On the compactifications of contractible affine threefolds and the Zariski Cancellation Problem

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The Zariski Cancellation Problem asks whether or not the condition $X \times \mathbf{A}^1 \cong \mathbf{A}^{n+1}$ implies $X \cong \mathbf{A}^n$. This problem is still unsolved for $n \geq 3$. Our main interest lies in the case $n = 3$. In this article, we consider the compactifications of some kinds of contractible smooth affine threefolds and the characterizations of the affine 3-space $\mathbf{A}^3$ keeping the 3-dimensional Zariski Cancellation Problem in mind. We classify the compactifications of contractible smooth affine threefolds with a certain condition concerning the numerical property of boundary divisors with respect to compactifications and, then, we show that if an affine threefold X satisfies $X \times \mathbf{A}^1 \cong \mathbf{A}^4$ and this numerical condition, then X is isomorphic to the affine 3-space $\mathbf{A}^3$.

1 Introduction

Throughout the present article we work over the field of complex numbers $\mathbf{C}$. In affine algebraic geometry, there are several important problems about the affine spaces, e.g. Abhyankar-Sathaye Embedding Problem, Linearization Problem, Jacobian Problem, the structure of automorphism groups of the affine spaces, Zariski Cancellation Problem and so on (cf. [43]). Our main interest in this article lies in Zariski Cancellation Problem in dimension n (we abbreviate (ZC;n) in the subsequent). This problem is stated as follows:

Problem 1.1 *Let X be an affine algebraic variety of dimension $n > 0$ satisfying the cancellation condition $X \times \mathbf{A}^1 \cong \mathbf{A}^{n+1}$. Then is X isomorphic to the affine n-space $\mathbf{A}^n$?*

The problem (ZC;1) is obviously true. The two-dimensional case (ZC;2) is non-trivial. But, by virtue of the results of Fujita, Miyanishi and Sugie (cf. [5], [33]), (ZC;2) was settled around 1980. An algebraic characterization of the affine plane $\mathbf{A}^2$ and the structure theorem of smooth affine surfaces with log Kodaira dimension $\overline{\kappa} = -\infty$ played important and crucial roles for the solution of (ZC;2). Meanwhile, the problem (ZC;n) is still open for $n \geq 3$. We are now interested in (ZC;3) and restrict ourselves to this problem. The main obstacles for trying the problem (ZC;3) seem to lie in the lack of the characterizations of the affine 3-space $\mathbf{A}^3$ and the structure theorem of smooth affine threefolds with log Kodaira dimension $\overline{\kappa} = -\infty$. Note that, nevertheless, there are some progress about the algebro-geometric characterizations of $\mathbf{A}^3$ due to Miyanishi, Kaliman, Saveliev and Zaidenberg (cf. [30], [31], [21], [17], [19]). In this article we shall investigate the compactifications of some kinds of smooth contractible affine threefolds and consider the characterization of the affine 3-space $\mathbf{A}^3$ keeping the cancellation problem (ZC;3) in mind.

Let X be an affine algebraic threefold satisfying the condition stated in (ZC;3), i.e., $X \times \mathbf{A}^1 \cong \mathbf{A}^4$. Then we can easily see the following:

(i) X is smooth and topologically contractible,

(ii) the log Kodaira dimension of X is $\overline{\kappa}(X) = -\infty$.

Note that (i) implies that the coordinate ring $A := \Gamma(\mathcal{O}_X)$ of X is factorial and the multiplicative group A^* of all units in A coincides with the non-zero constants $\mathbf{C}^*$ (cf. Fujita [6]). Unfortunately, only the above mentioned conditions (i) and (ii) can not deduce that X is isomorphic to $\mathbf{A}^3$. In fact, there are infinitely many affine algebraic threefolds satisfying (i) and (ii) but not isomorphic to $\mathbf{A}^3$. For instance, the followings are well-known and typical examples:

Example 1.1 (1) Let Z be a contractible smooth affine surface but not isomorphic to the affine plane $\mathbf{A}^2$. It is known that such a surface Z exists and the log Kodaira dimension $\overline{\kappa}(Z)$ of Z is either 1 or 2 (cf. [6], [32, Chap.3, Sect.4]). We put $X := Z \times \mathbf{A}^1$. Then X satisfies (i) and (ii), but $X \not\cong \mathbf{A}^3$ by virtue of the solution of (ZC;2).

(2) Let X be a hypersurface in $\mathbf{A}^4{}_{(x,y,z,t)}$ defined by $x + x^2y + z^2 + t^3 = 0$, which is called the *Russell cubic threefold* (cf. [43]). This X also satisfies (i) and (ii). In this case, it is not easy to distinguish X from the affine 3-space $\mathbf{A}^3$. In [27], Makar-Limanov showed that $\bigcap_\Delta \mathrm{Ker}(\Delta)$ is greater than $\mathbf{C}$ and to conclude that $X \not\cong \mathbf{A}^3$, where Δ moves all the locally nilpotent derivations on $\Gamma(\mathcal{O}_X)$.

(3) Kaliman developed the method to produce exotic $\mathbf{A}^3$'s (see the argument after this example for the definition) by making use of the theory of affine modifications (cf. [20]) in connection with the characterizations of the affine 3-space $\mathbf{A}^3$ and the 3-dimensional Abhyankar-Sathaye Embedding Problem (cf. [17]). In fact, the Russell cubic threefold in (2) is obtained from $\mathbf{A}^3$ by one affine modification (cf. [43]).

So, we need more other conditions to characterize the affine 3-space $\mathbf{A}^3$. The affine algebraic threefolds satisfying (i) are important and interesting subjects themselves. In fact, if an affine algebraic variety Y of dimension $n \geq 3$ is topologically contractible, then it is diffeomorphic to $\mathbf{R}^{2n}$ (cf. [4]). We say that Y is an *exotic* $\mathbf{A}^n$ if Y is diffeomorphic to $\mathbf{R}^{2n}$ but not isomorphic to $\mathbf{A}^n$. The above mentioned examples are all exotic $\mathbf{A}^3$. Note that such a phenomenon occurs only for higher dimensional case. Indeed, it is known that there are no exotic $\mathbf{A}^2$, namely, if a smooth affine surface is diffeomorphic to $\mathbf{R}^4$, then it is, in fact, isomorphic to the affine plane $\mathbf{A}^2$ (cf. [40]).

On the other hand, it is an important problem to classify the smooth compactifications of the affine spaces because it enables us to connect affine algebraic geometry with projective (birational) geometry. The smooth compactification of the affine line $\mathbf{A}^1$ is obviously the projective line $\mathbf{P}^1$ only. Concerning the smooth compactifications of the affine plane $\mathbf{A}^2$, Morrow essentially classified them by making use of the result of Ramanujam (cf. [40], [35], [24]). Meanwhile, the complete classification of smooth compactifications of the affine n-sapce $\mathbf{A}^n$ is still unknown and complicated to describe for $n \geq 3$. Note that in the case where $n = 3$ and the boundary is irreducible Furushima and Nakayama classified all of them (cf. [8]). We are now interested in the smooth contractible affine algebraic threefolds X with the following condition concerning the numerical property of the boundary divisors when X are embedded into smooth projective threefolds:

(iii) X can be embedded into a smooth projective threefold V in such a way that the reduced boundary divisor D with respect to $X \hookrightarrow V$ is simple normal crossing (=S.N.C.) and strictly nef, i.e., $D \cdot C > 0$ for every irreducible curve on V.

Note that, by the definition of log Kodaira dimension $\overline{\kappa}$ (cf. [15]), if X satisfies (ii), then $|m(K_V + D)| = \emptyset$ for all $m \in \mathbf{N}$. So, we consider, more slightly generally, the following (iv) instead of the conditions (ii) and (iii):

(iv) X can be embedded into a smooth projective threefold V in such a way that the reduced boundary divisor D with respect to $X \hookrightarrow V$ is normal crossing, strictly nef and $\kappa(V; K_V + D) = -\infty$, i.e., $|m(K_V + D)| = \emptyset$ for all $m \in \mathbf{N}$.

Note that, in (iv), we do not assume the normality of each of boundary components. It is clear that the conditions (ii) and (iii) implies (iv). Our main aim in this paper is to classify all the pairs (V, D) and the inside affine threefolds $X = V\backslash\mathrm{Supp}(D)$ satisfying (i) and (iv). After that we shall check whether or not the condition $X \times \mathbf{A}^1 \cong \mathbf{A}^4$ holds for each such an affine threefold. Namely, we obtain the following result and its corollary:

Theorem 1.1 *Let X be an affine algebraic threefold satisfying the following conditions (i) and (iv).*

(i) *X is smooth and topologically contractible,*

(iv) *X can be embedded into a smooth projective threefold V in such a way that the reduced boundary divisor D with respect to $X \hookrightarrow V$ is normal crossing, strictly nef and $\kappa(V; K_V + D) = -\infty$.*

Then the following is the complete list of the possibility of the pair (V, D) and the inside affine algebraic threefold $X = V\backslash\mathrm{Supp}\,(D)$:

Type	V	D	X
$C2$	*V: a $C2$-type Mori fiber space $g : V \to W$ over a smooth projective rational surface W.*	$D = D_1 + \cdots + D_r$ ($r = \rho(V)$) *s.t.* $g\|_{D_1} : D_1 \to W$ *is an isomorphism,* $g\|_{D_i}$ *is a* $\mathbf{P}^1$*-bundle structure over* $\mathbf{P}^1$ $(2 \le i \le r)$.	$X \cong Z \times \mathbf{A}^1$, *$Z$ is a smooth affine contractible surface with* $\overline{\kappa}(Z) = -\infty$, 1 *or* 2.
$D3a$	*V: a $D3$-type Mori fiber space over* $W \cong \mathbf{P}^1$.	$D = D_1 + D_2$ *s.t.* $D_1\|_F \sim \mathcal{O}_F(1)$, $D_2 \sim F$, $F \cong \mathbf{P}^2$*: a fiber of g.*	$X \cong \mathbf{A}^3$.
$D3b$	$\exists\varphi : V \to V'$, *$V'$: a $D3$-type Mori fiber space over* $W \cong \mathbf{P}^1$.	$D \xrightarrow{\varphi} D' \subset V'$, $D' = D'_1 + D'_2 + D'_3$ *s.t.* $D'_1\|_F \sim D'_2\|_F \sim \mathcal{O}_F(1)$, $D'_3 \sim F$, $F \cong \mathbf{P}^2$*: a fiber of g.*	$X \cong \mathbf{A}^3$.
$F2$	*$V = V_5$: a smooth Fano 3-fold with $\rho(V) = 1$ and Fano index 2 of degree* $(-\frac{1}{2}K_V)^3 = 5$.	$D \in \|\mathcal{O}_{V_5}(1)\|$ *is either* H_5^0 *or* H_5^∞.	$X \cong \mathbf{A}^3$.
$F3$	$V = \mathbf{Q}^3$*: a smooth quadric hypersurface in* $\mathbf{P}^4$.	$D \in \|\mathcal{O}_{\mathbf{Q}^3}(1)\|$ *is a quadric cone* $D \cong \mathbf{Q}_0^2$.	$X \cong \mathbf{A}^3$.
$F4a$	$V = \mathbf{P}^3$.	$D \sim \mathcal{O}_{\mathbf{P}^3}(1)$.	$X \cong \mathbf{A}^3$.
$F4b$	$\exists\varphi : V \to V'$, $V' = \mathbf{P}^3$.	$D \xrightarrow{\varphi} D' \subset V'$, $D' = D'_1 + D'_2$, $D'_1 \sim D'_2 \sim \mathcal{O}_{\mathbf{P}^3}(1)$.	$X \cong \mathbf{A}^3$.

In the above list $D3b$ and $F4b$, $\varphi : V \to V'$ is a composite of $E2$-type divisorial contractions and $D' := \varphi(D)$ is the proper transform of D on V'. Exactly one contraction appearing in φ contracts an exceptional divisor not contained in the proper transform of D.

Remark 1.1 We can, in fact, describe the procedure $\varphi : V \to V'$ in $D3b$ and $F4b$-types more explicitly. Namely, at first, by Lemma 2.6 in Section 2, the original pair (V, D) is obtained from (V', D') by blowing up finitely many different points on D'. Secondly, all these points on D' except for only one point are lying on the intersection curve of the components of D'. Finally, the remaining one point lies on exactly one boundary component.

Remark 1.2 In the two-dimensional case, we can easily obtain the similar result as in Theorem 1.1. Namely, if a smooth affine contractible surface Z can be embedded into a smooth projective surface W in such a way that the reduced bounday divisor B is normal crossing, strictly nef and $\kappa(W; K_W + B) = -\infty$, then $Z \cong \mathbf{A}^2$ and the the possibility of such a pair (W, B) is either $(W, B) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$, $(\mathbf{P}^1 \times \mathbf{P}^2, M_0 + L_0)$ or $(\mathbf{F}_d, s + L_d)$, where s (resp. L_d) is a section distinct from the minimal section (resp. a fiber) on $\mathbf{F}_d$ for $d > 0$.

As a corollary of Theorem 1.1 we obtain the following characterization of the affine 3-space $\mathbf{A}^3$ from a point of view of the cancellation problem (ZC;3):

Corollary 1.1 *Let X be an affine algebraic threefold such that $X \times \mathbf{A}^1 \cong \mathbf{A}^4$ and satisfies (iv). Then X is isomorphic to the affine 3-space $\mathbf{A}^3$.*

Proof. Let X be an affine algebraic threefold such that $X \times \mathbf{A}^1 \cong \mathbf{A}^4$ (in particular, X is contractible) and satisfies (iv). Then, by virtue of Theorem 1.1, it follows that $X \cong Z \times \mathbf{A}^1$, where Z is a contractible affine surface with $\overline{\kappa}(Z) = -\infty$, 1 or 2. Since $X \times \mathbf{A}^1 \cong Z \times \mathbf{A}^2 \cong \mathbf{A}^4$ we have $\overline{\kappa}(Z) = -\infty$. Hence $Z \cong \mathbf{A}^2$, and $X \cong \mathbf{A}^3$. □

We shall explain the scheme of this article. Let $X \hookrightarrow (V, D)$ be an embedding of an affine algebraic threefold X into a smooth projective threefold V satisfying (i) and (iv), i.e., $V \backslash \mathrm{Supp}\,(D) \cong X$ and D is normal crossing and strictly nef with $\kappa(V; K_V + D) = -\infty$.

In Section 2 we starts from such a pair (V, D) and continue the contractions of the extremal rays to reach a suitable Mori fiber space. By noting that the reduced boundary divisor D is strictly nef, we can fairly explicitly describe this contraction process. Let $g : V' \to W$ be a obtained Mori fiber space and let D' be a proper transform of D on V', where $\dim W \leq 2$. Note that the complement $X' := V' \backslash \mathrm{Supp}\,(D')$ may be smaller than the original X. It is then not difficult to see that the type of g is either $C2$, $D2$, $D3$ or V' is a smooth Fano threefold with Picard number $\rho(V') = 1$ and the Fano index ≥ 2. In the subsequent sections, we shall investigate the structure of inside affine threefold X' and the original X associated with the Mori fiber space $g : V' \to W$ according to the type of it, separately.

In Section 3, we first deal with the case where g is of $C2$-type. In this case, we can see that $X \cong X'$ and the restriction of g onto X yields an $\mathbf{A}^1$-bundle structure over a contractible smooth affine surface (cf. $C2$ in Theorem 1.1).

In Section 4, we shall deal with the case where g is of type $D2$. In fact, we prove that this case does not occur. The argument to prove this is a little complicated but mainly due to the straightforward calculations. It is worth mentioning that if we replace that condition that D is strictly nef by that D is nef, then we can construct a smooth compactification of the affine 3-sapce $\mathbf{A}^3$ with an extremal ray of $D2$-type.

In Section 5 we consider the case where g is of type $D3$. Then we can see that one of the following cases occurs by noting the factoriality of X: there exists exactly one component contained in D', say D'_1, such that $D'_1|_F \sim \mathcal{O}_F(1)$ or there exist two components in D', say D'_1 and D'_2, such that $D'_1|_F \sim D'_2|_F \sim \mathcal{O}_F(1)$ for a general fiber $F \cong \mathbf{P}^2$ of g, and the remaining boundary components are all the fibers of g. In any case we can show $X \cong \mathbf{A}^3$ (cf. $D3a$ and $D3b$ in Theorem 1.1).

Finally, in Section 6, we treat with the case where V' is a Fano threefold with $\rho(V') = 1$ and Fano index ≥ 2. In the Fano index 2 case we know, in fact, that $V = V' = V_5$ and $D \in |\mathcal{O}_{V_5}(1)|$ is either H_5^0 or H_5^∞ by virtue of the result of Furushima and Nakayama [7] and [9], and conclude that $X \cong V_5 \backslash D \cong \mathbf{A}^3$ (cf. $F2$ in Theorem 1.1). In the case of Fano index 3, i.e., $V' \cong \mathbf{Q}^3 \hookrightarrow \mathbf{P}^4$ or Fano index 4, i.e., $V' \cong \mathbf{P}^3$, we can restrict the possibilities of the boundary divisor D' by noting that the coordinate ring $\Gamma(\mathcal{O}_{X'})$ of X' is factorial and $\kappa(V'; K_{V'} + D') = -\infty$ and, then, observe X' and the original X carefully for each case (cf. $F3$, $F4a$ and $F4b$ in Theorem 1.1). Thus we obtained Theorem 1.1 and Corollary 1.1 as a corollary.

Notation.

- K_V : canonical divisor of a variety V;
- $F|_W$: scheme-theoretic restriction of a Cartier divisor F on V to a subvariety $W \subset V$;
- $F|_F$: self-intersection of an irreducible Cartier divisor F on V;
- $b_i(V)$: the i-th Betti number of V as a real topological manifold;
- $\chi_{\mathrm{top}}(V)$: the topological Euler number of a topological manifold V;
- $F \cdot C$: the intersection number of a ($\mathbf{Q}$-)Cartier divisor F and a curve C on a threefold;

- $(C_1 \cdot C_2)_Y$: the intersection number of curves C_1 and C_2 on a surface Y;
- $p_a(C)$: the arithmetic genus of a curve C;
- $F_1 \sim F_2$: F_1 is linearly equivalent to F_2;
- $F_1 \equiv F_2$: F_1 is numerically equivalent to F_2;
- $\mathbf{F}_a$: Hirzebruch surface of degree a, $(a \geq 0)$;
- M_a : the minimal section on $\mathbf{F}_a$;
- L_a : a fiber on $\mathbf{F}_a$;
- $\overline{\kappa}(U)$: the logarithmic Kodaira dimension of a quasi-projective variety U;
- $\mathbf{A}^n$: n-dimensional affine space;
- $\mathbf{P}^n$: n-dimensional projective space;
- $\mathbf{Q}_0^2$: a quadric cone in $\mathbf{P}^3$;
- $\mathbf{Q}^3$: a smooth quadric hypersurface in $\mathbf{P}^4$;
- $\mathbf{C}^{*k}$: the affine line with k points punctured;
- V_5 : a linear section $\mathrm{Gr}(2,5) \cap \mathbf{P}^6$ of the Grassmann variety $\mathrm{Gr}(2,5) \hookrightarrow \mathbf{P}^9$ by three hyperplanes in $\mathbf{P}^9$, which is a Fano threefold of Fano index 2 and degree $(-\frac{1}{2}K_{V_5})^3 = 5$ in $\mathbf{P}^6$;
- H_5^0 : a normal hyperplane section of V_5 with exactly one A_4-type Du Val singularity;
- H_5^∞ : a non-normal hyperplane section of V_5 whose singular locus is a line $L \cong \mathbf{P}^1$ with the normal bundle $\mathcal{N}_{L/V_5} \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$;
- $\overline{\mathrm{NE}}(V)$: the closure of the cone of effective 1-cycles on V.

Terminology. We use the standard terminologies in [36] as follows: An *extremal ray* R of $\overline{\mathrm{NE}}(V)$ is a one-dimensional edge of $\overline{\mathrm{NE}}(V)$ which is K_V-negative. An *$E2$-type divisorial contraction* is a birational morphism between 3-folds which contracts an exceptional divisor $E \cong \mathbf{P}^2$ with normal bundle $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbf{P}^2}(-1)$ to a smooth point. For *$C2$, $D2$ and $D3$-type extremal contractions*, see Lemma 2.5 for the detail.

2 Reduction Step

Let X be an affine algebraic threefold satisfying the two conditions (i) and (iv) in Theorem 1.1, namely,

(i) X is smooth and topologically contractible, and

(iv) X can be embedded into a smooth projective threefold V in such a way that the reduced boundary divisor D with respect to $X \hookrightarrow V$ is normal crossing, strictly nef and $\kappa(V; K_V + D) = -\infty$.

In this section, we begin with this pair (V, D) and shall bring it to a suitable Mori fiber space via the several contractions of extremal rays. But note that the original affine threefold X may be shrinked after this contraction procedure.

Remark 2.1 (1) Since X is smooth affine, we can find an ample Cartier divisor on V supported by the boundary $\mathrm{Supp}\,(D) = \cup_{i=1}^r D_i$ (cf. Goodman [10]). Hence D is big automatically, i.e., $mD \sim A + G$ for some $m \in \mathbf{N}$, where A is an ample and G is an effective Cartier divisor on V (cf. [26, Lemma 2.60]).

(2) Note that since X is a smooth contractible affine variety, its coordinate ring $A := \Gamma(\mathcal{O}_X)$ is factorial and $A^* = \mathbf{C}^*$, where A^* is the multiplicative group consisting of all units in A (cf. [6]). Hence the Picard group Pic (V) of V is free generated by the boundary compnents $D_1, \cdots, D_r$ of D, i.e., Pic $(V) = \oplus_{i=1}^r \mathbf{Z}[D_i]$.

Lemma 2.1 *$K_V + D$ is not nef.*

Proof. Assume to the contrary that $K_V + D$ is nef. Then, by Non-vanishing Theorem [26, Theorem 3.4], we have $|m(K_V + D)| \neq \emptyset$ for $m >> 0$, a contradiction to $\kappa(V; K_V + D) = -\infty$. □

We put $t_0 := \sup\{\lambda \in \mathbf{Q}\ ;\ D + \lambda K_V \text{ is nef}\}$, which is less than 1 by Lemma 2.1 and is greater than 0 by the strict nefness of D. There exists an extremal ray $R_0 \subset \overline{\mathrm{NE}}\,(V)$ contained in the hyperplane $\{D + t_0 K_V\}^{\perp}$. Let $\varphi_0 := \mathrm{cont}_{R_0}$ be the contraction of R_0. Then we have the following:

Lemma 2.2 *φ_0 is an $E2$-type divisorial contraction (cf.* [36]*), i.e., it contracts an exceptional divisor $E^0 \cong \mathbf{P}^2$ with $\mathcal{O}_{E^0}(E^0) \cong \mathcal{O}_{\mathbf{P}^2}(-1)$ to a smooth point, unless φ_0 yields a Mori fiber space.*

Proof. Suppose that φ_0 does not yield a Mori fiber space. Since V is a smooth projective threefold, φ_0 is not small (cf. [2]), hence it is divisorial. Let l_0 be an irreducible curve such that $[l_0] \in R_0$. Then we have $D \cdot l_0 = t_0(-K_V \cdot l_0) < (-K_V \cdot l_0)$. Since D is a strictly nef Cartier divisor, $D \cdot l_0$ is a positive integer. Hence $(-K_V \cdot l_0) \geq 2$, and so φ_0 is a divisorial contraction of $E2$-type. □

We shall investigate the configuration of D near the exceptional divisor $E^0 \cong \mathbf{P}^2$. In fact, we prove the following:

Lemma 2.3 *One of the following three cases* **A**, **B** *and* **C** *occurs concerning the configuration of D around the exceptional divisor E^0:*

A: *E^0 is contained in the boundary divisor D and there exists exactly one other boundary component intersecting E^0 in such a way that the intersection curve is a conic on $E^0 \cong \mathbf{P}^2$. The remaining boundary components are disjoint from E^0;*

B: *E^0 is contained in the boundary divisor D and there exist two other boundary components intersecting E^0 in such a way that the intersection curves are distinct lines on $E^0 \cong \mathbf{P}^2$. The remaining boundary components are disjoint from E^0;*

C: *E^0 is not contained in the boundary divisor D and there exists exactly one boundary component intersecting E^0 in such a way that the intersection curve is a line on $E^0 \cong \mathbf{P}^2$. The remaining boundary components are disjoint from E^0.*

Proof. Let l_0 be a line in the exceptional divisor $E^0 \cong \mathbf{P}^2$. Then $D \cdot l_0 = t_0(-K_V \cdot l_0) = 2t_0 \in \mathbf{N}$. Hence $t_0 = 1/2$ and $D \cdot l_0 = 1$. Then it is not difficult to obtain the assertion by noting that D is normal crossing. □

Via the $E2$-type divisorial contraction $\varphi_0 : V \to V^1$, we put $D^1 := \varphi_0(D)$. Then D^1 is still normal crossing. More precisely, if case **B** or **C** in Lemma 2.3 occurs, then the singularities of D^1 do not change compared with those of D, on the other hand, if case **A** occurs then a new A_1-type Du Val singular point, which is away from the intersection curves with other boundary components, occurs. We put $X^1 := V^1 \backslash \mathrm{Supp}\,(D^1)$. In the case of **A** or **B**, it is clear that $X^1 \cong X$, on the other hand, in the case of **C** the complement $X \backslash X^1$ is isomorphic to the affine plane $\mathbf{A}^2$.

Lemma 2.4 (1) *D^1 is strictly nef and big,*

(2) *$\kappa(V^1; K_{V^1} + D^1) = -\infty$ and $K_{V^1} + D^1$ is not nef.*

Proof. (1) Let C be any irreducible curve on V^1 and let $\widetilde{C} \subset V$ be such that $\varphi_0(\widetilde{C}) = C$. Then we have $D^1 \cdot C = \varphi_0^*(D^1) \cdot \widetilde{C} = (D + E^0) \cdot \widetilde{C} \geq D \cdot \widetilde{C} > 0$. Thus D^1 is strictly nef. Since $X^1 = V^1 \backslash \mathrm{Supp}\,(D^1)$ is smooth affine, we can readily see that D^1 is big by the same reason in Remark 2.1 (1).

(2) Note that $\varphi_0^*(K_{V^1}+D^1) = K_V + D - E^0$. Hence we have $\kappa(K_{V^1}; K_{V^1}+D^1) \leq \kappa(K_V; K_V+D) = \overline{\kappa}(X) = -\infty$ as desired. Assume that $K_{V^1}+D^1$ is nef. Then, by noting that D^1 is (strictly) nef and big, we have $|m(K_{V^1}+D^1)| \neq \emptyset$ for sufficiently large $m >> 0$ by Non-vanishing Theorem (cf. [26]). This is a contradiction to $\kappa(V^1; K_{V^1}+D^1) = -\infty$. □

From now on, by the induction on Picard number, we shall construct the following diagram $(*)$:

$$(*):\ (V,D) \stackrel{\varphi_0}{\to} (V^1,D^1) \to \cdots \to (V^j,D^j) \stackrel{\varphi_j}{\to} (V^{j+1},D^{j+1}) \to \cdots$$

$$\to (V^{s-1},D^{s-1}) \stackrel{\varphi_{s-1}}{\to} (V^s,D^s) =: (V',D') \stackrel{g}{\to} W,$$

where each (V^j,D^j) is a pair of a smooth projective threefold V^j and a strictly nef and big, reduced Cartier divisor D^j on V^j which is normal crossing. Furthermore, $\kappa(V^j; K_{V^j}+D^j) = -\infty$. We put $X^j := V^j \backslash \mathrm{Supp}\,(D^j)$ for $0 \leq j \leq s$. Then $X^j \backslash X^{j+1} \cong \mathbf{A}^2$ unless X^j is isomorphic to X^{j+1}. The right terminal $(V',D') \stackrel{g}{\to} W$ is a Mori fiber space.

Construction of a diagram $(*)$

We have already obtained the first step $\varphi_0 : (V,D) \to (V^1,D^1)$. Now suppose that we have the j-th step (V^j,D^j). Note that since $\kappa(V^j; K_{V^j}+D^j) = -\infty$ and D^j is (strictly) nef and big, $K_{V^j}+D^j$ can not be nef by Non-vanishing Theorem. We put $t_j := \sup\{\lambda \in \mathbf{Q}; D^j + \lambda K_{V^j}$ is nef$\}$, which satisfies $0 < t_j < 1$ because D^j is strictly nef and $K_{V^j}+D^j$ is not nef. There exists an extremal ray $R_j \subset \overline{\mathrm{NE}}\,(V^j) \cap \{D^j + t_j K_{V^j}\}^{\perp}$ and let $\varphi_j := \mathrm{cont}_{R_j}$ be the contraction of R_j. Let $l_j \subset V^j$ be an irreducible curve such that $[l_j] \in R_j$. Then we have $D^j \cdot l_j = t_j(-K_{V^j} \cdot l_j) < (-K_{V^j} \cdot l_j)$. By noting that $D^j \cdot l_j$ is a positive integer, we can see $(-K_{V^j} \cdot l_j) \geq 2$. If φ_j yields a Mori fiber space, we put $(V',D') := (V^j,D^j)$ and $g := \varphi_j$ to obtain $(*)$. On the other hand, if φ_j does not yield a Mori fiber space, then it is divisorial, say $\varphi_j : V^j \to V^{j+1}$. Since $(-K_{V^j} \cdot l_j) \geq 2$, φ_j is of $E2$-type (cf. [36]), i.e., it contracts an exceptional divisor $E^j \subset V^j$ with $E^j \cong \mathbf{P}^2$ and $\mathcal{O}_{E^j}(E^j) \cong \mathcal{O}_{\mathbf{P}^2}(-1)$ to a smooth point. We choose l_j as a line in E^j. In consideration of the equality $D^j \cdot l_j = t_j(-K_{V^j} \cdot l_j) = 2t_j < 2$, we see that $t_j = 1/2$ and $D^j \cdot l_j = 1$. Since D^j is normal crossing, we can see that one of the following three cases **A**, **B** and **C** occurs concerning the configuration of D^j near the exceptional divisor E^j:

A: E^j is contained in D^j and there exists exactly one other boundary component intersecting E^j in such a way that the intersection curve is a conic on $E^j \cong \mathbf{P}^2$. The remaining boundary components are disjoint from E^j;

B: E^j is contained in D^j and there exist two other boundary components intersecting E^j in such a way that the intersection curves are distinct lines on $E^j \cong \mathbf{P}^2$. The remaining boundary components are disjoint from E^j;

C: E^j is not contained in D^j and there exists exactly one boundary component intersecting E^j in such a way that the intersection curve is a line on $E \cong \mathbf{P}^2$. The remaining boundary components are disjoint from E^j.

We denote by D^{j+1} the proper transform of D^j on V^{j+1} and put $X^{j+1} := V^{j+1} \backslash \mathrm{Supp}\,(D^{j+1})$. In consideration of the configuration of D^j near E^j, it follows that D^{j+1} is normal crossing. Furthermore, if the case **A** or **B** occurs, then $X^j \cong X^{j+1}$, meanwhile, if **C** occurs X^{j+1} is an open affine subset of X^j such that $X^j \backslash X^{j+1} \cong \mathbf{A}^2$. D^{j+1} is strictly nef and big. In fact, let $C \subset V^{j+1}$ be an irreducible curve and let $\widetilde{C} \subset V^j$ be such that $\varphi_j(\widetilde{C}) = C$. We have $D^{j+1} \cdot C = \varphi_j^*(D^{j+1}) \cdot \widetilde{C} = (D^j + E^j) \cdot \widetilde{C} \geq D^j \cdot \widetilde{C} > 0$. Since $V^{j+1} \backslash \mathrm{Supp}\,(D^{j+1}) = X^{j+1}$ is smooth affine, there exists an effective ample Cartier divisor supported by the boundary $\mathrm{Supp}\,(D^{j+1})$ (cf. Goodman [10]). Hence the reduced boundary divisor D^{j+1} is big. We can readily see $\kappa(V^{j+1}; K_{V^{j+1}} + D^{j+1}) = -\infty$. Thus we obtain the $(j+1)$-th pair (V^{j+1}, D^{j+1}).

By induction on Picard number, after a finite number of above mentioned procedure, we finally reach a situation where the contraction of an extremal ray $R_s \subset \overline{\mathrm{NE}}\,(V^s) \cap \{D^s + t_s K_{V^s}\}^{\perp}$ yields a

Mori fiber space, say $g : V^s \to W$, where W is a smooth projective variety of $\dim W \leq 2$. We put $(V', D') := (V^s, D^s)$ and then obtain the desired diagram $(*)$.

We have the following concerning the type of $g : V' \to W$.

Lemma 2.5 *The type of the Mori fiber space $g : V' \to W$ is described as one of the following:*

Type $C2$: g is a $\mathbf{P}^1$-bundle structure over a smooth projective surface W;

Type $D2$: g is a Del Pezzo fibration over a smooth projective curve W with a general fiber isomorphic to a smooth quadric $\mathbf{P}^1 \times \mathbf{P}^1$ and possibly a singular fiber isomorphic to a quadric cone $\mathbf{Q}_0^2$;

Type $D3$: g is a $\mathbf{P}^2$-bundle structure over a smooth projective curve W;

Type F: V' is a smooth Fano threefold with $\rho(V') = 1$ and Fano index equal to 2, 3 or 4.

Proof. Let $l' \subset V'$ be any curve such that $[l'] \in R' := R_s$. Then we have $D' \cdot l' = t'(-K_{V'} \cdot l') < (-K_{V'} \cdot l')$. Since $D' \cdot l'$ is a positive integer, it follows that $(-K_{V'} \cdot l') \geq 2$. Then we can see the assertion by virtue of the classification of the extremal rays on smooth projective threefolds (cf. [36]). □

Let α (resp. β and γ) be the number of contractions of case **A** (resp. **B** and **C**) appearing in the procedure $\varphi = \varphi_{s-1} \circ \cdots \circ \varphi_0$. It is clear that $\alpha + \beta + \gamma = s$. We put $X' := V' \backslash \mathrm{Supp}\,(D')$. If $\gamma = 0$, then $X \cong X'$ obviously, on the other hand, if $\gamma > 0$ then X' is an open affine subset of X such that the complement $X \backslash X'$ is composed of γ irreducible components, say $S_1, \cdots, S_\gamma$.

We have seen that D^j is strictly nef on V^j in each step of $\varphi = \varphi_{s-1} \circ \cdots \circ \varphi_0$. Hence when we begin with the Mori fiber space $(V', D') = (V^s, D^s)$ to go back to the original pair (V, D) via blowing-ups, we have to keep the strictly nef property of the boundary divisors D^j for $0 \leq j \leq s$. This note enables us to describe the procedure of $\varphi = \varphi_{s-1} \circ \cdots \circ \varphi_0$ in detail. Namely, we have the following:

Lemma 2.6 *Let p_j^k be the point on V^k which is the image of the exceptional divisor $E^j \subset V^j$ of φ^j on V^k for $0 \leq j < s$ and $j < k \leq s$, that is, $p_j^k := (\varphi_{k-1} \circ \cdots \circ \varphi_j)(E^j)$. Then p_j^k is not contained in the exceptional divisor $E^k \subset V^k$ of φ_k for all $1 \leq j < s$ and $j < k \leq s$. In other words, the images of E_j on V' are mutually distinct points on D'.*

Proof. Note that it follows that $D^j = \varphi_j^*(D^{j+1}) - E^j$ for each $0 \leq j < s$. Assume to the contrary that the center p_j^{j+1} of the blowing-up $\varphi_j : V^j \to V^{j+1}$ is contained in the proper transform on V^{j+1} of the exceptional divisor E^k of φ_k for some $0 \leq j < s$ and $j < k$. We may and shall assume that this situation occurs for the first time after starting from $V' = V^s$. Let l be the proper image on V^j of the line l_k in $E^k \cong \mathbf{P}^2$ passing through the point p_j^k. Then we have $(D^j \cdot l) = (D^{j+1} \cdot \varphi_j(l)) - (E^j \cdot l) = (D^{j+1} \cdot \varphi_j(l)) - 1$. Note that $(D^{j+1} \cdot \varphi_j(l)) = (D^k \cdot l_k)$ by our choice of j and k. Since $D^k = \varphi^*(D^{k+1}) - E^k$ we have $(D^k \cdot l_k) = (-E_k \cdot l_k) = 1$. Therefore we have $(D^j \cdot l) = 0$, which is a contradiction as D^j is strictly nef on V^j. □

Remark 2.2 By Lemma 2.6 and by the construction of the diagram $(*)$, the complement $X \backslash X'$ is a disjoint union $X \backslash X' = \amalg_{k=1}^{\gamma} S_k$ of the affine planes $S_k \cong \mathbf{A}^2$ unless X is isomorphic to X'. Let a_k be a prime element in A such that $\mathrm{div}(a_k)_X = S_k$ for $1 \leq k \leq \gamma$ (note that A is factorial; see Remark 2.1) and put $a := \prod_{k=1}^{\gamma} a_k$. Then the coordinate ring $A' := \Gamma(\mathcal{O}_{X'})$ of $X' = V' \backslash \mathrm{Supp}\,(D')$ is obtained as the localization $A' = A[a^{-1}]$, in particular, it is factorial as so is A.

In the subsequent sections we consider the affine algebraic threefolds X' and X associated with the Mori fiber space $g : V' \to W$ according to the type of it separately (cf. Lemma 2.5).

3 The case $g : V' \to W$ is of $C2$-type

In this section we consider the case where $g : V' \to W$ is of $C2$-type, i.e., it is a $\mathbf{P}^1$-bundle structure over the smooth projective surface W. We keep the notations in Section 2. Let l' be the class of a fiber of g. Then we have $D' \cdot l' = t'(-K_{V'} \cdot l') = 2t'$. Since D' is a strictly nef Cartier divisor and $0 < t' < 1$, it follows that $t' = 1/2$ and $D' \cdot l' = 1$. This means the following:

Lemma 3.1 *There exists exactly one component of D' intersecting a general fiber of g in a single point transversally, and each of the remaining components of D' is mapped onto a curve via g.*

We can, in fact, see that the process of $\varphi = \varphi_{s-1} \circ \cdots \circ \varphi_0 : V \to V'$ is fairly simple in this case. Namely, we have the follwoing:

Lemma 3.2 *The process $\varphi : V \to V'$ is an identity, that is, the original smooth projective threefold V is already a $C2$-type Mori fiber space.*

Proof. Assume that φ is not an identity. Let l' be the fiber of g passing through the center of the blowing-up $\varphi_{s-1} : V^{s-1} \to V^s = V'$ and let $l \subset V^{s-1}$ be such that $\varphi_{s-1}(l) = l'$. Then we have $(D^{s-1} \cdot l) = (\varphi_{s-1}^*(D') - E^{s-1} \cdot l) = (D' \cdot l') - (E^{s-1} \cdot l) = 1 - 1 = 0$, a contradiction as D^{s-1} have to be strictly nef. □

By virtue of Lemmas 3.1 and 3.2, we may assume that V has a structure of $C2$-type Mori fiber space, say $g : V \to W$, and $g|_{D_1} : D_1 \to W$ is a birational morphism and $g|_{D_i} : D_i \to C_i$ is a $\mathbf{P}^1$-bundle structute, where $C_i := g(D_i)$ is a curve for $2 \le i \le r$. Let $S := W \backslash (\cup_{i=2}^r C_i)$, let $\Sigma := \{y \in S; g^{-1}(y) \subset D_1\}$ and let $S^\circ := S \backslash \Sigma$. It is easy to see that $\dim \Sigma \le 0$ and the restriction $g^\circ := g|_X$ of g onto X yields an $\mathbf{A}^1$-bundle structure over S°. Therefore there exists an open affine covering $\mathcal{U} = \{Z_\lambda\}$ of S° such that $U_\lambda := (g^\circ)^{-1}(Z_\lambda) \cong Z_\lambda \times \mathbf{A}^1$ and $g^\circ|_{U_\lambda}$ coincides with the projection onto the first factor for each λ. We shall fix one λ once and for all, and let $B_\lambda := \Gamma(\mathcal{O}_{Z_\lambda})$ be the coordinate ring of Z_λ. Then $\Gamma(\mathcal{O}_{U_\lambda}) = B_\lambda[t]$, where t is a variable over B_λ. Since U_λ is an open affine subset of the smooth affine variety X the complement $T := X \backslash U_\lambda$ is of pure codimension one. As $A = \Gamma(\mathcal{O}_X)$ is factorial (cf. Remark 2.1), there exists an element $b \in A$ such that $T = \mathrm{div}(b)_X$, that is, $A[b^{-1}] = B_\lambda[t]$.

Lemma 3.3 *The locally nilpotent derivation $\partial/\partial t$ on $A[b^{-1}] = B_\lambda[t]$ can be extended to that on A, say $\Delta_t = b^N(\partial/\partial t)$, after multiplying b^N with $\partial/\partial t$ for a sufficiently large $N >> 0$.*

Proof. Since A is a finitely generated $\mathbf{C}$-algebra, the assertion is easy to verify. □

Let $B := \mathrm{Ker}(\Delta_t) \subset A$, which is a finitely generated $\mathbf{C}$-algebra by Zariski's Lemma (cf. Nagata [38]). Moreover, B is an inert sub-algebra of A, i.e., for elements $b_1, b_2 \in A$, if $b_1 b_2 \in B$ then $b_1, b_2 \in B$ (cf. [28]). Hence B is factorial as so is A, and $B^* = A^* = \mathbf{C}^*$. Let $h : X \to Z := \mathrm{Spec}\,(B)$ be the morphism associated to the canonical inclusion $B \hookrightarrow A$. Note that, by virtue of the result of Kaliman [18, Prop. 3.2], h is surjective. It is clear that $b \in B$ and $B[b^{-1}] = B_\lambda$. Let C be an affine curve on Z defined by $b \in B$. By the construction, $Z \backslash C \cong Z_\lambda$ and the restriction $h|_{X \backslash T} : X \backslash T \to Z \backslash C$ coincides with the projection $g^\circ|_{U_\lambda} : U_\lambda \cong Z_\lambda \times \mathbf{A}^1 \to Z_\lambda$ under an isomorphsim $X \backslash T \cong U_\lambda$.

Lemma 3.4 *There exists an isomorphism $\theta : S^\circ \xrightarrow{\sim} Z$ such that $h = \theta \circ g^\circ$.*

Proof. We define the morphism $\theta : S^\circ \to Z$ as $\theta(y) := h((g^\circ)^{-1}(y))$ for a point y on S°. As g° is an $\mathbf{A}^1$-bundle this is well-defined and an injective morphism. By the construction, it is clear that for any point $z \in Z \backslash C$ there exists $y \in Z_\lambda \subset S^\circ$ such that $z = \theta(y)$. Now let z be a point on the curve $C \subset Z$ and let C_z be an affine curve on Z distinct from C passing through z. Let $T_z = h^{-1}(C_z) \subset X$. Note that $h|_{T_z} : T_z \to C_z$ is a surjective $\mathbf{A}^1$-fibration as h is surjective. We consider the morphism $g^\circ|_{T_z}$, which also gives rise to an $\mathbf{A}^1$-fibration and the general fibers of $h|_{T_z}$ coincides with those of $g^\circ|_{T_z}$ by the construction. Therefore $h^{-1}(z)$ is also a fiber of g°. Hence we have $\theta(y) = z$ for a point $y := g^\circ(h^{-1}(z))$. Thus θ is an isomorphism and it follows that $h = \theta \circ g^\circ$. □

By Lemma 3.4, we know that $\Sigma = \emptyset$ and $S^\circ = S \cong Z$ is a factorial smooth affine surface. Indeed, if $\Sigma \neq \emptyset$ then a factorial affine surface $S^\circ \cong Z$ is embedded into a smooth projective surface W with the boundary $(\cup_{i=2}^r C_i) \cup \Sigma$, which is not of pure codimension one. This is a contradiction. Since $g^\circ : X \to S$ is an $\mathbf{A}^1$-bundle structure and S is factorial, we conclude that it is trivial, that is, $X \cong S \times \mathbf{A}^1$ and g° coincides with the projection onto the first factor under this isomorphism. As X is contractible and $X \cong S \times \mathbf{A}^1$, we have that S is also contractible. Note that the log Kodaira dimension $\overline{\kappa}(S)$ of such a smooth contractible affine surface S is equal to either $-\infty$, 1 or 2 (cf. [6], [32]). In the case $\overline{\kappa}(S) = -\infty$, it is isomorphic to the affine plane $\mathbf{A}^2$ (cf. [32]).

Lemma 3.5 *$g|_{D_1} : D_1 \to W$ is an isomorphism and $g|_{D_i} : D_i \to C_i$ yields a $\mathbf{P}^1$-bundle structure over the smooth rational curve $C_i = g(D_i) \cong \mathbf{P}^1$ for $2 \leq i \leq r$. Furthermore, W is rational and $\cup_{i=1}^r C_i$ contains no circular chains.*

Proof. Since $\Sigma = \emptyset$ we can easily see the first assertion. Suppose that C_i is not smooth and let p be a singular point of it. Then D_i is singular along the fiber $(g^\circ)^{-1}(p)$. But then D is not normal crossing at the point $D_1 \cap (g^\circ)^{-1}(p)$, which is a contradiction. Thus C_i is smooth. Note that W is rational because any ($\mathbf{Q}$-)homology plane is rational (cf. [39],[13],[12]). Moreover, it follows that $\cup_{i=2}^r C_i$ is simply connected (cf. [32]). Hence $C_i \cong \mathbf{P}^1$ for all $2 \leq i \leq r$ and $g|_{D_i} : D_i \to C_i$ is a $\mathbf{P}^1$-bundle structure over the smooth rational curve $\mathbf{P}^1$. □

Thus we obtained the assertion in Theorem 1.1 in the case where g is of type $C2$.

4 The case $g : V' \to W$ is of $D2$-type

In this section we consider the case where $g : V' \to W$ is of $D2$-type, i.e., it is a Del Pezzo fibration with a general fiber isomorphic to a smooth quadric $\mathbf{P}^1 \times \mathbf{P}^1$ and possibly a singular fiber isomorphic to a quadric cone $\mathbf{Q}_0^2$ (cf. Lemma 2.5). In fact, we shall prove that this case does not occur. But note that if we replace the condition that D is strictly nef by that D is nef, then we can construct such an example of the compactification of the affine 3-space $\mathbf{A}^3$ (cf. Example 4.1).

Let l' be the class of a curve which is a ruling on a general fiber $F \cong \mathbf{P}^1 \times \mathbf{P}^1$. We have $D' \cdot l' = t'(-K_{V'} \cdot l') = 2t'$. Since D' is a strictly nef Cartier divisor and $0 < t' < 1$, we have $t' = 1/2$ and $D' \cdot l' = 1$. Hence we have the following:

Lemma 4.1 *There exists exactly one component in D' intersecting a general fiber $F \cong \mathbf{P}^1 \times \mathbf{P}^1$ of g along the (not necessarily irreducible) curve which is linearly equivalent to $\mathcal{O}_F(1,1)$, and each of the remaining components of D' is a fiber of g.*

Moreover, by noting that $D' \cdot l' = 1$, we can see the following by the similar argument as in Lemma 3.2.

Lemma 4.2 *The process $\varphi : V \to V'$ is an identity, that is, the original smooth projective threefold V is already a $D2$-type Mori fiber space.*

Thus we may and shall assume that V has a structure of $D2$-type Mori fiber space, say $g : V \to W$. Since $\rho(V) = 2$ the number of components in the boundary D equals two (cf. Remark 2.1), and let D_1 and D_2 be such components. We may assume that D_1 intersects a general fiber F of g in such a way that $D_1|_F \sim \mathcal{O}_F(1,1)$ and that D_2 is a fiber. Since D is normal crossing, the intersection $C_{1,2} := D_1 \cap D_2$ is a smooth curve. Furthermore, we can see that $(C_{1,2}^2)_{D_2} = 2$ and $(K_{D_2} \cdot C_{1,2})_{D_2} = -4$, i.e., $C_{1,2} \cong \mathbf{P}^1$ by an easy computation.

Lemma 4.3 (1) *The restriction $g|_{D_1} : D_1 \to W$ yields a $\mathbf{P}^1$-bundle structure over $W \cong \mathbf{P}^1$, that is, $D_1 \cong \mathbf{F}_d$ for some $d \geq 0$,*

(2) $D_2 \cong \mathbf{Q}_0^2$.

Proof. Our proof consists of three steps.

Step 1. First of all, we shall show that the base curve W is isomorphic to $\mathbf{P}^1$. We consider the cohomology exact sequence of the pair (V, D):

$$\cdots \to H^i(V, D) \to H^i(V) \to H^i(D) \to H^{i+1}(V, D) \to \cdots .$$

By noting that $H_{6-i}(X) \cong H^i(V, D) = 0$ for $0 \le i < 6$, we have $0 = H^5(D) \cong H^5(V) \cong H^1(V)$. Thus $H^1(\mathcal{O}_V) = 0$ by the Hodge theorem (cf. [11, p. 536]). On the other hand, since $g : V \to W$ is a Mori fiber space, it follows that $g_*\mathcal{O}_V = \mathcal{O}_W$ and $R^j g_*\mathcal{O}_V = 0$ for $j > 0$. Thus we obtain $0 = H^1(\mathcal{O}_V) \cong H^1(\mathcal{O}_W)$. This means that $W \cong \mathbf{P}^1$.

Step 2. Next we show that D_1 is normal. Since D_1 is Cartier, we have only to prove that the singular locus of D_1 is at most of dimension ≤ 0. Note that $C_F := D_1|_F \sim \mathcal{O}_F(1, 1)$, so C_F is either a diagonal or a union of two rulings, where $F \cong \mathbf{P}^1 \times \mathbf{P}^1$ is a general fiber of g. In the case C_F is a diagonal it is easy to see that D_1 is smooth along it, meanwhile, in the case C_F is a union of two rulings the possibility of the singular point of D_1 along C_F is only the intersection of these two lings, say P_F. Assume that the singular locus Σ of D_1 is one-dimensional. Then the above argument says $\Sigma = \bigcup P_F$, where F moves all the fibers of g. But, by considering the case $F = D_2$, we have a contradiction as $C_{1,2}$ is a smooth curve. Thus D_1 is normal.

Step 3. Note that the exact sequence in Step 1 implies $H^2(D) \cong \mathbf{Z}^{\oplus 2}$. We consider the following Mayer-Vietoris exact sequence:

$$\cdots \to H^1(C_{1,2}) \to H^2(D) \to H^2(D_1) \oplus H^2(D_2) \to H^2(C_{1,2}) \to \cdots .$$

Since $H^1(C_{1,2}) = 0$, $H^2(D) \cong \mathbf{Z}^{\oplus 2}$ and $H^2(C_{1,2}) \cong \mathbf{Z}$, this exact sequence implies $b_2(D_1) + b_2(D_2) \le 3$. But since $g|_{D_1} : D_1 \to W$ yields a $\mathbf{P}^1$-fibration, we have $b_2(D_1) \ge 2$. Thus we obtain $b_2(D_1) = 2$ and $b_2(D_2) = 1$. In particular, $D_2 \cong \mathbf{Q}_0^2$ and all the fibers of $g|_{D_1}$ are irreducible and reduced. If D_1 is not smooth, let P be a singular point of D_1 and let l be a fiber of $g|_{D_1}$ passing through P. Let $\mu : \hat{D}_1 \to D_1$ be the minimal resolution and let $\hat{g} : \hat{D}_1 \to W$ be the $\mathbf{P}^1$-fibration induced from $g|_{D_1}$. Then the fiber of $\hat{g}$ corresponding to l is composed of the proper transform $\hat{l}$ of l and the components of $\mu^{-1}(P)$. Since $\hat{l}$ is contained in the fiber with multiplicity 1 there exists a (-1)-curve in $\mu^{-1}(P)$ (cf. [28, Chap. 2, Lemma 2.2.]), which is absurd as μ is the minimal resolution. Therefore we conclude that D_1 is smooth and $g|_{D_1}$ yields a $\mathbf{P}^1$-bundle structure over $W \cong \mathbf{P}^1$, i.e., $D_1 \cong \mathbf{F}_d$ for some $d \ge 0$. □

Lemma 4.4 *With the above notations, we have the following:*

(1) $-K_V \sim 2D_1 + aD_2$ *for some* $a \in \mathbf{Z}$*,*

(2) $D_1|_{D_1} \sim 2M_d + (d - a + 2)L_d$ *(see* **Notation.***),*

(3) $d + a < 3$*,*

(4) *We put* $\Delta := D|_{D_1}$*. Then* Δ *is very ample on* D_1*.*

Proof. The assertions (1) and (2) are easy to see by noting that $(-K_V \cdot l) = 2$, where l is a ruling on a general fiber $F \cong \mathbf{P}^1 \times \mathbf{P}^1$ of g, and the adjunction formula. Let s be a class of the minimal section on $D_1 \cong \mathbf{F}_d$. Then $D \cdot s = (2M_d + (d-a+2)L_d \cdot M_d) + 1 = -d-a+3$. Since D is strictly nef, we have $d+a < 3$ as desired. By the straightforward computation, we have $(\Delta^2)_{D_1} = {D_1}^3 + 2{D_1}^2 \cdot D_2 + D_1 \cdot {D_2}^2 = 4(3-a)$, which is a positive integer by virtue of (3). On the other hand, it is clear that Δ is strictly nef on D_1. Thus Δ is ample on D_1 and, furthermore, very ample on D_1 by [14, Chap. V, Cor. 2.18.]. □

We consider the adjoint bundle $K_{D_1} + \Delta$ and ask whether or not this is nef. In the sebsequent of this section we consider according as $K_{D_1} + \Delta$ is nef or not, separately.

4.1 CASE: $K_{D_1} + \Delta$ is nef

We assume that $K_{D_1} + \Delta$ is nef. Then Bs $|m(K_{D_1} + \Delta)| = \emptyset$ for $m >> 0$ by Base Point Freeness Theorem (cf. [23]). Let $f : D_1 \to T$ be a morphism obtained as the Stein factorization of the morphim defined by $|m(K_{D_1} + \Delta)|$, where T is a normal variety of $\dim T \leq 2$.

Lemma 4.5 $\dim T = 0$ *or* 1.

Proof. In fact, we have $(K_{D_1} + \Delta)^2{}_{D_1} = (K_V + D_1 + D)^2 \cdot D_1 = (a-1)^2 D_1 \cdot D_2{}^2 = 0$. This implies the assertion. □

4.1.1 Case: $\dim T = 0$

First of all, we consider the case where $\dim T = 0$. Then $-K_{D_1} \sim \Delta$.

Lemma 4.6 *The case* $\dim T = 0$ *does not occur.*

Proof. We have $(-K_{D_1}{}^2)_{D_1} = (\Delta^2)_{D_1} = 4(3-a)$. On the other hand, since $D_1 \cong \mathbf{F}_d$ by Lemma 4.3, we have $a = 1$, that is, $-K_V \sim 2D_1 + D_2$.

Claim. $-K_V$ is ample.

Proof of Claim. Note that $\overline{\mathrm{NE}}\,(V) = \mathrm{NE}\,(V)$ because $\overline{\mathrm{NE}}\,(V)$ is a two-dimensional closed cone in $\mathbf{R}^2$. Since $V \backslash (D_1 \cup D_2) \cong X$ is affine, we have $(-K_V \cdot C) = (2D_1 + D_2 \cdot C) > 0$ for an irreducible curve C not contained in D. Hence we have only to consider curves contained in D. If $C \subset D_2$, then $D_1 \cdot C > 0$ and $D_2 \cdot C = 0$, so we have $(-K_V \cdot C) > 0$. Assume that $C \subset D_1$ and $C \sim \alpha M_d + \beta L_d$ on $D_1 \cong \mathbf{F}_d$, where $(\alpha, \beta) = (1, 0)$, $(0, 1)$ or $\beta \geq d\alpha$. Then, by the straightforward computation, we have $(-K_V \cdot C) = (3 - 2d)\alpha + 4\beta > 0$. Here we note that $d < 2$ by Lemma 4.4 (3). Thus we conclude that $-K_V$ is ample. □

Hence V is a smooth Fano threefold with $b_2(V) = 2$. By the direct calculation, we obtain $(-K_V)^3 = (2D_1 + D_2{}^3) = 8D_1{}^3 + 12D_1{}^2 \cdot D_2 + 6D_1 \cdot D_2{}^2 + D_2{}^3 = 56$. Then, by [37, Table 2], we know that $V \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1))$. But this does not have an extremal ray of $D2$-type, which is a contradiction. □

4.1.2 Case: $\dim T = 1$

Next, we shall consider the case where $\dim T = 1$. Then $K_{D_1} + \Delta = f^*(A)$ for some ample $\mathbf{Q}$-divisor $A \in \mathrm{Pic}\,(T) \otimes \mathbf{Q}$.

Lemma 4.7 (1) *$a = 0$, i.e., $-K_V \sim 2D_1$,*

(2) *$D_1|_{D_1} \sim 2M_d + (d+2)L_d$,*

(3) *$d = 0$, 1 or 2.*

Proof. Let δ be a general fiber of f. Then $(-K_{D_1} \cdot \delta)_{D_1} = (\Delta \cdot \delta)_{D_1} > 0$. Hence $(-K_{D_1} \cdot \delta)_{D_1} = (\Delta \cdot \delta)_{D_1} = 2$. On the other hand, since $D_1 \cong \mathbf{F}_d$ we have $8 = (-K_{D_1}{}^2)_{D_1} = (\Delta - f^*(A))^2_{D_1} = (\Delta^2)_{D_1} - 2(\Delta \cdot f^*(A))_{D_1} = 4(3-a) - 4\deg(A)$. Therefore $\deg(A) = 1 - a$, which is positive as A is ample $\mathbf{Q}$-divisor. Thus $a = 0$, i.e., $-K_V \sim 2D_1$ and $D_1|_{D_1} \sim 2M_d + (d+2)L_d$ (cf. Lemma 4.4). Note that the possibility of the value of d is either 0, 1 or 2 by Lemma 4.4 (3). □

Lemma 4.8 *With the above notations, we have the following:*

(1) *$-K_V$ is nef and big, but not ample,*

(2) *$(-K_V)^3 = 64$.*

Proof. Since $-K_V \sim 2D_1$, we have only to check $D_1 \cdot C \geq 0$ for an irreducible curve C contained in D_1 to see that $-K_V$ is nef. Let $C \subset D_1$ and let $C \sim \alpha M_d + \beta L_d$ with $(\alpha, \beta) = (1,0)$, $(0,1)$ or $\alpha \geq d\beta$ on $D_1 \cong \mathbf{F}_d$. Then we have $D_1 \cdot C = (2M_d + (d+2)L_d \cdot \alpha M_d + \beta L_d) = (2-d)\alpha + 2\beta \geq 0$. By the straightforward calculation, we see $(-K_V)^3 = 8D_1^3 = 64$. Thus $-K_V$ is nef and big. Now assume to the contrary that $-K_V$ is ample. Then V is a smooth Fano threefold with $b_2(V) = 2$ and $(-K_V)^3 = 64$. This is a contradiction by [37, Table 2]. Thus we obtain the assertion. □

Since $\overline{\mathrm{NE}}\,(V) = \mathrm{NE}\,(V)$ is a closed cone of dimension 2 in $\mathbf{R}^2$, it is spanned by two edges, say R and Q. We may assume that $R = \mathbf{R}_+[l]$ comes from $g : V \to W$ and the other edge $Q = \mathbf{R}_+[C]$ is $(-K_V)$-trivial. Note that any curve C from Q meets D_2, for otherwise, we have $D_2 \cdot C = 0$, so C is contained in a fiber of g and $[C] \in R$, which is absurd.

Lemma 4.9 *The case $d = 0$ and $d = 1$ do not occur.*

Proof. Our proof consists of five steps.

Step 1. Since $-K_V$ is nef and big, we have $\mathrm{Bs}\,|m(-K_V)| = \emptyset$ for $m >> 0$ (cf. [23]) and let $\psi : V \to \overline{V}$ be the Stein factorization of the morphism defined by $|m(-K_V)|$, which is birational.

Claim. ψ is small.

Proof of Claim. Note that each curve contracted by ψ is contained in Q. Moreover, we note that any curve from Q is disjoint from D_1. In fact, since $D_1|_{D_1}$ is ample in the case $d = 0$, 1 an irreducible curve in D_1 have positive intersection number with D_1 and hence also with $-K_V$. Now assume that ψ is not small. Then it contracts a divisor, say $E \subset V$. Since any curve from Q intersects D_2 we know that E meets D_2 along the curve. But as $C_{1,2} = D_1 \cap D_2$ is ample on D_2, E intersects $C_{1,2}$ and hence intersects D_1 also. But as $(-K_V \cdot Q) = 2(D_1 \cdot Q) = 0$ this is absurd. □

Step 2. We perform a $(-D_2)$-flop with respect to ψ in the sense of [25], say $\psi^+ : V^+ \to \overline{V}$, and let $\chi := (\psi^+)^{-1} \circ \psi : V \cdots \to V^+$. Note that V^+ is a smooth projective threefold. Let D_1^+ (resp. D_2^+) be the proper transform on V^+ of D_1 (resp. D_2). Since $\mathrm{Pic}\,(V) = \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ (cf. Remark 2.1) and χ is an isomorphism in codimension 1, it follows that $\mathrm{Pic}\,(V^+) = \mathbf{Z}[D_1^+] \oplus \mathbf{Z}[D_2^+]$. By the construction, it follows that $-K_V = \psi^*(-K_{\overline{V}})$ and $-K_{V^+} = (\psi^+)^*(-K_{\overline{V}})$. Note that $-K_{V^+} \sim 2D_1^+$ and $D_1^3 = {D_1^+}^3 = 8$. The closed cone $\overline{\mathrm{NE}}\,(V^+)$ is spanned by two edges one of which, say Q^+, comes from the flopped curve. Since K_V^+ is not nef, the other edge, say $R' := \mathbf{R}_+[l']$, is an extremal ray. Let $g' : V^+ \to W'$ be the contraction of R'.

Claim. $(D_1^+ \cdot l') = 1$ and g' is of type either $E2$, $C2$ or $D2$.

Proof of Claim. Since $(-K_{V^+} \cdot l') = 2(D_1^+ \cdot l') > 0$ is even, we can immediately obtain the assertion by the classification of extremal rays on smooth projective threefolds (cf. [36]). Note that since $b_2(V^+) = 2$, the Fano case with $\rho = 1$ can not occur. □

Step 3. At first we consider the case where $g' : V^+ \to W'$ is of $E2$-type, i.e., it contracts an exceptional divisor $E \subset V^+$ with $E|_E \sim \mathcal{O}_{\mathbf{P}^2}(-1)$ to a smooth point. Since $(D_1^+ \cdot l') = 1$, it follows that $D_1^+|_E$ is a line on $E \cong \mathbf{P}^2$. It is easy to see that $-K_{W'} \sim 2D_1'$, where D_1' is the proper transform on W' of D_1^+. Hence W' is a smooth Fano threefold with $\rho(V') = 1$ and of index either 2 or 4. On the other hand, by the straightforward computation, we have $(-K_{W'})^3 = (2D_1^+ + 2E)^3 = 8({D_1^+}^3 + 3{D_1^+}^2 \cdot E + 3D_1^+ \cdot E^2 + E^3) = 72$. This is a contradiction.

Step 4. Next we consider the case where $g' : V^+ \to W'$ is of $C2$-type. Since $\rho(W') = \rho(V^+) - 1 = 1$, we have $W' \cong \mathbf{P}^2$. Let $L' := {g'}^*\mathcal{O}_{\mathbf{P}^2}(1) \sim xD_1^+ + yD_2^+$ with $x, y \in \mathbf{Z}$. Since $2 = {L'}^2 \cdot (-K_{V^+}) = (xD_1^+ + yD_2^+)^2 \cdot (2D_1^+)$ and ${D_1^+}^3 = 8$, ${D_1^+}^2 \cdot D_2^+ = 2$ and $D_1^+ \cdot {D_2^+}^2 = 0$, we have $8x^2 + 4xy = 1$, which is absurd.

Step 5. Finally, we consider the case where $g' : V^+ \to W' \cong \mathbf{P}^1$ is of $D2$-type. As in Step 4, we write $L' := {g'}^*\mathcal{O}_{\mathbf{P}^1}(1) \sim xD_1^+ + yD_2^+$ with $x, y \in \mathbf{Z}$. We have $0 = (L' \cdot l') = x + y(D_2^+ \cdot l')$ and $0 = {L'}^2 \cdot (-K_{V+}) = 2D_1^+ \cdot (xD_1^+ + yD_2^+)^2 = 2(8x^2 + 4xy)$. Hence $x(y + 2x) = 0$ and $x + y(D_2^+ \cdot l') = 0$. If $x = 0$ then $y(D_2^+ \cdot l') = 0$. But since $y \neq 0$, we have $(D_2^+ \cdot l') = 0$, i.e., D_2^+ is a fiber of g'. This implies that two Mori fiber spaces $g : V \to W$ and $g : V^+ \to W'$ are isomorphic (cf. [3, (3.5) Proposition]). This is a contradiction. Hence $x \neq 0$ and $y = -2x$. We substitute $y = -2x$ for $x + y(D_2^+ \cdot l') = 0$ to have $(D_2^+ \cdot l') = 1/2$. This is absurd. □

Next we consider the case $d = 2$. Then the edge $Q = \mathbf{R}_+[C]$ of $\overline{\mathrm{NE}}(V)$, which is $(-K_V)$-trivial, is spanned by the minimal section $C = M_2$ on $D_1 \cong \mathbf{F}_2$. As in the proof of Lemma 4.9, we consider a morphism $\psi : V \to \overline{V}$ which is the Stein factorization of the morphism defined by $|m(-K_V)|$ for $m >> 0$. It is clear that ψ is birational as $-K_V$ is big. Let $\overline{D_1}$ (resp. $\overline{D_2}$) be the proper transform on $\overline{V}$ of D_1 (resp. D_2).

Lemma 4.10 *ψ is small.*

Proof. Assume to the contrary that ψ is not small, i.e., it contracts a divisor $E \subset V$. It is clear that E is distinct from D_1 and D_2. Furthermore, E meets D_1 along the minimal section C on $D_1 \cong \mathbf{F}_2$, that is, $E|_{D_1} = nC$ for some $n \in \mathbf{N}$. By noting that $\mathrm{Pic}\,(V) = \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ (cf. Remark 2.1), we can write $E \sim (\frac{n}{2})D_1 - 2nD_2$, in particular, n is even. Note that general curves from Q are mutually disjoint, so $\psi|_E$ yields a fibration. Let $\mu : \widetilde{E} \to E$ be the normalization and let $\mathcal{I} \subset \mathcal{O}_E$ be the conductor of μ defining closed subshemes $L := V_E(\mathcal{I}) \subset E$ and $\widetilde{L} := V_{\widetilde{E}}(\mathcal{I}) \subset \widetilde{E}$. Then it follows that $K_{\widetilde{E}} = \mu^*(K_E) - \widetilde{L}$ (cf. [36, pp. 165–167]). Let $\pi : \hat{E} \to \widetilde{E}$ be the minimal resolution with $\cup \hat{B}_i$ as an exceptional set. Then we have $K_{\hat{E}} = \pi^*(K_{\widetilde{E}}) - \sum \alpha_i \hat{B}_i$, where $\alpha_i \geq 0$. Since $nC = D_1|_E$ is Cartier on E, the pull-back $A := \mu^*(nC)$ is an effective Cartier divisor on $\widetilde{E}$. Let $\hat{A}$ be the proper transform on $\hat{E}$ of A. Note that $\psi|_E$ induces a fibration structure on $\hat{E}$ and each component of $\hat{A}$ is contained in a fiber of it. Hence $(\hat{A})^2_{\hat{E}} \leq 0$. On the other hand, we obtain $(-K_{\hat{E}} \cdot \hat{A})_{\hat{E}} = (\pi^*(-K_{\widetilde{E}}) + \sum \alpha_i \hat{B}_i \cdot \hat{A})_{\hat{E}} \geq (-K_{\widetilde{E}} \cdot A)_{\widetilde{E}} = (\mu^*(-K_E) + \widetilde{L} \cdot \mu^*(nC))_{\widetilde{E}} \geq (-K_E \cdot nC)_E$. By the adjunction, $(-K_E \cdot nC)_E = (2D_1 - (\frac{n}{2})D_1 + 2nD_2) \cdot (nC) = 2n^2$. Thus we have $0 \leq p_a(\hat{A}) \leq -n^2 + 1$. Hence $n = 1$, which is a contradiction as n is even. □

Lemma 4.11 *The case $d = 2$ does not occur.*

Proof. Our proof consists of four steps and is similar to that of Lemma 4.9. But in this present case the curve from Q is contained in D_1. By Lemma 4.10 we may assume that ψ is small.

Step 1. We perform a $(-D_2)$-flop $\psi^+ : V^+ \to \overline{V}$ with respect to $\psi : V \to \overline{V}$ and put $\chi := (\psi^+)^{-1} \circ \psi : V \cdots \to V^+$. Note that V^+ is a smooth projective threefold. Let D_1^+ (resp. D_2^+) be the proper transform on V^+ of D_1 (resp. D_2). By noting that $\mathrm{Pic}\,(V) = \mathbf{Z}[D_1] \oplus \mathbf{Z}[D_2]$ and χ is an isomorphims in codimension 1 we have $\mathrm{Pic}\,(V^+) = \mathbf{Z}[D_1^+] \oplus \mathbf{Z}[D_2^+]$. Furthermore, $-K_{V^+} \sim 2D_1^+$ as $-K_V \sim 2D_1$. The cone of curves $\overline{\mathrm{NE}}\,(V^+)$ is spanned by two edges one of which, say Q^+, is spanned by a flopped curve C^+ of C. Note that $D_1^+ \cdot C^+ = 0$ and $D_2^+ \cdot C^+ < 0$ via the $(-D_2)$-flop $\chi : V \cdots \to V^+$. In particular $C^+ \subset D_2^+$. Since K_{V^+} is not nef, the other edge, say $R' = \mathbf{R}_+[l']$, is an extremal ray. Let $g' : V^+ \to W'$ be the contraction of R'. By the same argument as in the proof of Lemma 4.9, g' is of type either $E2$, $C2$ or $D2$ and $D_1^+ \cdot l' = 1$. Furthermore, in the case where g' is of $E2$-type we can obtain a contradiction by the same argument as in Step 3 in the proof of Lemma 4.9. So, we have only to consider the case where g is of type $C2$ or $D2$.

Step 2. In any case we shall claim the following:

Claim. The restriction $\chi|_{D_1} : D_1 \cdots \to D_1^+$ is an isomorphism.

Proof of Claim. Note that $D_1 \backslash C$ and $D_1^+ \backslash (C^+ \cap D_1^+)$ is isomorphic. Now assume that $\chi|_{D_1}$ is not an isomorphism. Let $\mu : \widetilde{D_1^+} \to D_1^+$ and $\pi : \hat{D_1^+} \to \widetilde{D_1^+}$ be the normalization and the minimal resolution, respectively. It is clear that μ and π do not affect on $D_1^+ \backslash (C^+ \cap D_1^+)$. Let $D_1 \overset{h}{\leftarrow} Y \overset{\hat{h}}{\to} \hat{D_1^+}$ be the common resolution of the birational map $D_1 \cdots \to \hat{D_1^+}$. We may assume that there are no extra (-1)-curves on Y in the sense that the contractions of them induce also the common resolutions. After the process $h : Y \to D_1$ the possibility of (-1)-curve on Y which is contracted by $\hat{h}$ at the first time is either the last exceptional curve of h or the proper transform of C. But the former case is excluded by our choice of the resolution. On the other hand, the latter is impossible as C is the minimal section on $D_1 \cong \mathbf{F}_2$. Thus $\chi|_{D_1} : D_1 \cdots \to D_1^+$ is an isomorphism. □

Hence $D_1^+ \cong \mathbf{F}_2$ and C^+ is the minimal section on D_1^+. By the adjunction formula, it follows that $D_1^+|_{D_1^+} \sim 2M_2 + 4L_2$. Since $C^+ \subset D_2^+$, we see that D_1^+ and D_2^+ intersect each other along $C^+ \cup C_{1,2}^+$, where $C_{1,2}^+$ is the proper image of $C_{1,2} = D_1 \cap D_2$. It is clear that $C_{1,2}^+$ is a fiber on $D_1^+ \cong \mathbf{F}_2$. We can write $D_2^+|_{D_1^+} = nC^+ + C_{1,2}^+$ for some $n \in \mathbf{N}$.

Step 3. Assume that $g : V^+ \to W'$ is of $C2$-type. Since $\rho(W') = \rho(V^+) - 1 = 1$, we have $W' \cong \mathbf{P}^2$. Let $L' := g'^*\mathcal{O}_{\mathbf{P}^2}(1) \sim xD_1^+ + yD_2^+$ with $x, y \in \mathbf{Z}$. Then we have $2 = L'^2 \cdot (-K_{V^+}) = 2(x^2{D_1^+}^3 + 2xy{D_1^+}^2 \cdot D_2^+ + y^2 D_1^+ \cdot {D_2^+}^2)$. Here we note that ${D_1^+}^3 = 8$, ${D_1^+}^2 \cdot D_2^+ = (2M_2 + 4L_2 \cdot nM_2 + L_2)_{D_1^+} = 2$ and $D_1^+ \cdot {D_2^+}^2 = (nM_2 + L_2)^2_{D_1^+} = -2n^2 + 2n$. Hence we obatin the equation: $8x^2 + 4xy - 2n(n-1)y^2 = 1$, which is impossible.

Step 4. Assume that $g : V^+ \to W'$ is of $D2$-type. We can easily see that $W' \cong \mathbf{P}^1$. Let $L' := g'^*\mathcal{O}_{\mathbf{P}^1}(1) \sim xD_1^+ + yD_2^+$ with $x, y \in \mathbf{Z}$. Then we have $0 = L'^2 \cdot (-K_{V^+}) = 2(x^2{D_1^+}^3 + 2xy{D_1^+}^2 \cdot D_2^+ + y^2 D_1^+ \cdot {D_2^+}^2)$ and $8 = L' \cdot (-K_{V^+})^2 = 4(x{D_1^+}^3 + y{D_1^+}^2 \cdot D_2^+)$. By noting that ${D_1^+}^3 = 8$, ${D_1^+}^2 \cdot D_2^+ = 2$ and $D_1^+ \cdot {D_2^+}^2 = -2n^2 + 2n$, we have the equations: $(2x + ny)(2x - (n-1)y) = 0$ and $4x + y = 1$. Thus $(x, y) = (\frac{n}{2(2n-1)}, \frac{1}{1-2n})$ or $(x, y) = (\frac{n-1}{2(2n-1)}, \frac{1}{2n-1})$. The former has no integral roots, so this is impossible. On the other hand, the latter has only one integral root $(x, y) = (0, 1)$ when $n = 1$. This means that $L' \sim D_2^+$, hence D_2^+ is a fiber of g'. Thus $\chi : V/W \cdots \to V^+/W'$ is an isomorphism (cf. [3, (3.5) Proposition]). This is absurd. □

4.2 CASE: $K_{D_1} + \Delta$ is not nef

We consider the case where the adjoint bundle $K_{D_1} + \Delta$ is not nef. Note that Δ is very ample on D_1. Then, by Cone Theorem (cf. [23]), there exists an extremal ray $P = \mathbf{R}_+[s]$ which is $(K_{D_1} + \Delta)$-negative. Since $(-K_{D_1} \cdot s)_{D_1} > (\Delta \cdot s)_{D_1} > 0$, the only possibility is $(-K_{D_1} \cdot s)_{D_1} = 2$, $(\Delta \cdot s)_{D_1} = 1$ and the contraction of P yields a $\mathbf{P}^1$-bundle.

Lemma 4.12 *(1) $d = 0$, i.e., $D_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$,*

(2) $a = 2$, i.e., $-K_V \sim 2D_1 + 2D_2$,

(3) $D_1|_{D_1} \sim 2M_0$.

Proof. Note that $(K_{D_1} + \Delta \cdot L_d)_{D_1} = 0$. This means that L_d is not contained in the ray P, and the possibility of the degree of such a Hirzebruch surface $D_1 \cong \mathbf{F}_d$ is only $d = 0$. We have $1 = (\Delta \cdot s) = (\Delta \cdot M_0) = 3 - a$. Hence $a = 2$, i.e., $-K_V \sim 2D_1 + 2D_2$. The last assertion is immediate from Lemma 4.4 (2). □

Since $\overline{\mathrm{NE}}(V)$ is a two-dimensional closed cone in $\mathbf{R}^2$, it is spanned by two edges one of which $R = \mathbf{R}_+[l]$ comes from the Mori fiber space $g : V \to W$. Let $Q = \mathbf{R}_+[C]$ be the other edge. We write the class of a ruling $s = M_0$ on $D_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ as $[s] = \xi[l] + \eta[C]$ in $\overline{\mathrm{NE}}(V)$, where $\xi, \eta \in \mathbf{R}_+$.

Lemma 4.13 *(1) $\xi = 0$, i.e., $Q = \mathbf{R}_+[s]$,*

(2) $-K_V$ is ample,

(3) The contraction $g' := \mathrm{cont}_Q$ of the ray Q is of type either $E2$, $C2$ or $D2$.

Proof. We have the following equalities:

$$\begin{cases} 0 = (D_1 \cdot s) = (D_1 \cdot \xi l + \eta C) = \xi + (D_1 \cdot C)\eta, \\ 1 = (D_2 \cdot s) = (D_2 \cdot \xi l + \eta C) = (D_2 \cdot C)\eta. \end{cases}$$

Hence $\xi = -(D_1 \cdot C)/(D_2 \cdot C)$ and $\eta = 1/(D_2 \cdot C)$. Since $\xi, \eta \in \mathbf{R}_+$, we have $(D_2 \cdot C) > 0$, $(D_1 \cdot C) = 0$ and $\xi = 0$. Hence $Q = \mathbf{R}_+[s]$. By noting that $-K_V \sim 2D_1 + 2D_2$ (cf. Lemma 4.12), we have $(-K_V \cdot l) = (-K_V \cdot s) = 2$. Whence $-K_V$ is ample. Then Q is also an extremal ray. Since, for any curve from Q, the intersection number of this curve with $-K_V \sim 2D_1 + 2D_2$ is even, the last assertion follows from the classification of the extremal rays on the smooth projective threefolds (cf. [36]). □

Lemma 4.14 *The case $K_{D_1} + \Delta$ is not nef does not occur.*

Proof. Assume to the contrary that this case occurs and let $g' : V \to W'$ be the contraction of the extremal ray Q. Then the type of g is $E2$, $C2$ or $D2$ (cf. Lemma 4.13). First of all, let g' be of $E2$-type, that is, it contracts an exceptional divisor $E \cong \mathbf{P}^2$ to a smooth point. Since the curves from Q moves in D_1, it follows that $D_1 = E$, which is a contradiction. Next suppose that g' is of $C2$-type. Then $W' \cong \mathbf{P}^2$ as $\rho(W') = \rho(V) - 1 = 1$. Since $D_2 \cdot s = 1$, the restriction $g'|_{D_2} : D_2 \to W'$ is birational. After resolving the singular point of $D_2 \cong \mathbf{Q}_0^2$ (cf. Lemma 4.3), we have a birational morphism $\mathbf{F}_2 \to W' \cong \mathbf{P}^2$. This is absurd. Finally, suppose that g' is of $D2$-type. Since $D_2 \cdot s = 1$ the restriction $g'|_{D_2} : D_2 \to W' \cong \mathbf{P}^1$ yields a fibration. But as $D_2 \cong \mathbf{Q}_0^2$ this is impossible. □

Thus we know that the case where $g : V \to W$ is of $D2$-type does not occur. But if we replace the condition that D is strictly nef by that D is nef, we can construct a compactification of a contractible smooth affine threefold with an extremal ray of $D2$-type as the following example indicates.

Example 4.1 Let $f = x^2 + y^2 + z^2 \in \mathbf{C}[x, y, z]$ and let S_0 be the hypersurface in $\mathbf{A}^3{}_{(x,y,z)}$ defined by f. We embed $\mathbf{A}^3{}_{(x,y,z)}$ canonically into $\mathbf{P}^3{}_{(x:y:z:w)}$, where $H_\infty = (w = 0)$ is a hyperplane at infinity. We consider the linear pencil Λ on $\mathbf{P}^3$ spanned by S and $2H_\infty$, where S is the closure of S_0 in $\mathbf{P}^3$. We shall resolve the base locus $\mathrm{Bs}\,\Lambda = S \cap H_\infty =: C$ as follows (we shall use the same notations to represent the proper transforms in the subsequent): Let $\sigma_1 : V_1 \to \mathbf{P}^3$ be the blow-up of C with an exceptional divisor $E_1 \cong \mathbf{F}_2$ and let $\sigma_2 : V_2 \to V_1$ be the blow-up of the curve $S \cap E_1$ with an exceptional divisor $E_2 \cong \mathbf{P}^1 \times \mathbf{P}^1$. Let $\sigma_3 : V_2 \to V$ be the contraction of $H_\infty \cong \mathbf{P}^2$ to a smooth point. Then the proper transform Λ_V of Λ on V is spanned by S and E_1 and is base point free. In fact, by the construction, it defines a Mori fiber space $\Phi_{\Lambda_V} : V \to \mathbf{P}^1$ of $D2$-type. It is clear that $V \backslash (E_1 \cup E_2) \cong \mathbf{A}^3$. By the straightforward computation, it follows that $E_1 + E_2$ is nef. Let L be a ruling on $E_2 \cong \mathbf{P}^1 \times \mathbf{P}^1$ meeting E_1. Then $(E_1 + E_2) \cdot L = 0$, so $E_1 + E_2$ is not strictly nef.

5 The case $g : V' \to W$ is of $D3$-type

In this section we consider the case where $g : V' \to W$ is of $D3$-type, i.e., it is a $\mathbf{P}^2$-bundle structure over a smooth curve W. Let l' be the class of the line in a fiber $F \cong \mathbf{P}^2$ of g. Then we have $D' \cdot l' = t'(-K_{V'} \cdot l') = 3t'$. Since D' is strictly nef Cartier divisor and $0 < t' < 1$, we have either $t' = 1/3$, $D' \cdot l' = 1$ or $t' = 2/3$, $D' \cdot l' = 2$.

Lemma 5.1 *The base curve W is isomorphic to $\mathbf{P}^1$.*

Proof. It is easy to see that $H^1(\mathcal{O}_V) = 0$ by noting the contractibility of X. On the other hand, since $\varphi : V \to V'$ is a composite of $E2$-type contractions, we have $H^1(\mathcal{O}'_V) = 0$ also. As $g : V' \to W$ is a Mori fiber space, it follows that $g_* \mathcal{O}_{V'} = \mathcal{O}_W$ and $R^j g_* \mathcal{O}_{V'} = 0$. Hence $H^1(\mathcal{O}_W) = 0$, i.e., $W \cong \mathbf{P}^1$. □

In the subsequent we shall divide the situation into two cases according as $t' = 1/3$ or $t' = 2/3$.

5.1 CASE: $t' = 1/3$

Suppose $t' = 1/3$, so $D' \cdot l' = 1$. Then we have the following:

Lemma 5.2 *Suppose that $t' = 1/3$, $D' \cdot l' = 1$.*

(1) *The process $\varphi : V \to V'$ is an identity, i.e., V is already a $D3$-type Mori fiber space,*

(2) *D consists of two components D_1 and D_2. One of them, say D_1, is such that $D_1|_F \sim \mathcal{O}_F(1)$ and the other D_2 is a fiber of g, where $F \cong \mathbf{P}^2$ is a fiber of g,*

(3) *$g|_X$ yields an $\mathbf{A}^2$-bundle structure over $\mathbf{A}^1$, i.e., $X \cong \mathbf{A}^3$.*

Proof. The first assertion is ascertained by the similar argument as in Lemma 3.2. Since $D' \cdot l' = D \cdot l' = 1$ we can easily see (2). Thus the restriction $g|_X : X = V \backslash (D_1 \cup D_2) \to W \backslash g(D_2) \cong \mathbf{A}^1$ is an $\mathbf{A}^2$-bundle over the affine line $\mathbf{A}^1$. Since Pic $(\mathbf{A}^1) = 0$, it is, in fact, trivial, i.e., $X \cong \mathbf{A}^1 \times \mathbf{A}^2 \cong \mathbf{A}^3$. □

Thus we obtained the case $D3a$ in the statement of Theorem 1.1.

5.2 CASE: $t' = 2/3$

Next we suppose $t' = 2/3$, so $D' \cdot l' = 2$. Then we have the following:

Lemma 5.3 *Suppose that $t' = 2/3$, $D' \cdot l' = 2$.*

(1) *$\gamma = 1$ (cf. the argument after Lemma 2.5),*

(2) *D' consists of three components $D' = D'_1 + D'_2 + D'_3$. One of them, say D'_3 is a fiber of g, and the remainings $D'_1|_F$ and $D'_2|_F$ are the mutually distinct lines on every fiber $F \cong \mathbf{P}^2$ of g,*

(3) *The restriction $g|_{X'}$ yields an $(\mathbf{A}^1 \times \mathbf{C}^*)$-bundle over the affine line $\mathbf{A}^1$, i.e., $X' \cong \mathbf{A}^2 \times \mathbf{C}^*$.*

Proof. Note that $2 = \rho(V') = \rho(V) - s = r - s$, i.e., the number s of $E2$-type divisorial contractions appearing in the process φ equals $s = r - 2$, where r is the number of irreducible components in D (cf. Remark 2.1). Hence D' consists of $\gamma + 2$ components. Suppose that there exists exactly one component of D', say D'_1, such that $D'_1 \cdot l' = 2$. Then the remaining components $D'_2, \cdots, D'_{\gamma+2}$ are fibers of g. Then Pic (X') contains a 2-torsion element. But since X' is factorial (cf. Remark 2.2), this is a contradiction. Thus there exists two components, say D'_1 and D'_2, such that $D'_1 \cdot l' = D'_2 \cdot l' = 1$ and the remaining components $D'_3, \cdots, D'_{\gamma+2}$ are fibers of g. By noting that D' is normal crossing, $D'_1|_F$ and $D'_2|_F$ are mutually distinct lines on every fiber $F \cong \mathbf{P}^2$ of g. Then the restriction $g|_{X'}$ yields an $(\mathbf{A}^1 \times \mathbf{C}^*)$-bundle over the curve $W \backslash (\cup_{i=3}^{\gamma+2} g(D'_i)) \cong \mathbf{C}^{*\gamma-1}$. (Here we understand as $\mathbf{C}^{*0} = \mathbf{A}^1$ and $\mathbf{C}^{*-1} = \mathbf{P}^1$ if $\gamma = 1$ and $\gamma = 0$, respectively.) Hence we have $\chi_{\text{top}}(X') = \chi_{\text{top}}(\mathbf{C}^{*\gamma-1})\chi_{\text{top}}(\mathbf{A}^1 \times \mathbf{C}^*) = 0$. On the other hand, since $X \backslash X' \cong \amalg_{j=1}^{\gamma} S_j$ with $S_j \cong \mathbf{A}^2$ for $1 \leq j \leq \gamma$ (cf. Remark 2.2), we have $\chi_{\text{top}}(X') = \chi_{\text{top}}(X) - \gamma\chi_{\text{top}}(\mathbf{A}^2) = 1 - \gamma$. This means that $\gamma = 1$. By the above argument we obtain the remaining assertions also. □

By Lemma 5.3, $X \backslash X' = S_1 \cong \mathbf{A}^2$ and $X' \cong \mathbf{A}^2 \times \mathbf{C}^*$. Let $p : X' \cong \mathbf{A}^2 \times \mathbf{C}^* \to \mathbf{C}^*$ be a trivial $\mathbf{A}^2$-bundle structure.

Lemma 5.4 *$p : X' \to \mathbf{C}^*$ can be extended to an $\mathbf{A}^2$-bundle structure on X over the affine line $\mathbf{A}^1$, that is, $X \cong \mathbf{A}^3$.*

Proof. Let $a \in A$ be a prime element of A defining S_1. Since $X \backslash S_1 = X' \cong \mathbf{A}^2 \times \mathbf{C}^*$ we can write $A[a^{-1}] = \mathbf{C}[x, y, t, t^{-1}]$. Since A is a finitely generated $\mathbf{C}$-algebra, the locally nilpotent derivations $\partial/\partial x$ and $\partial/\partial y$ on $A[a^{-1}]$ can be extended to those on A, say Δ_1 and Δ_2 by multiplying a^N for a sufficiently large $N >> 0$. Let $B := \text{Ker}(\Delta_1) \cap \text{Ker}(\Delta_2)$, which is a finitely generated $\mathbf{C}$-algebra by Zariski's Lemma and an inert sub-algebra of A (cf. [38], [28]). B is of dimension 1, factorial and $B^* = \mathbf{C}^*$, hence $B = \mathbf{C}^{[1]}$: a polynomial ring in one variable. By the construction, it is easy to see that $a \in B$ and $B[a^{-1}] = \mathbf{C}[t, t^{-1}]$. Let $h : X \to Z := \text{Spec}\,(B) \cong \mathbf{A}^1$ be the morphism associated with the canonical inclusion $B \hookrightarrow A$. Every fiber of h over the point on $\mathbf{C}^*(\subset Z)$ is isomorphic to the affine plane $\mathbf{A}^2$. Furthermore, the fiber of h over the point $Z \backslash \mathbf{C}^*$ is $S_1 \cong \mathbf{A}^2$. Thus all the fibers of h are $\mathbf{A}^2$ and is, in fact, is a trivial $\mathbf{A}^2$-bundle structure over $\mathbf{A}^1$, whence $X \cong \mathbf{A}^1 \times \mathbf{A}^2 \cong \mathbf{A}^3$. □

Thus we obtained the case $D3b$ stated in Theorem 1.1.

6 The case V' is a Fano 3-fold with $\rho(V') = 1$ of index ≥ 2

In this section we treat with the case where V' is a smooth Fano threefold with $\rho(V') = 1$ of index ≥ 2. We consider according as the Fano index separately.

6.1 CASE: Fano index $= 2$

First of all, we consider the case of Fano index $= 2$.

Lemma 6.1 (1) *The process φ is an identity, i.e., V is already a smooth Fano threefold of index $= 2$,*

(2) *D is irreducible and $b_2(D) = 1$,*

(3) *$V = V_5$ (see* **Notation.***),*

(4) *If D is normal, then $D \cong H_5^0$ and $X = V \backslash D \cong \mathbf{A}^3$,*

(5) *If D is non-normal, then $D \cong H_5^\infty$ and $X = V \backslash D \cong \mathbf{A}^3$.*

Proof. Note that $D' \cdot l' = t'(-K_{V'} \cdot l') = 2t' \in \mathbf{N}$. Since $0 < t' < 1$, we have $t' = 1/2$ and $D' \cdot l' = 1$. Then the similar argument as in Lemma 3.2 implies that φ is an identity, i.e., $V = V'$. At the same time, we know that $D' = D$ is irreducible as $D \cdot l' = 1$. By the contractibility of $X = V \backslash D$, it follows that $b_2(D) = 1$. The remainings (3), (4) and (5) follow from [7, 9]. Here we note that the arguments in [7] and [9] can be applied to our present situation without any modification. In fact, these arguments use only the acyclicity of the inside smooth affine threefold X. □

Thus we obtained cases $F2$ in Theorem 1.1.

6.2 CASE: Fano index $= 3$

Next we consider the case Fano index $= 3$, i.e., $V' \cong \mathbf{Q}^3$. Then we have $(D' \cdot l') = t'(-K_{V'} \cdot l') = 3t' \in \mathbf{N}$. By noting that $0 < t' < 1$, either $t' = 1/3$, $D' \cdot l' = 1$ or $t' = 2/3$, $D' \cdot l' = 2$ holds. More precisely, we have the following:

Lemma 6.2 (1) *If $t' = 1/3$, $D' \cdot l' = 1$, then the process φ is an identity, i.e., $V = \mathbf{Q}^3$, and $D \sim \mathcal{O}_{\mathbf{Q}^3}(1)$ is isomorphic to $\mathbf{Q}_0^2$. Furthermore, $X \cong \mathbf{A}^3$,*

(2) *If $t' = 2/3$, $D' \cdot l' = 2$, then D' consists of two components D'_1 and D'_2 both of which are isomorphic to $\mathbf{Q}_0^2$. Furthermore, $\gamma = 1$, that is, $X \backslash X' = S_1 \cong \mathbf{A}^2$.*

Proof. (1) Since $D' \cdot l' = 1$ we can see that φ have to be an isomorphism by the similar argument as in Lemma 3.2. By noting the contractibility of $X = V \backslash D$, it follows that $D \sim \mathcal{O}_{\mathbf{Q}^3}(1)$ is irreducible and $b_2(D) = 1$. Thus $D \cong \mathbf{Q}_0^2$ and the complement $X = \mathbf{Q}^3 \backslash \mathbf{Q}_0^2$ is isomorphic to $\mathbf{A}^3$.

(2) Since $D' \cdot l' = 2$, the boundary D' consists of one or two components. If D' consists of only one component, then Pic (X') contains a 2-torsion element. This is absurd (cf. Remark 2.2). Thus we can write $D' = D'_1 + D'_2$. On the other hand, since $1 = \rho(V') = r - s$ the number s of contractions appearing in $\varphi : V \to V'$ equals $r - 1$, where r is the number of irreducible components of D. Thus D' is composed of $\gamma + 1$ components. Hence $\gamma = 1$. This implies that $X \backslash X' =: S_1 \cong \mathbf{A}^2$. By noting the topology of X' and considering cohomology exact sequence associated with the pair (V', D') we know that $H^2(D') = \mathbf{Z}$. On the other hand, we consider the Mayer-Vietoris exact sequence:

$$\cdots \to H^1(C'_{1,2}) \to H^2(D') \to H^2(D'_1) \oplus H^2(D'_2) \to H^2(C'_{1,2}) \to \cdots,$$

where $C'_{1,2} := D'_1 \cap D'_2$ is an intersection curve. Since D' is normal crossing, it is easy to see $C'_{1,2} \cong \mathbf{P}^1$. Thus we have $b_2(D'_1) = b_2(D'_2) = 1$. Therefore $D'_1 \cong D'_2 \cong \mathbf{Q}_0^2$ and they meet each other along $C'_{1,2}$ normally which does not pass through vertices of D'_1 and D'_2. □

We need to investigate the situation (2) in Lemma 6.2 in detail. As remarked in the proof of Lemma 6.2, the number s of contractions appearing in $\varphi = \varphi_{s-1} \circ \cdots \circ \varphi_0 : V \to V'$ equals $r - 1$, where r is the number of irreducible components of D. As seen in Lemma 6.2, we have $\gamma = 1$, i.e., there exists exactly one j such that $\varphi_j : V^j \to V^{j+1}$ contracts an exceptional divisor not contained in the proper transform D^j (cf. Section 2). When we consider this contraction process $\varphi : V \to V'$ as a composite of blow-ups starting from V' conversely, the images p'_j on V' of the centers p_j^{j+1} of the blow-ups $\varphi_j : V_j \to V_{j+1}$ are mutually distinct points on D' by virtue of Lemma 2.6. In fact, we shall show:

Lemma 6.3 *The case (2) in Lemma 6.2 does not occur.*

Proof. Since $\gamma = 1$, there exists exactly one j_0 $(0 \leq j_0 \leq s-1)$ such that an exceptional divisor E^{j_0} of the blow-up with center p'_{j_0} is not contained in the original boundary divisor D. We may assume that $p'_{j_0} \in D'_1 \backslash D'_2$ and p'_{j_0} is distinct from the vertex of $D'_1 \cong \mathbf{Q}^2_0$. Let (x, y, z) be the affine coordinates on the complement $V' \backslash D'_2 \cong \mathbf{A}^3$ and let $D^\circ_1 := D'_1 \cap \mathbf{A}^3$. We may assume that D°_1 is defined by the polynomial $f = x^2 + y^2 - z^2 \in \mathbf{C}[x, y, z]$ and $p := p'_{j_0} = (0, 1, 1)$ in $\mathbf{A}^3$. Then the original affine threefold X can be viewed as an affine modification of $\mathbf{A}^3{}_{(x,y,z)}$ with locus (m_p, f) in the sense of [17], where $m_p := (x, y-1, z-1)$ is the maximal ideal of the point $p \in \mathbf{A}^3$. Let $\sigma : X \to \mathbf{A}^3$ be this affine modification associated to the canonical inclusion $\mathbf{C}[x, y, z] \hookrightarrow A = \mathbf{C}[x, y, z][m_p/f]$. It is easy to see that the divisor of σ is D°_1 and the exceptional divisor of it is $E^\circ := E^{j_0} \cap X \cong \mathbf{A}^2$ (here we shall cite the terminology in [17]). We claim the following. This claim is informed by Kaliman.

Claim. Such an X is not acyclic, in particular, not contractible.

Proof of Claim. We put $Y := \mathbf{A}^3$ and $X^* := X \backslash E^\circ$, which is isomorphic to $Y^* := \mathbf{A}^3 \backslash D^\circ_1$. We write the homology exact sequence of the pair (X, X^*) with $\mathbf{Z}$-coefficients:

$$\cdots \to H_3(X, X^*) \to H_2(X^*) \to H_2(X) \cdots \to,$$

where we note that $H_3(X, X^*) \cong H_1(E^\circ) = 0$ by Thom isomorphims. On the other hand, by noting that $Y^* = (Y \backslash \{0\}) \backslash (D^\circ_1 \backslash \{0\})$, we write the homology exact sequence of the pair $(Y \backslash \{0\}, Y^*)$:

$$\cdots \to H_3(Y \backslash \{0\}) \to H_3(Y \backslash \{0\}, Y^*) \to H_2(Y^*) \cdots \to,$$

where we note that $H_3(Y \backslash \{0\}) = 0$ and, by virtue of Thom isomorphism, $H_3(Y \backslash \{0\}, Y^*) \cong H_1(D^\circ_1 \backslash \{0\}) \cong \mathbf{Z}/2\mathbf{Z}$. Thus $H_2(X^*) \overset{\sigma_*}{\cong} H_2(Y^*)$ is non-trivial and so is $H_2(X)$ by the first sequence. □

Thus X is not contractible and this is a contradiction to our assumption. Thus we complete the proof. □

Thus we obtained the case $F3$ in Theorem 1.1.

6.3 CASE: Fano index $= 4$

Finally we consider the case Fano index $= 4$, i.e., $V' \cong \mathbf{P}^3$. Then we have $(D' \cdot l') = t'(-K_{V'} \cdot l') = 4t' \in \mathbf{N}$. By noting that $0 < t' < 1$, either $t' = 1/4$, $D' \cdot l' = 1$ or $t' = 1/2$, $D' \cdot l' = 2$ or $t' = 3/4$, $D' \cdot l' = 3$ holds. More precisely, we have the following:

Lemma 6.4 (1) *If $t' = 1/4$, $D' \cdot l' = 1$, then the process φ is an identity, i.e., $V = \mathbf{P}^3$, and $D \sim \mathcal{O}_{\mathbf{P}^3}(1)$. Then $X \cong \mathbf{A}^3$,*

(2) *If $t' = 1/2$, $D' \cdot l' = 2$, then D' consists of two components D'_1 and D'_2. These D'_1 and D'_2 are hyperplanes. Furthermore, $\gamma = 1$, i.e., $X \backslash X' \cong \mathbf{A}^2$ and $X' \cong \mathbf{C}^* \times \mathbf{A}^2$,*

(3) *If $t' = 3/4$, $D' \cdot l' = 3$, then D' consists of two or three components. More precisely:*

- (i) *If D' consists of two components D'_1 and D'_2, then one of them, say D'_1 is a quadric cone $\mathbf{Q}^2_0$, and the other D'_2 is a hyperplane and they intersect each other along a smooth conic. Then $\gamma = 1$, i.e., $X \backslash X' \cong \mathbf{A}^2$;*
- (ii) *If D' consists of three components D'_1, D'_2 and D'_3, then they are hyperplanes and these three do not intersect along a common line. Then $\gamma = 2$, i.e., $X \backslash X' = S_1 \amalg S_2$ with $S_1 \cong S_2 \cong \mathbf{A}^2$ and $X' \cong \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{A}^1$.*

Proof. (1) Since $D' \cdot l' = 1$ we can see that φ have to be an isomorphism by the similar argument as in Lemma 3.2. The remaining assertion in (1) is easy to see.

(2) Since $D' \cdot l' = 2$, the boundary D' consists of one or two components. If D' consists of only one component, then Pic (X') contains a 2-torsion element. This is a contradiction (cf. Remark 2.2).

Thus we can write $D' = D'_1 + D'_2$, where D'_1 and D'_2 are hyperplanes. On the other hand, since $1 = \rho(V') = r - s$ the number s of contractions appearing in $\varphi : V \to V'$ equals $r-1$, where r is the number of irreducible components of D. Thus D' is composed of $\gamma+1$ components. Hence $\gamma = 1$. This implies that $X\backslash X' =: S_1 \cong \mathbf{A}^2$. The remaining assertion is easy to see.

(3) In the case $t' = 3/4$ and $D' \cdot l' = 3$, the boundary D' is composed of at most three irreducible components. But if D' is irreducible then Pic (X') contains 3-torsion element, which is a contradiction (cf. Remark 2.2). If D' consists of two components D'_1 and D'_2, then $\gamma = 1$ and we may assume that D'_1 is quadric and D'_2 is a hyperplane intersecting along a smooth conic. By noting the topology of X' (cf. the argument in the proof of Lemma 6.2), we have $b_2(D'_1) = b_2(D'_2) = 1$, in particular, D'_1 is a quadric cone $\mathbf{Q}^2_0$. If D' consists of three components then $\gamma = 2$ and it is clear that all of them are hyperplanes. Moreover, they do not have a common line as D' is normal crossing and $X' = \mathbf{P}^3\backslash\mathrm{Supp}\,(D') \cong \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{A}^1$. □

We need to investigate the situations (2) and (3) in Lemma 6.4 in detail.

Lemma 6.5 (1) *In the case (2) of Lemma 6.4, we have* $X \cong \mathbf{A}^3$,

(2) *The case (3)-(i) of Lemma 6.4 does not occur.*

Proof. (1) When we consider the process $\varphi = \varphi_{s-1} \circ \cdots \circ \varphi_0 : V \to V'$ as a composite of blow-ups starting from V', the centers p'_j of the blow-ups $\varphi_j : V^j \to V^{j+1}$ on V' are mutually distinct points on D'. Since $\gamma = 1$, there exists exactly one j_0 $(0 \le j_0 \le s-1)$ such that an exceptional divisor E^{j_0} of the blow-up with center p'_{j_0} is not contained in the original boundary divisor D. We may assume that $p'_{j_0} \in D'_1\backslash D'_2$. Let (x, y, z) be the system of affine coordinates on the affine 3-space $\mathbf{A}^3 = V'\backslash D'_2$. We may assume that $D^\circ_1 := D'_1 \cap \mathbf{A}^3$ is defined by $x = 0$ and $p := p'_{j_0} = (0, 0, 0)$ in $\mathbf{A}^3$. Then the original X can be viewed as an affine modification of $\mathbf{A}^3{}_{(x,y,z)}$ with locus (m_p, x) in the sense of [17], where $m_p := (x, y, z)$ is the maximal ideal of the point $p \in \mathbf{A}^3$. Hence $A = \Gamma(\mathcal{O}_X) = \mathbf{C}[x, y, z][m_p/x] \cong \mathbf{C}^{[3]}$, i.e., $X \cong \mathbf{A}^3$.

(2) By the same argument as in Lemma 6.3, we see the assertion. □

Finally we shall prove the following:

Lemma 6.6 *The case (3)-(ii) in Lemma 6.4 does not occur.*

Proof. Since $X' \cong \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{A}^1$ (cf. Lemma 6.4), we have $A' = \Gamma(\mathcal{O}_{X'}) = \mathbf{C}[x, y, z, x^{-1}, y^{-1}]$. Let $p : X' \cong \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{A}^1 \to \mathbf{C}^* \times \mathbf{C}^*$ be the canonical projection which is induced by the locally nilpotent derivation $\partial/\partial z$ on A'. On the other hand, let a_i be a prime element in A such that $\mathrm{div}(a_i)_X = S_i \cong \mathbf{A}^2$ for $i = 1, 2$, and let $a := a_1 a_2$. Since $A' = A[a^{-1}]$ (cf. Remark 2.2) and A is a finitely generated $\mathbf{C}$-algebra, $\partial/\partial z$ can be extended to a locally nilpotent derivation Δ on A after multiplying with a^N for a sufficiently large $N >> 0$. Let $B := \mathrm{Ker}(\Delta) \subset A$, which is finitely generated by Zariski's Lemma and an inert sub-algebra of A (cf. [38], [28]). Let $h : X \to Z := \mathrm{Spec}\,(B)$ be a morphism associated with the canonical inclusion $B \hookrightarrow A$. By the construction, it is not difficult to see that $a_1, a_2 \in B$ and $B[a^{-1}] = \mathbf{C}[x, y, x^{-1}, y^{-1}]$. Let C_i be an affine curve on Z defined by a_i for $i = 1, 2$. Note that Z is a smooth contractible surface and $h : X \to Z$ is surjective (cf. [18] and [19]).

Claim. $Z \cong \mathbf{A}^2$, C_1 *and* C_2 *are isomorphic to the affine line* $\mathbf{A}^1$ *meeting each other.*

Proof of Claim. Note that Z is a homology plane containing $\mathbf{C}^* \times \mathbf{C}^*$, hence $\overline{\kappa}(Z) \le \overline{\kappa}(\mathbf{C}^* \times \mathbf{C}^*) = 0$. Since there are no homology planes of $\overline{\kappa} = 0$ (cf. [6]), we have $\overline{\kappa}(Z) = -\infty$. Hence $Z \cong \mathbf{A}^2$ (cf. [32, Chap. 3]). Since C_i are the images of the affine planes $S_i \cong \mathbf{A}^2$, it is easy to see that C_i are topologically contractible. Now assume, for instance, that C_1 is not smooth and we put $Y := Z\backslash C_1$. The polynomial a_1 defining C_1 defines a morphism $f : Y \to \mathbf{C}^*$. Then, by the addition formula of log Kodaira dimension (cf. [22]), we have

$$\overline{\kappa}(Y) \ge \overline{\kappa}(F) + \overline{\kappa}(\mathbf{C}^*) = \overline{\kappa}(F),$$

where F is a general fiber of f. Since $Y \supset \mathbf{C}^* \times \mathbf{C}^*$, we have $\overline{\kappa}(Y) \le 0$, so $\overline{\kappa}(F) = -\infty$ or $= 0$. If $\overline{\kappa}(Y) = -\infty$ then $\overline{\kappa}(F) = -\infty$. So f is an $\mathbf{A}^1$-fibration and it extends to an $\mathbf{A}^1$-bundle on $\mathbf{A}^1$. This is a contradiction. If $\overline{\kappa}(Y)=0$ and $\overline{\kappa}(F)=-\infty$ then we have a contradiction by the same fashion. Assume $\overline{\kappa}(Y)=0$ and $\overline{\kappa}(F)=0$. Then f is a $\mathbf{C}^*$-fibration. It is known that any component of a fiber of $\mathbf{C}^*$-fibration on a smooth affine surface is smooth (cf. [32]). Hence C_1 is smooth, which is a contradiction. Thus we obtain that C_1 and C_2 are isomorphic to the affine line $\mathbf{A}^1$. We can choose a suitable affine coordinates (u,v) on $Z\cong\mathbf{A}^2$ such that $C_1=(u=0)$ by Abhyankar-Moh-Suzuki Embedding Theorem (cf. [1], [41]). If C_1 does not meet C_2, then $C_2=(u=c)$ for some $c\in\mathbf{C}^*$, so $Z\backslash(C_1\cup C_2)\cong\mathbf{C}^{*2}\times\mathbf{A}^1$, a contradiction. Hence C_1 meets C_2. □

Since $C_1\cap C_2\neq\emptyset$ and $h:X\to Z$ is surjective, we have $\emptyset\neq h^{-1}(C_1\cap C_2)=h^{-1}(C_1)\cap h^{-1}(C_2)=S_1\cap S_2$. But since $S_1\cap S_2=\emptyset$ (cf. Remark 2.2), this is absurd. □

Thus we complete the proof of Theorem 1.1.

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References

[1] S.S. Abhyankar and T.T. Moh, *Embeddings of the line in the plane*, J. reine angew. Math.,**276** (1975), pp. 148–166.

[2] H. Clemens, J. Kollár and S. Mori, *Higher dimensional complex geometry*, Astérisque **166** (1988).

[3] A. Corti, *Factoring birational maps of 3-folds after Sarkisov*, J. Alg. Geom., **4** (1995), pp. 223–254.

[4] A. Dimca, *Hypersurfaces in* $\mathbf{C}^{2n}$ *diffeomorphic to* $\mathbf{R}^{4n-2}$ $(n\geq 2)$, Max-Planck-Institut, preprint, 1991.

[5] T. Fujita, *On Zariski problem*, Proc. Japan Acad. Ser A Math. Sci., **55** (1979), pp. 106–110.

[6] T. Fujita, *On the topology of non complete algebraic surfaces*, J. Fac. Sci. Univ. Tokyo, Sect. IA, **29** (1982), pp. 503–566.

[7] M. Furushima, *Singular Del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space* $\mathbf{C}^3$, Nagoya Math. J., **104** (1986), pp. 1–28.

[8] M. Furushima, *The complete classification of compactifications of* $\mathbf{C}^3$ *which are projective manifolds with the second Betti number one*, Math. Ann., **297** (1993), pp. 627–662.

[9] M. Furushima and N. Nakayama, *The family of lines on the Fano threefold* V_5, Nagoya Math. J. **116** (1989), pp. 111-122.

[10] J. E. Goodman, *Affine open subsets of algebraic varieties and ample divisors*, Ann. of Math., **89** (1969), pp. 160–183.

[11] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons, New York, 1978.

[12] R. V. Gurjar and C. R. Pradeep, $\mathbf{Q}$*-homology planes are rational. III*, Osaka J. Math., **36** (1999), pp. 259–335.

[13] R. V. Gurjar, C. R. Pradeep and A. R. Shastri, *On rationality of logarithmic* $\mathbf{Q}$*-homology planes. II*, Osaka J. Math., **34** (1997), pp. 725–743.

[14] R. Hartshorne, *Algebraic geometry*, GTM **52**, Springer, 1977.

[15] S. Iitaka, *On logarithmic Kodaira dimension of algebraic varieties*, Complex Analysis and Algebraic Geometry (W. L. Baily, Jr. and T. Shioda, eds.), Iwanami Shoten, Tokyo, 1977, pp. 175–189.

[16] V. A. Iskovskikh, *Anticanonical models of three-dimensional algebraic varieties*, Current problems in Mathematics, VINITI, Moscow, **12** (1979), pp. 59–157 (Russian). [English transl.: J. Soviet Math., **13** (1980), pp. 745–814.]

[17] Sh. Kaliman, *Polynomials with general* $\mathbf{C}^2$*-fibers are variables*, Pacific J. Math., **203** (2002), pp. 161–190.

[18] Sh. Kaliman, *Free* $\mathbf{C}_+$*-actions on* $\mathbf{C}^3$ *are translations*, alg-geom/0207156 (to appear).

[19] Sh. Kaliman and N. Saveliev, $\mathbf{C}_+$*-actions on contractible threefolds*, alg-geom/0209306.

[20] Sh. Kaliman and M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism groups*, Transformation Groups., **4** (1999), pp. 53–95.

[21] Sh. Kaliman and M. Zaidenberg, *Families of affine planes: The existence of a cylinder*, Michigan Math. J., **49** (2001), pp. 353–367.

[22] Y. Kawamata, *Addition formula of logarithmic Kodaira dimensions for morphisms of relative dimension one*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto, 1977) (M. Nagata, ed.), Kinokuniya, Tokyo, 1978, pp. 207–217.

[23] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, in: Algebraic Geometry, Sendai, 1985, pp. 283–360, Adv. Stud. Pure Math., **10**, North Holland, Amsterdam, 1987.

[24] T. Kishimoto, *A new proof of a theorem of Ramanujam-Morrow*, J. Math. Kyoto Univ., **42** (2002), pp. 117–139.

[25] J. Kollár, *Flops*, Nagoya Math. J., **113** (1989), pp. 15–36.

[26] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press **134**, 1998.

[27] L. Makar-Limanov, *On the hypersurface $x+x^2y+z^2+t^3=0$ in $\mathbf{C}^4$ or a $\mathbf{C}^3$-like threefold which is not $\mathbf{C}^3$*, Israel J. Math., **96** (1996), pp. 419–429.

[28] M. Miyanishi, *Lectures on curves on rational and unirational surfaces*, Tata Inst. Fund. Res., Bombay, 1978.

[29] M. Miyanishi, *Non-complete algebraic surfaces*, Lectures Notes in Math., **857**, Berlin-Heidelberg-New York, Springer, 1981.

[30] M. Miyanishi, *An algebro-topological characterization of the affine space of dimension three*, Amer J. Math., **106** (1984), pp. 1469–1486.

[31] M. Miyanishi, *Algebraic characterization of the affine 3-space*, Proc. Algebraic Geom. Seminar, Singapore, World Scientific, (1987), pp. 53–67.

[32] M. Miyanishi, *Open algebraic surfaces*, CRM Monograph Series **12**, AMS Providence, RI, 2001.

[33] M. Miyanishi and T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto Univ., **20** (1980), pp. 11–42.

[34] M. Miyanishi and T. Sugie, *Homology planes with quotient singularities*, J. Math. Kyoto Univ., **31** (1991), pp. 755–788.

[35] J.A. Morrow, *Minimal normal compactification of $\mathbf{C}^2$*, Rice Univ. Studies, **59** (1973), pp. 97–112.

[36] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math., **115** (1982), pp. 133–176,

[37] S. Mori and S. Mukai, *Classification of Fano 3-folds with $B_2 \geq 2$*, Manuscripta Math., **36** (1981), pp. 147–162.

[38] M. Nagata, *Lectures on the fourteenth problem of Hilbert*, Tata Inst. Fund. Res., Bombay, 1965.

[39] C. R. Pradeep and A. R. Shastri, *On rationality of logarithmic $\mathbf{Q}$-homology planes. I*, Osaka J. Math., **34** (1997), pp. 429–456.

[40] C.P. Ramanujam, *A topological characterization of the affine plane as an algebraic variety*, Ann. of Math., **94** (1971), pp. 69–88.

[41] M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes et automorphismes algébriques de I'espace* $\mathbf{C}^2$, J. Math. Soc. Japan, **26** (1974), pp. 241–257.

[42] M. G. Zaidenberg, *Isotrivial families of curves on affine surfaces and characterizations of the affine plane*, Izv. Akad. Nauk SSSR Ser. Mat., **51** (1987), pp. 534–567 (Russian), English transl. Math. USSR-Izv., **30** (1988), no. 3, pp. 503–532.

[43] M. G. Zaidenberg, *Lectures on exotic algebraic structures on affine spaces*, Schriftenreihe des Graduiertenkollegs Geometrie und Mathematische Physik, Helt 24, Institut Für Mathematik, Ruhr-Universität Bochum, (Juli 1997), 56 p.

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