SUPPLEMENT TO: ON FOLIATIONS OF SEMI-SIMPLICIAL MANIFOLDS AND THEIR HOLONOMY; THE UNIQUENESS

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The aim of this supplement to [1] is to give a unique characterization of the holonomy groupoid Γ (of an arbitrary ss-foliation) together with the corresponding canonical ss-Γ-structure by the universal property established in [1],III, thm 2.2.

Theorem 1.1. Let F be an arbitrary foliation of an ss-manifold X.

(i) There is a groupoid of germs Γ_F and an ss-Γ_F-structure ω_F defining F on X such that for any groupoid of germs Γ and any ss-Γ-structure ω defining F on X there exists exactly one morphism of groupoids $\Sigma = \Sigma^{\omega}: \Gamma_F \longrightarrow \Gamma$ with the property $\Sigma_* \omega_F = \omega$.

(ii) If $\tilde{\Gamma}_F$ and $\tilde{\omega}_F$ are any other objects with the universal property (i), then there is exactly one equivalence of groupoids $I: \Gamma_F \xrightarrow{\approx} \tilde{\Gamma}_F$ such that $\tilde{\omega}_F = I_* \omega_F$.

Furthermore, the morphisms of groupoids which exist according to (i) - (ii) come from uniquely defined morphisms of the underlying pseudogroups of germs.

Proof. "(i)" By [1],III, thm 2.2 (i), good candidates for Γ_F and ω_F are: any holonomy groupoid $\Gamma_{F,T}$ of F with respect to an arbitrary complete transversal $T \hookrightarrow X_1$, and

the canonical ss-$\Gamma_{F,T}$-structure $\omega_{F,T}$ on X.

The only thing to be proved is uniqueness of a morphism $\Sigma: \Gamma_{F,T} \to \Gamma$ such that $\Sigma_* \omega_{F,T} = \omega$ for a given ss-Γ-structure ω defining F on X. Let $E \to X$ be an ss-Γ-bundle representing ω.

By construction of $\omega_{F,T}$ (cf [1],III,2.3 for the notation), it is more convenient to prove the uniqueness in the non-reduced case. So let us assume that a morphism $\Sigma': \Gamma_{F,T^0 \sqcup T^1} \longrightarrow \Gamma$ is subject to the condition

$$E \cong \Sigma'_* E^!_{F,T} = (\Sigma' \times_{\Gamma_{F,T^0 \sqcup T^1}} (E^!_{F,T})_n)_{n \geq 0}.$$

In particular,

$$E_0 \cong \Sigma' \times_{\Gamma_{F_0,T^0 \sqcup T^1}} E_{F_0,T^0 \sqcup T^1}$$

and we may consider first a simpler, non-semi-simplicial case.

Namely, the restriction Σ'' of Σ' to $\Gamma_{F_0,T^0\sqcup T^1} \subset \Gamma_{F,T^0\sqcup T^1}$ represents a morphism $\Gamma_{F_0,T^0\sqcup T^1} \to \Gamma$ which describes the Γ-structure on X_0 represented by E_0 as the one induced from the canonical $\omega_{F_0,T^0\sqcup T^1}$. If Γ_{F_0} denotes the graph of F_0, then for the canonical equivalence $J: \Gamma_{F_0} \to \Gamma_{F_0,T^0\sqcup T^1}$ one has

$$J \cong E_{F_0,T^0\sqcup T^1} \text{ - as principal } \Gamma_{F_0,T^0\sqcup T^1}\text{-bundles over } X_0.$$

Hence

$$E_0 \cong \Sigma'' \times_{\Gamma_{F_0,T^0\sqcup T^1}} J = \Sigma'' \circ J$$

as (left) principal Γ-bundles over X_0.

We know already that Γ_{F_0} acts on E_0 (cf [1],I,2.4) and it is evident that this action is the only continuous one. Therefore, the restriction Σ'' of Σ' is isomorphic to the composition

$$(\Sigma'' \circ J) \circ J^{-1} \cong E_0 \times_{\Gamma_{F_0}} J$$

and it turns out that the morphism $\Sigma'': \Gamma_{F_0,T^0\sqcup T^1} \to \Gamma$ is unique and comes from a morphism $\Psi: G_{F_0,T^0\sqcup T^1} \to G$ of the underlying pseudogroups. It remains to show that Σ'' admits a unique extension to $\Gamma_{F,T^0\sqcup T^1}$.

Any extension requires (cf [1],II,3.12)

$$[\gamma,\phi(x)]\cdot[\phi,x] = [\psi^{(x)},x] \quad \text{in} \quad |\Psi|$$

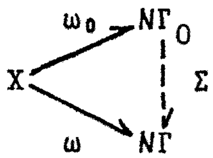
for arbitrary $\gamma \in G_{F,T^0\sqcup T^1}$ and $\phi \in G$, and a suitable $\psi^{(x)} \in G$, $x \in \text{domain}(\gamma\phi)$. By comparing the targets we see that the only continuous (left) action of $\Gamma_{F,T^0\sqcup T^1}$ on $|\Psi|$ is given by the product of germs. The uniqueness of $\Sigma = \Sigma' \circ |\Phi_T|$ as well as its shape are now completely established.

"(ii)" This is an immediate consequence of (i).

Theorem 1.1 allows us to propose the following definition, evidently more elegant than def. 1.2, Ch III of [1].

Definition 1.2. By a holonomy groupoid of an arbitrary ss-foliation F we shall mean every groupoid of germs Γ_F which, together with some ss-Γ_F-structure ω_F, verifies the universal condition (i) of theorem 1.1. The ss-Γ_F-structure ω_F is then a minimal Γ_F-structure for F.

1.3 It is sometimes convenient to have a "diagram-to-close" form of the universal property formulated in theorem 1.1 (i). Such a possibility is supplied by the formalism of ss-morphisms:

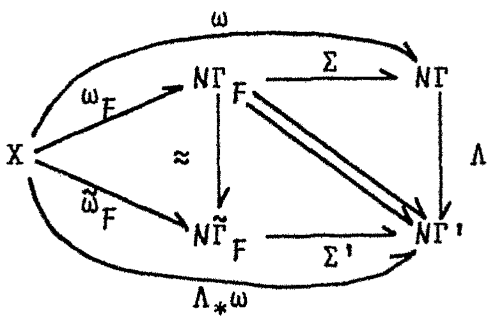


(cf [1],IV,prop. 1.5 & 1.8, coroll. 1.8.1). Note that ω defines $\mathcal{F}$ on X iff the corresponding ss-morphism $X \rightharpoonup N\Gamma$ is transverse to the discrete (pointwise) foliation of $N\Gamma$ and $\mathcal{F}$ is the pull-back foliation (cf [1],IV, coroll. 2.3.1). We shall call every ss-morphism $\pi_{\mathcal{F}}: X \rightharpoonup N\Gamma_{\mathcal{F}}$ that corresponds to some minimal ss-$\Gamma_{\mathcal{F}}$-structure a <u>transverse projection</u> of X (along the leaves of $\mathcal{F}$).

In view of the uniqueness, those statements of [1] which deal with consistency of the canonically constructed objects become now straightforward consequences of theorem 1.1. We list the most important of them.

<u>2.1</u> (cf [1],III, thm 2.2 (ii)) The canonical equivalences $\Gamma_{\mathcal{F},T} \approx \Gamma_{\mathcal{F},\overline{T}}$ and $\Gamma_{\mathcal{F},T} \approx \Gamma_{\mathcal{F},T^0 \sqcup T^1}$ between holonomy groupoids are uniquely characterized as the only morphisms which transfer to each other the corresponding minimal $\Gamma_{\mathcal{F}}$-structures. Hence any triangle of the equivalences commutes.

<u>2.2</u> (cf [1],III, thm 2.2 (iii)) Let $\Gamma_{\mathcal{F}}$ and $\tilde{\Gamma}_{\mathcal{F}}$ be any two holonomy groupoids of $\mathcal{F}$, $\omega_{\mathcal{F}}$ a minimal $\Gamma_{\mathcal{F}}$-structure, and $\tilde{\omega}_{\mathcal{F}}$ a minimal $\tilde{\Gamma}_{\mathcal{F}}$-structure. For any ss-$\Gamma$-structure ω defining $\mathcal{F}$ and an arbitrary morphism of groupoids of germs $\Lambda: \Gamma \rightharpoonup \Gamma'$ that comes from a morphism of pseudogroups, there is a commuting diagram



in which the two compositions $\Gamma_{\mathcal{F}} \rightharpoonup \Gamma'$ induce from $\omega_{\mathcal{F}}$ the same ss-Γ'-structure $\Lambda_*\omega$. In particular, the square

$$\begin{array}{ccc} \Gamma_F & \xrightarrow{\Sigma} & \Gamma \\ \approx\big\downarrow & & \big\downarrow\Lambda \\ \tilde{\Gamma}_F & \xrightarrow[\Sigma']{} & \Gamma' \end{array}$$

commutes.

<u>2.3</u> (cf [1],III,ex. 3.2) If Γ is an arbitrary groupoid of germs then the identity ss-morphism

$$1: N\Gamma \longrightarrow N\Gamma$$

induces on $N\Gamma$ the discrete (pointwise) foliation and evidently verifies the uniqueness property of theorem 1.1 (i). In particular, Γ is a holonomy groupoid of the foliation, and 1 is a transverse projection.

<u>2.4</u> (cf [1],IV, thm 3.2) Assume that an ss-morphism $f: Y \to X$ is transverse to a foliation F of X. Then, for any holonomy groupoids Γ_F and $\Gamma_{F'}$ of, respect., F and $F' = f^*F$ (and any transverse projections Π_F and $\Pi_{F'}$) there is a unique morphism $\Gamma_{F'} \to \Gamma_F$ such that the square

$$\begin{array}{ccc} Y & \xrightarrow{\Pi_{F'}} & N\Gamma_{F'} \\ f\big\downarrow & & \big\downarrow \\ X & \xrightarrow{\Pi_F} & N\Gamma_F \end{array}$$

commutes. In particular, part (iii) of theorem 3.2, ch. IV of [1] is now an immediate consequence of the uniqueness.

The last paragraph of the supplement is devoted to flags of ss-foliations and presents a refined version of the uniqueness theorem.

<u>Definition</u> 3.1. A pair (F,F') of foliations of an ss-manifold $X = (X_n)$ is a <u>flag</u> if for every n the topology F_n is coarser than F'_n.

<u>Proposition</u> 3.2. Let F be any foliation of an ss-manifold X, Γ_F an arbitrary holonomy groupoid of F, and $\Pi_F: X \to N\Gamma_F$ a transverse projection along leaves of F. Then, the assignment

$$\tilde{F} \longmapsto F' := \Pi_F{}^* \tilde{F}$$

establishes a bijective correspondence between foliations $\tilde{F}$ of the nerve $N\Gamma_F$ and foliations F' of X such that (F,F') is a flag.

Remark 3.2.1. The minimality of $\Pi_F\colon X \to N\Gamma_F$ allows one to treat the nerve $N\Gamma_F$ as a "quotient" of X by the leaves of F. Proposition 3.2 assures that for any flag (F,F') the foliation F' descends to a uniquely determined foliation of the "quotient".

We begin the proof with a general lemma.

Lemma 3.2.2. If (F,F') is any flag of foliations of an ss-manifold X and $f\colon Y \to X$ is an arbitrary ss-morphism transverse to F, then f is also transverse to F', and the pull-back foliations form a flag on Y.

Proof. The lemma follows immediately from the analogous property of ss-maps $Y_U \to X$ and from lemma 2.1, ch. IV of [1].

3.2.3 Proof of proposition 3.2. By lemma 3.2.2, the pair $(F,\Pi_F{}^*\tilde{F})$ is always a flag. So let F' be an arbitrary foliation of X such that (F,F') is a flag and let $T \hookrightarrow X_1$ be a fixed complete transversal for F. We first prove that there is a foliation F_T' of $N\Gamma_{F,T^0\sqcup T^1}$ such that $F' = \Pi^* F_T'$, for the canonical ss-morphism

$$\Pi = \Pi_{F,T^0\sqcup T^1}\colon X \longrightarrow N\Gamma_{F,T^0\sqcup T^1}$$

that corresponds to the canonical minimal ss-$\Gamma_{F,T^0\sqcup T^1}$-structure $\omega_{F,T}^!$.

Indeed, the restricted foliation $F_0'|T^0\sqcup T^1$ is clearly $\Gamma_{F,T^0\sqcup T^1}$-invariant; let F_T' be its unique extension to $N\Gamma_{F,T^0\sqcup T^1}$ (cf [1],II,2.3 One needs to compare the foliations F' and Π^*F_T' at the 0-level. Unfortunately, we still have no cocycle description of the ss-morphism Π.

Lemma 3.2.4. Let $U = \{U_a;\, a \in A\}$ be an open covering of X_0 such that there exist holonomy projections $H_a\colon U_a \to T^0\sqcup T^1$, $a \in A$, along leaves of F_0. Then the transverse projection $\Pi_{F,T^0\sqcup T^1}$ admits a representant $\gamma\colon X_U \to N\Gamma_{F,T^0\sqcup T^1}$ such that γ_0 is the map

$$\coprod_a U_a \ni (a,x) \longrightarrow H_a(x) \in T^0\sqcup T^1.$$

Assuming the lemma we see that the induced foliation $\gamma_0^*(F_0'|T^0\sqcup T^1)$ is exactly the localization of F_0', and thus conclude that $\gamma^*F_T' = (F')_U$. This means precisely $\Pi^*F_T' = F'$. Furthermore, any foliation F'' of $N\Gamma_{F,T^0\sqcup T^1}$ which induces F' must verify the same condition $\gamma^* F'' = (F')_{\mathcal{U}}$. By comparing the two foliations at the 0-level, one immediately gets $F'' = F'_T$.

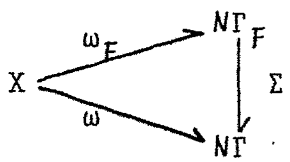
In order to transfer the above existence and uniqueness of the desired foliation to the general case $\Pi_F\colon X \longrightarrow N\Gamma_F$ it now sufficies to consider the uniquely determined equivalence $J\colon \Gamma_F \longrightarrow \Gamma_{F,T^0 \sqcup T^1}$ which induces on $N\Gamma_F$ the only foliation $\tilde{F}$ such that $\Pi_F^* \tilde{F} = F'$.

<u>3.2.5</u> Proof of lemma 3.2.4. The holonomy projections give rise to local sections

$$U_a \ni x \xrightarrow{\sigma_a} [H_a, x] \in E_{F_0, T^0 \sqcup T^1} \subset E_{F,0}$$

where $(E_{F,n})_{n\geq 0}$ is an ss-$\Gamma_{F,T^0 \sqcup T^1}$-bundle representing the minimal $\omega^!_{F,T}$ (cf [1],III,2.3). The collection of sections yields a $\Gamma_{F,T^0 \sqcup T^1}$-cocycle $\{\gamma_{ab}\}$ such that $\gamma_{aa} \circ \eta_0 = H_a$ for $a \in A$. Since the only extension of the cocycle to an ss-map $\gamma\colon X_{\mathcal{U}} \to N\Gamma_{F,T^0 \sqcup T^1}$ acts on $(X_{\mathcal{U}})_0$ as the map $\bigsqcup \gamma_{aa} \circ \eta_0$ (cf [1],IV,(1.1.1)), we are done.

<u>Theorem</u> 3.3. Let (F,F') be any flag of foliations of an ss-manifold X. For every holonomy groupoid Γ_F of F, any minimal ss-Γ_F-structure ω_F on X, and an arbitrary ss-Γ-structure ω defining the foliation F', there exists <u>exactly one</u> morphism of groupoids $\Sigma\colon \Gamma_F \longrightarrow \Gamma$ such that $\omega = \Sigma_* \omega_F$, ie the the triangle of ss-morphisms



commutes.

<u>Proof</u>. Clearly, it sufficies to consider the particular case $\omega_F = \omega^!_{F,T}$ where $T \overset{i}{\hookrightarrow} X_1$ is a fixed complete transversal for F. Similarly, we fix a principal ss-Γ-bundle $E \longrightarrow X$ representing ω.

Suppose there is a morphism of groupoids $\Sigma\colon \Gamma_{F,T^0 \sqcup T^1} \longrightarrow \Gamma$ such that $\Sigma_* \omega^!_{F,T} = \omega$. The ss-morphism determined by Σ is transverse to the discrete (pointwise) foliation of $N\Gamma$ (as such is the composition $X \longrightarrow N\Gamma_{F,T^0 \sqcup T^1} \longrightarrow N\Gamma$; cf [1],IV, coroll. 2.3.2); by proposition 3.2, the foliation it induces on $N\Gamma_{F,T^0 \sqcup T^1}$ is exactly F'_T. This implies that the principal Γ-bundle $\Sigma \longrightarrow T^0 \sqcup T^1$ (we forget the action of $\Gamma_{F,T^0 \sqcup T^1}$) defines on $T^0 \sqcup T^1$ the foliation $F' | T^0 \sqcup T^1$. Consequently, Σ admits a canonical form $\tilde{\Sigma}$ (cf [1],I,1.11),which consists of all germs of the distinguished submersions for Σ, ie of the maps $\beta \circ \sigma$, where σ ranges over local sections of Σ and β is the target map from Σ to the units of Γ (it is convenient to assume that Γ, a groupoid of germs, acts from the left). We claim that the submersions $\beta \circ \sigma$ are restrictions to $T^0 \sqcup T^1 \hookrightarrow X_0$ of the distinguished submersions for $E_0 \to X_0$ (which defines the foliation $F_0^!$ of X_0).

Indeed, the equality $\Sigma_* \omega^!_{F,T} = \omega$ yields an ss-Γ-bundle isomorphism

$$\Sigma_* E^!_{F,T} \cong E \ ;$$

in particular,

$$E_0 \cong \Sigma \times_{\Gamma_{F,T^0 \sqcup T^1}} E_{F,0}$$

$$\cong \Sigma \times_{\Gamma_{F_0,T^0 \sqcup T^1}} E_{F_0,T^0 \sqcup T^1}.$$

Therfore, for any local section σ of Σ and an arbitrary holonomy projection $H: U \to T^0 \sqcup T^1$ along leaves of F_0, there is a section

$$x \to (Hx)\ [H,x]$$

of the Γ-bundle isomorphic to E_0. As any isomorphism preserves the target map β, we conclude that the composition $\beta \circ \sigma H$ is a distinguished submersion for E_0. For an arbitrary t in domain σ and a suitable holonomy projection H, one has $H|T^0 \sqcup T^1 = \mathrm{id}$ over a neighbourhood of t, and thus

$$\beta\sigma = (\beta \circ \sigma H)|T^0 \sqcup T^1 \quad \text{around } t.$$

Since $\tilde{\Sigma}$, as a manifold of germs, is closed under the action of Γ, it must coincide with the (left) principal Γ-bundle of all the germs of the distinguished submersions for E_0 restricted to the transversal $T^0 \sqcup T^1 \hookrightarrow X_0$. By continuity, the (right) action of $\Gamma_{F,T^0 \sqcup T^1}$ on $\tilde{\Sigma}$ induced from the analogous action on Σ is unique and must be the standard product of germs. This completes our proof of uniqueness of the morphism

$$(\Sigma: \Gamma_{F,T^0 \sqcup T^1} \to \Gamma) = (\tilde{\Sigma}: \Gamma_{F,T^0 \sqcup T^1} \to \Gamma).$$

In order to prove that a morphism Σ with the required property does exist one ought to check that the collection Φ of submersions

$$T^0 \sqcup T^1 \supset U_\phi \xrightarrow{\phi} N_0\Gamma$$

which are locally restrictions of the distinguished submersions for $E_0 \to X_0$ is closed under compositions with elements of the holonomy pseudogroup $G_{F,T^0\sqcup T^1}$. There are two essential cases:

1° For any path c connecting two points of $T^0 \sqcup T^1 \hookrightarrow X_0$ in a leaf of F_0, the holonomy $h'_{c^{-1}}$ (" ' " means: with respect to F'_0) acts on the canonical form $\tilde{E}_0$ of E_0 by the formula (cf [1],I,2.4)

$$(h'_{c^{-1}}[\phi,c(1)])|T'_0 = [(\phi|T'_1)\circ h'^{T'_0}_{c,T'_1},c(0)]$$

where T'_i stands for any local transversal (for F'_0) at $c(i)$, $i = 0,1$. If one choses T'_0 inside $T^0 \sqcup T^1$ and defines $T'_1 \subset T^0 \sqcup T^1$ as the image $\gamma(T'_0)$ of T'_0 under the holonomy translation $\gamma = h^{T^0\sqcup T^1}_{c,T^0\sqcup T^1}$ (with respect to F_0), then evidently $h'^{T'_0}_{c,T'_1} = \gamma|T'_0$, and thus

$$[(\phi|T^0 \sqcup T^1)\circ \gamma|T'_0,c(0)] = h'_{c^{-1}}[\phi,c(1)]|T'_0$$

$$= [\psi|T'_0,c(0)]$$

for some distinguished submersion ψ. Since T'_0 is also a transversal for $F'_0|T^0 \sqcup T^1$, and both the maps $\psi|T^0 \sqcup T^1$ and $(\phi|T^0 \sqcup T^1)\gamma$ are locally constant on leaves of $F'_0|T^0 \sqcup T^1$, we conclude that

$$(\phi|T^0 \sqcup T^1)\gamma = \psi|T^0 \sqcup T^1 \text{ over a nbhd of } c(0) \in T^0 \sqcup T^1.$$

In other words, Φ is closed under composition with elements of the pseudogroup $G_{F_0,T^0\sqcup T^1}$.

2° By definition, $G_{F,T^0\sqcup T^1}$ is generated by the above holonomy pseudogroup of F_0 and by the identification map $id^0_1: T^0 \to T^1$. We claim that for any distinguished submersion ϕ one has locally

$$\phi|T^1 \circ id^0_1 = \tilde{\phi}|T^0$$

where $\tilde{\phi}$ is another distinguished submersion. Indeed, by reasoning as in [1],III,2.6, we obtain what follows:

If t is an arbitrary element of T such that $\varepsilon_1\iota t \in$ domain ϕ and ψ is any distinguished submersion for E_0 defined over a neighbourhood of $\varepsilon_0\iota t$, then there is a local diffeomorphism $\gamma \in G$ (G - the pseudogroup underlying Γ) such that

$$(\varepsilon_1,\pi)^{-1}([\phi,\varepsilon_1\iota t],\iota t) = [\gamma,\psi\varepsilon_0\iota t]\ (\varepsilon_0,\pi)^{-1}([\psi,\varepsilon_0\iota t],\iota t)$$

in E_1, ie

$$[\phi\varepsilon_1, \iota t] = [\gamma, \psi\varepsilon_0\iota t]\,[\psi\varepsilon_0, \iota t]$$

in the canonical form $\tilde{E}_1$ (cf [1],I,1.12). This implies that $\phi\varepsilon_1 = (\gamma \circ \psi)\varepsilon_0$ over a neighbourhood of ιt, and thus $\tilde{\phi} = \gamma\psi$ is the desired distinguished submersion. Clearly, the analogous property of $id_0^1 : T^1 \to T^0$ can be obtained in the same way.

By 1°-2°, the principal Γ-bundle $|\Phi| := \{[\phi,x];\ \phi \in \Phi,\ x \in U_\phi\}$ admits a natural action of the holonomy groupoid $\Gamma_{F,T^0 \sqcup T^1}$. It remains to prove that the morphism $\Sigma = |\Phi| : \Gamma_{F,T^0\sqcup T^1} \to \Gamma$ has the required property $\omega = \Sigma_* \omega^!_{F,T}$.

We define an isomorphism (on the canonical form $\tilde{E}_0$ of E_0)

$$I_0 : \tilde{E}_0 \longrightarrow |\Phi|_* E_{F,0} \cong |\Phi| \times_{\Gamma_{F_0,T^0\sqcup T^1}} E_{F_0,T^0\sqcup T^1}$$

by the formula

$$[\phi,x] \xrightarrow{I_0} [\psi|T^0 \sqcup T^1, t]\,[H_c, x] \qquad \text{for} \quad [\phi,x] \in \tilde{E}_0$$

where c stands for any path connecting x to a $t \in T^0 \sqcup T^1 \hookrightarrow X_0$ in a leaf of F_0, and $[\psi, \iota t] = h_c[\phi,x]$. By (2.4.1), ch. I of [1], the right-hand side does not depend on the choices involved.

In view of coroll. 3.6, ch. II of [1], I_0 admits an extension to an isomorphism of ss-Γ-bundles $E \to |\Phi|_* E^!_{F,T}$ iff the diagram

$$\begin{array}{ccccc}
\varepsilon_0{}^*\tilde{E}_0 & \xrightarrow{\cong} & \tilde{E}_1 & \xrightarrow{\varepsilon_1} & \tilde{E}_0 \\
{\scriptstyle (I_0\times id)}\downarrow & & \downarrow & & \downarrow {\scriptstyle I_0} \\
\varepsilon_0{}^*|\Phi|_*E_{F,0} & \longrightarrow & |\Phi|_*E_{F,1} & \xrightarrow{\varepsilon_1} & |\Phi|_*E_{F,0}
\end{array}$$

commutes. So, let the pair $([\phi,\varepsilon_0 x],x)$ denote an arbitrary element of $\varepsilon_0{}^*E_0$. If $\tilde{\phi}$ is a distinguished submersion for E_0 such that $\phi\varepsilon_0 = \tilde{\phi}\varepsilon_1$ over a neighbourhood of $x \in X_1$(such a $\tilde{\phi}$ exists by the earlier part of proof) and c is any path connecting x to a $t \in T$ in a leaf of F_1, then the maps that compose the diagram act on the arbitrary element of $\varepsilon_0{}^*E_0$ according to the following scheme

$$\begin{array}{ccc}
([\phi,\varepsilon_0 x],x) \longrightarrow [\phi\varepsilon_0, x] = [\tilde{\phi}\varepsilon_1, x] & \xrightarrow{\varepsilon_1} & [\tilde{\phi}, \varepsilon_1 x] \\
\downarrow & & {\scriptstyle I_0}\downarrow \\
(h_{\varepsilon_0 c}[\phi,\varepsilon_0 x]|_{T^0}[H_{\varepsilon_0 c}, \varepsilon_0 x], x) & & h_{\varepsilon_1 c}[\tilde{\phi},\varepsilon_1 x]|_{T^1}[H_{\varepsilon_1 c}, \varepsilon_1 x] \\
\downarrow & &
\end{array}$$

$$\Big\downarrow \varepsilon_1$$

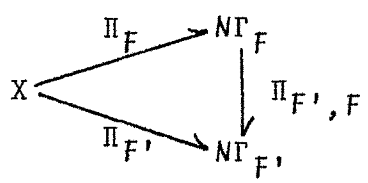
$$h_{\varepsilon_0 c}[\phi,\varepsilon_0 x]\big|_{T^0}\ \mathrm{id}_0^1\ [H_{\varepsilon_1 c},\varepsilon_1 x].$$

The fact that the resulted two elements of $|\Phi|_* E_{F,0}$ coincide follows from the relations

$$\begin{aligned} h_{\varepsilon_0 c}[\phi,\varepsilon_0 x]\big|_{T^0}\ \mathrm{id}_0^1 &= h_c[\phi\varepsilon_0, x]\big|_T\ \mathrm{id}^1 \\ &= h_c[\tilde{\phi}\varepsilon_1, x]\big|_T\ \mathrm{id}^1 \\ &= h_{\varepsilon_1 c}[\tilde{\phi},\varepsilon_1 x]\big|_{T^1} \end{aligned}$$

where id^1 means the suitable germ of the identification map $T^1 \to T$. This concludes our proof of the theorem.

<u>Corollary</u> 3.4. If (F,F') is a flag of foliations of X and $\Pi_F : X \to N\Gamma_F$ (resp., $\Pi_{F'} : X \to N\Gamma_{F'}$) is any transverse projection along leaves of F (resp., of F'), then the only ss-morphism $\Pi_{F',F}$ which closes the triangle



is a transverse projection of $N\Gamma_F$ along leaves of the only foliation induced by F'.

<u>References</u>

[1] G. Andrzejczak, On foliations of semi-simplicial manifolds and their holonomy. Preprint MPI Bonn/SFB 84-55.