Mutation of knots

by

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Abstract. In general, mutation does not preserve the Alexander module or the concordance class of a knot.

For a discussion of mutation of classical links, and the invariants which it is known to preserve, the reader is referred to [LM, APR, MT]. Suffice it here to say that mutation of knots preserves the polynomials of Alexander, Jones, and HOMFLY, and also the signature. <u>Mutation</u> of an oriented link k can be described as follows. Take a diagram of k and a tangle T with two outputs and two inputs, as in Figure 1.

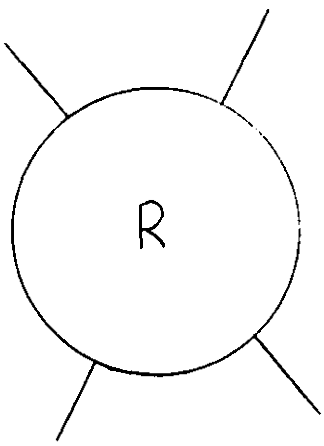

Fig. 1

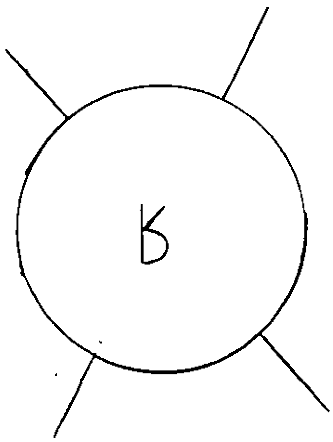

Fig. 2

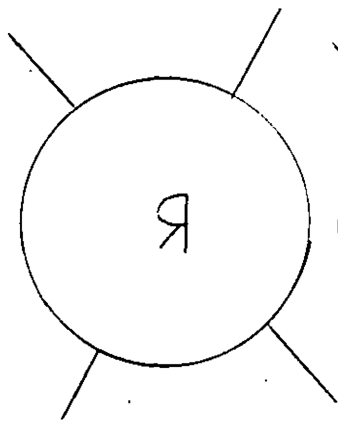

Fig. 3

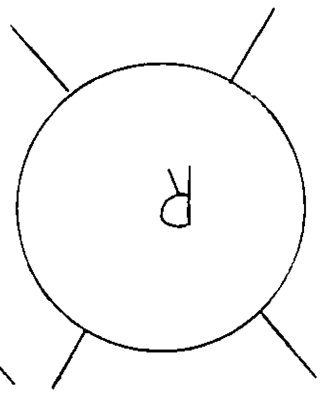

Fig. 4

Rotate the tangle about the east-west axis to obtain Figure 2, or about the north-south axis to obtain Figure 3, or about the axis perpendicular to the paper to obtain Figure 4. Keep or reverse all the orientations of T as dictated by the rest of k. Each of the links so obtained is a <u>mutant</u> of k.

The <u>reverse</u> k' of a link k is obtained by reversing the orientation of each component of k. Let us adopt the convention that a <u>knot</u> is a link of one component, and that k+l denotes the connected sum of two knots k and l .

<u>Lemma</u>. For any knot k , the knot k+k' is a mutant of k+k .

<u>Proof</u>. Shrink one of the summands in k+k to a small knot, and arrange a diagram of k+k to have a tangle as in Figure 5.

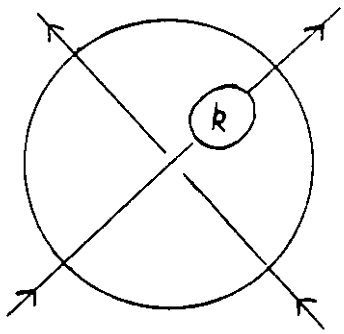

Fig. 5

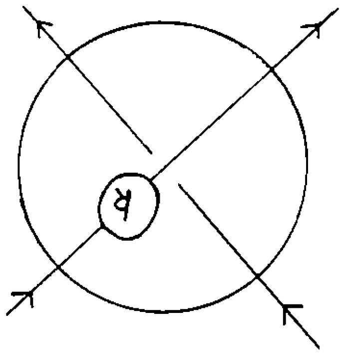

Fig. 6

Rotate about the axis perpendicular to the page, to obtain Figure 6 , which represents k+k' . Note that whatever convention we make about orientations, we always obtain k+k' .

Q.E.D.

By a result of C. Livingston [L], there exist knots k which are not concordant to their reverses k' . It follows at once that k+k is not concordant to k+k' , and hence that mutation does not preserve the concordance class in general. I should like to thank Cameron Gordon for reminding me of Livingston's result.

In [K] there is an example of a knot k, in fact the pretzel knot (25, -3, 13), whose Steinitz-Fox-Smythe row ideal class ρ does not satisfy $\rho^2 = 1$. The row ideal class of k', as pointed out in [K], is τ, the column ideal class of k. Of course, $\rho\tau = 1$, and so we see that the row ideal class of $k+k$ is $\rho^2 \neq 1$, whereas the row ideal class of $k+k'$ is $\rho\tau = 1$. Thus we have an example in which the knot module of $k+k$ is not isomorphic to that of $k+k'$. Another example can be obtained from [BHK, § 4], and other examples can be found using [B] and number theory tables.

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