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TWISTED BURNSIDE-FROBENIUS THEORY FOR ENDOMORPHISMS

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Abstract. We prove that the number $R(\varphi)$ of φ-conjugacy (or Reidemeister) classes of an endomorphism φ (and its iterations) of a group G from several classes of groups (including polycyclic) is equal to the number of fixed points of the induced map $\widehat{\varphi}$ (respectively, its iterations) of an appropriate subspace of the unitary dual $\widehat{G}$, when $R(\varphi) < \infty$ (respectively, $R(\varphi^n) < \infty$). This implies Gauss congruences for Reidemeister numbers.

In contrast with the case of automorphisms, studied previously, we have a plenty of examples, even among groups with R_∞ property.

Introduction

The *Reidemeister number* or *φ-conjugacy number* of an endomorphism φ of a group G is the number of its *Reidemeister* or *φ-conjugacy classes*, defined by the equivalence

$$g \sim xg\varphi(x^{-1}).$$

They play an important role in several fields of Mathematics, including Algebraic Geometry and Dynamics (see e.g. an exposition in [16, 23] and basic sources [44, 1, 29, 32, 9]).

An important problem in the field is to identify the Reidemeister numbers with numbers of fixed points on an appropriate space in a way respecting iterations. This opens possibility of obtaining congruences for Reidemeister numbers and other important information.

For the role of the above "appropriate space" typically some versions of unitary dual can be taken. This desired construction is called the twisted Burnside-Frobenius theory (TBFT), because in the case of a finite group and identity automorphism we arrive to the classical Burnside-Frobenius theorem on enumerating of (usual) conjugacy classes.

In the case of automorphism this problem was solved for polycyclic-by-finite groups in [16, 23]. Preliminary and related results, examples and counter-examples can be found in [11, 8, 9, 15, 17, 12, 14, 47, 49].

The importance of obtaining the present results namely for endomorphisms is justified by a plenty of examples (in contrast with the case of automorphisms) (see, in particular, 1.1, 1.4, and 6.4 below).

A related fact is that, in contrast with TBFT for automorphisms, TBFT for endomorphisms is weakly connected with the theory of R_∞-groups (see e.g. Example 6.4). A group is called R_∞ if any its *automorphism* has infinite Reidemeister number. This was the subject of an intensive recent research and for many groups this property was established, see the following partial bibliography and the literature therein: [10, 36, 18, 19, 46, 30, 13, 2, 26, 4,

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20, 33, 37, 40, 38, 6, 21, 27, 28, 43, 7, 5, 22, 31, 48]. In some situations the property R_∞ has some direct topological consequences (see e.g. [28]).

Decision problems for twisted conjugacy classes of endomorphisms of polycyclic groups were studied in [42].

The paper is organized in the following way.

In Section 1 we show how drastically the theory of twisted conjugacy classes for endomorphisms differs from the theory for automorphisms.

In Section 2 we introduce and investigate a dual object for a pair (G, φ).

In Section 3 we prove Gauss congruences for Reidemeister numbers.

In Section 4 we discuss proof of twisted Burnside–Frobenius theorem for endomorphisms of any finite group.

Section 5 is technical.

In Section 6 we proof the twisted Burnside–Frobenius theorem for endomorphisms of polycyclic groups. We finish the paper with a series of examples of R_∞ groups admitting endomorphisms with finite Reidemeister numbers .

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1. Preliminaries

First of all, let us make the following observation, showing how drastically the Reidemeister numbers world for endomorphisms differs from the Reidemeister world for automorphisms.

Proposition 1.1. *For any group G there exists an endomorphism $\varphi : G \to G$ with $R(\varphi) < \infty$, namely $R(\varphi) = 1$.*

Proof. Take φ to be the trivial map $\varphi(g) = e$ for any $g \in G$. □

This observation can be enforced.

Proposition 1.2. *Suppose, $\varphi : G \to G$ is an endomorphism and $K := \operatorname{Ker} \varphi$. Then all Reidemeister classes are some unions of K-cosets.*

Proof. Let g_1 and g_2 be in a K-coset, i.e. $g_1 g_2^{-1} = k \in K$. Then $g_1 = kg_2 = kg_2\varphi(k^{-1})$. □

Using Lemma 1.9 we immediately obtain

Corollary 1.3. *The map $p_\varphi : G \to G/K$ gives a bijection of Reidemeister numbers.*

Corollary 1.4. *Any endomorphism with finite image has a finite Reidemeister number.*

Definition 1.5. Denote by $\widehat{G}$ the *unitary dual* of G, by $\widehat{G}_f$ the part of the unitary dual formed by irreducible finite-dimensional representations, and by $\widehat{G}_{ff}$ the part of $\widehat{G}_f$ formed by *finite* representations, i.e. representations that factorize through a finite group.

Definition 1.6. Let us call *o.t. commutant* the operator theoretical commutant of a set D of bounded operators on a Hilbert space, i.e. the subset of the algebra of all bounded operators on this space, formed by all operators that commute with all elements of D. Denote it $D^\star$.

Lemma 1.7. *A representation is irreducible if and only if the o.t. commutant of the set of representing operators is formed just by scalar operators* [34, Theorem 2, p. 114].

In particular, if ρ is irreducible and φ is an epimorphism, $\rho \circ \varphi$ is also irreducible.

Lemma 1.8. *If representations π and ρ of G are equivalent, then $\pi \circ \varphi$ and $\rho \circ \varphi$ are equivalent for any endomorphism $\varphi : G \to G$.*

Proof. Indeed: use the same intertwining operator. □

The following statement is well known

Lemma 1.9. *Suppose, $\varphi : G \to G$ is an endomorphism and $H \subset G$ is a normal φ-invariant subgroup, then $p : G \to G/H$ induces an epimorphism of Reidemeister classes.*

Proof. Indeed, suppose, $p(g') = p(\widetilde{g})p(g)p(\varphi(\widetilde{g}^{-1}))$. Then it is equal to $p(\widetilde{g}g\varphi(\widetilde{g}^{-1}))$. □

Also, we need the following

Lemma 1.10. *Any Reidemeister class of φ is φ-invariant.*

Proof. Indeed, $\varphi(x) = x^{-1}x\varphi(x)$. □

The following fact can be extracted from [25, Prop. 1.6].

Lemma 1.11. *In the above situation $R(\varphi|_H) \leqslant R(G) \cdot |C(\varphi_{G/H})|$ where $C(\varphi_{G/H})$ is the fixed point subgroup for the induced map $\varphi_{G/H} : G/H \to G/H$.*

The following statement is well known in the field.

Lemma 1.12. *A right shift by $g \in G$ maps Reidemeister classes of φ onto Reidemeister classes of $\tau_{g^{-1}} \circ \varphi$, where τ_g is the inner automorphism: $\tau_g(x) = gxg^{-1}$.*

Proof. This follows immediately from the equality

$$xy\varphi(x^{-1})g = x(yg)g^{-1}\varphi(x^{-1})g = x(yg)(\tau_{g^{-1}} \circ \varphi)(x^{-1}).$$

□

2. Dual object for a pair (G, φ)

Definition 2.1. We will call a representation ρ a $\widehat{\varphi}$-**f**-point, if ρ is equivalent to $\rho \circ \phi$ (we avoid to say that it is a fixed point, because we can not define the corresponding dynamical system).

Definition 2.2. An element $[\rho] \in \widehat{G}$ (respectively, in $\widehat{G}_f$ or $\widehat{G}_{ff}$) is called *φ-irreducible* if $\rho \circ \varphi^n$ is irreducible for any $n = 0, 1, 2, \dots$.

Denote the corresponding subspaces of $\widehat{G}$ (resp., $\widehat{G}_f$ or $\widehat{G}_{ff}$) by $\widehat{G}^\varphi$ (resp., $\widehat{G}_f^\varphi$ or $\widehat{G}_{ff}^\varphi$).

In some important cases these subspaces coincide with the entire spaces:

Proposition 2.3. *(1) If G is abelian, $\widehat{G} = \widehat{G}_f = \widehat{G}^\varphi = \widehat{G}_f^\varphi$.*

(2) If φ is an epimorphism, $\widehat{G} = \widehat{G}^\varphi$ and $\widehat{G}_f = \widehat{G}_f^\varphi$

Proof. The first statement immediately follows from the fact that a representation of an abelian group is irreducible if and only if it is 1-dimensional.

The second one follows from Lemma 1.7 keeping in mind that φ^n is an epimorphism if and only if φ is. □

Evidently continuous (w.r.t. the topology of weak containment) maps $\widehat{\varphi^n} : [\rho] \mapsto [\rho \circ \varphi^n]$ are defined for both subspaces (generally not a homeomorphism!) and $\widehat{\varphi^n} = (\widehat{\varphi})^n$ Thus we obtain a dynamical system as the corresponding action of the semigroup $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ (we will reserve $\mathbb{N}$ for $\{1, 2, \dots\}$).

The key observation is the following one:

Lemma 2.4. *Let $[\rho] \in \widehat{G}$ be an φ^n-$\mathbf{f}$-point for some $n \geq 1$, i.e. the representations $[\rho]$ and $[\rho \circ \varphi^n]$ are equivalent. Then $[\rho] \in \widehat{G}^\varphi$.*

Proof. By Lemma 1.8 we obtain

$$\rho \sim \rho \circ \varphi^n \sim \dots \sim \rho \circ \varphi^{kn} \sim \dots$$

In particular, all they are irreducible. Now consider an arbitrary m and choose k such that $m \leq kn$. Then $\operatorname{Im} \rho \circ \varphi^m \supseteq \operatorname{Im} \rho \circ \varphi^{kn}$ and $(\operatorname{Im} \rho \circ \varphi^m)^\star \subseteq (\operatorname{Im} \rho \circ \varphi^{kn})^\star$. It remains to apply Lemma 1.7 and conclude that $\rho \circ \varphi^m$ is irreducible. □

Corollary 2.5. *So, we have no dynamical system generated by φ on $\widehat{G}$ (resp, $\widehat{G}_f$, or $\widehat{G}_{ff}$) generally, but we have the well-defined notion of a $\widehat{\varphi}^n$-$\mathbf{f}$-point.*

The corresponding well-defined dynamical system exists on $\widehat{G}^\varphi$ (resp, $\widehat{G}^\varphi_f$, or $\widehat{G}^\varphi_{ff}$) and its n-periodic points are exactly $\widehat{\varphi}^n$-$\boldsymbol{f}$-points.

Definition 2.6. Denote the number of $\widehat{\varphi}^n$-$\mathbf{f}$-points by $\mathbf{F}(\widehat{\varphi}^n)$.

3. TBFT implies congruences

Definition 3.1. We say that TBFT (resp., TBFT$_f$, TBFT$_{ff}$) takes place for an endomorphism $\varphi : G \to G$ and its iterations, if $R(\varphi^n) < \infty$ and $R(\varphi^n)$ coincides with the number of $\widehat{\varphi}^n$-$\mathbf{f}$-points in $\widehat{G}$ (resp., in $\widehat{G}_f$, $\widehat{G}_{ff}$) for all $n \in \mathbb{N}$.

Similarly, one can give a definition for a single endomorphism (without iterations).

Definition 3.2. Denote by $\mu(d)$, $d \in \mathbb{N}$, be the *Möbius function*, i.e.

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } d \text{ is not square} - \text{free.} \end{cases}$$

Theorem 3.3. *Suppose, TBFT (resp., TBFT$_f$ or TBFT$_{ff}$) takes place for an endomorphism $\varphi : G \to G$ and its iterations. In particular, $R(\varphi^n) < \infty$ for any n. Then one has the following congruences for Reidemeister numbers:*

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n$$

for any n.

Proof. This follows from 2.5 and the general theory of congruences for periodic points (cf. [45, 50]).

More precisely, let P_n be the number of periodic points of least period n of the dynamical system of 2.5. Then $R(\varphi^n) = \mathbf{F}(\widehat{\varphi}^n) = \sum_{d|n} P_d$. By the Möbius inversion formula,

$$\sum_{d|n} \mu(d) R(\varphi^{n/d}) = P_n \equiv 0 \mod n,$$

since each orbit brings to P_n just n points. □

Remark 3.4. In [24] it is shown that TBFT_f and TBFT_{ff} are equivalent for finitely generated groups (see also [49]).

4. TBFT for endomorphisms of abelian and finite groups

For these classes all irreducible representations are finite-dimensional. That is why TBFT is the same as TBFT_f. In [11] using Pontryagin duality the following statement was proved.

Proposition 4.1. *$TBFT_f$ holds for any endomorphism of an abelian group.*

For polycyclic groups below we need also

Proposition 4.2. *$TBFT_{ff}$ holds for any endomorphism of an abelian group.*

Proof. Indeed, in an abelian group

$$x\varphi(x^{-1})y\varphi(y^{-1}) = (xy)\varphi((xy)^{-1}), \qquad (x\varphi(x^{-1}))^{-1} = x^{-1}\varphi(x),$$

$$xg\varphi(x^{-1})(yg\varphi(y^{-1}))^{-1} = (xy^{-1})\varphi((x^{-1}y)).$$

This shows that the Reidemeister class of e is a subgroup H, and the other classes are H-cosets. Being a Reidemeister class, H is φ-invariant (see Lemma 1.10 above) and the factorization $p : G \to G/H$ gives a bijection of Reidemeister classes. The induced action on G/H is trivial, as well as on $\widehat{G/H}$. The fixed representations $\rho \circ p$, where ρ runs over $\widehat{G/H}$, are desired finite representations. □

Theorem 4.3 (cf. [11])**.** *Let $\varphi : G \to G$ be an endomorphism of a finite group G. Then the Reidemeister number $R(\varphi)$ coincides with the number of $\widehat{\varphi}$-$\mathbf{f}$-points on $\widehat{G}$, i.e. TBFT is true in this situation.*

Proof. Let us note that $R(\varphi)$ is equal to the dimension of the space of φ-class functions (i.e. those functions that are constant on Reidemeister classes). They can be also described as fixed elements of the action $a \mapsto ga\varphi(g^{-1})$ on the group algebra $\mathbb{C}[G]$. For the latter algebra we have the Peter-Weyl decomposition

$$\mathbb{C}[G] \cong \bigoplus_{[\rho]\in\widehat{G}} \operatorname{End} V_\rho, \qquad \rho : G \to U(V_\rho),$$

which respects the left and right G-actions. Hence,

$$R(\varphi) = \mathop{+}_{[\rho]\in\widehat{G}} \dim T_\rho, \qquad T_\rho := \{a \in \operatorname{End} V_\rho \mid a = \rho(g)a\rho(\varphi(g^{-1}) \text{ for all } g \in G\}.$$

Thus, if $0 \neq a \in T_\rho$, a is an intertwining operator between the irreducible representation ρ and some representation $\rho \circ \varphi$. This implies that ρ is equivalent to some (irreducible) subrepresentation π of $\rho\circ\varphi$ (cf. [39, VI, p.57]). Hence, $\dim \rho = \dim \pi$, while $\dim \rho = \dim \rho\circ\varphi$. Thus, $\pi = \rho \circ \varphi$, and is irreducible. In this situation $\dim T_\rho = 1$ by the Schur lemma. Evidently, vice versa, if $\rho \sim \rho \circ \varphi$ then $\dim T_\rho = 1$. Hence,

$$R(\varphi) = \mathop{+}_{[\rho]\in\widehat{G}} \begin{cases} 1, & \text{if } \rho \sim \rho\circ\varphi \\ 0, & \text{if } \rho \not\sim \rho\circ\varphi \end{cases} = \text{ number of } \widehat{\varphi}\text{-}\mathbf{f}\text{-points}.$$

□

5. Technical lemmas

We will need the following

Lemma 5.1. *Let ρ be a finite representation. It is a φ-**f**-point if and only if there exists a non-zero φ class function being a matrix coefficient of ρ.*

In this situation this function is unique up to scaling and is defined by the formula

$$T_{S,\rho} : g \mapsto \mathrm{Tr}(S \circ \rho(g)), \tag{1}$$

where S is an intertwining operator between ρ and $\rho \circ \varphi$:

$$\rho(\varphi(x))S = S\rho(x) \quad \text{for any } x \in G.$$

In particular, $TBFT_{ff}$ is true for φ if and only if these matrix coefficients form a base of the space of φ-class functions.

Proof. First, let us note that (1) defines a class function:

$$T_{S,\rho}(xg\varphi(x^{-1}) = \mathrm{Tr}(S\rho(xg\varphi(x^{-1}))) = \mathrm{Tr}(\rho(\varphi(x))S\rho(g)\rho(\varphi(x^{-1})) = \mathrm{Tr}(S\rho(g)).$$

If $S \neq 0$, then $\rho(a) = S^*$ for some $a \in \ell^1(G)$, and $\mathrm{Tr}(SS^*) \neq 0$. Thus, the φ-class function is non-zero. On the other hand, any matrix coefficient of ρ, i.e. a functional $T : \mathrm{End}(V_\rho) \to \mathbb{C}$ has the form $g \mapsto \mathrm{Tr}(D\rho(g))$ for some fixed matrix $D \neq 0$. If it is a φ-class function, then for any $g \in G$, or similarly, $a \in \ell^1(G)$,

$$\mathrm{Tr}(D\rho(a)) = \mathrm{Tr}(D\rho(xa\varphi(x^{-1}))) = \mathrm{Tr}(\rho(\varphi(x^{-1}))D\rho(x)\rho(a)).$$

Since $\rho(a)$ runs over all the matrix algebra, this implies $D = \rho(\varphi(x^{-1}))D\rho(x)$, or $\rho(\varphi(x))D = D\rho(x)$, i.e. D is the desired non-zero intertwining operator.

The uniqueness up to scaling follows now from the explicit formula and the Shur lemma.

The last statement follows from linear independence of matrix coefficients of non-equivalent representations. □

Lemma 5.2. *A group G satisfies $TBFT_{ff}$ for an endomorphism $\varphi : G \to G$ if and only if there exists a φ-equivariant factorization of G onto a finite group F, such that Reidemeister classes on G map onto distinct classes on F.*

Proof. Let $\rho_1, \ldots \rho_k$ be all finite φ-**f**-representations, $F_1, \ldots F_k$, $F_i = \rho_i(G)$ the corresponding finite groups. Suppose, $g \in \operatorname{Ker} \rho_i =: K_i$. Then $\rho(\varphi(g)) = S\rho(g)S^{-1} = SS^{-1} = e$. Hence, K_i is a normal φ-invariant subgroup. Define $K := \cap K_i$. It is still a normal φ-invariant subgroup of finite index. Let $F := G/K$. By Lemma 1.9 the Reidemeister number of the induced map φ_F satisfies $R(\varphi_F) \le R(\varphi)$. On the other hand, ρ_i define disjoint irreducible φ_F-**f**-representations of F. By Lemma 5.1 this implies $R(\varphi_F) \ge R(\varphi)$. Thus, $G \to F$ gives a bijection of Reidemeister classes.

The opposite statement follows from Lemma 5.1 and Theorem 4.3. Indeed, if $p : G \to F$ is the mentioned epimorphism with $r = R(\varphi) = R(\varphi_F)$, then by Theorem 4.3 we have exactly r pairwise non-equivalent irreducible unitary φ_F-**f**-representations $\rho_1, \ldots, \rho_r$ of F. Then $\rho_i \circ p$ play the same role for G. Finally G can not have "additional" φ-**f**-representations by linear independence in Lemma 5.1. □

We can enforce this statement.

Lemma 5.3. *Let $\varphi : G \to G$ be an endomorphism with $R(\varphi) < \infty$. Suppose there exists a (not necessarily equivariant) factorization p of G onto a finite group F, such that Reidemeister classes on G map onto distinct subsets (not necessarily classes) on F. Then G has $TBFT_{ff}$ for φ.*

Proof. For F characteristic functions of *any* sets are matrix coefficients of some (finite) representations. Thus, characteristic functions of Reidemeister classes on G are matrix coefficients of some finite representations (coming from F). Lemma 5.1 completes the proof. □

Lemma 5.4. *Let φ be an endomorphism of a group G with $R(\varphi) < \infty$. Then G has $TBFT_{ff}$ for φ if and only if the right shifts of all Reidemeister classes form a finite number of subsets of G.*

Proof. First, let us note, that if we have a bijection for Reidemeister classes of φ, then we have a bijection for Reidemeister classes of any $\tau_g \circ \varphi$ by Lemma 1.12.

Now the "only if" direction is evident, because these sets are pre-images of some sets in finite F.

Suppose, there are finitely many shifts. This means, that the stabilizers under right shifts of any Reidemeister class of φ have finite index. Since $R(\varphi) < \infty$, their intersection is a subgroup H in G of finite index. By Lemma 1.12, its elements stabilize Reidemeister classes of any $\tau_g \circ \varphi$.

Suppose, $h \in H$, then

$$\begin{aligned} yg\varphi(y^{-1})zhz^{-1} &= y(gz)z^{-1}\varphi(y^{-1})zhz^{-1} = y(gz)(\tau_{z^{-1}} \circ \varphi)(y^{-1})hz^{-1} \\ &= x(gz)(\tau_{z^{-1}} \circ \varphi)(x^{-1})z^{-1} = xg\varphi(x^{-1}). \end{aligned}$$

Thus, H is normal (and φ-invariant, if φ is an automorphism). The projection $G \to G/H =: F$ maps Reidemeister classes of φ to disjoint sets in F. Indeed, if $h \in H$, then for any g,

$$e \cdot h = xe(\tau_g \circ \varphi)(x^{-1}).$$

This means, that H entirely is in the Reidemeister class of e of any $\tau_g \circ \varphi$. By Lemma 1.12 this means that all Reidemeister classes of ϕ are formed by H-cosets and we are done.

It remains to apply Lemma 5.3. □

Lemma 5.5. *Suppose, G has only finitely many inner automorphisms and an endomorphism $\varphi : G \to G$ with $R(\phi) < \infty$. Then G has $TBFT_{ff}$ for φ.*

Proof. By Lemma 1.12, the number of right shifts of Reidemeister classes is not more than $R(\phi) \cdot |Inn(G)|$. It remains to apply Lemma 5.4. □

Lemma 5.6. *Let $\varphi : A \to A$ be an endomorphism of a finitely generated abelian group A with $R(\varphi) < \infty$. Then the number of fixed elements $C(\varphi)$ on A is finite.*

Proof. The torsion subgroup is finite and totally invariant. Factorizing we reduce the problem to the case of $\mathbb{Z}^n$ by Lemma 1.9. In this case we will show that φ has only the trivial fixed element 0. By [11] in this case $\det(\mathrm{Id} - \varphi) \neq 0$, considered as $n \times n$ integer matrix. Diagonalising this matrix by left and right multiplication by unimodular matrix (as it was done e.g. in [3]) we see that it can not have (non-zero) eigenvector with eigenvalue zero, i.e. there is no non-trivial φ-fixed point. □

6. TBFT$_{ff}$ for endomorphisms of polyciclic groups

Consider a polycyclic group G. Its (finite) derived series is formed by fully invariant subgroups G_i with abelian quotients $A_i = G_i/G_{i+1}$. The key difference from a general finitely generated solvable group is the following: all G_i and A_i are finitely generated (see [41] for details). Let $G_n \neq \{e\}$, $G_{n+1} = \{e\}$.

We will argue by induction. For the basis of this induction let us observe that for the abelian group G_n and any its automorphism with finite Reidemeister number we have TBFT$_f$ by Prop. 4.1. Now suppose by induction that same is true for G_{i+1} and prove it for G_i. Denote $G_{i+1} =: H$, $G_i =: \Gamma$, $A_i =: A$.

Consider an endomorphism $\varphi : \Gamma \to \Gamma$ with $R(\varphi) < \infty$ and induced endomorphisms $\varphi_H : H \to H$ and $\varphi_A : A \to A$. Then by Lemma 1.9 $R(\varphi_A) < \infty$ and by Lemma 5.6 the number of fixed elements of φ_A on A is finite. Thus, by Lemma 1.11 $R(\varphi_H) < \infty$. Let $H_0 \subset H$ be a normal φ-invariant subgroup of finite index such that $H \to H/H_0$ gives a bijection on Reidemeister classes (see Lemma 5.2). This means that classes of φ_H are some unions of H_0 cosets. Consider automorphisms $\tau_g : H \to H$, $\tau_g(h) = ghg^{-1}$. $g \in G$, and define $H_1 := \cap_{g \in \Gamma} \tau_g(H_0)$. All subgroups in the intersection have the same finite index in H. Since H is finitely generated, H_1 also has a finite index. By construction, H_1 is normal in Γ. Also,

$$\varphi(\tau_g(h_0)) = \varphi(g)\varphi(h_0)(\varphi(g))^{-1} = \tau_{\varphi(g)}(\varphi(h_0)), \qquad h_0,\ \varphi(h_0) \in H_0,$$

i.e. $\varphi(\tau_g(H_0)) \subset \tau_{\varphi(g)}(H_0)$. Hence, H_1 is φ-invariant and $H_1 \subset H_0$, thus H_1 can play the same role as H_0 with an additional property of being normal in Γ. In particular, classes of φ_H are some unions of H_1 cosets. The same is true for intersections of Reidemeister classes of φ with H (because they are unions of some classes of φ_H). This means that $\Gamma \to \Gamma/H_1$ separates these classes, i.e. the classes which map on the Reidemeister class of $e \in A$ under $\Gamma \to A$. Similarly we can find subgroups $H_2, \dots H_{R(\varphi_A)}$ which separate classes over other classes of A. For this purpose, suppose $g \in \Gamma$ be over some other class of A. Then the classes of $\tau_g \circ \varphi$ (with the same finite Reidemeister number) that intersect with H are obtained from the classes under consideration by a shift by g. Then a group H_1^g constructed for $\tau_g \circ \varphi|_H$ in the same way as H_1 for φ_H, will separate these classes. Moreover, it will be $\tau_g \circ \varphi$-invariant for this specific g and normal in Γ, i.e. $\tau_{g'}$-invariant for any $g' \in \Gamma$. Taking $g' = g^{-1}$ we see that H_1^g is φ-invariant. Taking the (finite !) intersection of H_1, H_1^g,... we obtain a φ-invariant subgroup $H' \subset H$, which is normal in Γ, and $p : \Gamma \to \Gamma/H' = \Gamma'$ gives a bijection on Reidemeister classes. $F := p(\Gamma) \subset \Gamma'$ is a finite normal subgroup and $\Gamma'/F \cong A$.

Thus, Γ' is finitely generated finite-by-abelian group, and it has finitely many inner automorphisms. Lemma 5.5 completes the proof of the following statement.

Theorem 6.1. *Let $\varphi : G \to G$ be an endomorphism of a polycyclic group with $R(\varphi) < \infty$. Then TBFT$_{ff}$ is true for φ.*

Remark 6.2. The results of this section can be extended to some virtually polycyclic groups (under supposition of some polycyclic subgroup to be φ-invariant).

More precisely the following statement can be proved by word-by-word rewriting of the above argument.

Theorem 6.3. *Let G be an almost polycyclic group admitting a fully invariant polycyclic subgroup of finite index. Then TBFT$_{ff}$ is true for any endomorphism $\varphi : G \to G$ with $R(\phi) < \infty$.*

Let us illustrate this theorem by the following

Example 6.4. In [31] seven series of almost polycyclic (polycyclic-by-finite) groups with the property R_∞ (any automorphism has infinite Reidemeister number) were found. They have a fully invariant polycyclic subgroup of finite index ([31, Remark 2.1], [35]). Thus, they are covered by Theorem 6.3.

On the other hand, they evidently have endomorphisms with finite Reidemeister number (see Prop. 1.1 and Cor. 1.4).

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