathcal{O} = P_{\mathcal{V}} P_{\mathcal{W}}$. We may set $g = h = 1$ in definition (2.1.4) of the filtration

$$X_r^m = V^{(m-1)} + (W_{(r)} \cap V^{(m)}) = \Big(\bigoplus_{n=1}^{m-1} \bigoplus_{s=1}^{l} V_s^n \Big) \oplus \Big(\bigoplus_{s=1}^{r} V_s^m \Big).$$

This filtration coincides with $V^=_- = \mathcal{U}$ obtained from (2.2.1). The computed orbit $\mathcal{O} = P_{\mathcal{V}} P_{\mathcal{W}}$ corresponds via bijection of Section 2.1.1 to the term in the left hand side of (1.1.2) indexed by $(v_r^m = \dim V_r^m)$.

An impression might be created, that to a matrix (v_r^m) correspond many operations $\mathcal{K}^{\mathcal{O};\mathcal{V}}_{\mathcal{W}}$, when $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{O}$ vary. However, up to natural equivalences and isomorphisms they coincide.

The general coherence isomorphism is built as the composition



The three components of the coherence isomorphism are defined next. All cells in the following diagrams represent isomorphisms, defined in Section 4.1.

$$
\begin{tikzcd}
\frac{\prod E_{Y_s^m}}{\prod L_{\mathcal{Y}^m}} \arrow[r, "\phi^*"] \arrow[d, "\phi^*"'] & \frac{\prod \mathcal{O}'_m \times F_{\mathcal{Y}^m}}{\prod P_{\mathcal{V}^m} \times P_{\mathcal{Y}^m}} \arrow[r, "\pi_*"] \arrow[d, "\phi^*"] & \frac{\prod \mathcal{O}'_m \times_{P_{\mathcal{Y}^m}} F_{\mathcal{Y}^m}}{\prod P_{\mathcal{V}^m}} \arrow[r, "\alpha_!"] \arrow[d, "\phi^*"] & \frac{\prod E_{V^m}}{\prod P_{\mathcal{V}^m}} \arrow[d, "\phi^*"] \\
\frac{\mathcal{O} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{Y}}} \arrow[r, "\pi^*"] \arrow[dr, Rightarrow, no head, "\mathrm{Id}"'] & \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times \overline{\mathcal{O}} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}} \times P_{\mathcal{Y}}} \arrow[r, "\pi_*"] \arrow[d, "\pi_*"] & \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times \overline{\mathcal{O}} \times_{P_{\mathcal{Y}}} F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}}} \arrow[r, "1 \times \alpha_!"] \arrow[d, "\pi_*"] & \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times F_{\mathcal{V}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}}} \arrow[d, "\pi_*"] \\
 & \frac{\mathcal{O} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{Y}}} \arrow[r, "\pi_*"] & \frac{\mathcal{O} \times_{P_{\mathcal{Y}}} F_{\mathcal{Y}}}{P_{\mathcal{Z}}} \arrow[r, "\beta_!"] \arrow[dr, "\alpha_!"'] & \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times_{P_{\mathcal{U}}} F_{\mathcal{V}}}{P_{\mathcal{Z}}} \arrow[d, "\alpha_!"] \\
 & & & \frac{E_V}{P_{\mathcal{Z}}}
\end{tikzcd}
$$

assoc

$$
\begin{tikzcd}
\frac{E_V}{P_{\mathcal{Z}}} \arrow[r, "\iota^*"] \arrow[dr, "\iota^*"'] & \frac{F_{\mathcal{W}}}{\prod_r P_{\mathcal{Z}_r}} \arrow[r, "\kappa_!"] \arrow[d, "\iota^*"] & \frac{\prod_r E_{W_r}}{\prod_r P_{\mathcal{Z}_r}} \arrow[d, "\iota^*"] \\
 & \frac{F_{\mathcal{Z}}}{L_{\mathcal{Z}}} \arrow[r, "\kappa_!"] \arrow[dr, "\kappa_!"'] & \frac{\prod_r F_{\mathcal{Z}_r}}{\prod_r L_{\mathcal{Z}_r}} \arrow[d, "\kappa_!"] \\
 & & \frac{\prod_{n,r} E_{Z_r^n}}{L_{\mathcal{Z}}}
\end{tikzcd}
$$

coass

The isomorphism coher is presented in Figure 1, where the numbers

$$
A = \sum_{m<n;r>s} \sum_{i \in I} \dim V_r^m(i) \cdot \dim V_s^n(i),
$$
$$
B = \sum_{m>n;r>s} \sum_{h \in H} \dim V_r^m(h') \cdot \dim V_s^n(h'')
$$

are nothing else but dimensions of the spaces

$$
U_{\mathcal{W}}/(U_{\mathcal{W}} \cap P_{\mathcal{V}}) \simeq \oplus_{m<n;r>s} \operatorname{Hom}(V_r^m, V_s^n),
$$
$$
f_{\mathcal{W}} \cap f_{\mathcal{V}} = \oplus_{h \in H; m>n; r>s} \operatorname{Hom}_{\mathbb{C}}(V_r^m(h'), V_s^n(h'')) \subset F_{\mathcal{W}} \cap F_{\mathcal{V}}.
$$

A v) type isomorphism $T^{-2B} \to \sigma_! \sigma^*$ comes from affine linear $P_{\mathcal{W}} \cap P_{\mathcal{V}}$-bundle $F_{\mathcal{W}} \cap F_{\mathcal{V}} \to (F_{\mathcal{W}} \cap F_{\mathcal{V}})/(f_{\mathcal{W}} \cap f_{\mathcal{V}})$.

A V) type isomorphism $T^{-2A} \to t_! t^*$ is obtained from a sequence of affine linear $\prod P_{\mathcal{W}_r}$-bundles

$$
U_{\mathcal{W}}/(U_{\mathcal{W}} \cap P_{\mathcal{V}}) \longrightarrow \ldots \longrightarrow U_{\mathcal{W}}/[U_{\mathcal{W}}^{(k+1)}(U_{\mathcal{W}} \cap P_{\mathcal{V}})] \longrightarrow U_{\mathcal{W}}/[U_{\mathcal{W}}^{(k)}(U_{\mathcal{W}} \cap P_{\mathcal{V}})] \longrightarrow \ldots pt
$$

constructed via Young tableau (B.2.2) in Section B.2.8. The group $\prod P_{\mathcal{W}_r}$ acts by conjugation. Pull-back of this sequence along $\prod \mathcal{O}''_r \times F_{\mathcal{W}_r} | \prod P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r} \to pt | \prod P_{\mathcal{W}_r}$ gives a sequence of affine linear $\prod P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r}$-bundles

$$
\prod \mathcal{O}''_r \times U_{\mathcal{W}}/(U_{\mathcal{W}} \cap P_{\mathcal{V}}) \times F_{\mathcal{W}_r} \longrightarrow \ldots \longrightarrow \prod \mathcal{O}''_r \times U_{\mathcal{W}}/[U_{\mathcal{W}}^{(k+1)}(U_{\mathcal{W}} \cap P_{\mathcal{V}})] \times F_{\mathcal{W}_r}
$$
$$
\longrightarrow \prod \mathcal{O}''_r \times U_{\mathcal{W}}/[U_{\mathcal{W}}^{(k)}(U_{\mathcal{W}} \cap P_{\mathcal{V}})] \times F_{\mathcal{W}_r} \longrightarrow \ldots \longrightarrow \prod \mathcal{O}''_r \times F_{\mathcal{W}_r}.
$$

$$
\begin{tikzcd}
\frac{\prod E_{Vm}}{\prod P_{\mathcal{V}m}} \arrow[rr, "\prod \iota^*"] \arrow[dr, "\iota^*"] \arrow[dddd, "\phi^*"'] & & \frac{\prod F_{\mathcal{V}m}}{\prod G_{V_r^m}} \arrow[rr, "\prod \kappa_!"] & & \frac{\prod E_{V_r^m}}{\prod G_{V_r^m}} \arrow[dl, "p^*"] \arrow[ddl, "p^*"] \arrow[dd, "\phi^*"] \\
& \frac{\prod F_{\mathcal{V}m}}{\prod P_{\mathcal{V}m}} \arrow[ur, "\mathrm{Res}"] \arrow[r, "\prod \kappa_!"'] \arrow[dd, "\phi^*"'] & \frac{\prod E_{V_r^m}}{\prod P_{\mathcal{V}m}} \arrow[urr, "\mathrm{Res}=i^*"] \arrow[r, equal, "\mathrm{Id}"'] & \frac{\prod E_{V_r^m}}{\prod P_{\mathcal{V}m}} \arrow[u, phantom, "\boxed{\mathrm{S}}"] \arrow[ddl, "p^*"'] & \\
& & & \frac{\underline{\mathcal{O}}/(U_{\mathcal{W}}\cap P_{\mathcal{V}})\times\prod F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}\times P_{\mathcal{W}_r}} \arrow[dl, "q^*"'] \arrow[d, equal, "\mathrm{Id}"] & \frac{\prod \mathcal{O}''_r\times F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}\times P_{\mathcal{W}_r}} \arrow[dl, "t^*"'] \arrow[d, "{[-2A]}"] \\
& \frac{\underline{\mathcal{O}}\times(F_{\mathcal{W}}\cap F_{\mathcal{V}})/(f_{\mathcal{W}}\cap f_{\mathcal{V}})}{(\prod P_{\mathcal{Z}_r})\times(P_{\mathcal{W}}\cap P_{\mathcal{V}})} \arrow[r, "r_!"] \arrow[d, "\sigma^*"'] \arrow[dr, "q_*"] & \frac{\underline{\mathcal{O}}\times\prod F_{\mathcal{W}_r}}{(\prod P_{\mathcal{Z}_r})\times(P_{\mathcal{W}}\cap P_{\mathcal{V}})} \arrow[r, "q_*"] \arrow[dr, "q_*"] \arrow[ur, phantom, "\boxed{\mathrm{F}}"] & \frac{\underline{\mathcal{O}}/(U_{\mathcal{W}}\cap P_{\mathcal{V}})\times\prod F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}\times P_{\mathcal{W}_r}} \arrow[r, "t_!"'] \arrow[d, "q_*"] \arrow[ur, phantom, "\boxed{\mathrm{V}}"] & \frac{\prod \mathcal{O}''_r\times F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}\times P_{\mathcal{W}_r}} \arrow[d, "\pi_*"] \\
\frac{\underline{\mathcal{O}}P_{\mathcal{U}}\times F_{\mathcal{V}}}{P_{\mathcal{Z}}\times P_{\mathcal{U}}} \arrow[r, "i^*"] \arrow[d, "\pi_*"'] \arrow[dr, phantom, "\boxed{\mathrm{F}^{-1}}"] & \frac{\underline{\mathcal{O}}\times(F_{\mathcal{W}}\cap F_{\mathcal{V}})}{(\prod P_{\mathcal{Z}_r})\times(P_{\mathcal{W}}\cap P_{\mathcal{V}})} \arrow[d, "q_*"'] \arrow[dr, phantom, "\boxed{\mathrm{v}}"] & \frac{\underline{\mathcal{O}}\times_{P_{\mathcal{W}}\cap P_{\mathcal{V}}}(F_{\mathcal{W}}\cap F_{\mathcal{V}})/(f_{\mathcal{W}}\cap f_{\mathcal{V}})}{\prod P_{\mathcal{Z}_r}} \arrow[r, "r_!"] \arrow[dl, "\sigma^*"'] \arrow[d, "{[-2B]}"] & \frac{\underline{\mathcal{O}}\times_{P_{\mathcal{W}}\cap P_{\mathcal{V}}}\prod F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}} \arrow[r, "t_!"] \arrow[d, "{[-2B]}"] & \frac{\prod \mathcal{O}''_r\times_{P_{\mathcal{W}_r}} F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}} \arrow[d, "{[-2B]}"] \\
\frac{\underline{\mathcal{O}}P_{\mathcal{U}}\times_{P_{\mathcal{U}}} F_{\mathcal{V}}}{P_{\mathcal{Z}}} \arrow[r, "\iota^*"] \arrow[d, "\alpha_!"'] & \frac{\underline{\mathcal{O}}\times_{P_{\mathcal{W}}\cap P_{\mathcal{V}}}(F_{\mathcal{W}}\cap F_{\mathcal{V}})}{\prod P_{\mathcal{Z}_r}} \arrow[r, "\sigma_!"] \arrow[d, "\alpha_!"'] \arrow[drrr, "\alpha_!"] & \frac{\underline{\mathcal{O}}\times_{P_{\mathcal{W}}\cap P_{\mathcal{V}}}(F_{\mathcal{W}}\cap F_{\mathcal{V}})/(f_{\mathcal{W}}\cap f_{\mathcal{V}})}{\prod P_{\mathcal{Z}_r}} \arrow[r, "r_!"] & \frac{\underline{\mathcal{O}}\times_{P_{\mathcal{W}}\cap P_{\mathcal{V}}}\prod F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}} \arrow[r, "t_!"] & \frac{\prod \mathcal{O}''_r\times_{P_{\mathcal{W}_r}} F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}} \arrow[d, "\alpha_!"] \\
\frac{E_V}{P_{\mathcal{Z}}} \arrow[r, "\iota^*"] & \frac{F_{\mathcal{W}}}{\prod P_{\mathcal{Z}_r}} \arrow[rrr, "\kappa_!"] & & & \frac{\prod E_{W_r}}{\prod P_{\mathcal{Z}_r}}
\end{tikzcd}
$$

Figure 1: Coherence isomorphism coher

Multiplication map $\prod \mathcal{O}''_r \times U_{\mathcal{W}} \to \underline{\mathcal{O}}$ is an isomorphism of $\prod P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r}$-spaces, hence, we may replace one with another. Finally, we take quotient by $\prod P_{\mathcal{Z}_r}$ and get a sequence of affine linear $\prod P_{\mathcal{W}_r}$-bundles

$$\begin{aligned}(\prod P_{\mathcal{Z}_r})\backslash\underline{\mathcal{O}}/(U_{\mathcal{W}} \cap P_{\mathcal{V}}) \times F_{\mathcal{W}_r} \longrightarrow \dots \longrightarrow (\prod P_{\mathcal{Z}_r})\backslash\underline{\mathcal{O}}/[U_{\mathcal{W}}^{(k+1)}(U_{\mathcal{W}} \cap P_{\mathcal{V}})] \times F_{\mathcal{W}_r} \\ \longrightarrow (\prod P_{\mathcal{Z}_r})\backslash\underline{\mathcal{O}}/[U_{\mathcal{W}}^{(k)}(U_{\mathcal{W}} \cap P_{\mathcal{V}})] \times F_{\mathcal{W}_r} \longrightarrow \dots \longrightarrow \prod P_{\mathcal{Z}_r}\backslash\mathcal{O}''_r \times F_{\mathcal{W}_r}.\end{aligned}$$

2.2.1. Lemma. *The group $P_{\mathcal{W}} \cap P_{\mathcal{U}} = P_{\mathcal{W}} \cap P_{\mathcal{V}}$ is a semidirect product of its normal subgroup $U_{\mathcal{W}} \cap P_{\mathcal{U}} = U_{\mathcal{W}} \cap P_{\mathcal{V}}$ and diagonally embedded subgroup $\prod_r P_{\mathcal{W}_r}$.*

Proof. For an element $g \in G_V$ the following conditions are equivalent:

$$\begin{aligned}
g \in P_{\mathcal{V}} &\iff \forall \mu, \rho \quad (g-1)(V^\mu_\rho) \subset \oplus_{m,r}\{V^m_r \mid m \leqslant \mu\}, \\
g \in P_{\mathcal{U}} &\iff \forall \mu, \rho \quad (g-1)(V^\mu_\rho) \subset \oplus_{m,r}\{V^m_r \mid m < \mu, \text{ or } (m = \mu, r \leqslant \rho)\}, \\
g \in P_{\mathcal{W}} &\iff \forall \mu, \rho \quad (g-1)(V^\mu_\rho) \subset \oplus_{m,r}\{V^m_r \mid r \leqslant \rho\}, \\
g \in U_{\mathcal{W}} &\iff \forall \mu, \rho \quad (g-1)(V^\mu_\rho) \subset \oplus_{m,r}\{V^m_r \mid r < \rho\}, \\
g \in P_{\mathcal{W}} \cap P_{\mathcal{U}} &\iff \forall \mu, \rho \quad (g-1)(V^\mu_\rho) \subset \oplus_{m,r}\{V^m_r \mid m \leqslant \mu, r \leqslant \rho\} \iff g \in P_{\mathcal{W}} \cap P_{\mathcal{V}}, \\
g \in U_{\mathcal{W}} \cap P_{\mathcal{U}} &\iff \forall \mu, \rho \quad (g-1)(V^\mu_\rho) \subset \oplus_{m,r}\{V^m_r \mid m \leqslant \mu, r < \rho\} \iff g \in U_{\mathcal{W}} \cap P_{\mathcal{V}}, \\
g \in \prod_r P_{\mathcal{W}_r} &\iff \forall \mu, \rho \quad (g-1)(V^\mu_\rho) \subset \oplus_{m,r}\{V^m_r \mid m \leqslant \mu, r = \rho\}
\end{aligned}$$

This implies the lemma. □

Diagram in Fig. 2 shows how three isomorphisms combine into full coherence isomorphism Coher. Its complete description including types of all cells is given in diagram on page 12.

3. Coherence equations

3.1. General description

The introduced coherence isomorphisms have to satisfy certain coherence equations. Let us describe these equations.

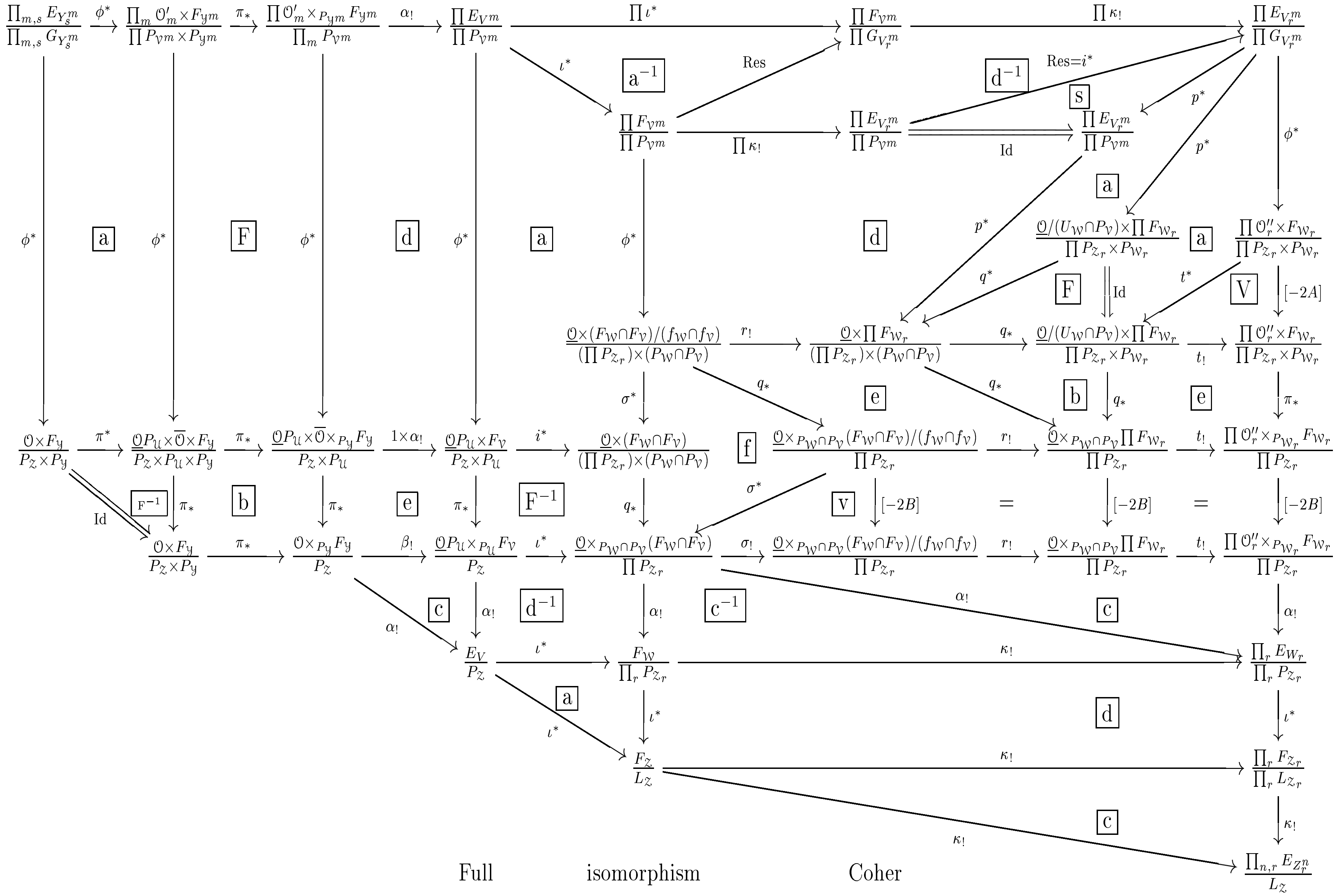
Assume that k, n are positive integers, $(a_1, \dots, a_k)$, $(b_1, \dots, b_n)$ are sequences of positive integers, and $(c^1_1, \dots, c^{a_1}_1)$, …, $(c^1_k, \dots, c^{a_k}_k)$, $(d^1_1, \dots, d^{b_1}_1)$, …, $(d^1_n, \dots, d^{b_n}_n)$ are sequences of positive integers. Set

$$m = \sum_{s=1}^k a_s, \qquad l = \sum_{r=1}^n b_r,$$
$$c_s = \sum_{t=1}^{a_s} c^t_s, \text{ for } 1 \leqslant s \leqslant k, \text{ and } c = \sum_{s=1}^k c_s,$$
$$d_r = \sum_{p=1}^{b_r} d^p_r, \text{ for } 1 \leqslant r \leqslant n, \text{ and } d = \sum_{r=1}^n d_r.$$

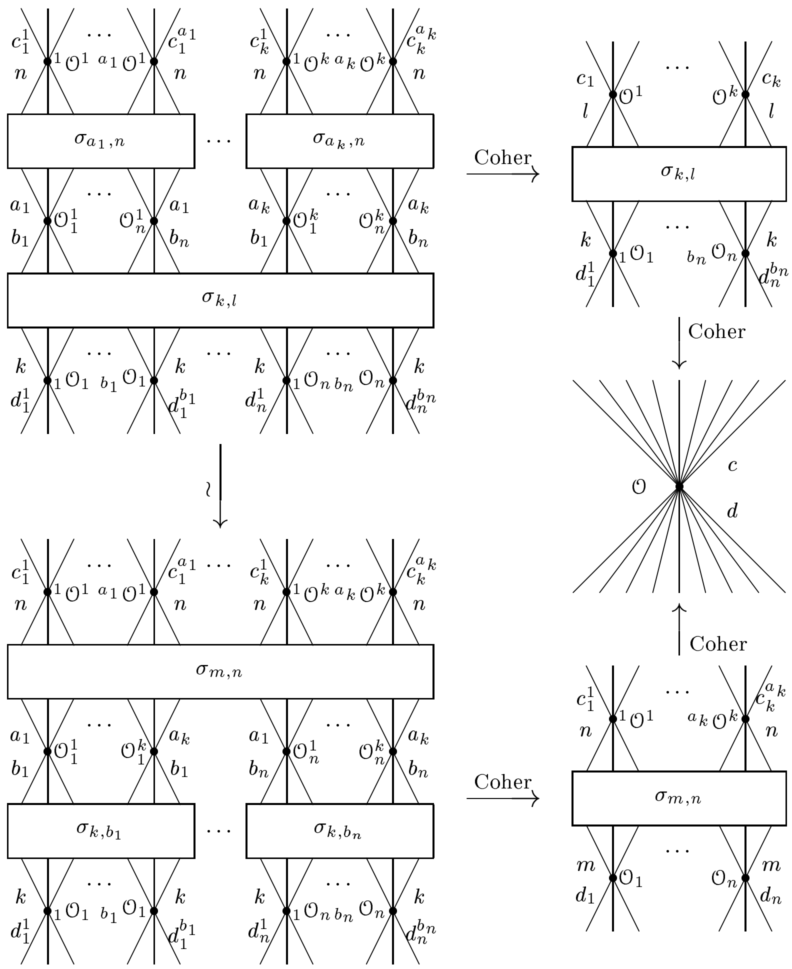
These integers are numbers of incoming or outgoing edges for vertices (operations $\mathcal{K}^{\mathcal{O};\mathcal{V}}_{\mathcal{W}}$) in the following diagram made of composite functors as well as isomorphisms between them. Braiding

$$
\begin{tikzcd}
\frac{\prod_{m,s} E_{Y_s^m}}{\prod_m L_{\mathcal{Y} m}} \arrow[r, "\phi^*"] \arrow[dd, "\phi^*"'] & \frac{\prod_m \mathcal{O}'_m \times F_{\mathcal{Y} m}}{\prod_m P_{\mathcal{V} m} \times P_{\mathcal{Y} m}} \arrow[r, "\pi_*"] \arrow[dd, "\phi^*"] & \frac{\prod_m \mathcal{O}'_m \times_{P_{\mathcal{Y} m}} F_{\mathcal{Y} m}}{\prod_m P_{\mathcal{V} m}} \arrow[r, "\alpha_!"] \arrow[dd, "\phi^*"] & \frac{\prod_m E_{\mathcal{V} m}}{\prod_m P_{\mathcal{V} m}} \arrow[r, "\prod \iota^*"] \arrow[dd, "\phi^*"] & \frac{\prod_m F_{\mathcal{V} m}}{\prod_m L_{\mathcal{V} m}} \arrow[r, "\prod \kappa_!"] & \frac{\prod_{m,r} E_{V_r^m}}{\prod_m L_{\mathcal{V} m}} \arrow[d, "\phi^*"] \\
 & & & & & \frac{\prod_r \mathcal{O}''_r \times F_{\mathcal{W}_r}}{\prod_r P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r}} \arrow[d, "{[-2A]}"] \\
\frac{\mathcal{O} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{Y}}} \arrow[r, "\pi^*"] \arrow[ddr, Rightarrow, "\mathrm{Id}"'] & \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times \overline{\mathcal{O}} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}} \times P_{\mathcal{Y}}} \arrow[r, "\pi_*"] \arrow[dd, "\pi_*"] & \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times \overline{\mathcal{O}} \times_{P_{\mathcal{Y}}} F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}}} \arrow[r, "1 \times \alpha_!"] \arrow[dd, "\pi_*"] & \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times F_{\mathcal{V}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}}} \arrow[dd, "\pi_*"] & \boxed{\text{coher}} & \frac{\prod_r \mathcal{O}''_r \times F_{\mathcal{W}_r}}{\prod_r P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r}} \arrow[d, "\pi_*"] \\
 & & & & & \frac{\prod_r \mathcal{O}''_r \times_{P_{\mathcal{W}_r}} F_{\mathcal{W}_r}}{\prod_r P_{\mathcal{Z}_r}} \arrow[d, "{[-2B]}"] \\
 & \frac{\mathcal{O} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{Y}}} \arrow[r, "\pi_*"] & \frac{\mathcal{O} \times_{P_{\mathcal{Y}}} F_{\mathcal{Y}}}{P_{\mathcal{Z}}} \arrow[r, "\beta_!"] \arrow[dr, "\alpha_!"'] & \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times_{P_{\mathcal{U}}} F_{\mathcal{V}}}{P_{\mathcal{Z}}} \arrow[d, "\alpha_!"] & & \frac{\prod_r \mathcal{O}''_r \times_{P_{\mathcal{W}_r}} F_{\mathcal{W}_r}}{\prod_r P_{\mathcal{Z}_r}} \arrow[d, "\alpha_!"] \\
 & & & \frac{E_V}{P_{\mathcal{Z}}} \arrow[r, "\iota^*"] \arrow[dr, "\iota^*"'] & \frac{F_{\mathcal{W}}}{\prod_r P_{\mathcal{Z}_r}} \arrow[r, "\kappa_!"] \arrow[d, "\iota^*"] & \frac{\prod_r E_{W_r}}{\prod_r P_{\mathcal{Z}_r}} \arrow[d, "\iota^*"] \\
 & & & & \frac{F_{\mathcal{Z}}}{L_{\mathcal{Z}}} \arrow[r, "\kappa_!"] \arrow[dr, "\kappa_!"'] & \frac{\prod_r F_{\mathcal{Z}_r}}{\prod_r L_{\mathcal{Z}_r}} \arrow[d, "\kappa_!"] \\
 & & & & & \frac{\prod_{n,r} E_{Z_r^n}}{L_{\mathcal{Z}}}
\end{tikzcd}
$$

Figure 2: Scheme of the composite coherence isomorphism Coher

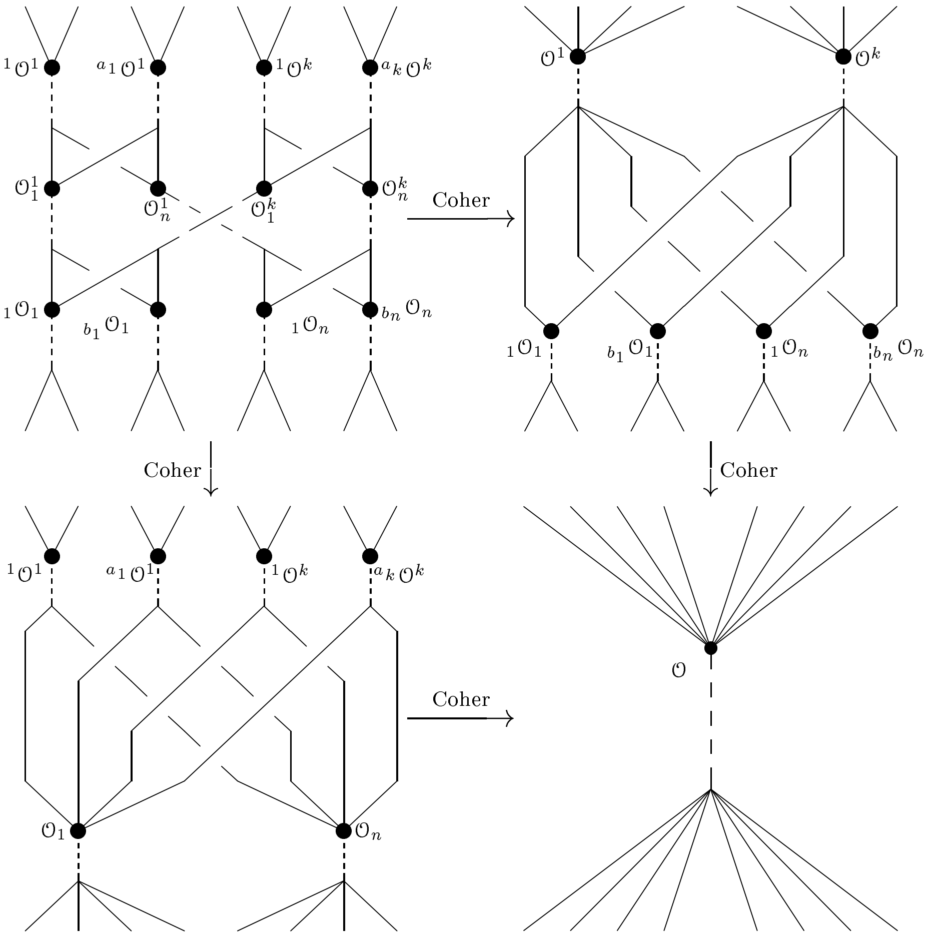

$\frac{\prod_{m,s} E_{Y_s^m}}{\prod_{m,s} G_{Y_s^m}} \xrightarrow{\phi^*} \frac{\prod_m \mathcal{O}'_m \times F_{\mathcal{Y}^m}}{\prod P_{\mathcal{V}^m} \times P_{\mathcal{Y}^m}} \xrightarrow{\pi_*} \frac{\prod \mathcal{O}'_m \times_{P_{\mathcal{Y}^m}} F_{\mathcal{Y}^m}}{\prod_m P_{\mathcal{V}^m}} \xrightarrow{\alpha_!} \frac{\prod E_{V^m}}{\prod P_{\mathcal{V}^m}} \xrightarrow{\prod \iota^*} \frac{\prod F_{\mathcal{V}^m}}{\prod G_{V_r^m}} \xrightarrow{\prod \kappa_!} \frac{\prod E_{V_r^m}}{\prod G_{V_r^m}}$
ι^*, Res, Res$=i^*$, p^*, ϕ^*
a^{-1}, d^{-1}, s
$\frac{\prod F_{\mathcal{V}^m}}{\prod P_{\mathcal{V}^m}} \xrightarrow{\prod \kappa_!} \frac{\prod E_{V_r^m}}{\prod P_{\mathcal{V}^m}} \xrightarrow{\mathrm{Id}} \frac{\prod E_{V_r^m}}{\prod P_{\mathcal{V}^m}}$
a, F, d, a, d, a, a
$\frac{\underline{\mathcal{O}}/(U_{\mathcal{W}} \cap P_{\mathcal{V}}) \times \prod F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r}}$, $\frac{\prod \mathcal{O}''_r \times F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r}}$
q^*, F, Id, t^*, V, $[-2A]$
$\frac{\underline{\mathcal{O}} \times (F_{\mathcal{W}} \cap F_{\mathcal{V}})/(f_{\mathcal{W}} \cap f_{\mathcal{V}})}{(\prod P_{\mathcal{Z}_r}) \times (P_{\mathcal{W}} \cap P_{\mathcal{V}})} \xrightarrow{r_!} \frac{\underline{\mathcal{O}} \times \prod F_{\mathcal{W}_r}}{(\prod P_{\mathcal{Z}_r}) \times (P_{\mathcal{W}} \cap P_{\mathcal{V}})} \xrightarrow{q_*} \frac{\underline{\mathcal{O}}/(U_{\mathcal{W}} \cap P_{\mathcal{V}}) \times \prod F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r}} \xrightarrow{t_!} \frac{\prod \mathcal{O}''_r \times F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r} \times P_{\mathcal{W}_r}}$
σ^*, q_*, e, q_*, b, q_*, e, π_*
$\frac{\mathcal{O} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{Y}}} \xrightarrow{\pi^*} \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times \overline{\mathcal{O}} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}} \times P_{\mathcal{Y}}} \xrightarrow{\pi_*} \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times \overline{\mathcal{O}} \times_{P_{\mathcal{Y}}} F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}}} \xrightarrow{1 \times \alpha_!} \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times F_{\mathcal{V}}}{P_{\mathcal{Z}} \times P_{\mathcal{U}}} \xrightarrow{i^*} \frac{\underline{\mathcal{O}} \times (F_{\mathcal{W}} \cap F_{\mathcal{V}})}{(\prod P_{\mathcal{Z}_r}) \times (P_{\mathcal{W}} \cap P_{\mathcal{V}})}$ f $\frac{\underline{\mathcal{O}} \times_{P_{\mathcal{W}} \cap P_{\mathcal{V}}} (F_{\mathcal{W}} \cap F_{\mathcal{V}})/(f_{\mathcal{W}} \cap f_{\mathcal{V}})}{\prod P_{\mathcal{Z}_r}} \xrightarrow{r_!} \frac{\underline{\mathcal{O}} \times_{P_{\mathcal{W}} \cap P_{\mathcal{V}}} \prod F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}} \xrightarrow{t_!} \frac{\prod \mathcal{O}''_r \times_{P_{\mathcal{W}_r}} F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}}$
Id, F^{-1}, π_*, b, π_*, e, π_*, F^{-1}, q_*, σ^*, V, $[-2B]$, $=$, $[-2B]$, $=$, $[-2B]$
$\frac{\mathcal{O} \times F_{\mathcal{Y}}}{P_{\mathcal{Z}} \times P_{\mathcal{Y}}} \xrightarrow{\pi_*} \frac{\mathcal{O} \times_{P_{\mathcal{Y}}} F_{\mathcal{Y}}}{P_{\mathcal{Z}}} \xrightarrow{\beta_!} \frac{\underline{\mathcal{O}} P_{\mathcal{U}} \times_{P_{\mathcal{U}}} F_{\mathcal{V}}}{P_{\mathcal{Z}}} \xrightarrow{\iota^*} \frac{\underline{\mathcal{O}} \times_{P_{\mathcal{W}} \cap P_{\mathcal{V}}} (F_{\mathcal{W}} \cap F_{\mathcal{V}})}{\prod P_{\mathcal{Z}_r}} \xrightarrow{\sigma_!} \frac{\underline{\mathcal{O}} \times_{P_{\mathcal{W}} \cap P_{\mathcal{V}}} (F_{\mathcal{W}} \cap F_{\mathcal{V}})/(f_{\mathcal{W}} \cap f_{\mathcal{V}})}{\prod P_{\mathcal{Z}_r}} \xrightarrow{r_!} \frac{\underline{\mathcal{O}} \times_{P_{\mathcal{W}} \cap P_{\mathcal{V}}} \prod F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}} \xrightarrow{t_!} \frac{\prod \mathcal{O}''_r \times_{P_{\mathcal{W}_r}} F_{\mathcal{W}_r}}{\prod P_{\mathcal{Z}_r}}$
$\alpha_!$, c, $\alpha_!$, d^{-1}, $\alpha_!$, c^{-1}, $\alpha_!$, c, $\alpha_!$
$\frac{E_V}{P_{\mathcal{Z}}} \xrightarrow{\iota^*} \frac{F_{\mathcal{W}}}{\prod_r P_{\mathcal{Z}_r}} \xrightarrow{\kappa_!} \frac{\prod_r E_{W_r}}{\prod_r P_{\mathcal{Z}_r}}$
a, ι^*, ι^*, d, ι^*
$\frac{F_{\mathcal{Z}}}{L_{\mathcal{Z}}} \xrightarrow{\kappa_!} \frac{\prod_r F_{\mathcal{Z}_r}}{\prod_r L_{\mathcal{Z}_r}}$
c, $\kappa_!$, $\kappa_!$
$\frac{\prod_{n,r} E_{Z_r^n}}{L_{\mathcal{Z}}}$
Full isomorphism Coher

functors correspond to braids which are liftings of permutations of (2.2.2) type. The functors and isomorphisms in this diagram are already constructed.



The Hopf category property requires that the two composite isomorphisms were equal, so that the diagram is commutative. Here the left vertical isomorphism comes from the permutation isomorphism of spaces and groups, which corresponds to an isotopy of the tangle diagram. The isotopy consists of pulling down the middle row of vertices along the strings of $\sigma_{k,l}$ closer to the lower row.

The same equation can be rewritten less formally as follows.



The left top corner functor is obviously isomorphic to both functors on the left in the previous diagram. So we replace the mentioned two functors with the one above.

To prove this equation we use presentation of Coher as composition of three isomorphisms coher, assoc and coass. We divide the above square into several polygons, and try to prove commutativity of each of them separately. To divide the square we introduce new isomorphisms, which are similar to already used coher, assoc and coass, however, do not formally follows from them. Squares marked with = commute for trivial reasons.

Main Theorem. *The following relations, marked with !=! in diagram on the next page, hold true: coassociativity–coassociativity, associativity–associativity and coherence–coassociativity.*

Concrete expressions for all functors and morphisms in this diagram are provided below. Proof of the theorem will appear elsewhere.

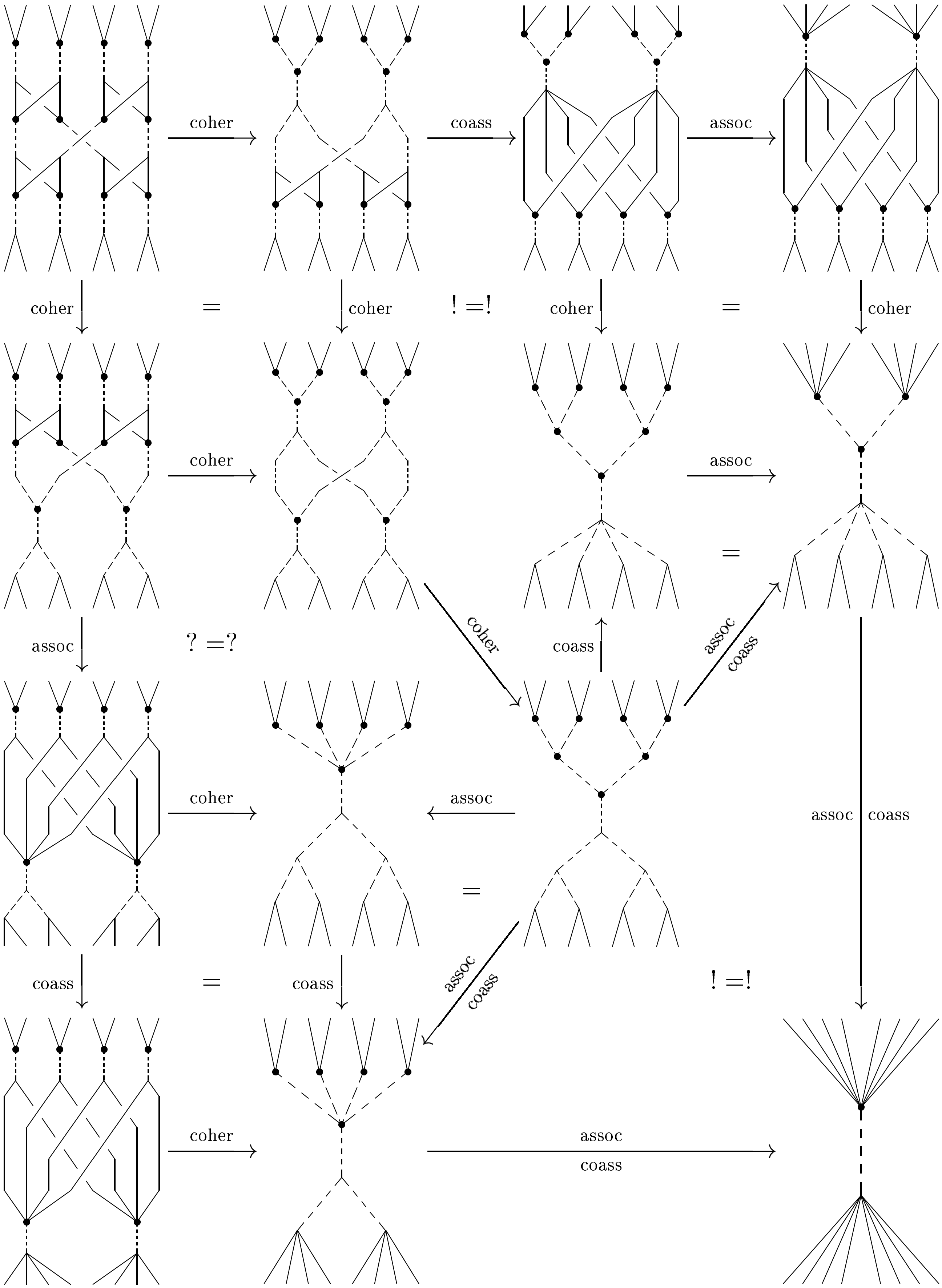
Main Missing Theorem. *The coherence–associativity relation, marked with ?=? in diagram on the following page also holds?*

The general case of isomorphisms between direct sums reduces to the case of subspaces of a common fixed space. So let a vector space V be given, together with its subspaces ${}^t_pX^s$, ${}^tV^s_r$, ${}_jY^s_r$, ${}^i_jZ_r$, such that all sums

$$\sum_{p,t,s} {}^t_pX^s = \sum_{t,s,r} {}^tV^s_r = \sum_{s,r,j} {}_jY^s_r = \sum_{r,j,i} {}^i_jZ_r = V$$

are direct sums. Introduce also subspaces

$$V^s_r = \sum_t {}^tV^s_r, \qquad {}^tV_r = \sum_s {}^tV^s_r, \qquad {}^tV^s = \sum_r {}^tV^s_r,$$
$${}^tV = \sum_{s,r} {}^tV^s_r, \qquad V^s = \sum_{r,t} {}^tV^s_r, \qquad V_r = \sum_{t,s} {}^tV^s_r \tag{3.1.1}$$


coher
coass
assoc
coher
=
coher
! =!
coher
=
coher
coher
assoc
=
assoc
? =?
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and similarly for other letters X, Y, Z. We require the following subspaces to coincide pairwise:

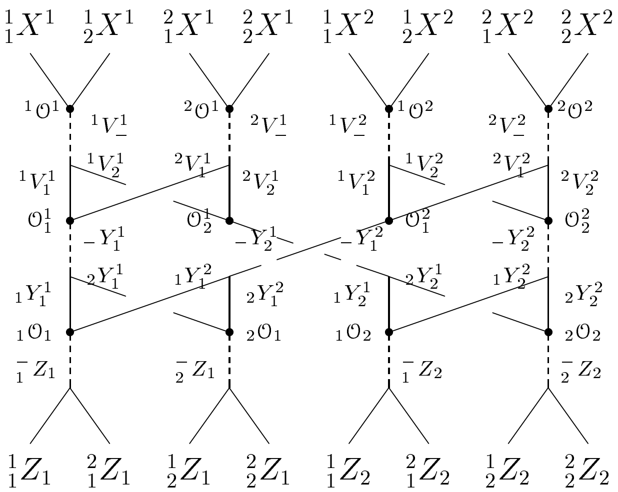
$$ {}^{t}X^{s} = {}^{t}V^{s}, \qquad V_r^s = Y_r^s, \qquad {}_jY_r = {}_jZ_r. $$

This implies, of course, the equations

$$ {}^{t}X = {}^{t}V, \quad X^s = V^s = Y^s, \quad V_r = Y_r = Z_r, \quad {}_jY = {}_jZ. $$

Let us refer to the functors in diagram as Φ_{ij}, $1 \leqslant i, j \leqslant 4$; i is the row and j is the column number. An example of Φ_{11} is given below.

$\Phi_{11} =$



We use the lexicographical order of subsequences of $({}^t_pX^s)$, $({}^tV_r^s)$, $({}_jY_r^s)$, $({}^i_jZ_r)$ or (3.1.1) or similar for other letters. The subsequence is fixed by fixing 2, 1 or 0 indices. It is denoted by the corresponding letter X, V, Y or Z with these fixed indices, where the remaining indices are replaced with symbols $-$, $=$ or $\equiv$. These symbols indicate places at which indices are compared to establish the lexicographical order. We compare two triples of indices first at place indicated by $\equiv$ (if present). If the values coincide or are not specified, we compare at place indicated by $=$ (if present). If these coincide or are not specified, we compare values at $-$. For instance,

$$
\begin{aligned}
{}^t_-X^s &= ({}^t_1X^s, {}^t_2X^s, \dots),\\
{}_-Y^s_= &= ({}_1Y^s_1, {}_2Y^s_1, \dots, {}_1Y^s_2, {}_2Y^s_2, \dots),\\
Y_r^- &= (Y_r^1, Y_r^2, \dots),\\
{}^-_=Z_\equiv &= ({}^1_1Z_1, {}^2_1Z_1, \dots, {}^1_2Z_1, {}^2_2Z_1, \dots, {}^1_1Z_2, {}^2_1Z_2, \dots, {}^1_2Z_2, {}^2_2Z_2, \dots)\\
{}_-Z_= &= ({}_1Z_1, {}_2Z_1, \dots, {}_1Z_2, {}_2Z_2, \dots),\\
Y^- &= (Y^1, Y^2, \dots).
\end{aligned}
$$

From so ordered sequence we produce an increasing filtration of the space $V = X = Y = Z$. The filtration is denoted by the same symbols as the sequence. The m-th term of the filtration is the sum of first m elements of the sequence.

3.2. Detailed description

The locally closed subsets parametrising operations are the following:

$$
\begin{aligned}
G_{{}^tV^s} \supset {}^t\mathcal{O}^s &\in P_{{}^tV^s_-}\text{-sets-}P_{{}^t_-X^s},\\
G_{Y_r^s} \supset \mathcal{O}_r^s &\in P_{{}_-Y_r^s}\text{-sets-}P_{{}^-V_r^s},\\
G_{{}_jY_r} \supset {}_j\mathcal{O}_r &\in P_{{}^-_jZ_r}\text{-sets-}P_{{}_jY_r^-}.
\end{aligned}
$$

All parametrising subsets obtained from these will be also locally closed as shown in Section 2.2.

We have to deal with the following functors.

$$\Phi_{11}=\frac{\prod E_{{}^{t}_{p}X^{s}}}{\prod G_{{}^{t}_{p}X^{s}}}\xrightarrow{\phi^{*}}\frac{\prod {}^{t}\mathcal{O}^{s}\times F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}}\times P_{{}^{t}_{-}X^{s}}}\xrightarrow{\pi_{*}}\frac{\prod {}^{t}\mathcal{O}^{s}\times_{P_{{}^{t}_{-}X^{s}}}F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{{}^{t}V^{s}}}{\prod P_{{}^{t}V^{s}_{-}}}\xrightarrow{\iota^{*}}\frac{\prod F_{{}^{t}V^{s}_{-}}}{\prod L_{{}^{t}V^{s}_{-}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}^{t}V^{s}_{r}}}{\prod G_{{}^{t}V^{s}_{r}}}$$
$$\xrightarrow{\phi^{*}}\frac{\prod \mathcal{O}^{s}_{r}\times F_{-V^{s}_{r}}}{\prod P_{-Y^{s}_{r}}\times P_{-V^{s}_{r}}}\xrightarrow{\pi_{*}}\frac{\prod \mathcal{O}^{s}_{r}\times_{P_{-V^{s}_{r}}}F_{-V^{s}_{r}}}{\prod P_{-Y^{s}_{r}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{Y^{s}_{r}}}{\prod P_{-Y^{s}_{r}}}\xrightarrow{\iota^{*}}\frac{\prod F_{-Y^{s}_{r}}}{\prod L_{-Y^{s}_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}_{j}Y^{s}_{r}}}{\prod G_{{}_{j}Y^{s}_{r}}}$$
$$\xrightarrow{\phi^{*}}\frac{\prod_{j}\mathcal{O}_{r}\times F_{{}_{j}Y^{-}_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}\times P_{{}_{j}Y^{-}_{r}}}\xrightarrow{\pi_{*}}\frac{\prod_{j}\mathcal{O}_{r}\times_{P_{{}_{j}Y^{-}_{r}}}F_{{}_{j}Y^{-}_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{{}_{j}Z_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}}\xrightarrow{\iota^{*}}\frac{\prod F_{{}^{-}_{j}Z_{r}}}{\prod L_{{}^{-}_{j}Z_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}\xrightarrow{\text{shift}}\frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}.$$

$$\Phi_{12}=\frac{\prod E_{{}^{t}_{p}X^{s}}}{\prod G_{{}^{t}_{p}X^{s}}}\xrightarrow{\phi^{*}}\frac{\prod {}^{t}\mathcal{O}^{s}\times F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}}\times P_{{}^{t}_{-}X^{s}}}\xrightarrow{\pi_{*}}\frac{\prod {}^{t}\mathcal{O}^{s}\times_{P_{{}^{t}_{-}X^{s}}}F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{{}^{t}V^{s}}}{\prod P_{{}^{t}V^{s}_{-}}}$$
$$\xrightarrow{\phi^{*}}\frac{\prod \mathcal{O}^{s}_{-}\times F_{-V^{s}}}{\prod P_{-Y^{s}_{=}}\times P_{=V^{s}_{-}}}\xrightarrow{\pi_{*}}\frac{\prod \mathcal{O}^{s}_{-}\times_{P_{=V^{s}_{-}}}F_{-V^{s}}}{\prod P_{-Y^{s}_{=}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{Y^{s}}}{\prod P_{-Y^{s}_{=}}}$$
$$\xrightarrow{\iota^{*}}\frac{\prod F_{Y^{s}_{-}}}{\prod P_{-Y^{s}_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{Y^{s}_{r}}}{\prod P_{-Y^{s}_{r}}}\xrightarrow{\iota^{*}}\frac{\prod F_{-Y^{s}_{r}}}{\prod L_{-Y^{s}_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}_{j}Y^{s}_{r}}}{\prod G_{{}_{j}Y^{s}_{r}}}$$
$$\xrightarrow{\phi^{*}}\frac{\prod_{j}\mathcal{O}_{r}\times F_{{}_{j}Y^{-}_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}\times P_{{}_{j}Y^{-}_{r}}}\xrightarrow{\pi_{*}}\frac{\prod_{j}\mathcal{O}_{r}\times_{P_{{}_{j}Y^{-}_{r}}}F_{{}_{j}Y^{-}_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{{}_{j}Z_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}}\xrightarrow{\iota^{*}}\frac{\prod F_{{}^{-}_{j}Z_{r}}}{\prod L_{{}^{-}_{j}Z_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}\xrightarrow{\text{shift}}\frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}$$

Here

$$P_{V^{s}_{-}}\supset\underline{\mathcal{O}}^{s}_{-}\overset{\text{def}}{=}U_{V^{s}_{-}}\cdot\prod_{r}\mathcal{O}^{s}_{r}=\prod_{r}\mathcal{O}^{s}_{r}\cdot U_{V^{s}_{-}}\in P_{-Y^{s}_{=}}\text{-sets-}P_{-V^{s}_{=}},$$
$$G_{V^{s}}\supset\mathcal{O}^{s}_{-}\overset{\text{def}}{=}\underline{\mathcal{O}}^{s}_{-}\cdot P_{=V^{s}_{-}}\in P_{-Y^{s}_{=}}\text{-sets-}P_{=V^{s}_{-}},\qquad \mathcal{O}^{s}_{-}=\underline{\mathcal{O}}^{s}_{-}\times_{P_{V^{s}_{-}}\cap P_{-V^{s}}}P_{=V^{s}_{-}}.$$

$$\Phi_{13}=\frac{\prod E_{{}^{t}_{p}X^{s}}}{\prod G_{{}^{t}_{p}X^{s}}}\xrightarrow{\phi^{*}}\frac{\prod {}^{t}\mathcal{O}^{s}\times F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}}\times P_{{}^{t}_{-}X^{s}}}\xrightarrow{\pi_{*}}\frac{\prod {}^{t}\mathcal{O}^{s}\times_{P_{{}^{t}_{-}X^{s}}}F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{{}^{t}V^{s}}}{\prod P_{{}^{t}V^{s}_{-}}}$$
$$\xrightarrow{\phi^{*}}\frac{\prod \mathcal{O}^{s}_{-}\times F_{-V^{s}}}{\prod P_{-Y^{s}_{=}}\times P_{=V^{s}_{-}}}\xrightarrow{\pi_{*}}\frac{\prod \mathcal{O}^{s}_{-}\times_{P_{=V^{s}_{-}}}F_{-V^{s}}}{\prod P_{-Y^{s}_{=}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{Y^{s}}}{\prod P_{-Y^{s}_{=}}}\xrightarrow{\iota^{*}}\frac{\prod F_{-Y^{s}_{=}}}{\prod L_{-Y^{s}_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}_{j}Y^{s}_{r}}}{\prod G_{{}_{j}Y^{s}_{r}}}$$
$$\xrightarrow{\phi^{*}}\frac{\prod_{j}\mathcal{O}_{r}\times F_{{}_{j}Y^{-}_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}\times P_{{}_{j}Y^{-}_{r}}}\xrightarrow{\pi_{*}}\frac{\prod_{j}\mathcal{O}_{r}\times_{P_{{}_{j}Y^{-}_{r}}}F_{{}_{j}Y^{-}_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{{}_{j}Z_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}}\xrightarrow{\iota^{*}}\frac{\prod F_{{}^{-}_{j}Z_{r}}}{\prod L_{{}^{-}_{j}Z_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}\xrightarrow{\text{shift}}\frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}$$

The isomorphism coass : $\Phi_{12}\to\Phi_{13}$ is essentially given by

$$\begin{tikzcd}
\frac{\prod E_{Y^{s}}}{\prod P_{-Y^{s}_{=}}} \arrow[r, "\iota^{*}"] \arrow[dr, "\iota^{*}"'] & \frac{\prod F_{Y^{s}_{-}}}{\prod P_{-Y^{s}_{r}}} \arrow[r, "\kappa_{!}"] \arrow[d, "\iota^{*}"] & \frac{\prod E_{Y^{s}_{r}}}{\prod P_{-Y^{s}_{r}}} \arrow[d, "\iota^{*}"] \\
& \frac{\prod F_{-Y^{s}_{=}}}{\prod L_{-Y^{s}_{r}}} \arrow[r, "\kappa_{!}"] \arrow[dr, "\kappa_{!}"'] & \frac{\prod F_{-Y^{s}_{r}}}{\prod L_{-Y^{s}_{r}}} \arrow[d, "\kappa_{!}"] \\
& & \frac{\prod E_{{}_{j}Y^{s}_{r}}}{\prod G_{{}_{j}Y^{s}_{r}}}
\end{tikzcd}$$

$$\Phi_{14}=\frac{\prod E_{{}^{t}_{p}X^{s}}}{\prod G_{{}^{t}_{p}X^{s}}}\xrightarrow{\phi^{*}}\frac{\prod \mathcal{O}^{s}\times F_{=X^{s}}}{\prod P_{-Y^{s}_{=}}\times P_{=X^{s}}}\xrightarrow{\pi_{*}}\frac{\prod \mathcal{O}^{s}\times_{P_{=X^{s}}}F_{=X^{s}}}{\prod P_{-Y^{s}_{=}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{Y^{s}}}{\prod P_{-Y^{s}_{=}}}\xrightarrow{\iota^{*}}\frac{\prod F_{-Y^{s}_{=}}}{\prod L_{-Y^{s}_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}_{j}Y^{s}_{r}}}{\prod G_{{}_{j}Y^{s}_{r}}}$$
$$\xrightarrow{\phi^{*}}\frac{\prod_{j}\mathcal{O}_{r}\times F_{{}_{j}Y^{-}_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}\times P_{{}_{j}Y^{-}_{r}}}\xrightarrow{\pi_{*}}\frac{\prod_{j}\mathcal{O}_{r}\times_{P_{{}_{j}Y^{-}_{r}}}F_{{}_{j}Y^{-}_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}}\xrightarrow{\alpha_{!}}\frac{\prod E_{{}_{j}Z_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}}\xrightarrow{\iota^{*}}\frac{\prod F_{{}^{-}_{j}Z_{r}}}{\prod L_{{}^{-}_{j}Z_{r}}}\xrightarrow{\kappa_{!}}\frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}\xrightarrow{\text{shift}}\frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}$$

Here

$$
\begin{gathered}
P_{-V^s} \supset {}^-\mathcal{O}^s \overset{\text{def}}{=} U_{-V^s} \cdot \prod_t {}^t\mathcal{O}^s = \prod_t {}^t\mathcal{O}^s \cdot U_{-V^s} \in P_{=V^s_-}\text{-sets-}P_{\substack{=\\-}X^s},\\
G_{Y^s} \supset \mathcal{O}^s \overset{\text{def}}{=} \underline{\mathcal{O}^s_-} \cdot {}^-\mathcal{O}^s = \mathcal{O}^s_- \cdot {}^-\mathcal{O}^s \in P_{-Y^s_=}\text{-sets-}P_{\substack{=\\-}X^s},\\
\mathcal{O}^s = \mathcal{O}^s_- \times_{P_{=V^s_-}} {}^-\mathcal{O}^s = \underline{\mathcal{O}^s_-} \times_{P_{V^s_-} \cap P_{-V^s}} {}^-\mathcal{O}^s.
\end{gathered}
$$

The associativity isomorphism assoc : $\Phi_{13} \to \Phi_{14}$ (as well as assoc : $\Phi_{23} \to \Phi_{24}$ and the associativity part of the isomorphism assoc : $\Phi_{33} \to \Phi_{24}$) is essentially given by

$$
\begin{tikzcd}
\frac{\prod E_{\substack{t\\p}X^s}}{\prod G_{\substack{t\\p}X^s}} \arrow[r, "\phi^*"] \arrow[d, "\phi^*"'] & \frac{\prod {}^t\mathcal{O}^s \times F_{t_-X^s}}{\prod P_{tV^s_-} \times P_{t_-X^s}} \arrow[r, "\pi_*"] \arrow[d, "\phi^*"] & \frac{\prod {}^t\mathcal{O}^s \times_{P_{t_-X^s}} F_{t_-X^s}}{\prod P_{tV^s_-}} \arrow[r, "\alpha_!"] \arrow[d, "\phi^*"] & \frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \arrow[d, "\phi^*"] \\
\frac{\prod \mathcal{O}^s \times F_{\substack{=\\-}X^s}}{\prod P_{-Y^s_=} \times P_{\substack{=\\-}X^s}} \arrow[r, "\pi^*"] \arrow[dr, Rightarrow, "\mathrm{Id}"'] & \frac{\prod \mathcal{O}^s_- \times {}^-\mathcal{O}^s \times F_{\substack{=\\-}X^s}}{\prod P_{-Y^s_=} \times P_{=V^s_-} \times P_{\substack{=\\-}X^s}} \arrow[r, "\pi_*"] \arrow[d, "\pi_*"] & \frac{\prod \mathcal{O}^s_- \times {}^-\mathcal{O}^s \times_{P_{\substack{=\\-}X^s}} F_{\substack{=\\-}X^s}}{\prod P_{-Y^s_=} \times P_{=V^s_-}} \arrow[r, "1\times\alpha_!"] \arrow[d, "\pi_*"] & \frac{\prod \mathcal{O}^s_- \times F_{-V^s}}{\prod P_{-Y^s_=} \times P_{=V^s_-}} \arrow[d, "\pi_*"] \\
& \frac{\prod \mathcal{O}^s \times F_{\substack{=\\-}X^s}}{\prod P_{-Y^s_=} \times P_{\substack{=\\-}X^s}} \arrow[r, "\pi_*"] & \frac{\prod \mathcal{O}^s \times_{P_{\substack{=\\-}X^s}} F_{\substack{=\\-}X^s}}{\prod P_{-Y^s_=}} \arrow[r, "\beta_!"] \arrow[dr, "\alpha_!"'] & \frac{\prod \mathcal{O}^s_- \times_{P_{=V^s_-}} F_{-V^s}}{\prod P_{-Y^s_=}} \arrow[d, "\alpha_!"] \\
& & & \frac{\prod E_{Y^s}}{\prod P_{-Y^s_=}}
\end{tikzcd}
$$

$$
\begin{aligned}
\Phi_{21} = \frac{\prod E_{\substack{t\\p}X^s}}{\prod G_{\substack{t\\p}X^s}} &\xrightarrow{\phi^*} \frac{\prod {}^t\mathcal{O}^s \times F_{t_-X^s}}{\prod P_{tV^s_-} \times P_{t_-X^s}} \xrightarrow{\pi_*} \frac{\prod {}^t\mathcal{O}^s \times_{P_{t_-X^s}} F_{t_-X^s}}{\prod P_{tV^s_-}} \xrightarrow{\alpha_!} \frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \xrightarrow{\iota^*} \frac{\prod F_{tV^s_-}}{\prod L_{tV^s_-}} \xrightarrow{\kappa_!} \frac{\prod E_{tV^s_r}}{\prod G_{tV^s_r}} \\
&\xrightarrow{\phi^*} \frac{\prod \mathcal{O}^s_r \times F_{-V^s_r}}{\prod P_{-Y^s_r} \times P_{-V^s_r}} \xrightarrow{\pi_*} \frac{\prod \mathcal{O}^s_r \times_{P_{-V^s_r}} F_{-V^s_r}}{\prod P_{-Y^s_r}} \xrightarrow{\alpha_!} \frac{\prod E_{Y^s_r}}{\prod P_{-Y^s_r}} \\
&\xrightarrow{\phi^*} \frac{\prod {}_-\mathcal{O}_r \times F_{Y_r^-}}{\prod P_{\substack{-\\=}Z_r} \times P_{-Y_r^=}} \xrightarrow{\pi_*} \frac{\prod {}_-\mathcal{O}_r \times_{P_{-Y_r^=}} F_{Y_r^-}}{\prod P_{\substack{-\\=}Z_r}} \xrightarrow{\alpha_!} \frac{\prod E_{Z_r}}{\prod P_{\substack{-\\=}Z_r}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{-Z_r}}{\prod P_{\substack{-\\j}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{jZ_r}}{\prod P_{\substack{-\\j}Z_r}} \xrightarrow{\iota^*} \frac{\prod F_{\substack{-\\j}Z_r}}{\prod L_{\substack{-\\j}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{\substack{i\\j}Z_r}}{\prod G_{\substack{i\\j}Z_r}} \xrightarrow{\text{shift}} \frac{\prod E_{\substack{i\\j}Z_r}}{\prod G_{\substack{i\\j}Z_r}}.
\end{aligned}
$$

Here

$$
\begin{gathered}
P_{-Y_r} \supset {}_-\underline{\mathcal{O}}_r \overset{\text{def}}{=} U_{-Y_r} \cdot \prod_j {}_j\mathcal{O}_r = \prod_j {}_j\mathcal{O}_r \cdot U_{-Y_r} \in P_{\substack{-\\=}Z_r}\text{-sets-}P_{=Y_r^-},\\
G_{Y_r} \supset {}_-\mathcal{O}_r \overset{\text{def}}{=} {}_-\underline{\mathcal{O}}_r \cdot P_{-Y_r^=} \in P_{\substack{-\\=}Z_r}\text{-sets-}P_{-Y_r^=}, \qquad {}_-\mathcal{O}_r = {}_-\underline{\mathcal{O}}_r \times_{P_{-Y_r} \cap P_{Y_r^-}} P_{-Y_r^=}.
\end{gathered}
$$

$$
\begin{aligned}
\Phi_{22} = \frac{\prod E_{\substack{t\\p}X^s}}{\prod G_{\substack{t\\p}X^s}} &\xrightarrow{\phi^*} \frac{\prod {}^t\mathcal{O}^s \times F_{t_-X^s}}{\prod P_{tV^s_-} \times P_{t_-X^s}} \xrightarrow{\pi_*} \frac{\prod {}^t\mathcal{O}^s \times_{P_{t_-X^s}} F_{t_-X^s}}{\prod P_{tV^s_-}} \xrightarrow{\alpha_!} \frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \\
&\xrightarrow{\phi^*} \frac{\prod \mathcal{O}^s_- \times F_{-V^s}}{\prod P_{-Y^s_=} \times P_{=V^s_-}} \xrightarrow{\pi_*} \frac{\prod \mathcal{O}^s_- \times_{P_{=V^s_-}} F_{-V^s}}{\prod P_{-Y^s_=}} \xrightarrow{\alpha_!} \frac{\prod E_{Y^s}}{\prod P_{-Y^s_=}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{V^s_-}}{\prod P_{-Y^s_r}} \xrightarrow{\kappa_!} \frac{\prod E_{Y^s_r}}{\prod P_{-Y^s_r}} \xrightarrow{\phi^*} \frac{\prod {}_-\mathcal{O}_r \times F_{Y_r^-}}{\prod P_{\substack{-\\=}Z_r} \times P_{-Y_r^=}} \xrightarrow{\pi_*} \frac{\prod {}_-\mathcal{O}_r \times_{P_{-Y_r^=}} F_{Y_r^-}}{\prod P_{\substack{-\\=}Z_r}} \xrightarrow{\alpha_!} \frac{\prod E_{Z_r}}{\prod P_{\substack{-\\=}Z_r}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{-Z_r}}{\prod P_{\substack{-\\j}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{jZ_r}}{\prod P_{\substack{-\\j}Z_r}} \xrightarrow{\iota^*} \frac{\prod F_{\substack{-\\j}Z_r}}{\prod L_{\substack{-\\j}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{\substack{i\\j}Z_r}}{\prod G_{\substack{i\\j}Z_r}} \xrightarrow{\text{shift}} \frac{\prod E_{\substack{i\\j}Z_r}}{\prod G_{\substack{i\\j}Z_r}}.
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{23} = \frac{\prod E_{{}^{t}_{p}X^{s}}}{\prod G_{{}^{t}_{p}X^{s}}} &\xrightarrow{\phi^*} \frac{\prod {}^{t}\mathcal{O}^{s} \times F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}} \times P_{{}^{t}_{-}X^{s}}} \xrightarrow{\pi_*} \frac{\prod {}^{t}\mathcal{O}^{s} \times_{P_{{}^{t}_{-}X^{s}}} F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}}} \xrightarrow{\alpha_!} \frac{\prod E_{{}^{t}V^{s}}}{\prod P_{{}^{t}V^{s}_{-}}} \\
&\xrightarrow{\phi^*} \frac{\prod \mathcal{O}^{s}_{-} \times F_{{}_{-}V^{s}}}{\prod P_{{}_{-}Y^{s}_{\equiv}} \times P_{{}^{=}V^{s}_{-}}} \xrightarrow{\pi_*} \frac{\prod \mathcal{O}^{s}_{-} \times_{P_{{}^{=}V^{s}_{-}}} F_{{}_{-}V^{s}}}{\prod P_{{}_{-}Y^{s}_{\equiv}}} \xrightarrow{\alpha_!} \frac{\prod E_{Y^{s}}}{\prod P_{{}_{-}Y^{s}_{\equiv}}} \\
&\xrightarrow{\phi^*} \frac{{}_{-}\mathcal{O}_{=} \times F_{Y^{-}}}{P_{{}^{-}_{=}Z_{\equiv}} \times P_{{}_{-}Y_{\equiv}}} \xrightarrow{\pi_*} \frac{{}_{-}\mathcal{O}_{=} \times_{P_{{}_{-}Y_{\equiv}}} F_{Y^{-}}}{P_{{}^{-}_{=}Z_{\equiv}}} \xrightarrow{\alpha_!} \frac{E_{Z}}{P_{{}^{-}_{=}Z_{\equiv}}} \\
&\xrightarrow{\iota^*} \frac{F_{{}_{-}Z_{=}}}{\prod P_{{}^{-}_{j}Z_{r}}} \xrightarrow{\kappa_!} \frac{\prod E_{{}_{j}Z_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}} \xrightarrow{\iota^*} \frac{\prod F_{{}^{-}_{j}Z_{r}}}{\prod L_{{}^{-}_{j}Z_{r}}} \xrightarrow{\kappa_!} \frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}} \xrightarrow{\text{shift}} \frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}
\end{aligned}
$$

Here

$$
P_{{}_{-}Y_{=}} \supset {}_{-}\underline{\mathcal{O}}_{=} \overset{\text{def}}{=} U_{{}_{-}Y_{=}} \cdot \prod_{j,r} {}_{j}\mathcal{O}_{r} = \prod_{j,r} {}_{j}\mathcal{O}_{r} \cdot U_{{}_{-}Y_{=}} \in P_{{}^{-}_{=}Z_{\equiv}}\text{-sets-}P_{{}_{=}Y^{-}_{\equiv}},
$$

$$
G_{Y} \supset {}_{-}\mathcal{O}_{=} \overset{\text{def}}{=} {}_{-}\underline{\mathcal{O}}_{=} \cdot P_{{}_{-}Y_{\equiv}} \in P_{{}^{-}_{=}Z_{\equiv}}\text{-sets-}P_{{}_{-}Y_{\equiv}}, \qquad {}_{-}\mathcal{O}_{=} = {}_{-}\underline{\mathcal{O}}_{=} \times_{P_{{}_{-}Y_{=}} \cap P_{Y^{-}}} P_{{}_{-}Y_{\equiv}}.
$$

$$
\begin{aligned}
\Phi_{24} = \frac{\prod E_{{}^{t}_{p}X^{s}}}{\prod G_{{}^{t}_{p}X^{s}}} &\xrightarrow{\phi^*} \frac{\prod \mathcal{O}^{s} \times F_{{}^{=}_{-}X^{s}}}{\prod P_{{}_{-}Y^{s}_{\equiv}} \times P_{{}^{=}_{-}X^{s}}} \xrightarrow{\pi_*} \frac{\prod \mathcal{O}^{s} \times_{P_{{}^{=}_{-}X^{s}}} F_{{}^{=}_{-}X^{s}}}{\prod P_{{}_{-}Y^{s}_{\equiv}}} \xrightarrow{\alpha_!} \frac{\prod E_{Y^{s}}}{\prod P_{{}_{-}Y^{s}_{\equiv}}} \\
&\xrightarrow{\phi^*} \frac{{}_{-}\mathcal{O}_{=} \times F_{Y^{-}}}{P_{{}^{-}_{=}Z_{\equiv}} \times P_{{}_{-}Y_{\equiv}}} \xrightarrow{\pi_*} \frac{{}_{-}\mathcal{O}_{=} \times_{P_{{}_{-}Y_{\equiv}}} F_{Y^{-}}}{P_{{}^{-}_{=}Z_{\equiv}}} \xrightarrow{\alpha_!} \frac{E_{Z}}{P_{{}^{-}_{=}Z_{\equiv}}} \\
&\xrightarrow{\iota^*} \frac{F_{{}_{-}Z_{=}}}{\prod P_{{}^{-}_{j}Z_{r}}} \xrightarrow{\kappa_!} \frac{\prod E_{{}_{j}Z_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}} \xrightarrow{\iota^*} \frac{\prod F_{{}^{-}_{j}Z_{r}}}{\prod L_{{}^{-}_{j}Z_{r}}} \xrightarrow{\kappa_!} \frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}} \xrightarrow{\text{shift}} \frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{31} = \frac{\prod E_{{}^{t}_{p}X^{s}}}{\prod G_{{}^{t}_{p}X^{s}}} &\xrightarrow{\phi^*} \frac{\prod {}^{t}\mathcal{O}^{s} \times F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}} \times P_{{}^{t}_{-}X^{s}}} \xrightarrow{\pi_*} \frac{\prod {}^{t}\mathcal{O}^{s} \times_{P_{{}^{t}_{-}X^{s}}} F_{{}^{t}_{-}X^{s}}}{\prod P_{{}^{t}V^{s}_{-}}} \xrightarrow{\alpha_!} \frac{\prod E_{{}^{t}V^{s}}}{\prod P_{{}^{t}V^{s}_{-}}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{{}^{t}V^{s}_{-}}}{\prod L_{{}^{t}V^{s}_{-}}} \xrightarrow{\kappa_!} \frac{\prod E_{{}^{t}V^{s}_{r}}}{\prod G_{{}^{t}V^{s}_{r}}} \xrightarrow{\phi^*} \frac{\prod \mathcal{O}_{r} \times F_{{}_{-}V^{=}_{r}}}{\prod P_{{}^{-}_{=}Z_{r}} \times P_{{}_{-}V^{=}_{r}}} \xrightarrow{\pi_*} \frac{\prod \mathcal{O}_{r} \times_{P_{{}_{-}V^{=}_{r}}} F_{{}_{-}V^{=}_{r}}}{\prod P_{{}^{-}_{=}Z_{r}}} \xrightarrow{\alpha_!} \frac{\prod E_{Z_{r}}}{\prod P_{{}^{-}_{=}Z_{r}}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{{}_{-}Z_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}} \xrightarrow{\kappa_!} \frac{\prod E_{{}_{j}Z_{r}}}{\prod P_{{}^{-}_{j}Z_{r}}} \xrightarrow{\iota^*} \frac{\prod F_{{}^{-}_{j}Z_{r}}}{\prod L_{{}^{-}_{j}Z_{r}}} \xrightarrow{\kappa_!} \frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}} \xrightarrow{\text{shift}} \frac{\prod E_{{}^{i}_{j}Z_{r}}}{\prod G_{{}^{i}_{j}Z_{r}}}.
\end{aligned}
$$

Here

$$
P_{V^{-}_{r}} \supset \mathcal{O}^{-}_{r} \overset{\text{def}}{=} U_{V^{-}_{r}} \cdot \prod_{s} \mathcal{O}^{s}_{r} = \prod_{s} \mathcal{O}^{s}_{r} \cdot U_{V^{-}_{r}} \in P_{{}_{-}Y^{=}_{r}}\text{-sets-}P_{{}_{-}V^{=}_{r}},
$$

$$
\begin{gathered}
G_{Y_{r}} \supset \mathcal{O}_{r} \overset{\text{def}}{=} {}_{-}\underline{\mathcal{O}}_{r} \cdot \mathcal{O}^{-}_{r} = {}_{-}\mathcal{O}_{r} \cdot \mathcal{O}^{-}_{r} \in P_{{}^{-}_{=}Z_{r}}\text{-sets-}P_{{}_{-}V^{=}_{r}}, \\
\mathcal{O}_{r} = {}_{-}\mathcal{O}_{r} \times_{P_{{}_{-}Y^{=}_{r}}} \mathcal{O}^{-}_{r} = {}_{-}\underline{\mathcal{O}}_{r} \times_{P_{{}_{-}Y_{r}} \cap P_{Y^{-}_{r}}} \mathcal{O}^{-}_{r}.
\end{gathered}
$$

The associativity isomorphism $\operatorname{assoc} : \Phi_{21} \to \Phi_{31}$ is essentially the following.

$$
\begin{tikzcd}
\frac{\prod E_{tV_r^s}}{\prod G_{tV_r^s}} \arrow[r, "\phi^*"] \arrow[d, "\phi^*"'] & \frac{\prod \mathcal{O}_r^s \times F_{-V_r^s}}{\prod P_{-Y_r^s} \times P_{-V_r^s}} \arrow[r, "\pi_*"] \arrow[d, "\phi^*"] & \frac{\prod \mathcal{O}_r^s \times_{P_{-V_r^s}} F_{-V_r^s}}{\prod P_{-Y_r^s}} \arrow[r, "\alpha_!"] \arrow[d, "\phi^*"] & \frac{\prod E_{Y_r^s}}{\prod P_{-Y_r^s}} \arrow[d, "\phi^*"] \\
\frac{\prod \mathcal{O}_r \times F_{-V_r^=}}{\prod P_{\substack{-\\=}Z_r} \times P_{-V_r^=}} \arrow[r, "\pi^*"] \arrow[dr, Rightarrow, "\mathrm{Id}"'] & \frac{\prod {}_-\mathcal{O}_r \times \mathcal{O}_r^- \times F_{-V_r^=}}{\prod P_{\substack{-\\=}Z_r} \times P_{-Y_r^=} \times P_{-V_r^=}} \arrow[r, "\pi_*"] \arrow[d, "\pi_*"] & \frac{\prod {}_-\mathcal{O}_r \times \mathcal{O}_r^- \times_{P_{-V_r^=}} F_{-V_r^=}}{\prod P_{\substack{-\\=}Z_r} \times P_{-Y_r^=}} \arrow[r, "1\times\alpha_!"] \arrow[d, "\pi_*"] & \frac{\prod {}_-\mathcal{O}_r \times F_{Y_r^-}}{\prod P_{\substack{-\\=}Z_r} \times P_{-Y_r^=}} \arrow[d, "\pi_*"] \\
 & \frac{\prod \mathcal{O}_r \times F_{-V_r^=}}{\prod P_{\substack{-\\=}Z_r} \times P_{-V_r^=}} \arrow[r, "\pi_*"] & \frac{\prod \mathcal{O}_r \times_{P_{-V_r^=}} F_{-V_r^=}}{\prod P_{\substack{-\\=}Z_r}} \arrow[r, "\beta_!"] \arrow[dr, "\alpha_!"'] & \frac{\prod {}_-\mathcal{O}_r \times_{P_{-Y_r^=}} F_{Y_r^-}}{\prod P_{\substack{-\\=}Z_r}} \arrow[d, "\alpha_!"] \\
 & & & \frac{\prod E_{Z_r}}{\prod P_{\substack{-\\=}Z_r}}
\end{tikzcd}
$$

$$
\begin{aligned}
\Phi_{32} = \frac{\prod E_{\substack{t\\p}X^s}}{\prod G_{\substack{t\\p}X^s}} &\xrightarrow{\phi^*} \frac{\prod {}^t\mathcal{O}^s \times F_{\substack{t\\-}X^s}}{\prod P_{tV^s_-} \times P_{\substack{t\\-}X^s}} \xrightarrow{\pi_*} \frac{\prod {}^t\mathcal{O}^s \times_{P_{\substack{t\\-}X^s}} F_{\substack{t\\-}X^s}}{\prod P_{tV^s_-}} \xrightarrow{\alpha_!} \frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \\
&\xrightarrow{\phi^*} \frac{\mathcal{O}_- \times F_{-V^=}}{P_{\substack{-\\=}Z_\equiv} \times P_{=V^{\equiv}_-}} \xrightarrow{\pi_*} \frac{\mathcal{O}_- \times_{P_{=V^{\equiv}_-}} F_{-V^=}}{P_{\substack{-\\=}Z_\equiv}} \xrightarrow{\alpha_!} \frac{E_V}{P_{\substack{-\\=}Z_\equiv}} \xrightarrow{\iota^*} \frac{F_{Z_-}}{\prod P_{\substack{-\\=}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{Z_r}}{\prod P_{\substack{-\\=}Z_r}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{-Z_r}}{\prod P_{\substack{-\\j}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{jZ_r}}{\prod P_{\substack{-\\j}Z_r}} \xrightarrow{\iota^*} \frac{\prod F_{\substack{-\\j}Z_r}}{\prod L_{\substack{-\\j}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{\substack{i\\j}Z_r}}{\prod G_{\substack{i\\j}Z_r}} \xrightarrow{\text{shift}} \frac{\prod E_{\substack{i\\j}Z_r}}{\prod G_{\substack{i\\j}Z_r}}.
\end{aligned}
$$

Here

$$
P_{V_-} \supset \underline{\mathcal{O}}_- \overset{\text{def}}{=} U_{V_-} \cdot \prod_r \mathcal{O}_r = \prod_r \mathcal{O}_r \cdot U_{V_-} \in P_{\substack{-\\=}Z_\equiv}\text{-sets-}P_{-V^{\equiv}},
$$

$$
G_V \supset \mathcal{O}_- \overset{\text{def}}{=} \underline{\mathcal{O}}_- \cdot P_{=V^{\equiv}_-} \in P_{\substack{-\\=}Z_\equiv}\text{-sets-}P_{=V^{\equiv}_-}, \qquad \mathcal{O}_- = \underline{\mathcal{O}}_- \times_{P_{V_-} \cap P_{-V^=}} P_{=V^{\equiv}_-}.
$$

$$
\begin{aligned}
\Phi_{33} = \frac{\prod E_{\substack{t\\p}X^s}}{\prod G_{\substack{t\\p}X^s}} &\xrightarrow{\phi^*} \frac{\prod {}^t\mathcal{O}^s \times F_{\substack{t\\-}X^s}}{\prod P_{tV^s_-} \times P_{\substack{t\\-}X^s}} \xrightarrow{\pi_*} \frac{\prod {}^t\mathcal{O}^s \times_{P_{\substack{t\\-}X^s}} F_{\substack{t\\-}X^s}}{\prod P_{tV^s_-}} \xrightarrow{\alpha_!} \frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \\
&\xrightarrow{\phi^*} \frac{\prod \mathcal{O}^s_- \times F_{-V^s}}{\prod P_{-Y^{s}_{=}} \times P_{=V^s_-}} \xrightarrow{\pi_*} \frac{\prod \mathcal{O}^s_- \times_{P_{=V^s_-}} F_{-V^s}}{\prod P_{-Y^{s}_{=}}} \xrightarrow{\alpha_!} \frac{\prod E_{Y^s}}{\prod P_{-Y^{s}_{=}}} \\
&\xrightarrow{\phi^*} \frac{{}_-\mathcal{O}_= \times F_{Y^-}}{P_{\substack{-\\=}Z_\equiv} \times P_{-Y^{\underline{\equiv}}}} \xrightarrow{\pi_*} \frac{{}_-\mathcal{O}_= \times_{P_{-Y^{\underline{\equiv}}}} F_{Y^-}}{P_{\substack{-\\=}Z_\equiv}} \xrightarrow{\alpha_!} \frac{E_V}{P_{\substack{-\\=}Z_\equiv}} \xrightarrow{\iota^*} \frac{F_{Z_-}}{\prod P_{\substack{-\\=}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{Z_r}}{\prod P_{\substack{-\\=}Z_r}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{-Z_r}}{\prod P_{\substack{-\\j}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{jZ_r}}{\prod P_{\substack{-\\j}Z_r}} \xrightarrow{\iota^*} \frac{\prod F_{\substack{-\\j}Z_r}}{\prod L_{\substack{-\\j}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{\substack{i\\j}Z_r}}{\prod G_{\substack{i\\j}Z_r}} \xrightarrow{\text{shift}} \frac{\prod E_{\substack{i\\j}Z_r}}{\prod G_{\substack{i\\j}Z_r}}.
\end{aligned}
$$

Denote

$$
P_{V^-} \supset \mathcal{O}^=_- \overset{\text{def}}{=} U_{V^-} \cdot \prod_s \mathcal{O}^s_- = \prod_s \mathcal{O}^s_- \cdot U_{V^-} \in P_{-Y^{\underline{\equiv}}}\text{-sets-}P_{=V^{\equiv}_-}.
$$

3.2.1. Lemma. *The following equations hold*

$$
\mathcal{O}_- = {}_-\underline{\mathcal{O}}_= \cdot \mathcal{O}^=_- = {}_-\mathcal{O}_= \cdot \mathcal{O}^=_- \subset G_V, \qquad \mathcal{O}_- = {}_-\mathcal{O}_= \times_{P_{-Y^{\underline{\equiv}}}} \mathcal{O}^=_-.
$$

Proof. Clearly, ${}_-\underline{\mathcal{O}}_= \cdot \mathcal{O}^=_- = {}_-\mathcal{O}_= \cdot \mathcal{O}^=_-$. This is the image of the multiplication map

$$mult : {}_-\underline{\mathcal{O}}_= \cdot P_{-Y_{\equiv}} \times_{P_{-Y_{\equiv}}} \mathcal{O}^=_- \to G_Y. \tag{3.2.1}$$

Let us prove that this map is injective.

Suppose that two elements are mapped to the same element $p_1p_2p_3 = \bar{p}_1\bar{p}_2\bar{p}_3$ in G_Y, where $p_1, \bar{p}_1 \in {}_-\mathcal{O}_= \subset P_{-Y_=}$, $p_2, \bar{p}_2 \in P_{-Y_{\equiv}} \subset P_{Y^=_-}$, $p_3, \bar{p}_3 \in \mathcal{O}^=_- \subset P_{Y^=_-}$. Hence, $p_2p_3, \bar{p}_2\bar{p}_3 \in P_{Y^=_-}$. Then the element

$$r := \bar{p}_1^{-1}p_1 = \bar{p}_2\bar{p}_3(p_2p_3)^{-1}$$

belongs simultaneously to $P_{-Y_=}$ and to $P_{Y^=_-}$.

Notice that

$$P_{-Y_=} \cap P_{Y^=_-} \subset P_{-Y_{\equiv}}.$$

Indeed, for any $p \in P_{-Y_=}$ we have for all possible ι, σ, ρ

$$p({}_\iota Y^\sigma_\rho) \subset \oplus_{j,s,r}\{{}_jY^s_r \mid r < \rho \text{ or } (r = \rho, j \leqslant \iota)\},$$

and for any $p \in P_{Y^=_-}$ we have

$$p({}_\iota Y^\sigma_\rho) \subset \oplus_{j,s,r}\{{}_jY^s_r \mid s < \sigma \text{ or } (s = \sigma, r \leqslant \rho)\}.$$

Thus, for any $p \in P_{-Y_=} \cap P_{Y^=_-}$ we have

$$p({}_\iota Y^\sigma_\rho) \subset \oplus_{j,s,r}\{{}_jY^s_r \mid s < \sigma \text{ or } (s = \sigma, r < \rho) \text{ or } (s = \sigma, r = \rho, j \leqslant \iota)\},$$

which means that $p \in P_{-Y_{\equiv}}$.

Since $\bar{p}_2\bar{p}_3 = rp_2p_3$, $\bar{p}_1 = p_1r^{-1}$ and $r \in P_{-Y_{\equiv}}$, we get in ${}_-\underline{\mathcal{O}}_= \cdot P_{-Y_{\equiv}} \times_{P_{-Y_{\equiv}}} \mathcal{O}^=_-$

$$(\bar{p}_1\bar{p}_2, \bar{p}_3) = (\bar{p}_1, \bar{p}_2\bar{p}_3) = (p_1r^{-1}, rp_2p_3) = (p_1, p_2p_3) = (p_1p_2, p_3).$$

Therefore, map (3.2.1) is injective.

Now we wish to prove that the image of (3.2.1) coincides with $\mathcal{O}_-$, that is,

$${}_-\underline{\mathcal{O}}_= \cdot \mathcal{O}^=_- = \underline{\mathcal{O}}_- \cdot P_{=V^{\equiv}_-}.$$

Recall the definitions of these subsets:

$${}_-\underline{\mathcal{O}}_= = \prod_{j,r} {}_j\mathcal{O}_r \cdot U_{-Y_=}, \qquad \mathcal{O}^=_- = \prod_s \mathcal{O}^s_- \cdot U_{V^-} = \prod_s U_{V^s_-} \cdot \prod_{s,r} \mathcal{O}^s_r \cdot \prod_s P_{=V^s_-} \cdot U_{V^-},$$

$$\begin{aligned}\underline{\mathcal{O}}_- &= \prod_r \mathcal{O}_r \cdot U_{V_-} = \prod_r {}_-\underline{\mathcal{O}}_r \cdot \mathcal{O}^-_r \cdot U_{V_-} = \prod_r {}_-\underline{\mathcal{O}}_r \cdot U_{V_-} \cdot \prod_r \mathcal{O}^-_r \\ &= \prod_{j,r} {}_j\mathcal{O}_r \cdot \prod_r U_{-Y_r} \cdot U_{V_-} \cdot \prod_{s,r} \mathcal{O}^s_r \cdot \prod_r U_{V^-_r}.\end{aligned}$$

We have to check that

$$U_{-Y_=} \cdot \prod_s U_{V^s_-} \cdot \prod_{s,r} \mathcal{O}^s_r \cdot \prod_s P_{=V^s_-} \cdot U_{V^-} = \prod_r U_{-Y_r} \cdot U_{V_-} \cdot \prod_{s,r} \mathcal{O}^s_r \cdot \prod_r U_{V^-_r} \cdot P_{=V^{\equiv}_-}.$$

This equation is implied by the following facts

$$\prod_s U_{Y^s_-} \subset U_{-Y_=}, \qquad U_{-Y_=} = \prod_r U_{-Y_r} \cdot U_{Y_-}, \qquad \prod_r U_{V^-_r} \subset U_{=V^{\equiv}_-}, \qquad \prod_s P_{=V^s_-} \cdot U_{V^-} = P_{=V^{\equiv}_-},$$

which are easy to verify. □

Now we may describe the associativity isomorphism assoc : $\Phi_{33} \to \Phi_{32}$ as well as the associativity part of the isomorphism assoc : $\Phi_{33} \to \Phi_{42}$ as the following.

$$\begin{tikzcd}
\frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \arrow[r, "\phi^*"] \arrow[d, "\phi^*"'] & \frac{\prod \mathcal{O}^s_- \times F_{-V^s}}{\prod P_{-Y^s_{\equiv}} \times P_{=V^s_-}} \arrow[r, "\pi_*"] \arrow[d, "\phi^*"] & \frac{\prod \mathcal{O}^s_- \times_{P_{=V^s_-}} F_{-V^s}}{\prod P_{-Y^s_{\equiv}}} \arrow[r, "\alpha_!"] \arrow[d, "\phi^*"] & \frac{\prod E_{Y^s}}{\prod P_{-Y^s_{\equiv}}} \arrow[d, "\phi^*"] \\
\frac{\mathcal{O}_- \times F_{-V^=}}{P_{\underset{=}{-}Z_{\equiv}} \times P_{=V^{\equiv}_-}} \arrow[r, "\pi^*"] \arrow[dr, equal, "\mathrm{Id}"'] & \frac{{}_-\mathcal{O}_= \times \mathcal{O}^=_- \times F_{-V^=}}{P_{\underset{=}{-}Z_{\equiv}} \times P_{-Y^{\equiv}_{\equiv}} \times P_{=V^{\equiv}_-}} \arrow[r, "\pi_*"] \arrow[d, "\pi_*"] & \frac{{}_-\mathcal{O}_= \times \mathcal{O}^=_- \times_{P_{=V^{\equiv}_-}} F_{-V^=}}{P_{\underset{=}{-}Z_{\equiv}} \times P_{-Y^{\equiv}_{\equiv}}} \arrow[r, "1\times\alpha_!"] \arrow[d, "\pi_*"] & \frac{{}_-\mathcal{O}_= \times F_{Y^-}}{P_{\underset{=}{-}Z_{\equiv}} \times P_{-Y^{\equiv}_{\equiv}}} \arrow[d, "\pi_*"] \\
 & \frac{\mathcal{O}_- \times F_{-V^=}}{P_{\underset{=}{-}Z_{\equiv}} \times P_{=V^{\equiv}_-}} \arrow[r, "\pi_*"] & \frac{\mathcal{O}_- \times_{P_{=V^{\equiv}_-}} F_{-V^=}}{P_{\underset{=}{-}Z_{\equiv}}} \arrow[r, "\beta_!"] \arrow[dr, "\alpha_!"'] & \frac{{}_-\mathcal{O}_= \times_{P_{-Y^{\equiv}_{\equiv}}} F_{Y^-}}{P_{\underset{=}{-}Z_{\equiv}}} \arrow[d, "\alpha_!"] \\
 & & & \frac{E_V}{P_{\underset{=}{-}Z_{\equiv}}}
\end{tikzcd}$$

The isomorphism coass : $\Phi_{33} \to \Phi_{23}$ is essentially given by

$$\begin{tikzcd}
\frac{E_V}{P_{\underset{=}{-}Z_{\equiv}}} \arrow[r, "\iota^*"] \arrow[dr, "\iota^*"'] & \frac{F_{Z_-}}{\prod P_{\underset{=}{-}Z_r}} \arrow[r, "\kappa_!"] \arrow[d, "\iota^*"] & \frac{\prod E_{Z_r}}{\prod P_{\underset{=}{-}Z_r}} \arrow[d, "\iota^*"] \\
 & \frac{F_{-Z_=}}{\prod P_{\underset{j}{-}Z_r}} \arrow[r, "\kappa_!"] \arrow[dr, "\kappa_!"'] & \frac{\prod F_{-Z_r}}{\prod P_{\underset{j}{-}Z_r}} \arrow[d, "\kappa_!"] \\
 & & \frac{\prod E_{jZ_r}}{\prod P_{\underset{j}{-}Z_r}}
\end{tikzcd}$$

$$\begin{aligned}
\Phi_{41} = \frac{\prod E_{\underset{p}{t}X^s}}{\prod G_{\underset{p}{t}X^s}} &\xrightarrow{\phi^*} \frac{\prod {}^t\mathcal{O}^s \times F_{\underset{-}{t}X^s}}{\prod P_{tV^s_-} \times P_{\underset{-}{t}X^s}} \xrightarrow{\pi_*} \frac{\prod {}^t\mathcal{O}^s \times_{P_{\underset{-}{t}X^s}} F_{\underset{-}{t}X^s}}{\prod P_{tV^s_-}} \xrightarrow{\alpha_!} \frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{tV^s_-}}{\prod L_{tV^s_-}} \xrightarrow{\kappa_!} \frac{\prod E_{tV^s_r}}{\prod G_{tV^s_r}} \xrightarrow{\phi^*} \frac{\prod \mathcal{O}_r \times F_{-V^=_r}}{\prod P_{\underset{=}{-}Z_r} \times P_{-V^=_r}} \xrightarrow{\pi_*} \frac{\prod \mathcal{O}_r \times_{P_{-V^=_r}} F_{-V^=_r}}{\prod P_{\underset{=}{-}Z_r}} \xrightarrow{\alpha_!} \frac{\prod E_{Z_r}}{\prod P_{\underset{=}{-}Z_r}} \\
&\xrightarrow{\iota^*} \frac{\prod F_{\underset{=}{-}Z_r}}{\prod L_{\underset{=}{-}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{\underset{j}{i}Z_r}}{\prod G_{\underset{j}{i}Z_r}} \xrightarrow{\text{shift}} \frac{\prod E_{\underset{j}{i}Z_r}}{\prod G_{\underset{j}{i}Z_r}}.
\end{aligned}$$

$$\begin{aligned}
\Phi_{42} = \frac{\prod E_{\underset{p}{t}X^s}}{\prod G_{\underset{p}{t}X^s}} &\xrightarrow{\phi^*} \frac{\prod {}^t\mathcal{O}^s \times F_{\underset{-}{t}X^s}}{\prod P_{tV^s_-} \times P_{\underset{-}{t}X^s}} \xrightarrow{\pi_*} \frac{\prod {}^t\mathcal{O}^s \times_{P_{\underset{-}{t}X^s}} F_{\underset{-}{t}X^s}}{\prod P_{tV^s_-}} \xrightarrow{\alpha_!} \frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \\
&\xrightarrow{\phi^*} \frac{\mathcal{O}_- \times F_{-V^=}}{P_{\underset{=}{-}Z_{\equiv}} \times P_{=V^{\equiv}_-}} \xrightarrow{\pi_*} \frac{\mathcal{O}_- \times_{P_{=V^{\equiv}_-}} F_{-V^=}}{P_{\underset{=}{-}Z_{\equiv}}} \xrightarrow{\alpha_!} \frac{E_V}{P_{\underset{=}{-}Z_{\equiv}}} \\
&\xrightarrow{\iota^*} \frac{F_{Z_-}}{\prod P_{\underset{=}{-}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{Z_r}}{\prod P_{\underset{=}{-}Z_r}} \xrightarrow{\iota^*} \frac{\prod F_{\underset{=}{-}Z_r}}{\prod L_{\underset{=}{-}Z_r}} \xrightarrow{\kappa_!} \frac{\prod E_{\underset{j}{i}Z_r}}{\prod G_{\underset{j}{i}Z_r}} \xrightarrow{\text{shift}} \frac{\prod E_{\underset{j}{i}Z_r}}{\prod G_{\underset{j}{i}Z_r}}.
\end{aligned}$$

$$\begin{aligned}
\Phi_{44} = \frac{\prod E_{\underset{p}{t}X^s}}{\prod G_{\underset{p}{t}X^s}} &\xrightarrow{\phi^*} \frac{\mathcal{O} \times F_{\underset{-}{=}X^{\equiv}}}{P_{\underset{=}{-}Z_{\equiv}} \times P_{\underset{-}{=}X^{\equiv}}} \xrightarrow{\pi_*} \frac{\mathcal{O} \times_{P_{\underset{-}{=}X^{\equiv}}} F_{\underset{-}{=}X^{\equiv}}}{P_{\underset{=}{-}Z_{\equiv}}} \xrightarrow{\alpha_!} \frac{E_V}{P_{\underset{=}{-}Z_{\equiv}}} \\
&\xrightarrow{\iota^*} \frac{F_{\underset{=}{-}Z_{\equiv}}}{L_{\underset{=}{-}Z_{\equiv}}} \xrightarrow{\kappa_!} \frac{\prod E_{\underset{j}{i}Z_r}}{\prod G_{\underset{j}{i}Z_r}} \xrightarrow{\text{shift}} \frac{\prod E_{\underset{j}{i}Z_r}}{\prod G_{\underset{j}{i}Z_r}}.
\end{aligned}$$

We introduce

$$P_{Y^-} \supset \mathcal{O}^- \overset{\text{def}}{=} U_{Y^-} \cdot \prod_s \mathcal{O}^s = \prod_s \mathcal{O}^s \cdot U_{Y^-} \in P_{{}_-Y^{\equiv}_{=}}\text{-sets-}P_{{}^{=}_{-}X^{\equiv}},$$

$$G_V \supset \mathcal{O} \overset{\text{def}}{=} {}_-\underline{\mathcal{O}}_= \cdot \mathcal{O}^- = {}_-\mathcal{O}_= \cdot \mathcal{O}^- \in P_{{}^{-}_{=}Z_{\equiv}}\text{-sets-}P_{{}^{=}_{-}X^{\equiv}},$$

$$\mathcal{O} = {}_-\mathcal{O}_= \times_{P_{{}_-Y^{\equiv}_{=}}} \mathcal{O}^- = {}_-\underline{\mathcal{O}}_= \times_{P_{{}_-Y_=} \cap P_{Y^-}} \mathcal{O}^-.$$

Denote also

$$\begin{aligned} P_{{}^-V^=} \supset {}^-\mathcal{O}^= \overset{\text{def}}{=} U_{V^-} \cdot \prod_s {}^-\mathcal{O}^s = \prod_s {}^-\mathcal{O}^s \cdot U_{V^-} \\ = U_{{}^-V^=} \cdot \prod_{t,s} {}^t\mathcal{O}^s = \prod_{t,s} {}^t\mathcal{O}^s \cdot U_{{}^-V^=} \in P_{{}^=V^{\equiv}_{-}}\text{-sets-}P_{{}^{=}_{-}X^{\equiv}}. \end{aligned}$$

3.2.2. Lemma. *The following equations hold*

$$\mathcal{O}^- = \mathcal{O}^=_- \cdot {}^-\mathcal{O}^= = \mathcal{O}^=_- \times_{P_{{}^=V^{\equiv}_{-}}} {}^-\mathcal{O}^=.$$

Proof. We verify first the equation $\mathcal{O}^- = \mathcal{O}^=_- \cdot {}^-\mathcal{O}^=$. The definitions of these subsets are as follows:

$$\begin{aligned} \mathcal{O}^- = \prod_s \mathcal{O}^s \cdot U_{Y^-} = \prod_s \mathcal{O}^s_- \cdot {}^-\mathcal{O}^s \cdot U_{Y^-} = \prod_s \mathcal{O}^s_- \cdot U_{Y^-} \cdot \prod_s {}^-\mathcal{O}^s \\ = \prod_s \mathcal{O}^s_- \cdot U_{Y^-} \cdot \prod_s U_{{}^-V^s} \cdot \prod_{t,s} {}^t\mathcal{O}^s, \end{aligned}$$

$$\mathcal{O}^=_- = \prod_s \mathcal{O}^s_- \cdot U_{V^-}, \qquad {}^-\mathcal{O}^= = U_{{}^-V^=} \cdot \prod_{t,s} {}^t\mathcal{O}^s.$$

The required equation is implied by the following facts

$$U_{V^-} \subset U_{{}^-V^=} = U_{V^-} \cdot \prod_s U_{{}^-V^s}.$$

Now we prove the second equation. Recall that $P_{V^=_-} \supset \mathcal{O}^=_- \in P_{{}_-Y^{\equiv}_{=}}\text{-sets-}P_{{}^=V^{\equiv}_{-}}$ and $P_{{}^-V^=} \supset {}^-\mathcal{O}^= \in P_{{}^=V^{\equiv}_{-}}\text{-sets-}P_{{}^{=}_{-}X^{\equiv}}$. The multiplication map $mult : \mathcal{O}^=_- \times_{P_{{}^=V^{\equiv}_{-}}} {}^-\mathcal{O}^= \to \mathcal{O}^=_- \cdot {}^-\mathcal{O}^=$ is obviously surjective. We have to prove its injectivity. Suppose that two elements (g_1, h_1) and (g_2, h_2) are mapped to the same element $g_1 h_1 = g_2 h_2$ in G_V, where $g_1, g_2 \in \mathcal{O}^=_-$, $h_1, h_2 \in {}^-\mathcal{O}^=$. Then the element $r := g_2^{-1} g_1 = h_2 h_1^{-1}$ belongs simultaneously to $P_{V^=_-}$ and to $P_{{}^-V^=}$. Notice that

$$P_{V^=_-} \cap P_{{}^-V^=} \subset P_{{}^=V^{\equiv}_{-}}.$$

Indeed, for any $p \in P_{V^=_-} \cap P_{{}^-V^=}$ we have for all possible τ, σ, ρ

$$p({}^\tau V^\sigma_\rho) \subset \oplus_{t,s,r}\{{}^t V^s_r \mid s < \sigma \text{ or } (s = \sigma, t \leqslant \tau, r \leqslant \rho)\}.$$

This implies also that

$$p({}^\tau V^\sigma_\rho) \subset \oplus_{t,s,r}\{{}^t V^s_r \mid s < \sigma \text{ or } (s = \sigma, t < \tau) \text{ or } (s = \sigma, t = \tau, r \leqslant \rho)\},$$

which means that $p \in P_{{}^=V^{\equiv}_{-}}$. Therefore, in $\mathcal{O}^=_- \times_{P_{{}^=V^{\equiv}_{-}}} {}^-\mathcal{O}^=$ we have

$$(g_1, h_1) = (g_2 r, r^{-1} h_2) = (g_2, h_2),$$

which is the desired injectivity. $\square$

3.2.3. Lemma. *The following equations hold*

$$\mathcal{O} = \underline{\mathcal{O}_-} \cdot {}^-\mathcal{O}^= = \mathcal{O}_- \cdot {}^-\mathcal{O}^= \subset G_V, \qquad \mathcal{O} = \mathcal{O}_- \times_{P_{=\underline{V}\equiv}} {}^-\mathcal{O}^=.$$

Proof. The definition and Lemmas 3.2.2 and 3.2.1 imply the equations

$$\mathcal{O} = {}_-\mathcal{O}_{\equiv} \cdot \mathcal{O}^- = {}_-\mathcal{O}_{\equiv} \cdot \mathcal{O}^{\underline{=}} \cdot {}^-\mathcal{O}^= = \mathcal{O}_- \cdot {}^-\mathcal{O}^=.$$

The equation $\underline{\mathcal{O}_-} \cdot {}^-\mathcal{O}^= = \mathcal{O}_- \cdot {}^-\mathcal{O}^=$ is obvious. The equation $\mathcal{O}_- \times_{P_{=\underline{V}\equiv}} {}^-\mathcal{O}^= = \mathcal{O}_- \cdot {}^-\mathcal{O}^=$ follows from the existence of the associativity part of the isomorphism assoc : $\Phi_{42} \to \Phi_{44}$ presented below. □

Now we may describe the associativity part of the isomorphism assoc : $\Phi_{42} \to \Phi_{44}$ as follows.

$$\begin{tikzcd}
\frac{\prod E_{{}^t_pX^s}}{\prod G_{{}^t_pX^s}} \arrow[r, "\phi^*"] \arrow[d, "\phi^*"'] & \frac{\prod {}^t\mathcal{O}^s \times F_{t_X^s}}{\prod P_{tV^s_-} \times P_{t_X^s}} \arrow[r, "\pi_*"] \arrow[d, "\phi^*"] & \frac{\prod {}^t\mathcal{O}^s \times_{P_{t_X^s}} F_{t_X^s}}{\prod P_{tV^s_-}} \arrow[r, "\alpha_!"] \arrow[d, "\phi^*"] & \frac{\prod E_{tV^s}}{\prod P_{tV^s_-}} \arrow[d, "\phi^*"] \\
\frac{\mathcal{O} \times F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{\underline{=}X\equiv}} \arrow[r, "\pi^*"] \arrow[dr, Rightarrow, "\mathrm{Id}"'] & \frac{\mathcal{O}_- \times {}^-\mathcal{O}^= \times F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{=\underline{V}\equiv} \times P_{\underline{=}X\equiv}} \arrow[r, "\pi_*"] \arrow[d, "\pi_*"] & \frac{\mathcal{O}_- \times {}^-\mathcal{O}^= \times_{P_{\underline{=}X\equiv}} F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{=\underline{V}\equiv}} \arrow[r, "1\times\alpha_!"] \arrow[d, "\pi_*"] & \frac{\mathcal{O}_- \times F_{-V=}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{=\underline{V}\equiv}} \arrow[d, "\pi_*"] \\
 & \frac{\mathcal{O} \times F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{\underline{=}X\equiv}} \arrow[r, "\pi_*"] & \frac{\mathcal{O} \times_{P_{\underline{=}X\equiv}} F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv}} \arrow[r, "\beta_!"] \arrow[dr, "\alpha_!"'] & \frac{\mathcal{O}_- \times_{P_{=\underline{V}\equiv}} F_{-V=}}{P_{\overset{-}{\equiv}Z\equiv}} \arrow[d, "\alpha_!"] \\
 & & & \frac{E_Z}{P_{\overset{-}{\equiv}Z\equiv}}
\end{tikzcd}$$

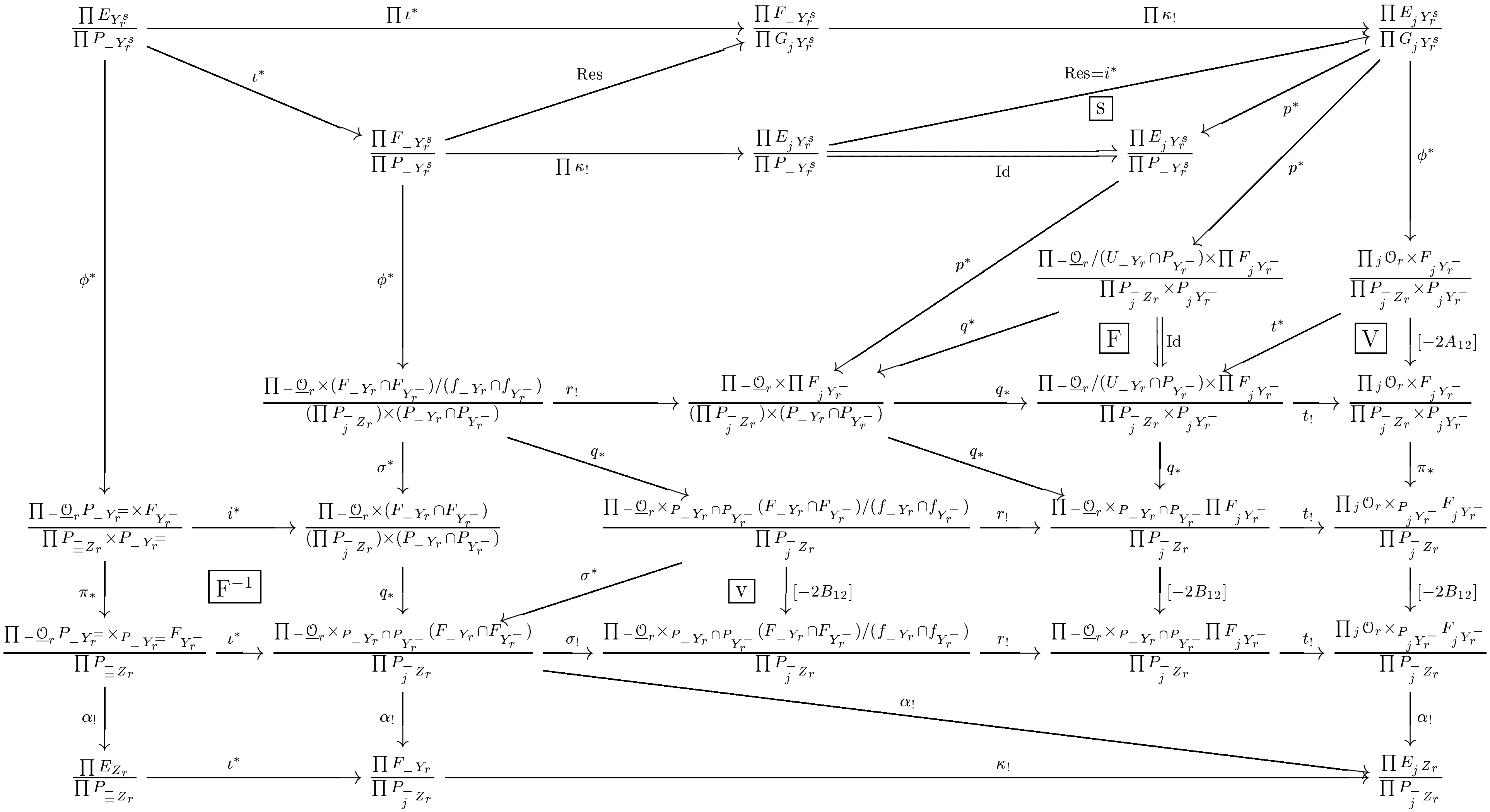
The associativity part of the isomorphism assoc : $\Phi_{24} \to \Phi_{44}$ is essentially the following.

$$\begin{tikzcd}
\frac{\prod E_{{}^t_pX^s}}{\prod G_{{}^t_pX^s}} \arrow[r, "\phi^*"] \arrow[d, "\phi^*"'] & \frac{\prod \mathcal{O}^s \times F_{\underline{=}X^s}}{\prod P_{-Y\underline{\underline{s}}} \times P_{\underline{=}X^s}} \arrow[r, "\pi_*"] \arrow[d, "\phi^*"] & \frac{\prod \mathcal{O}^s \times_{P_{\underline{=}X^s}} F_{\underline{=}X^s}}{\prod P_{-Y\underline{\underline{s}}}} \arrow[r, "\alpha_!"] \arrow[d, "\phi^*"] & \frac{\prod E_{Y^s}}{\prod P_{-Y\underline{\underline{s}}}} \arrow[d, "\phi^*"] \\
\frac{\mathcal{O} \times F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{\underline{=}X\equiv}} \arrow[r, "\pi^*"] \arrow[dr, Rightarrow, "\mathrm{Id}"'] & \frac{{}_-\mathcal{O}_= \times \mathcal{O}^- \times F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{-Y\underline{\equiv}} \times P_{\underline{=}X\equiv}} \arrow[r, "\pi_*"] \arrow[d, "\pi_*"] & \frac{{}_-\mathcal{O}_= \times \mathcal{O}^- \times_{P_{\underline{=}X\equiv}} F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{-Y\underline{\equiv}}} \arrow[r, "1\times\alpha_!"] \arrow[d, "\pi_*"] & \frac{{}_-\mathcal{O}_= \times F_{Y-}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{-Y\underline{\equiv}}} \arrow[d, "\pi_*"] \\
 & \frac{\mathcal{O} \times F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv} \times P_{\underline{=}X\equiv}} \arrow[r, "\pi_*"] & \frac{\mathcal{O} \times_{P_{\underline{=}X\equiv}} F_{\underline{=}X\equiv}}{P_{\overset{-}{\equiv}Z\equiv}} \arrow[r, "\beta_!"] \arrow[dr, "\alpha_!"'] & \frac{{}_-\mathcal{O}_= \times_{P_{-Y\underline{\equiv}}} F_{Y-}}{P_{\overset{-}{\equiv}Z\equiv}} \arrow[d, "\alpha_!"] \\
 & & & \frac{E_Z}{P_{\overset{-}{\equiv}Z\equiv}}
\end{tikzcd}$$

For the morphism coher : $\Phi_{12} \to \Phi_{22}$ we use notation

$$A_{12} = \dim \prod_r U_{-Y_r}/(U_{-Y_r} \cap P_{Y_r^-}), \quad B_{12} = \dim \prod_r f_{-Y_r} \cap f_{Y_r^-}.$$

Notice that $P_{-Y_r} \cap P_{Y_r^-} = P_{-Y_r} \cap P_{-Y_r^=} \subset P_{-Y_r^=}$ and $U_{-Y_r} \cap P_{Y_r^-} = U_{-Y_r} \cap P_{-Y_r^=}$ by Lemma 2.2.1.



12 − 22

$\Phi_{12} \xrightarrow{\text{coher}} \Phi_{22}$

$$
\begin{tikzcd}
\dfrac{\prod E_{Y^s}}{\prod P_{\_Y\underline{\underline{s}}}} \arrow[rr, "\prod\iota^*"] \arrow[dr, "\iota^*"] \arrow[dddd, "\phi^*"'] & & \dfrac{\prod F_{\_Y\underline{\underline{s}}}}{\prod G_{{}_jY_r^s}} \arrow[rr, "\prod\kappa_!"] & & \dfrac{\prod E_{{}_jY_r^s}}{\prod G_{{}_jY_r^s}} \arrow[dl, "p^*"'] \arrow[ddl, "p^*"] \arrow[dd, "\phi^*"] \\
& \dfrac{\prod F_{\_Y\underline{\underline{s}}}}{\prod P_{\_Y\underline{\underline{s}}}} \arrow[ur, "\mathrm{Res}"] \arrow[r, "\prod\kappa_!"'] \arrow[dd, "\phi^*"'] & \dfrac{\prod E_{{}_jY_r^s}}{\prod P_{\_Y\underline{\underline{s}}}} \arrow[urr, "\mathrm{Res}=i^*"] \arrow[r, Rightarrow, "\mathrm{Id}"'] \arrow[ur, phantom, "\boxed{\mathrm{S}}"] & \dfrac{\prod E_{{}_jY_r^s}}{\prod P_{\_Y\underline{\underline{s}}}} \arrow[ddl, "p^*"'] & \\
& & & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}/(U_{\_Y_{=}}\cap P_{Y_-})\times\prod F_{{}_jY_r^-}}{\prod P_{{}^-_jZ_r}\times P_{{}_jY_r^-}} \arrow[dl, "q^*"'] \arrow[d, Rightarrow, "\mathrm{Id}"] \arrow[d, phantom, "\boxed{\mathrm{F}}"', shift right=8] & \dfrac{\prod_j\mathcal{O}_r\times F_{{}_jY_r^-}}{\prod P_{{}^-_jZ_r}\times P_{{}_jY_r^-}} \arrow[dl, "t^*"'] \arrow[d, "{[-2A_{13}]}"] \arrow[d, phantom, "\boxed{\mathrm{V}}"', shift right=8] \\
& \dfrac{{}_{-}\underline{\mathcal{O}}_{=}\times(F_{\_Y_{=}}\cap F_{Y_-})/(f_{\_Y_{=}}\cap f_{Y_-})}{(\prod P_{{}^-_jZ_r})\times(P_{\_Y_{=}}\cap P_{Y_-})} \arrow[r, "r_!"] \arrow[dr, "q_*"] \arrow[d, "\sigma^*"'] & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}\times\prod F_{{}_jY_r^-}}{(\prod P_{{}^-_jZ_r})\times(P_{\_Y_{=}}\cap P_{Y_-})} \arrow[r, "q_*"] \arrow[dr, "q_*"] & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}/(U_{\_Y_{=}}\cap P_{Y_-})\times\prod F_{{}_jY_r^-}}{\prod P_{{}^-_jZ_r}\times P_{{}_jY_r^-}} \arrow[r, "t_!"'] \arrow[d, "q_*"] & \dfrac{\prod_j\mathcal{O}_r\times F_{{}_jY_r^-}}{\prod P_{{}^-_jZ_r}\times P_{{}_jY_r^-}} \arrow[d, "\pi_*"] \\
\dfrac{{}_{-}\underline{\mathcal{O}}_{=}P_{\_Y_{\equiv}}\times F_{Y_-}}{P_{{}^-_{\equiv}Z_{\equiv}}\times P_{\_Y_{\equiv}}} \arrow[r, "i^*"] \arrow[d, "\pi_*"'] \arrow[dr, phantom, "\boxed{\mathrm{F}^{-1}}"] & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}\times(F_{\_Y_{=}}\cap F_{Y_-})}{(\prod P_{{}^-_jZ_r})\times(P_{\_Y_{=}}\cap P_{Y_-})} \arrow[d, "q_*"'] & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}\times_{P_{\_Y_{=}}\cap P_{Y_-}}(F_{\_Y_{=}}\cap F_{Y_-})/(f_{\_Y_{=}}\cap f_{Y_-})}{\prod P_{{}^-_jZ_r}} \arrow[dl, "\sigma^*"'] \arrow[r, "r_!"] \arrow[d, "{[-2B_{13}]}"] \arrow[d, phantom, "\boxed{\mathrm{V}}"', shift right=8] & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}\times_{P_{\_Y_{=}}\cap P_{Y_-}}\prod F_{{}_jY_r^-}}{\prod P_{{}^-_jZ_r}} \arrow[r, "t_!"] \arrow[d, "{[-2B_{13}]}"] & \dfrac{\prod_j\mathcal{O}_r\times_{P_{{}_jY_r^-}}F_{{}_jY_r^-}}{\prod P_{{}^-_jZ_r}} \arrow[d, "{[-2B_{13}]}"] \\
\dfrac{{}_{-}\underline{\mathcal{O}}_{=}P_{\_Y_{\equiv}}\times_{P_{\_Y_{\equiv}}}F_{Y_-}}{P_{{}^-_{\equiv}Z_{\equiv}}} \arrow[r, "\iota^*"] \arrow[d, "\alpha_!"'] & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}\times_{P_{\_Y_{=}}\cap P_{Y_-}}(F_{\_Y_{=}}\cap F_{Y_-})}{\prod P_{{}^-_jZ_r}} \arrow[r, "\sigma_!"] \arrow[d, "\alpha_!"'] \arrow[drrr, "\alpha_!"] & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}\times_{P_{\_Y_{=}}\cap P_{Y_-}}(F_{\_Y_{=}}\cap F_{Y_-})/(f_{\_Y_{=}}\cap f_{Y_-})}{\prod P_{{}^-_jZ_r}} \arrow[r, "r_!"] & \dfrac{{}_{-}\underline{\mathcal{O}}_{=}\times_{P_{\_Y_{=}}\cap P_{Y_-}}\prod F_{{}_jY_r^-}}{\prod P_{{}^-_jZ_r}} \arrow[r, "t_!"] & \dfrac{\prod_j\mathcal{O}_r\times_{P_{{}_jY_r^-}}F_{{}_jY_r^-}}{\prod P_{{}^-_jZ_r}} \arrow[d, "\alpha_!"] \\
\dfrac{E_Y}{P_{{}^-_{\equiv}Z_{\equiv}}} \arrow[r, "\iota^*"] & \dfrac{F_{\_Y_{=}}}{\prod P_{{}^-_jZ_r}} \arrow[rrr, "\kappa_!"] & & & \dfrac{\prod E_{{}_jZ_r}}{\prod P_{{}^-_jZ_r}}
\end{tikzcd}
$$

$13-23$

$$\Phi_{13}\xrightarrow{\text{coher}}\Phi_{23}$$

For the morphism coher : $\Phi_{13} \to \Phi_{23}$ we use notation

$$A_{13} = \dim U_{-Y_=}/(U_{-Y_=} \cap P_{Y^-}), \quad B_{13} = \dim f_{-Y_=} \cap f_{Y^-}.$$

Notice that $P_{-Y_=} \cap P_{Y^-} = P_{-Y_=} \cap P_{-Y_{\underline{\equiv}}} \subset P_{-Y_{\underline{\equiv}}}$ and $U_{-Y_=} \cap P_{Y^-} = U_{-Y_=} \cap P_{-Y_{\underline{\equiv}}}$ by Lemma 2.2.1.
For the morphism coher : $\Phi_{21} \to \Phi_{22}$ we use notation

$$A_{21} = \dim \prod_s U_{V^s_-}/(U_{V^s_-} \cap P_{-V^s}), \quad B_{21} = \dim \prod_s f_{V^s_-} \cap f_{-V^s}.$$

Notice that $P_{V^s_-} \cap P_{-V^s} = P_{V^s_-} \cap P_{=V^s_-} \subset P_{=V^s_-}$ and $U_{V^s_-} \cap P_{-V^s} = U_{V^s_-} \cap P_{=V^s_-}$ by Lemma 2.2.1.
For the morphism coher : $\Phi_{31} \to \Phi_{32}$ we use notation

$$A_{31} = \dim U_{V_-}/(U_{V_-} \cap P_{-V^=}), \quad B_{31} = \dim f_{V_-} \cap f_{-V^=}.$$

Notice that $P_{V_-} \cap P_{-V^=} = P_{V_-} \cap P_{=V^{\equiv}_-} \subset P_{=V^{\equiv}_-}$ and $U_{V_-} \cap P_{-V^=} = U_{V_-} \cap P_{=V^{\equiv}_-}$ by Lemma 2.2.1.
For the morphism coher : $\Phi_{22} \to \Phi_{33}$ we use notation

$$A_{22} = \dim U_{Y_-}/(U_{Y_-} \cap P_{Y^-}), \quad B_{22} = \dim f_{Y_-} \cap f_{Y^-}.$$

3.2.4. Lemma. *The group $P_{-Y_{\underline{\equiv}}} \cap P_{Y^-} = P_{Y_-} \cap P_{-Y_{\underline{\equiv}}}$ is a semidirect product of its normal subgroup $U_{Y_-} \cap P_{Y^-}$ and diagonally embedded subgroup $\prod_r P_{-Y_r^=}$.*

Proof. For an element $g \in G_V$ the following conditions are equivalent:

$$\begin{aligned} g \in U_{Y_-} \cap P_{Y^-} &\iff (g-1)({}_\iota Y^\sigma_\rho) \subset \oplus_{j,s,r}\{{}_j Y^s_r \mid s \leqslant \sigma, r < \rho\}, \\ g \in \prod_r P_{-Y_r^=} &\iff (g-1)({}_\iota Y^\sigma_\rho) \subset \oplus_{j,s,r}\{{}_j Y^s_r \mid (s < \sigma, r = \rho) \text{ or } (j \leqslant \iota, s = \sigma, r = \rho)\} \end{aligned} \tag{3.2.2}$$

(for all ι, σ, ρ).
Semisplit exact sequence obtained in Lemma 2.2.1 gives the lower row in the following diagram.

$$\begin{tikzcd} 1 \arrow[r] & U_{Y_-} \cap P_{Y^-} \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, hook] & \prod_r P_{-Y_r^=} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & U_{Y_-} \cap P_{Y^-} \arrow[r] & P_{Y_-} \cap P_{Y^-} \arrow[r] & \prod_r P_{Y_r^-} \arrow[r] & 1 \end{tikzcd}$$

The subgroup X in the upper row is determined as a preimage. By construction X is a semidirect product of its normal subgroup $U_{Y_-} \cap P_{Y^-}$ and diagonally embedded subgroup $\prod_r P_{-Y_r^=}$.
An element $g \in G_V$ is in X iff for all ι, σ, ρ

$$\begin{aligned} (g-1)({}_\iota Y^\sigma_\rho) \subset \oplus_{j,s,r}\{{}_j Y^s_r \mid (s < \sigma, r < \rho) \text{ or } (s = \sigma, r < \rho) \\ \text{or } (s < \sigma, r = \rho) \text{ or } (j \leqslant \iota, s = \sigma, r = \rho)\}. \end{aligned} \tag{3.2.3}$$

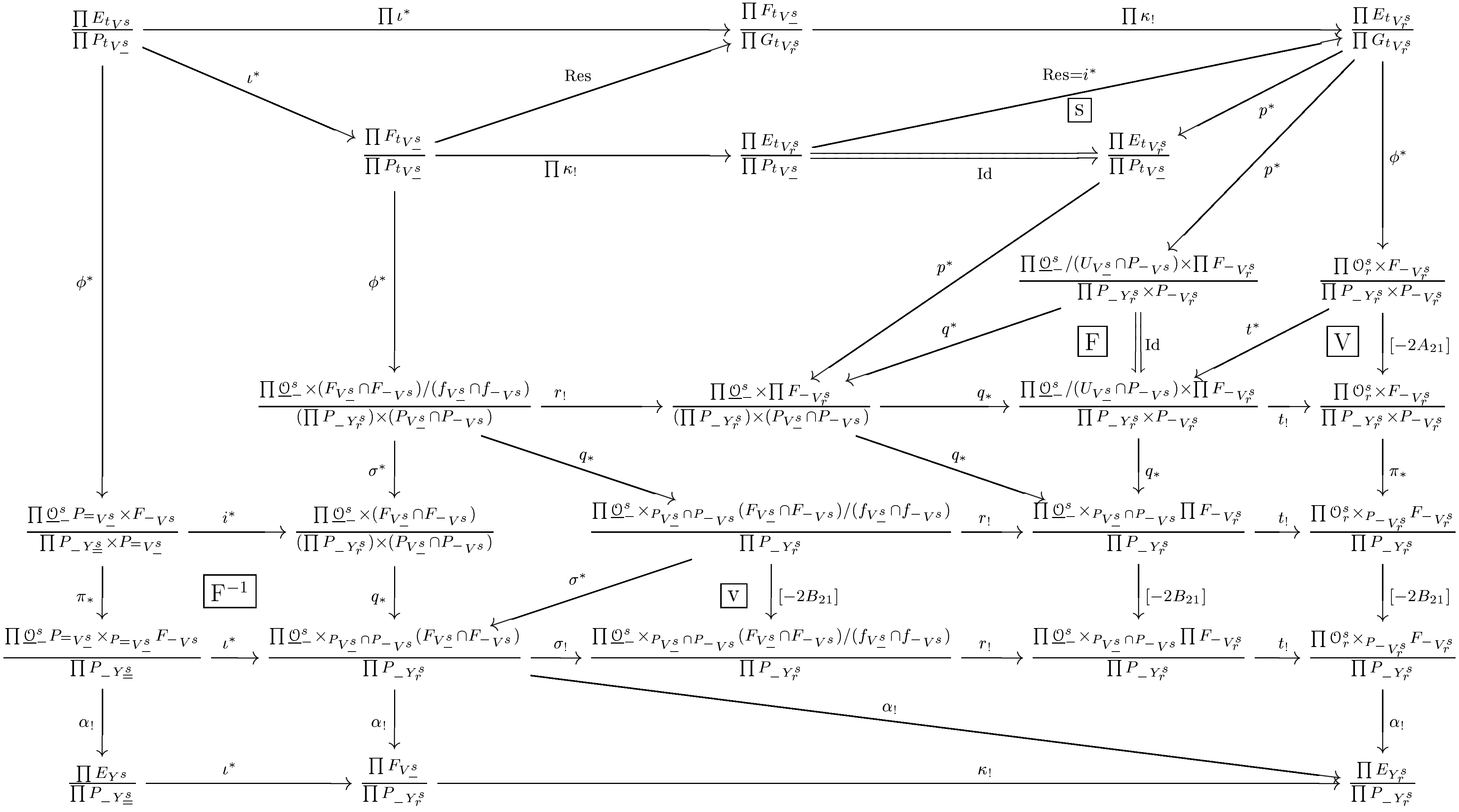
An element $g \in G_V$ is in $P_{-Y_{\underline{\equiv}}}$ iff for all ι, σ, ρ

$$(g-1)({}_\iota Y^\sigma_\rho) \subset \oplus_{j,s,r}\{{}_j Y^s_r \mid r < \rho \text{ or } (s < \sigma, r = \rho) \text{ or } (j \leqslant \iota, s = \sigma, r = \rho)\}.$$

Hence, $X = P_{-Y_{\underline{\equiv}}} \cap P_{Y^-}$. Symmetrically, an element $g \in G_V$ is in $P_{-Y_{\underline{\equiv}}}$ iff

$$(g-1)({}_\iota Y^\sigma_\rho) \subset \oplus_{j,s,r}\{{}_j Y^s_r \mid s < \sigma \text{ or } (s = \sigma, r < \rho) \text{ or } (j \leqslant \iota, s = \sigma, r = \rho)\}. \tag{3.2.4}$$

Hence, $X = P_{Y_-} \cap P_{-Y_{\underline{\equiv}}}$. $\square$



$21-22$

$$\Phi_{21} \xrightarrow{\text{coher}} \Phi_{22}$$

$$
\begin{tikzcd}
\frac{\prod E_{t_{V^s}}}{\prod P_{t_{V_-^s}}} \arrow[rr, "\prod \iota^*"] \arrow[dr, "\iota^*"] \arrow[dddd, "\phi^*"'] & & \frac{\prod F_{t_{V_-^s}}}{\prod G_{t_{V_r^s}}} \arrow[rr, "\prod \kappa_!"] & \arrow[d, phantom, "\boxed{\mathrm{S}}"] & \frac{\prod E_{t_{V_r^s}}}{\prod G_{t_{V_r^s}}} \arrow[dl, "p^*"] \arrow[ddl, "p^*"] \arrow[dd, "\phi^*"] \\
& \frac{\prod F_{t_{V_-^s}}}{\prod P_{t_{V_-^s}}} \arrow[ur, "\mathrm{Res}"] \arrow[r, "\prod \kappa_!"'] \arrow[dd, "\phi^*"'] & \frac{\prod E_{t_{V_r^s}}}{\prod P_{t_{V_-^s}}} \arrow[urr, "\mathrm{Res}=i^*"] \arrow[r, Rightarrow, "\mathrm{Id}"'] & \frac{\prod E_{t_{V_r^s}}}{\prod P_{t_{V_-^s}}} \arrow[ddl, "p^*"'] & \\
& & \arrow[dr, phantom, "\boxed{\mathrm{F}}"] & \frac{\underline{\mathcal{O}}_-/(U_{V_-}\cap P_{-V=})\times\prod F_{-V_r=}}{\prod P_{\overset{-}{=}Z_r}\times P_{-V_r=}} \arrow[dl, "q^*"'] \arrow[d, equal, "\mathrm{Id}"] \arrow[dr, phantom, "\boxed{\mathrm{V}}"] & \frac{\prod \mathcal{O}_r\times F_{-V_r=}}{\prod P_{\overset{-}{=}Z_r}\times P_{-V_r=}} \arrow[dl, "t^*"'] \arrow[d, "{[-2A_{31}]}"] \\
& \frac{\underline{\mathcal{O}}_-\times(F_{V_-}\cap F_{-V=})/(f_{V_-}\cap f_{-V=})}{(\prod P_{\overset{-}{=}Z_r})\times(P_{V_-}\cap P_{-V=})} \arrow[r, "r_!"] \arrow[d, "\sigma^*"'] \arrow[dr, "q_*"] & \frac{\underline{\mathcal{O}}_-\times\prod F_{-V_r=}}{(\prod P_{\overset{-}{=}Z_r})\times(P_{V_-}\cap P_{-V=})} \arrow[r, "q_*"] \arrow[dr, "q_*"] & \frac{\underline{\mathcal{O}}_-/(U_{V_-}\cap P_{-V=})\times\prod F_{-V_r=}}{\prod P_{\overset{-}{=}Z_r}\times P_{-V_r=}} \arrow[r, "t_!"'] \arrow[d, "q_*"] & \frac{\prod \mathcal{O}_r\times F_{-V_r=}}{\prod P_{\overset{-}{=}Z_r}\times P_{-V_r=}} \arrow[d, "\pi_*"] \\
\frac{\underline{\mathcal{O}}_-\,P_{=V_-^{\equiv}}\times F_{-V=}}{P_{\overset{-}{=}Z_{\equiv}}\times P_{=V_-^{\equiv}}} \arrow[r, "i^*"] \arrow[d, "\pi_*"'] \arrow[dr, phantom, "\boxed{\mathrm{F}^{-1}}"] & \frac{\underline{\mathcal{O}}_-\times(F_{V_-}\cap F_{-V=})}{(\prod P_{\overset{-}{=}Z_r})\times(P_{V_-}\cap P_{-V=})} \arrow[d, "q_*"'] & \frac{\underline{\mathcal{O}}_-\times_{P_{V_-}\cap P_{-V=}}(F_{V_-}\cap F_{-V=})/(f_{V_-}\cap f_{-V=})}{\prod P_{\overset{-}{=}Z_r}} \arrow[r, "r_!"] \arrow[dl, "\sigma^*"'] \arrow[d, "{[-2B_{31}]}"] \arrow[d, phantom, shift right=6ex, "\boxed{\mathrm{v}}"] & \frac{\underline{\mathcal{O}}_-\times_{P_{V_-}\cap P_{-V=}}\prod F_{-V_r=}}{\prod P_{\overset{-}{=}Z_r}} \arrow[r, "t_!"] \arrow[d, "{[-2B_{31}]}"] & \frac{\prod \mathcal{O}_r\times_{P_{-V_r=}} F_{-V_r=}}{\prod P_{\overset{-}{=}Z_r}} \arrow[d, "{[-2B_{31}]}"] \\
\frac{\underline{\mathcal{O}}_-\,P_{=V_-^{\equiv}}\times_{P_{=V_-^{\equiv}}} F_{-V=}}{P_{\overset{-}{=}Z_{\equiv}}} \arrow[r, "\iota^*"] \arrow[d, "\alpha_!"'] & \frac{\underline{\mathcal{O}}_-\times_{P_{V_-}\cap P_{-V=}}(F_{V_-}\cap F_{-V=})}{\prod P_{\overset{-}{=}Z_r}} \arrow[r, "\sigma_!"] \arrow[d, "\alpha_!"'] \arrow[drrr, "\alpha_!"] & \frac{\underline{\mathcal{O}}_-\times_{P_{V_-}\cap P_{-V=}}(F_{V_-}\cap F_{-V=})/(f_{V_-}\cap f_{-V=})}{\prod P_{\overset{-}{=}Z_r}} \arrow[r, "r_!"] & \frac{\underline{\mathcal{O}}_-\times_{P_{V_-}\cap P_{-V=}}\prod F_{-V_r=}}{\prod P_{\overset{-}{=}Z_r}} \arrow[r, "t_!"] & \frac{\prod \mathcal{O}_r\times_{P_{-V_r=}} F_{-V_r=}}{\prod P_{\overset{-}{=}Z_r}} \arrow[d, "\alpha_!"] \\
\frac{E_V}{P_{\overset{-}{=}Z_{\equiv}}} \arrow[r, "\iota^*"] & \frac{F_{V_-}}{\prod P_{\overset{-}{=}Z_r}} \arrow[rrr, "\kappa_!"] & & & \frac{\prod E_{Z_r}}{\prod P_{\overset{-}{=}Z_r}}
\end{tikzcd}
$$

$$31-32 \qquad \Phi_{31} \xrightarrow{\text{coher}} \Phi_{32}$$

3.2.5. Lemma. *We have*

$$P_{Y_-} \supset \Big(\prod {}_{-}\mathcal{O}_r\Big)U_{Y_-} = {}_{-}\underline{\mathcal{O}}_{=} \cdot P_{{}_{-}Y_{\underline{\equiv}}} \in P_{\underline{\equiv}Z_{\equiv}}\text{-sets-}P_{{}_{-}Y_{\underline{\equiv}}},$$
$${}_{-}\mathcal{O}_{=} = \Big(\prod {}_{-}\mathcal{O}_r\Big)U_{Y_-}P_{{}_{-}Y_{\underline{\equiv}}} \simeq \Big(\prod {}_{-}\mathcal{O}_r\Big)U_{Y_-} \times_{P_{Y_-}\cap P_{{}_{-}Y_{\underline{\equiv}}}} P_{{}_{-}Y_{\underline{\equiv}}}. \tag{3.2.5}$$

Proof. In the first equation both sides are equal to $(\prod {}_{-}\underline{\mathcal{O}}_r P_{{}_{-}Y_r^{=}})U_{Y_-}$ as follows from

$$\Big(\prod {}_{-}\underline{\mathcal{O}}_r\Big)U_{Y_-} = \prod_r\Big(\prod_j {}_j\mathcal{O}_r \cdot U_{{}_{-}Y_r}\Big)U_{Y_-} = \prod_{j,r} {}_j\mathcal{O}_r \cdot U_{{}_{-}Y_{=}} = {}_{-}\underline{\mathcal{O}}_{=}. \tag{3.2.6}$$

Hence, we find

$$\Big(\prod {}_{-}\mathcal{O}_r\Big)U_{Y_-}P_{{}_{-}Y_{\underline{\equiv}}} = \Big(\prod {}_{-}\underline{\mathcal{O}}_r P_{{}_{-}Y_r^{=}}\Big)U_{Y_-}P_{{}_{-}Y_{\underline{\equiv}}} = {}_{-}\underline{\mathcal{O}}_{=}\Big(\prod P_{{}_{-}Y_r^{=}}\Big)P_{{}_{-}Y_{\underline{\equiv}}} = {}_{-}\underline{\mathcal{O}}_{=}P_{{}_{-}Y_{\underline{\equiv}}} = {}_{-}\mathcal{O}_{=}$$

since $\prod P_{{}_{-}Y_r^{=}} \subset P_{{}_{-}Y_{\underline{\equiv}}}$ by (3.2.2) and (3.2.4).

The isomorphism in (3.2.5) follows from the fact that $(\prod {}_{-}\mathcal{O}_r)U_{Y_-} \subset P_{Y_-}$. □

To prove the existence of some of the pull-back squares in diagram on page 32 we use the following lemmas.

3.2.6. Lemma. *There is a pull-back square*

$$\begin{tikzcd}
\prod_{s,r} E_{Y_r^s} & \prod_r F_{Y_r^-} \arrow[l] \\
\prod_s F_{Y_-^s} \arrow[u] & (F_{Y_-} \cap F_{Y^-})/(f_{Y_-} \cap f_{Y^-}) \arrow[l] \arrow[u] \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

Proof. This square is the sum over all arrows $h \in H$ of the pull-back squares

$$\begin{tikzcd}
\oplus_{s=\sigma;r=\rho}\mathrm{Hom} & \oplus_{s\leqslant\sigma;r=\rho}\mathrm{Hom} \arrow[l] \\
\oplus_{s=\sigma;r\leqslant\rho}\mathrm{Hom} \arrow[u] & \oplus_{s\leqslant\sigma;r\leqslant\rho}\mathrm{Hom}\big/\oplus_{s<\sigma;r<\rho}\mathrm{Hom} \arrow[l] \arrow[u] \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

made of direct sums of spaces $\mathrm{Hom} = \mathrm{Hom}_{\mathbb{C}}(Y_\rho^\sigma(h'), Y_r^s(h''))$. □

3.2.7. Lemma. *There is a pull-back square*

$$\begin{tikzcd}
{}_{-}\mathcal{O}_{=} \times_{P_{{}_{-}Y_{\underline{\equiv}}}} F_{Y^-} \arrow[d, "\alpha"'] & ({}_{-}\underline{\mathcal{O}}_{=}P_{{}_{-}Y_{\underline{\equiv}}}) \times_{P_{Y_-}\cap P_{{}_{-}Y_{\underline{\equiv}}}} (F_{Y_-} \cap F_{Y^-}) \arrow[l, hook', "\iota"'] \arrow[d, "\alpha"] \arrow[dl, phantom, very near start, "\llcorner"] \\
E_Y & F_{Y_-} \arrow[l, hook', "\iota"']
\end{tikzcd}$$

Proof. By (3.2.5) we have

$${}_{-}\mathcal{O}_{=} \times_{P_{{}_{-}Y_{\underline{\equiv}}}} F_{Y^-} \simeq ({}_{-}\underline{\mathcal{O}}_{=}P_{{}_{-}Y_{\underline{\equiv}}}) \times_{P_{Y_-}\cap P_{{}_{-}Y_{\underline{\equiv}}}} F_{Y^-}.$$

This identification gives naturally the upper row embedding in the square. The action map α in the right column makes sense, since ${}_{-}\underline{\mathcal{O}}_{=}P_{{}_{-}Y_{\underline{\equiv}}} \subset P_{Y_-}$ by Lemma 3.2.5. Commutativity of the square is immediate.

Let us prove the pull-back property. Let $g \in {}_{-}\underline{\mathcal{O}}_{=} \cdot P_{{}_{-}Y_{\underline{\equiv}}}$ and let $f \in F_{Y^-}$. Assume that $g.f \in F_{Y_-}$. Then $g^{-1} \in P_{Y_-}$, $f = g^{-1}.(g.f) \in F_{Y_-}$, hence, $f \in F_{Y_-} \cap F_{Y^-}$. □

3.2.8. Lemma. *There exists a mapping*

$$\iota : P_{-Y\underline{\equiv}} \times_{P_{Y_-} \cap P_{-Y\underline{\equiv}}} \left(({}_{-}\underline{\mathcal{O}}_{=} P_{-Y\underline{\equiv}}) \times (F_{Y_-} \cap F_{Y^-})\right) \\ \to ({}_{-}\underline{\mathcal{O}}_{=} P_{-Y\underline{\equiv}}) \times_{P_{Y_-} \cap P_{-Y\underline{\equiv}}} P_{-Y\underline{\equiv}} \times F_{Y^-} \simeq {}_{-}\mathcal{O}_{=} \times F_{Y^-},$$

$$\iota : (p, g, f) \mapsto (g, p^{-1}, p.f) \mapsto (gp^{-1}, p.f),$$

where $p \in P_{-Y\underline{\equiv}}$, $g \in {}_{-}\underline{\mathcal{O}}_{=} P_{-Y\underline{\equiv}}$, $f \in F_{Y_-} \cap F_{Y^-}$. *It is an embedding and it makes*

$$\begin{tikzcd}
{}_{-}\mathcal{O}_{=} \times F_{Y^-} \arrow[d, "\pi"'] & P_{-Y\underline{\equiv}} \times_{P_{Y_-} \cap P_{-Y\underline{\equiv}}} \left(({}_{-}\underline{\mathcal{O}}_{=} P_{-Y\underline{\equiv}}) \times (F_{Y_-} \cap F_{Y^-})\right) \arrow[l, hook', "\iota"'] \arrow[d, "\pi"] \arrow[dl, phantom, "\llcorner", very near start] \\
{}_{-}\mathcal{O}_{=} \times_{P_{-Y\underline{\equiv}}} F_{Y^-} & ({}_{-}\underline{\mathcal{O}}_{=} P_{-Y\underline{\equiv}}) \times_{P_{Y_-} \cap P_{-Y\underline{\equiv}}} (F_{Y_-} \cap F_{Y^-}) \arrow[l, hook', "\iota"']
\end{tikzcd} \tag{3.2.7}$$

into a pull-back square.

Proof. For any $r \in P_{Y_-} \cap P_{-Y\underline{\equiv}}$ the constructed mapping takes a point $(pr^{-1}, gr^{-1}, r.f)$, equivalent to (p, g, f), into $(gr^{-1}rp^{-1}, pr^{-1}r.f) = (gp^{-1}, p.f)$. Hence, the mapping ι exists. It is an equivariant mapping of free $P_{-Y\underline{\equiv}}$-spaces.

Commutativity of square (3.2.7) is clear. Notice that the vertical mappings are quotient by $P_{-Y\underline{\equiv}}$ maps. Therefore, there is a quotient by $P_{-Y\underline{\equiv}}$ map $\bar{\iota}$ of the constructed mapping ι. It also closes up the square, making it into a pull-back square by [BL94, Lemma 2.1.1]. Since the right π is surjective, there exists only one map in the lower row, which makes the square commutative. Therefore, $\bar{\iota} = \iota$ in the lower row. Since in this pull-back square the lower ι is an embedding, the same holds for the upper ι. $\square$

4. Elementary coherence relations

4.1. Standard isomorphisms

Let $\mathfrak{D}$ be a 2-subcategory of the 2-category of categories, whose objects are equivariant derived categories $\frac{X}{G}$, 1-morphisms are compositions of functors of the following 3 types:

- inverse image functor $\frac{f}{\phi}^*$;
- direct image functor $\frac{f}{\phi}_*$, where $f|\phi$ is a quotient map;
- direct image functor with proper supports $\frac{f}{1}_!$,

and 2-morphisms are pastings of isomorphisms and their inverses of the following 6 types:

a) $\frac{g}{\psi}^* \frac{f}{\phi}^* \xrightarrow{\sim} \left(\frac{f}{\phi}\frac{g}{\psi}\right)^*$;

b) $\frac{f}{\phi}_* \frac{g}{\psi}_* \xrightarrow{\sim} \left(\frac{f}{\phi}\frac{g}{\psi}\right)_*$;

c) $\frac{f}{1}_! \frac{g}{1}_! \xrightarrow{\sim} \left(\frac{f}{1}\frac{g}{1}\right)_!$;

$$
\begin{tikzcd}[column sep=small]
\dfrac{\prod E_{Y^s}}{\prod P_{-Y^{s}_{\equiv}}} \arrow[rr, "\prod \iota^*"] \arrow[dr, "\iota^*"] \arrow[dddd, "\phi^*"'] & & \dfrac{\prod F_{Y^{s}_{-}}}{\prod P_{-Y^{s}_{r}}} \arrow[rr, "\prod \kappa_!"] & & \dfrac{\prod E_{Y^{s}_{r}}}{\prod P_{-Y^{s}_{r}}} \arrow[dl, "p^*"'] \arrow[ddl, "p^*"] \arrow[dd, "\phi^*"] \\
& \dfrac{\prod F_{Y^{s}_{-}}}{\prod P_{-Y^{s}_{\equiv}}} \arrow[ur, "\mathrm{Res}"] \arrow[r, "\prod \kappa_!"'] \arrow[dd, "\phi^*"'] & \dfrac{\prod E_{Y^{s}_{r}}}{\prod P_{-Y^{s}_{\equiv}}} \arrow[urr, "\mathrm{Res}=i^*"] \arrow[r, Rightarrow, "\mathrm{Id}"'] \arrow[ur, phantom, "\boxed{\mathrm{S}}"] & \dfrac{\prod E_{Y^{s}_{r}}}{\prod P_{-Y^{s}_{\equiv}}} \arrow[dl, "p^*"'] & \\
& & & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})/(U_{Y_-}\cap P_{Y-})\times\prod F_{Y_r-}}{\prod P_{\equiv Z_r}\times P_{-Y_r=}} \arrow[dl, "q^*"'] \arrow[d, Rightarrow, "\mathrm{Id}"] \arrow[dl, phantom, "\boxed{\mathrm{F}}"] & \dfrac{\prod -\mathcal{O}_r\times F_{Y_r-}}{\prod P_{\equiv Z_r}\times P_{-Y_r=}} \arrow[dl, "t^*"'] \arrow[d, "{[-2A_{22}]}"] \arrow[dl, phantom, "\boxed{\mathrm{V}}"] \\
& \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})\times(F_{Y_-}\cap F_{Y-})/(f_{Y_-}\cap f_{Y-})}{(\prod P_{\equiv Z_r})\times(P_{Y_-}\cap P_{-Y_{\equiv}})} \arrow[r, "r_!"] \arrow[d, "\sigma^*"'] \arrow[dr, "q_*"] & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})\times\prod F_{Y_r-}}{(\prod P_{\equiv Z_r})\times(P_{Y_-}\cap P_{-Y_{\equiv}})} \arrow[r, "q_*"] \arrow[dr, "q_*"] & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})/(U_{Y_-}\cap P_{Y-})\times\prod F_{Y_r-}}{\prod P_{\equiv Z_r}\times P_{-Y_r=}} \arrow[r, "t_!"] \arrow[d, "q_*"] & \dfrac{\prod -\mathcal{O}_r\times F_{Y_r-}}{\prod P_{\equiv Z_r}\times P_{-Y_r=}} \arrow[d, "\pi_*"] \\
\dfrac{-\mathcal{O}_{=}\times F_{Y-}}{P_{\equiv Z_{\equiv}}\times P_{-Y_{\equiv}}} \arrow[r, "i^*"] \arrow[d, "\pi_*"'] \arrow[dr, phantom, "\boxed{\mathrm{F}^{-1}}"] & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})\times(F_{Y_-}\cap F_{Y-})}{(\prod P_{\equiv Z_r})\times(P_{Y_-}\cap P_{-Y_{\equiv}})} \arrow[d, "q_*"'] & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})\times_{P_{Y_-}\cap P_{-Y_{\equiv}}}(F_{Y_-}\cap F_{Y-})/(f_{Y_-}\cap f_{Y-})}{\prod P_{\equiv Z_r}} \arrow[r, "r_!"] \arrow[dl, "\sigma^*"'] \arrow[d, "{[-2B_{22}]}"] \arrow[d, phantom, "\boxed{\mathrm{v}}"'] & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})\times_{P_{Y_-}\cap P_{-Y_{\equiv}}}\prod F_{Y_r-}}{\prod P_{\equiv Z_r}} \arrow[r, "t_!"] \arrow[d, "{[-2B_{22}]}"] & \dfrac{\prod -\mathcal{O}_r\times_{P_{-Y_r=}}F_{Y_r-}}{\prod P_{\equiv Z_r}} \arrow[d, "{[-2B_{22}]}"] \\
\dfrac{-\mathcal{O}_{=}\times_{P_{-Y_{\equiv}}}F_{Y-}}{P_{\equiv Z_{\equiv}}} \arrow[r, "\iota^*"] \arrow[d, "\alpha_!"'] & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})\times_{P_{Y_-}\cap P_{-Y_{\equiv}}}(F_{Y_-}\cap F_{Y-})}{\prod P_{\equiv Z_r}} \arrow[r, "\sigma_!"] \arrow[d, "\alpha_!"'] \arrow[drrr, "\alpha_!"] & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})\times_{P_{Y_-}\cap P_{-Y_{\equiv}}}(F_{Y_-}\cap F_{Y-})/(f_{Y_-}\cap f_{Y-})}{\prod P_{\equiv Z_r}} \arrow[r, "r_!"] & \dfrac{(-\underline{\mathcal{O}}_{=}P_{-Y_{\equiv}})\times_{P_{Y_-}\cap P_{-Y_{\equiv}}}\prod F_{Y_r-}}{\prod P_{\equiv Z_r}} \arrow[r, "t_!"] & \dfrac{\prod -\mathcal{O}_r\times_{P_{-Y_r=}}F_{Y_r-}}{\prod P_{\equiv Z_r}} \arrow[d, "\alpha_!"] \\
\dfrac{E_Y}{P_{\equiv Z_{\equiv}}} \arrow[r, "\iota^*"] & \dfrac{F_{Y_-}}{\prod P_{\equiv Z_r}} \arrow[rrr, "\kappa_!"] & & & \dfrac{\prod E_{Z_r}}{\prod P_{\equiv Z_r}}
\end{tikzcd}
$$

$22-33$ $\quad\quad$ $\Phi_{22} \xrightarrow{\text{coher}} \Phi_{33}$

d) base change isomorphism

$$
\begin{tikzcd}
\frac{X}{G} \arrow[r, "\frac{f}{1}_!"] \arrow[d, "\frac{h}{\phi}^*"'] & \frac{Y}{G} \arrow[d, "\frac{g}{\phi}^*"] \arrow[dl, Rightarrow] \\
\frac{W}{H} \arrow[r, "\frac{j}{1}_!"] & \frac{Z}{H}
\end{tikzcd}
$$

where $W = X \times_Y Z$, and h, j are the projections;

e) the isomorphism of $\frac{X}{G} \xrightarrow{\frac{f}{1}_!} \frac{Y}{G} \xrightarrow{\frac{g}{\phi}_*} \frac{Z}{H}$ with $\frac{X}{G} \xrightarrow{\frac{j}{\phi}_*} \frac{W}{H} \xrightarrow{\frac{h}{1}_!} \frac{Z}{H}$, where $K = \operatorname{Ker}(\phi : G \longrightarrow H)$, $W = K\backslash X$, $h = K\backslash f : W = K\backslash X \longrightarrow K\backslash Y = Z$, and j is the quotient map; it is given by the pasting

$$
\begin{tikzcd}
\frac{X}{G} \arrow[r, Rightarrow] \arrow[d, "\frac{j}{\phi}_*"'] & \frac{X}{G} \arrow[r, "\frac{f}{1}_!"] \arrow[d, Rightarrow] & \frac{Y}{G} \arrow[d, "\frac{g}{\phi}_*"] \\
\frac{W}{H} \arrow[r, "\frac{h}{1}_!"'] \arrow[ur, "\frac{j}{\phi}^*"] \arrow[u, phantom, shift left=6, "\Downarrow \varepsilon^{-1}"] & \frac{Z}{H} \arrow[r, Rightarrow] \arrow[ur, "\frac{g}{\phi}^*"] & \frac{Z}{H} \arrow[u, phantom, shift right=6, "\Downarrow \eta^{-1}"]
\end{tikzcd}
$$

f) the isomorphism of $\frac{X}{G} \xrightarrow{\frac{f}{\phi}_*} \frac{Y}{H} \xrightarrow{\frac{g}{\psi}^*} \frac{Z}{B}$ with $\frac{X}{G} \xrightarrow{\frac{h}{\xi}^*} \frac{W}{K} \xrightarrow{\frac{j}{\chi}_*} \frac{Z}{B}$, where $W = X \times_Y Z$, $K = G_\phi \underset{H}{\times}{}_\psi B$ and h, j, ξ, χ are the projections; it is given by the pasting

$$
\begin{tikzcd}
\frac{X}{G} \arrow[rr, "\frac{f}{\phi}_*"] \arrow[d, Rightarrow] & & \frac{Y}{H} \arrow[d, "\frac{g}{\psi}^*"] \arrow[dll, "\frac{f}{\phi}^*"'] \arrow[ddll] \\
\frac{X}{G} \arrow[d, "\frac{h}{\xi}^*"'] \arrow[u, phantom, shift right=4, "\Leftarrow \varepsilon"] & \Nwarrow \quad \Nwarrow & \frac{Z}{B} \arrow[d, Rightarrow] \arrow[dll, "\frac{j}{\chi}^*"] \\
\frac{W}{K} \arrow[rr, "\frac{j}{\chi}_*"'] & & \frac{Z}{B} \arrow[u, phantom, shift left=4, "\Leftarrow \eta"]
\end{tikzcd}
$$

4.1.1. Lemma. *Any 1-morphism $\frac{X}{G} \longrightarrow \frac{Y}{H}$ in $\mathfrak{D}$ is isomorphic to a composition of the form*

$$\frac{X}{G} \xrightarrow{\frac{f}{\phi}^*} \frac{Z}{B} \xrightarrow{\frac{g}{\psi}_*} \frac{W}{H} \xrightarrow{\frac{h}{1}_!} \frac{Y}{H}. \tag{4.1.1}$$

Proof. Isomorphisms d)–f) allow to place all functors in a given composition in the correct order: inverse image functors are followed by direct image functors and then by direct image functors with proper supports. Isomorphisms a)–c) reduce the sequence of functors of similar type to only one functor. □

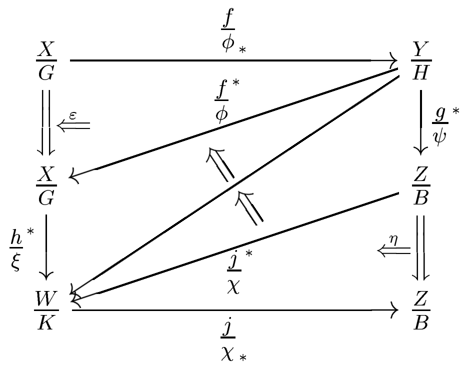
4.1.2. Theorem. *The category $\mathfrak{D}$ is a 2-groupoid, that is, for any 1-morphisms F and G with the common source and target the set $\operatorname{Hom}(F, G)$ either is empty or has exactly one element (and all 2-morphisms are invertible).*

The coherence isomorphism uses 3 more types of elementary isomorphisms.

F) A generalization of type f) isomorphism: the isomorphism of $\frac{X}{G} \xrightarrow{\frac{f}{\phi_*}} \frac{Y}{H} \xrightarrow{\frac{g^*}{\psi}} \frac{Z}{B}$ with $\frac{X}{G} \xrightarrow{\frac{h^*}{\xi}} \frac{W}{K} \xrightarrow{\frac{j}{\chi_*}} \frac{Z}{B}$, whenever the following square commutes

$$\begin{tikzcd}
X|G \arrow[r, "f|\phi"] & Y|H \\
W|K \arrow[u, "h|\xi"] \arrow[r, "j|\chi"] & Z|B \arrow[u, "g|\psi"']
\end{tikzcd}$$

the isomorphism is given by the pasting



s) Whenever a linear complex algebraic group P is a semidirect product of its subgroups U and L, such that U is normal in P and the manifold U is an affine space, and the action of U in a P-variety E is trivial, there is an isomorphism

$$\begin{tikzcd}
 & \frac{E}{L} \arrow[dr, "p^*"] \arrow[d, phantom, "\Downarrow"] & \\
\frac{E}{P} \arrow[ur, "i^*"] \arrow[rr, equal] & {} & \frac{E}{P}
\end{tikzcd}$$

constructed in Appendix A.1 as a corollary of Proposition A.1.1 for $Y = E$.

v) Whenever $h : E \to B$ is a complex affine linear G-bundle (h and linear combination maps are G-equivariant), then there is an isomorphism

$$\begin{tikzcd}
 & \frac{B}{G} \arrow[dl, "h^*"'] \arrow[dd, "T^{-2\,\mathrm{rk}_{\mathbb{C}}\, h}"] \\
\frac{E}{G} \arrow[dr, "h_!"'] \arrow[r, phantom, "\Leftarrow"] & {} \\
 & \frac{B}{G}
\end{tikzcd}$$

which is the inverse of isomorphism λ, constructed in Theorem B.2.5.

A more general V) type isomorphism is a composition of such. Let

$$E_1 \xrightarrow{h_1} E_2 \xrightarrow{h_2} \dots \xrightarrow{h_{n-1}} E_n \xrightarrow{h_n} B$$

be a sequence of affine linear G-bundles h_i of ranks d_i. Let h denote the composition $h_n \circ \cdots \circ h_2 \circ h_1$, and let $s^k = \sum_{i=1}^{k} d_i$, $d = s^n$. With this sequence of bundles we associate an isomorphism of V) type, inverse to the following isomorphism Λ:

$$
\begin{tikzcd}
 & & & & & & & \frac{B}{G} \arrow[ddlllllll, bend right, "h^*"'] \arrow[dl, "h_n^*"'] \arrow[dd, "{[-2d_n]}"] \\
 & \frac{E_2}{G} \arrow[dl, "h_1^*"'] \arrow[dd, "{[-2d_1]}"] \arrow[dr, "h_{2!}"] & \frac{E_3}{G} \arrow[l, "h_2^*"'] \arrow[d, "{[-2d_2]}"] \arrow[dr, "h_{3!}"] & \arrow[l, "h_3^*"'] & \cdots & \frac{E_{n-1}}{G} \arrow[d, "{[-2d_{n-2}]}"'] \arrow[dr, "h_{n-1!}"] & \frac{E_n}{G} \arrow[l, "h_{n-1}^*"'] \arrow[d, "{[-2d_{n-1}]}"] \arrow[dr, "h_{n!}"] \arrow[r, phantom, "\overset{\lambda_n}{\Longrightarrow}"] & \\
\frac{E_1}{G} \arrow[r, phantom, "\overset{\lambda_1}{\Longrightarrow}"] \arrow[dr, "h_{1!}"'] \arrow[ddrrrrrrr, bend right, "h_!"] & \arrow[r, phantom, "="] & \frac{E_3}{G} \arrow[d, "{[-2d_1]}"] \arrow[u, phantom, "\overset{\lambda_2}{\Longrightarrow}"'] & & & \frac{E_{n-1}}{G} \arrow[d, "{[-2s^{n-3}]}"] \arrow[r, phantom, "="] & \frac{E_n}{G} \arrow[d, "{[-2s^{n-2}]}"] \arrow[u, phantom, "\overset{\lambda_{n-1}}{\Longrightarrow}"'] & \frac{B}{G} \arrow[dd, "{[-2s^{n-1}]}"] \\
 & \frac{E_2}{G} \arrow[r, "h_{2!}"'] & \frac{E_3}{G} \arrow[r, "h_{3!}"'] & {} & \cdots & \frac{E_{n-1}}{G} \arrow[r, "h_{n-1!}"'] & \frac{E_n}{G} \arrow[dr, "h_{n!}"'] \arrow[r, phantom, "="] & {} \\
 & & & & & & & \frac{B}{G}
\end{tikzcd}
\qquad (4.1.2)
$$

A. Technical results

A.1. An equivalence of equivariant derived categories

Let P be a linear complex algebraic group, let U, L be its subgroups such that U is normal in P and the manifold U is an affine space. Assume that P is a semidirect product of U and L, that is, the multiplication map $\cdot : U \times L \to P$ is an isomorphism of manifolds. Denote by $\iota : L \hookrightarrow P$ and $\kappa : P \twoheadrightarrow L$ the natural inclusion and projection. Let Y be a P-variety, let E be an L-variety, let $i : E \to Y$ be a ι-map and let $p : Y \to E$ be a κ-map identifying E with the quotient $U \backslash Y$ such that $p \circ i = \mathrm{id}_E$. Consider

$$W = P \times_L E = (P \times E)/L, \quad (p,e).l \overset{\text{def}}{=} (pl, l^{-1}e).$$

The variety W is a P-variety via left translations on P. As a P-variety W is isomorphic to $U \times E$, where P acts by

$$(u,l).(v,e) = (u \cdot (lvl^{-1}), l.e).$$

There is a surjective map of P-varieties

$$\pi : W \simeq U \times E \to Y, \quad (u,e) \mapsto u.i(e).$$

Assume furthermore that π is smooth (a locally trivial fibration), and its fibres are affine spaces. Then $\pi : W \to Y$ is ∞-acyclic in the sense of [BL94].

A.1.1. Proposition ([Lyu]). *The functors*

$$\begin{aligned} \tfrac{\pi}{1}^* &: \tfrac{Y}{P} \to \tfrac{W}{P}, \\ \tfrac{p}{\kappa}^* &: \tfrac{E}{L} \to \tfrac{Y}{P}, \\ \tfrac{i}{\iota}^* &: \tfrac{Y}{P} \to \tfrac{E}{L} \end{aligned}$$

are equivalences, and the last two are quasi-inverse to each other.

Using the fact that $\frac{i}{\iota}^*$ is an equivalence, we define unambiguously an isomorphism

$$\begin{tikzcd}
\frac{Y}{P} \arrow[r, "\frac{i}{\iota}^*"] \arrow[dr, Rightarrow, no head, "\mathrm{Id}"'] \arrow[dr, phantom, shift left=3, "\boxed{\mathrm{S}}"] & \frac{E}{L} \arrow[d, "\frac{p}{\kappa}^*"] \\
& \frac{Y}{P}
\end{tikzcd}$$

by the equation

$$\begin{tikzcd}
\frac{Y}{P} \arrow[r, "\frac{i}{\iota}^*"] \arrow[d, "\frac{i}{\iota}^*"'] \arrow[dr, Rightarrow, no head] \arrow[dr, phantom, shift left=3, "\boxed{\mathrm{S}}"] \arrow[dr, phantom, shift right=3, "="] & \frac{E}{L} \arrow[d, "\frac{p}{\kappa}^*"] \\
\frac{E}{L} & \frac{Y}{P} \arrow[l, "\frac{i}{\iota}^*"]
\end{tikzcd}
=
\begin{tikzcd}
\frac{Y}{P} \arrow[r, "\frac{i}{\iota}^*"] \arrow[d, "\frac{i}{\iota}^*"'] \arrow[dr, phantom, shift right=4, "="] & \frac{E}{L} \arrow[d, "\frac{p}{\kappa}^*"] \arrow[dl, Rightarrow, no head] \arrow[dl, phantom, shift right=3, "\boxed{\mathrm{a}}"] \\
\frac{E}{L} & \frac{Y}{P} \arrow[l, "\frac{i}{\iota}^*"]
\end{tikzcd}
\tag{A.1.1}$$

B. Affine linear bundles

B.1. Non-equivariant case

An *affine linear bundle* of rank d over a complex manifold B is a morphism of complex manifolds $h : E \to B$, together with additional data consisting of an open covering $\{U_i\}$ of B, and isomorphisms $\psi_i : h^{-1}(U_i) \to \mathbb{C}^d \times U_i$ over U_i, such that for any i, j, and for $U = U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{C}^d \times U$ is fiberwise affine-linear, i.e., $\psi(x, u) = (A(u)x + b(u), u)$, $u \in U$, for some $\mathbb{C}$-linear maps $A(u) : \mathbb{C}^d \to \mathbb{C}^d$ and some vectors $b(u) \in \mathbb{C}^d$.

We are going to discuss a functor in derived category, which is a kind of braiding in a monoidal 2-category.

B.1.1. Theorem. *Let $h : E \to B$ be an affine linear bundle of dimension d over a complex manifold B. Then there is an isomorphism of functors*

$$\lambda_B : \big[D^{b,c}(B) \xrightarrow{h^*} D^{b,c}(E) \xrightarrow{h_!} D^{b,c}(B)\big] \simeq \big[D^{b,c}(B) \xrightarrow{[-2d]} D^{b,c}(B)\big].$$

B.1.2. Proposition. *Let $f : F \to B$ and $h : E \to B$ be affine linear bundles, and let $y : F \to E$ be a surjective map of affine linear bundles over B with its natural structure of an affine linear bundle. Then the following isomorphisms are equal*

$$\begin{tikzcd}
& D^{b,c}(E) \arrow[dl, "y^*"'] \arrow[dd, "{[-2\,\mathrm{rk}\,y]}"] \arrow[dr, "h_!"'] & D^{b,c}(B) \arrow[l, "h^*"'] \arrow[d, "{[-2\,\mathrm{rk}\,h]}"] \arrow[dl, phantom, shift right=3, "\overset{\lambda_h}{\Longrightarrow}"] \\
D^{b,c}(F) \arrow[r, phantom, "\overset{\lambda_y}{\Longrightarrow}"] \arrow[dr, "y_!"'] & {} & D^{b,c}(B) \arrow[d, "{[-2\,\mathrm{rk}\,y]}"] \arrow[dl, phantom, "="] \\
& D^{b,c}(E) \arrow[r, "h_!"'] & D^{b,c}(B)
\end{tikzcd}
=
\begin{tikzcd}
& D^{b,c}(E) \arrow[dl, "y^*"'] & D^{b,c}(B) \arrow[l, "h^*"'] \arrow[dll, "f^*"] \arrow[dd, "{[-2\,\mathrm{rk}\,f]}"] \\
D^{b,c}(F) \arrow[rr, phantom, "\overset{\lambda_f}{\Longrightarrow}"] \arrow[dr, "y_!"'] \arrow[drr, "f_!"] & & {} \\
& D^{b,c}(E) \arrow[r, "h_!"'] & D^{b,c}(B)
\end{tikzcd}$$

B.2. Equivariant affine linear bundles

B.2.1. Definition. An affine linear bundle $h : E \to B$, which is a G-equivariant map, is called an affine linear G-bundle if elements of G map fibers to fibers, preserving affine linear structure.

B.2.2. Lemma. *Let $h : E \to B$ be an affine linear G-bundle, and let K be a normal subgroup of G. Assume that B is a free K-space. Then the quotient $K\backslash h : K\backslash E \to K\backslash B$ is an affine linear G/K-bundle.*

One of the sources of affine linear G-bundles is the following. Let M be a G-module over $\mathbb{C}$, and let N be its G-submodule. Then the projection $\pi : M \to M/N$ equipped with the linear combination map

$$\psi : \mathbb{C} \times M \times_{M/N} M \to M, \quad (\lambda, m, m') \mapsto \lambda m + (1-\lambda)m',$$

is an affine linear G-bundle. Indeed, denote $L = M/N$ and fix $\lambda \in \mathbb{C}$. To prove that

$$\begin{tikzcd} M \times_L M \arrow[r, "\psi(\lambda)"] \arrow[d, "g\times g"'] & M \arrow[d, "g"] \\ M \times_L M \arrow[r, "\psi(\lambda)"] & M \end{tikzcd}$$

commutes choose a splitting $i : L \to M$ of π, so that $M = i(L) \oplus N$ as a vector space, and use $g.(i(l), n) = (i(g.l), \nu(g, l) + n)$. (If N has a direct complement G-submodule in M, then $\pi : M \to M/N$ has an obvious structure of a vector G-bundle.)

The above consideration apply, in particular, to $P_{\mathcal{W}} \cap P_{\mathcal{V}}$-modules $N = f_{\mathcal{W}} \cap f_{\mathcal{V}} \subset F_{\mathcal{W}} \cap F_{\mathcal{V}} = M$, giving an affine $P_{\mathcal{W}} \cap P_{\mathcal{V}}$-bundle

$$F_{\mathcal{W}} \cap F_{\mathcal{V}} \to (F_{\mathcal{W}} \cap F_{\mathcal{V}})/(f_{\mathcal{W}} \cap f_{\mathcal{V}}),$$

or, among many others, an affine $P_{Y_-} \cap P_{Y^-}$-bundle

$$F_{Y_-} \cap F_{Y^-} \to (F_{Y_-} \cap F_{Y^-})/(f_{Y_-} \cap f_{Y^-}).$$

B.2.3. A tower of affine line bundles. Identifying U_{Y_-} with $1 + \mathfrak{n}_{Y_-}$, where $\mathfrak{n}_{Y_-}$ is a Lie algebra of U_{Y_-} (strictly uppertriangular matrices), we could make U_{Y_-} into a P_{Y_-}-module via conjugation. We need, however, to deal with the $P_{Y_-} \cap P_{Y^-}$-variety $U_{Y_-}/(U_{Y_-} \cap P_{Y^-})$. The polynomial action of $P_{Y_-} \cap P_{Y^-}$ can be presented as action in a tower of affine linear bundles as follows.

Consider the partition (Young diagram) $\lambda = (m-1, m-2, \dots, 2, 1)$, identified with the shape of strictly uppertriangular $m \times m$ matrices. Consider a (non-standard, row strict) *Young tableau* for this diagram, which we define specialising the usual term [Ful97, Mac95] to our needs as an $m \times m$ matrix T with integer coefficients, such that (i) for some $n \in \mathbb{Z}$ the set of values $\{T_{kl}\}$ is $[0, n]$, (ii) $T_{kl} = 0$ iff $k \geqslant l$, (iii) $l < l'$ implies $T_{kl} < T_{kl'}$, and (iv) $k < k'$ implies $T_{kl} \geqslant T_{k'l}$. A Young tableau is standard if each value from $[1, n]$ occurs exactly once in (T_{kl}). To a Young tableau T corresponds a series of normal subgroups of $U = U_{Y_-}$, namely,

$$U = U^{(1)} \supset U^{(2)} \supset \dots \supset U^{(n)} \supset U^{(n+1)} = 1, \tag{B.2.1}$$

where

$$U^{(p)} = 1 + \{A = (A_{kl})_{k,l=1}^m,\ A_{kl} \in \mathrm{Hom}_{\mathbb{C}}(Y_l, Y_k) \mid A_{kl} \neq 0 \implies T_{kl} \geqslant p\}.$$

For instance, the lower central series $U = U_1 \supset U_2 \supset \dots \supset U_m = 1$, where $U_{k+1} = \{U, U_k\}$, corresponds to $n = m - 1$ and

$$T = \begin{pmatrix} 0 & 1 & \ddots & m-2 & m-1 \\ 0 & 0 & \ddots & m-3 & m-2 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & \ddots & 0 & 0 \end{pmatrix} \tag{B.2.2}$$

A Young tableau T defines a sequence of Young diagrams

$$\lambda = T_1 \supset \cdots \supset T_k \supset T_{k+1} \supset \cdots \supset T_m = \varnothing,$$

where T_k is the union of places (i,j) for which $T_{ij} \geqslant k$. The corresponding ideals of the algebra of triangular matrices are

$$\mathfrak{n}_{V_-}^{T_k} = \{A = (A_{ij})_{i,j=1}^m,\ A_{ij} \in \mathrm{Hom}_{\mathbb{C}}(V_j, V_i) \mid A_{ij} \neq 0 \implies (i,j) \in T_k\},$$

and the corresponding normal subgroups of U_{V_-} are $U_{V_-}^{T_k} = 1 + \mathfrak{n}_{V_-}^{T_k}$. It follows from our definition of Young tableau that

$$T_k - T_{k+1} \textit{ has no more than 1 element in each row.} \tag{B.2.3}$$

B.2.4. Lemma. *A pair of embedded Young diagrams $T_k \supset T_{k+1}$ satisfying condition* (B.2.3) *gives rise to an affine linear $P_{V_-} \cap P_{V^-}$-bundle*

$$h_k : E_{k+1} = U_{V_-}/[U_{V_-}^{T_{k+1}}(U_{V_-} \cap P_{V^-})] \to U_{V_-}/[U_{V_-}^{T_k}(U_{V_-} \cap P_{V^-})] = E_k.$$

Proof. Let $e, f \in E_{k+1}$ lie over the same point $a = h_k(e) = h_k(f) \in E_k$. Their linear combination $\lambda e + (1-\lambda) f \in E_{k+1}$, $\lambda \in \mathbb{C}$, is defined as follows. Consider $\tilde e, \tilde f \in U_{V_-}$ such that $e = \tilde e U_{V_-}^{T_{k+1}}(U_{V_-} \cap P_{V^-})$, $f = \tilde f U_{V_-}^{T_{k+1}}(U_{V_-} \cap P_{V^-})$. Then we define $\lambda e + (1-\lambda) f \in E_{k+1}$ as

$$(\lambda \tilde e + (1-\lambda)\tilde f) U_{V_-}^{T_{k+1}}(U_{V_-} \cap P_{V^-}). \tag{B.2.4}$$

Let us check consistency of this definition.

We have $a = h_k(e) = \tilde e U_{V_-}^{T_k}(U_{V_-} \cap P_{V^-})$, $a = h_k(f) = \tilde f U_{V_-}^{T_k}(U_{V_-} \cap P_{V^-})$. Let $g \in U_{V_-}^{T_k}(U_{V_-} \cap P_{V^-})$ be such that $\tilde e = \tilde f g$. Then

$$\lambda \tilde e + (1-\lambda)\tilde f = \tilde f(\lambda g + 1 - \lambda).$$

Hence, point (B.2.4) lies over

$$(\lambda \tilde e + (1-\lambda)\tilde f) U_{V_-}^{T_k}(U_{V_-} \cap P_{V^-}) = \tilde f(\lambda g + 1 - \lambda) U_{V_-}^{T_k}(U_{V_-} \cap P_{V^-}) = a,$$

since $b \overset{\text{def}}{=} \lambda g + 1 - \lambda \in U_{V_-}^{T_k}(U_{V_-} \cap P_{V^-})$.

Let $e' = \tilde e u$, $f' = \tilde f v$ be another representatives of e, f for $u, v \in U_{V_-}^{T_{k+1}}(U_{V_-} \cap P_{V^-})$. Then

$$\begin{aligned} &\lambda e' + (1-\lambda) f' = \lambda \tilde e u + (1-\lambda)\tilde f v = \tilde f[\lambda g u + (1-\lambda) v] = \tilde f[\lambda g + (1-\lambda) v u^{-1}] u \\ &= \tilde f(\lambda g + 1 - \lambda)(\lambda g + 1 - \lambda)^{-1}[\lambda g + (1-\lambda) + (1-\lambda) m] u = [\lambda \tilde e + (1-\lambda)\tilde f][1 + (1-\lambda) b^{-1} m] u, \end{aligned}$$

where we denote $vu^{-1} = 1 + m$, $m \in \mathfrak{n}_{V_-}^{T_{k+1}} + (\mathfrak{n}_{V_-} \cap P_{V^-})$. Let us decompose some terms in this formula as follows: $(1-\lambda) m = m' + m''$, $m' \in \mathfrak{n}_{V_-}^{T_{k+1}}$, $m'' \in \mathfrak{n}_{V_-} \cap P_{V^-}$, and $b^{-1} = 1 + b' + b''$, $b' \in \mathfrak{n}_{V_-}^{T_k}$, $b'' \in \mathfrak{n}_{V_-} \cap P_{V^-}$. Then

$$\begin{aligned} b^{-1}(1-\lambda) m = (1 + b' + b'')(m' + m'') = (1 + b' + b'') m' + (1 + b'') m'' + b' m'' \\ \in \mathfrak{n}_{V_-}^{T_{k+1}} + \mathfrak{n}_{V_-} \cap P_{V^-} + \mathfrak{n}_{V_-}^{T_k} \cdot \mathfrak{n}_{V_-}. \end{aligned}$$

Condition (B.2.3) implies that $\mathfrak{n}_{V_-}^{T_k} \cdot \mathfrak{n}_{V_-} \subset \mathfrak{n}_{V_-}^{T_{k+1}}$, hence,

$$\lambda e' + (1-\lambda) f' \in [\lambda \tilde e + (1-\lambda)\tilde f] U_{V_-}^{T_{k+1}}(U_{V_-} \cap P_{V^-}).$$

Therefore, linear combinations of elements are well-defined.

The obtained affine linear bundle is a $P_{V_-} \cap P_{V^-}$-bundle, on which the group acts by conjugation. Indeed, let $p \in P_{V_-} \cap P_{V^-}$. If $h_k(e) = h_k(f)$ as above, then $h_k(p.e) = h_k(p.f)$. Furthermore,

$$\begin{aligned} p.[\lambda e + (1-\lambda)f] &= p[\lambda\tilde{e} + (1-\lambda)\tilde{f}]p^{-1}U_{V_-}^{T_{k+1}}(U_{V_-} \cap P_{V^-}) \\ &= [\lambda p\tilde{e}p^{-1} + (1-\lambda)p\tilde{f}p^{-1}]U_{V_-}^{T_{k+1}}(U_{V_-} \cap P_{V^-}) = \lambda(p.e) + (1-\lambda)(p.f). \end{aligned}$$

Thus, linear combinations are equivariant. ☐

B.2.5. Theorem. *Let $h : E \to B$ be an affine linear bundle G-bundle of dimension d over a complex manifold B. Then there is an isomorphism of functors*

$$\lambda : \left[D_G^{b,c}(B) \xrightarrow{h^*} D_G^{b,c}(E) \xrightarrow{h_!} D_G^{b,c}(B)\right] \xrightarrow{\sim} \left[D_G^{b,c}(B) \xrightarrow{[-2d]} D_G^{b,c}(B)\right].$$

B.2.6. Proposition. *Let $f : F \to B$ and $h : E \to B$ be affine linear G-bundles, and let $y : F \to E$ be a G-equivariant surjective map of affine linear bundles over B with its natural structure of an affine linear G-bundle. Then the following isomorphisms are equal*

$$\begin{tikzcd} & \frac{E}{G} \arrow[dl, "y^*"'] \arrow[dr, "h_!"] \arrow[dd, "{[-2\,\mathrm{rk}\,y]}"] & \frac{B}{G} \arrow[l, "h^*"'] \arrow[d, "{[-2\,\mathrm{rk}\,h]}"] \arrow[d, phantom, shift right=4, "\overset{\lambda_h}{\Longrightarrow}"] \\ \frac{F}{G} \arrow[r, phantom, "\overset{\lambda_y}{\Longrightarrow}"] \arrow[dr, "y_!"'] & {} \arrow[r, phantom, "="] & \frac{B}{G} \arrow[d, "{[-2\,\mathrm{rk}\,y]}"] \\ & \frac{E}{G} \arrow[r, "h_!"'] & \frac{B}{G} \end{tikzcd} = \begin{tikzcd} & \frac{E}{G} \arrow[dl, "y^*"'] & \frac{B}{G} \arrow[l, "h^*"'] \arrow[dll, "f^*"] \arrow[dd, "{[-2\,\mathrm{rk}\,f]}"] \\ \frac{F}{G} \arrow[rr, phantom, "\overset{\lambda_f}{\Longrightarrow}"] \arrow[dr, "y_!"'] \arrow[drr, "f_!"] & & {} \\ & \frac{E}{G} \arrow[r, "h_!"'] & \frac{B}{G} \end{tikzcd}$$

Proof. To define inverse image functors it suffices to work with pull-back resolutions. These will be used also for direct image with proper support functors. Thus, for an arbitrary G-resolution $p : P \to B$ we pull back the affine linear G-bundle structures and take their quotients by G:

$$\begin{tikzcd} \overline{P \times_B F} \arrow[dr, "\overline{1\times y}"] \arrow[dd, "\overline{\mathrm{pr}_1}"'] & & P \times_B F \arrow[ll] \arrow[rr] \arrow[dr, "1\times y"] \arrow[dd, "\mathrm{pr}_1"'] & & F \arrow[dr, "y"] \arrow[dd, "f"'] & \\ & \overline{P \times_B E} \arrow[dl, "\overline{\mathrm{pr}_1}"] & & P \times_B E \arrow[ll] \arrow[rr] \arrow[dl, "\mathrm{pr}_1"] & & E \arrow[dl, "h"] \\ \overline{P} & & P \arrow[ll] \arrow[rr] & & B & \end{tikzcd}$$

Therefore, the proposition reduces to its non-equivariant version Proposition B.1.2. ☐

B.2.7. Example. Let $K \subset N \subset M$ be embedded G-modules. There are quotient maps

$$\begin{tikzcd} M \arrow[rr, "y"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "="] & M/K \arrow[dl, "h"] \\ & M/N & \end{tikzcd}$$

According to Section B.2.2 each of these three maps has a natural structure of an affine linear G-bundle. Furthermore, $y : M \to M/K$ is a surjective map between affine linear G-bundles f and h over M/N. The arising structure of y as an affine linear G-bundle coincides with that of Section B.2.2, since both come, in the final account, from the vector space structure of M. Therefore, Proposition B.2.6 applies in this situation.

B.2.8. Relations between towers of affine line bundles. Let

$$h = \big(E_1 \xrightarrow{h_1} E_2 \xrightarrow{h_2} \dots \xrightarrow{h_{n-1}} E_n \xrightarrow{h_n} B\big)$$

be a sequence of affine linear G-bundles. Suppose that each map $h_k : E_k \to E_{k+1}$ equals to a composition of several maps

$$h_k = \big(E_k = E_k^0 \xrightarrow{h_k^1} E_k^1 \xrightarrow{h_k^2} \dots E_k^{m-1} \xrightarrow{h_k^m} E_k^m = E_{k+1}\big)$$

(m can be 1 as well as bigger integer), where each composite map $E_k^j \to E_{k+1}$ is an affine linear G-bundle, and $h_k^j : E_k^{j-1} \to E_k^j$ is a surjection of affine linear bundles. Then h_k^j is given its natural structure of affine linear G-bundle, and we can form a sequence $(h_k^j) : E_1 \to \dots \to B$ of affine linear G-bundles. We may associate type V^{-1}) isomorphisms $\Lambda : h_! h^* \to [-2d]$ given by (4.1.2) to both sequences (h_k) and (h_k^j). Repeated application of Proposition B.2.6 shows that both isomorphisms Λ coincide.

Examples of such situations are provided by Section B.2.3. To normal series (B.2.1) corresponds a sequence of affine linear P_{Y_-}-bundles

$$U = U/U^{(n+1)} \longrightarrow \dots \longrightarrow U/U^{(k+1)} \longrightarrow U/U^{(k)} \longrightarrow \dots \longrightarrow U/U^{(1)} = pt$$

and a series of affine linear $P_{Y_-} \cap P_{Y^-}$-bundles

$$h = \big(U/(U \cap P_{Y^-}) \longrightarrow \dots \longrightarrow U/[U^{(k+1)}(U \cap P_{Y^-})] \longrightarrow U/[U^{(k)}(U \cap P_{Y^-})] \longrightarrow \dots pt\big).$$

Let Γ be the following graph, whose vertices are Young tableaux of shape λ. An edge connects tableaux T and T' if for some $1 \leqslant p < n$ we have $\{T_{kl}\} = [1, n]$, $\{T'_{kl}\} = [1, n+1]$, $T_{kl} < p$ implies $T'_{kl} = T_{kl}$, $T_{kl} > p$ implies $T'_{kl} = T_{kl}+1$, and $T_{kl} = p$ implies that $T'_{kl} \in \{p, p+1\}$. Notice that if two tableaux T and T' are connected by an edge, then the corresponding isomorphisms $\Lambda : h_! h^* \to [-2d]$ are equal by Proposition B.2.6.

The graph Γ is connected. Indeed, any tableau is connected by a path with a standard tableau. Any standard tableau is connected to the tableau

$$\overline{T} = \begin{pmatrix} 0 & 1 & m & \ddots & n-2 & n \\ 0 & 0 & 2 & \ddots & n-4 & n-1 \\ 0 & 0 & 0 & \ddots & \ddots & n-3 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & 0 & m-1 \\ 0 & 0 & 0 & \ddots & 0 & 0 \end{pmatrix}, \tag{B.2.5}$$

where $n = m(m-1)/2$, by a sequence of double moves – pairs of edges, which replace values p, $p+1$ of a pair of entries (not contained in the same row or column) with values p, p, and then with $p+1$, p. Let us say that the rating of entry (k,l) is $\overline{T}_{kl}$. Start with the highest rated entry for which $T_{kl} \neq \overline{T}_{kl}$, hence, $T_{kl} < \overline{T}_{kl}$. Replace T_{kl} with $T_{kl}+1$ using a double move, then with $T_{kl}+2$ and so on until T_{kl} becomes equal to $\overline{T}_{kl}$. Iterating this procedure we transform T in $\overline{T}$.

Since Γ is connected, *all* isomorphisms $\Lambda : h_! h^* \to [-2d]$ obtained above from Young tableaux are equal.

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