Hypersurfaces with constant scalar curvature in a hyperbolic space form*

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Abstract

Let M^n be a complete hypersurface with constant normalized scalar curvature R in a hyperbolic space form H_1^{n+1}. We prove that if $\bar{R} = R+1 \geq 0$ and the norm square $|h|^2$ of the second fundamental form of M^n satisfies $n\bar{R} \leq sup\,|h|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n]$, then either $sup\,|h|^2 = n\bar{R}$ and M^n is a totally umbilical hypersurface; or $sup\,|h|^2 = \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R}+n]$, and M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$, for some $r > 0$.

Key words and phrases: hypersurface, hyperbolic space form, scalar curvature

1991 Mathmatics Subject Classification: 53C42, 53A10.

1 Introduction

Let $R^{n+1}(c)$ be an $(n+1)$-dimensional Riemannian manifold with constant sectional curvature c. We also call it a space form. When $c > 0$, $R^{n+1}(c) = S^{n+1}(c)$ (i.e. $(n+1)$-dimensioanl sphere space); when $c = 0$, $R^{n+1}(c) = R^{n+1}$ (i.e. $(n+1)$-dimensioanl Euclidean space); when $c < 0$, $R^{n+1}(c) = H^{n+1}(c)$ (i.e. $(n+1)$-dimensioanl hyperbolic space). We simply denote $H^{n+1}(-1)$ by H^{n+1}. Let M^n be an n-dimensional hypersurface in $R^{n+1}(c)$, and $e_1, \ldots, e_n$ a local orthonormal frame field on M^n, $\omega_1, \ldots, \omega_n$ its dual coframe field. Then the second fundamental form of M^n is

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j. \tag{1}$$

Further, near any given point $p \in M^n$, we can choose a local frame field $e_1, \ldots, e_n$ so that at p, $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j = \sum_i k_i\omega_i \otimes \omega_j$, then the Gauss equation says

$$R_{ijij} = c + k_ik_j, \quad i \neq j. \tag{2}$$

*This work is supported in part by the National Natural Science Foundation of China

$$n(n-1)(R-c) = n^2H^2 - |h|^2, \qquad (3)$$

where R is the normalized scalar curvature, $H = \frac{1}{n}\sum_i k_i$ the mean curvature and $|h|^2 = \sum_i k_i^2$ the norm square of the second fundamental form of M^n.

As it is well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature H in $R^{n+1}(c)$ $(c \geq 0)$ by use of J. Simons' method, for example, see [1], [4], [5], [8], [12] etc., but less of that were obtained for hypersurfaces immersed into a hyperbolic space from. Walter [13] gave a classification for non-negatively curved compact hypersurfaces in a space form under the assumption that the rth mean curvature is constant. Morvan-Wu [7], Wu [14] also proved some rigidity theorems for complete hypersurfaces M^n in a hyperbolic space form $H^{n+1}(c)$ under the assumption that the mean curvature is constant and the Ricci curvature is non-negative. Moreover, they proved that M^n is a geodesic distance sphere in $H^{n+1}(c)$ provided that it is compact.

On the other hand, Cheng-Yau [3] introduced a new self-adjoint differential operator $\Box$ to study the hypersurfaces with constant scalar curvature. Later, Li [6] obtained interesting rigidity results for compact hypersurfaces with constant scalar curvature in space-forms using the Cheng-Yau's self-adjoint operator $\Box$.

In the present paper, we would like to use Cheng-Yau's self-adjoint operator $\Box$ to study the complete hypersurfaces in a hyperbolic space form with constant scalar curvature, and prove the following theorem:

Theorem. Let M^n be an n-dimensional $(n \geq 3)$ complete hypersurface with constant normalized scalar curvature R in H^{n+1}. If

(1) $\bar{R} = R + 1 \geq 0$,

(2) the norm square $|h|^2$ of the second fundamental form of M^n satisfies

$$n\bar{R} \leq sup\,|h|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n],$$

then either

$$sup\,|h|^2 = n\bar{R}$$

and M^n is a totally umbilical hypersurface; or

$$sup\,|h|^2 = \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n],$$

and M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$, for some $r > 0$.

2 Preliminaries

Let M^n be an n-dimensional hypersurface in H^{n+1}. We choose a local orthonormal frame $e_1, \ldots, e_{n+1}$ in H^{n+1} such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of M^n and form an orthonormal frame there. Let $\omega_1, \ldots, \omega_{n+1}$ be its dual coframe. In this paper, we use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+1; \quad 1 \leq i, j, k, \ldots \leq n.$$

Then the structure equations of H^{n+1} are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{4}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{5}$$

$$K_{ABCD} = -(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{6}$$

Restrict these form to M^n, we have

$$\omega_{n+1} = 0. \tag{7}$$

From Cartan's lemma we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{8}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{9}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{10}$$

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{11}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j \tag{12}$$

is the second fundamental form of M^n. We also have

$$R_{ij} = -(n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{13}$$

$$n(n-1)(R+1) = n^2H^2 - |h|^2, \tag{14}$$

where R is the normalized scalar curvature, and H the mean curvature.

Define the first and the second covariant derivatives of h_{ij}, say h_{ijk} and h_{ijkl} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{15}$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}. \tag{16}$$

Then we have the Codazzi equation

$$h_{ijk} = h_{ikj}, \tag{17}$$

and the Ricci's identity

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{18}$$

For a C^2-function f defined on M^n, we defined its gradient and Hessian (f_{ij}) by the following formulas

$$df = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}. \tag{19}$$

The Laplacian of f is defined by $\Delta f = \sum_i f_{ii}$.

Let $\phi = \sum_{ij}\phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n, where

$$\phi_{ij} = nH\delta_{ij} - h_{ij}. \tag{20}$$

Following Cheng-Yau [3], we introduce an opertator $\Box$ associated to ϕ acting on any C^2-function f by

$$\Box f = \sum_{i,j}\phi_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{21}$$

Since ϕ_{ij} is divergence-free, it follows [3] that the operator $\Box$ is self-adjoint relative to the L^2 inner product of M^n, i.e.

$$\int_{M^n} f\Box g = \int_{M^n} g\Box f. \tag{22}$$

We can choose a local frame field $e_1, \ldots . e_n$ at any point $p \in M^n$, such that $h_{ij} = k_i\delta_{ij}$ at p, by use of (21) and (14), we have

$$\begin{aligned}
\Box(nH) &= nH\Delta(nH) - \sum_i k_i(nH)_{ii} \\
&= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i(nH)_{ii} \\
&= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|h|^2 - n^2|\nabla H|^2 - \sum_i k_i(nH)_{ii}.
\end{aligned} \tag{23}$$

On the other hand, through a standard calculation by use of (17) and (18), we get

$$\frac{1}{2}\Delta|h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i k_i(nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{24}$$

Putting (24) into (23), we have

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{25}$$

From (11), we have $R_{ijij} = -1 + k_i k_j, \quad i \neq j$, and by putting this into (25), we obtain

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 - n|h|^2 + n^2H^2 - |h|^4 + nH\sum_i k_i^3. \tag{26}$$

Let $\mu_i = k_i - H$ and $|Z|^2 = \sum_i \mu_i^2$, we have

$$\sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2, \tag{27}$$

$$\sum_i k_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3, \tag{28}$$

From (26)-(28), we get

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + |Z|^2(-n + nH^2 - |Z|^2) + nH\sum_i \mu_i^3. \tag{29}$$

We need the following algebraic lemma due to M. Okumura [9] (see also [1]).

Lemma 2.1. Let μ_i, $i = 1, \ldots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = constant \geq 0$. then

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{30}$$

and the equality holds in (30) if and only if at least $(n-1)$ of the μ_i are equal.

By use of Lemma 2.1, we have

$$\begin{aligned}\Box(nH) \geq {} & \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + \\ & (|h|^2 - nH^2)(-n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{|h|^2 - nH^2}).\end{aligned} \tag{31}$$

3 Umbilical hypersurface in a hyperbolic space form

In this section, we consider some special hypersurfaces in a hyperbolic space form which we will need in our latter discussions.

First we want to give a description of the real hyperbolic space-form $H^{n+1}(c)$ of constant curvature c (< 0). For any two vectors x and y in R^{n+2}, we set

$$g(x,y) = \sum_{i=1}^{n+1} x_i y_i - x_{n+2} y_{n+2}.$$

(R^{n+2}, g) is the so-called Minkowski space-time. Denote $\rho = \sqrt{-1/c}$. We define

$$H^{n+1}(c) = \{x \in R^{n+2} \mid x_{n+2} > 0,\ g(x,x) = -\rho^2\}.$$

Then $H^{n+1}(c)$ is a connected simply-connected hypersurface of R^{n+2}. It is not hard to check that the restriction of g to the tangent space of $H^{n+1}(c)$ yields a complete Riemannian metric of constant curvature c. Here we obtain a model of a real hyperbolic space form.

We are interested in those complete hypersurfaces with at most two distinct constant principal curvatures in $H^{n+1}(c)$. this kind of hypersurfaces was described by Lawson [5] and completely classified by Ryan [11].

Lemma 3.1 [11]. Let M^n be a complete hypersurface in $H^{n+1}(c)$. Suppose that, under a suitable choice of a local orthonormal tangent frame field of TM^n, the shape operator over TM^n is expressed as a matrix A. If M^n has at most two distinct constant principal curvatures, then it is congruent to one of the following:

(1) $M_1 = \{x \in H^{n+1}(c) \mid x_1 = 0\}$. In this case, $A = 0$, and M_1 is totally geodesic. Hence M_1 is isometric to $H^n(c)$;

(2) $M_2 = \{x \in H^{n+1}(c) \mid x_1 = r > 0\}$. In this case, $A = \frac{1/\rho^2}{\sqrt{1/\rho^2+1/r^2}} I_n$, where I_n denotes the identity matrix of degree n, and M_2 is isometric to $H^n(-1/(r^2+\rho^2))$;

(3) $M_3 = \{x \in H^{n+1}(c) \mid x_{n+2} = x_{n+1} + \rho\}$. In this case, $A = \frac{1}{\rho} I_n$, and M_3 is isometric to a Euclidean space E^n;

(4) $M_4 = \{x \in H^{n+1}(c) \mid \sum_{i=1}^{n+1} x_i^2 = r^2 > 0\}$. In this case, $A = \sqrt{1/r^2 + 1/\rho^2} I_n$, and M_4 is isometric to a round sphere $S^n(r)$ of radius r;

(5) $M_5 = \{x \in H^{n+1}(c) \mid \sum_{i=1}^{k+1} x_i^2 = r^2 > 0,\ \sum_{j=k+2}^{n+1} x_j^2 - x_{n+2}^2 = -\rho^2 - r^2\}$. In this case, $A = \lambda I_k \oplus \nu I_{n-k}$, where $\lambda = \sqrt{1/\rho^2 + 1/r^2}$, and $\nu = \frac{1/\rho^2}{\sqrt{1/r^2+1\rho^2}}$, M_5 is isometric to $S^k(r) \times H^{n-k}(-1/(r^2+\rho^2))$.

Remark 3.1. $M_1, \ldots, M_5$ are often called the standard examples of complete hypersurfaces in $H^{n+1}(c)$ with at most two distinct constant principal curvatures. It is obvious that $M_1, \ldots, M_4$ are totally umbilical. In the sence of Chen [2], they are called the hyperspheres of $H^{n+1}(c)$. M_3 is called the horosphere and M_4 the geodesic distance sphere of $H^{n+1}(c)$.

Remark 3.2. Ryan [11] stated that the shape operator of M_2 is $A = \sqrt{1/r^2 - 1/\rho^2} I_n$, and M_2 is isometric to $H^n(-1/r^2)$, where $r \le \rho$. This is incorrect and we have corrected it here.

4 The proof of Theorem

The following lemma essentially due to Cheng-Yau [3].

Lemma 4.1. Let M^n be an n-dimensional hypersurface in H^{n+1}. Suppose that the normalized scalar curvature $R = constant$ and $R \geq -1$. Then $|\nabla h|^2 \geq n^2 |\nabla H|^2$.

Proof. From (14),

$$n^2 H^2 - \sum_{i,j} h_{ij}^2 = n(n-1)(R+1).$$

Taking the covariant derivative of the above expression, and using the fact $R = constant$, we get

$$n^2 H H_k = \sum_{i,j} h_{ij} h_{ijk}.$$

By Cauchy-Schwarz inequality, we have

$$\sum_k n^4 H^2 (H_k)^2 = \sum_k (\sum_{i,j} h_{ij} h_{ijk})^2 \leq (\sum_{i,j} h_{ij}^2) \sum_{i,j,k} h_{ijk}^2,$$

that is

$$n^4 H^2 \|\nabla H\|^2 \leq |h|^2 |\nabla h|^2.$$

On the other hand, from $R + 1 \geq 0$, we have $n^2 H^2 - |h|^2 \geq 0$. Thus

$$H^2 |\nabla h|^2 \geq n^2 H^2 |\nabla H|^2$$

and Lemma 4.1 follows.

From the assumption of the Theorem that R is constant and $\bar{R} = R + 1 \geq 0$ and Lemma 4.1 we have

$$\square(nH) \geq (|h|^2 - nH^2)(-n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{|h|^2 - nH^2}). \tag{32}$$

By Gauss equation (14) we know that

$$|Z|^2 = |h|^2 - nH^2 = \frac{n-1}{n}(|h|^2 - n\bar{R}). \tag{33}$$

From (32) and (33) we have

$$\square(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_H(|h|), \tag{34}$$

where

$$\phi_H(|h|) = -n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{|h|^2 - nH^2}.$$

By (33) we can write $\phi_H(|h|)$ as

$$\phi_{\bar{R}}(|h|) = -n+2(n-1)\bar{R}-\frac{n-2}{n}|h|^2-\frac{n-2}{n}\sqrt{(n(n-1)\bar{R}+|h|^2)(|h|^2-n\bar{R})}. \quad (35)$$

Therefore (34) becomes

$$\Box(nH) \geq \frac{n-1}{n}(|h|^2-n\bar{R})\phi_{\bar{R}}(|h|), \quad (36)$$

It is a direct check that our assumption

$$sup\,|h|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2-4(n-1)\bar{R}+n] \quad (37)$$

is equivalent to

$$(-n+2(n-1)\bar{R}-\frac{n-2}{n}sup|h|^2)^2 \geq \frac{(n-2)^2}{n^2}(n(n-1)\bar{R}+sup|h|^2)(sup|h|^2-n\bar{R}). \quad (38)$$

But it is clear from (37) that (38) is equivalent to

$$-n+2(n-1)\bar{R}-\frac{n-2}{n}sup|h|^2 \geq \frac{n-2}{n}\sqrt{(n(n-1)\bar{R}+sup|h|^2)(sup|h|^2-n\bar{R})}. \quad (39)$$

So under the hyperthesis that

$$sup\,|h|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2-4(n-1)\bar{R}+n],$$

we have

$$\phi_{\bar{R}}(\sqrt{sup|h|^2}) \geq 0. \quad (40)$$

On the other hand,

$$\begin{aligned}\Box(nH) &= \sum_{i,j}(nH\delta_{ij}-nh_{ij})(nH)_{ij} = \sum_i(nH-nh_{ii})(nH)_{ii} \\ &= n\sum_i H(nH)_{ii} - n\sum_i k_i(nH)_{ii} \leq (|H|_{max}-C)\Delta(nH), \end{aligned} \quad (41)$$

where $|H|_{max}$ is the maximum of the mean curvature H and $C = min\,k_i$ is the minimum of the principal curvatures of M^n.

Now we need the following maximum principle at infinity for complete manifolds due to Omori [10] and Yau [15]:

Lemma 4.2. Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $f: M^n \to R$ a smooth function bounded from below. Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M^n$ such that

(i) $|\nabla f|(p_\varepsilon) < \varepsilon$,

(ii) $\Delta f(p_\varepsilon) > -\varepsilon$,
(iii) $inf\, f \leq\ f(p_\varepsilon) \leq inf\, f + \varepsilon$.

From the hypothesis of the Theorem and Gauss equation, we know that the Ricci curvature is bounded below. So we may apply Lemma 4.2 to the following smooth function f on M^n defined by

$$f = \frac{1}{\sqrt{1+(nH)^2}}.$$

It is immediate to check that

$$|\nabla f|^2 = \frac{1}{4}\frac{|\nabla (nH)^2|^2}{(1+(nH)^2)^3} \tag{42}$$

and that

$$\Delta f = -\frac{1}{2}\frac{\Delta (nH)^2}{(1+(nH)^2)^{3/2}} + \frac{3}{4}\frac{|\nabla (nH)^2|^2}{(1+(nH)^2)^{5/2}}. \tag{43}$$

By Lemma 4.2 we can find a sequence of points p_k, $k \in N$ in M^n, such that

$$lim_{k\to\infty} f(p_k) = inf\, f,\ \ \Delta f(p_k) > -\frac{1}{k},\ \ \ |\nabla f|^2(p_k) < \frac{1}{k^2}. \tag{44}$$

Using (44) in the equations (42) and (43) and the fact that

$$lim_{k\to\infty}(nH)(p_k) = sup_{p\in M^n}(nH)(p), \tag{45}$$

we get

$$-\frac{1}{k} \leq -\frac{1}{2}\frac{\Delta (nH)^2}{(1+(nH)^2)^{3/2}}(p_k) + \frac{3}{k^2}(1+(nH)^2(p_k))^{1/2}. \tag{46}$$

Hence we obtain

$$\frac{\Delta (nH)^2}{(1+(nH)^2)^2}(p_k) < \frac{2}{k}\left(\frac{1}{\sqrt{1+(nH)^2(p_k)}} + \frac{3}{k}\right). \tag{47}$$

On the other hand, by (36) and (41), we have

$$\frac{n-1}{n}(|h|^2 - n\bar{R})\phi_{\bar{R}}(|h|) \leq \Box(nH) \leq n(|H|_{max} - C)\Delta(nH). \tag{48}$$

At points p_k of the sequence given in (44), this becomes

$$\begin{aligned}&\frac{n-1}{n}(|h|^2(p_k) - n\bar{R})\phi_{\bar{R}}(|h|(p_k)) \leq \Box(nH(p_k))\\ &\leq n(|H|_{max} - C)\Delta(nH)(p_k).\end{aligned} \tag{49}$$

Let $k \to \infty$ and use (47) we have that the right hand side of (49) goes to zero, so we have either $\frac{n-1}{n}(sup|h|^2 - n\bar{R}) = 0$, i.e. $sup\,|h|^2 = n\bar{R}$ or $\phi_{\bar{R}}(\sqrt{sup|h|^2}) = 0$.

If $sup|h|^2 = n\bar{R}$, by (33) $|Z|^2 = \frac{n-1}{n}(|h|^2 - n\bar{R})$ we have $sup|Z|^2 = \frac{n-1}{n}(sup|h|^2 - n\bar{R}) = 0$, then $|Z|^2 = 0$ and M^n is totally umbilical.

If $\phi_{\bar{R}}(\sqrt{sup|h|^2}) = 0$, it is easy to prove that

$$sup\, H^2 = \frac{1}{n^2}[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}],$$

then equalities hold in (30) and Lemma 4.1, we follow that $k_i = constant$ for all i and $(n-1)$ of the k_i are equal. After renumberation if necessary, we can assume

$$k_1 = k_2 = \cdots = k_{n-1} \quad k_1 \neq k_n.$$

Therefore, from Lemma 3.1, we know that M^n is a hypersurface in H^{n+1} with two distinct principal curvatures, and M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$, for some $r > 0$. This completes the proof of Theorem.

When M^n is compact, we can prove

Corollary 1. Let M^n be an n-dimensional ($n \geq 3$) compact hypersurface with constant normalized scalar curvature R in H^{n+1}. If

(1) $\bar{R} = R + 1 \geq 0$,

(2) the norm square $|h|^2$ of the second fundamental form of M^n satisfies

$$n\bar{R} \leq |h|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n], \tag{50}$$

then M^n is a totally umbilical hypersurface.

Proof. From (36) we have

$$\begin{aligned}\Box(nH) \geq &\frac{n-1}{n}(|h|^2 - n\bar{R})[-n + 2(n-1)\bar{R} - \frac{n-2}{n}|h|^2 \\ &-\frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R})}],\end{aligned} \tag{51}$$

It is a direct check that our assumption condition (50) is equivalent to

$$(-n + 2(n-1)\bar{R} - \frac{n-2}{n}|h|^2)^2 \geq \frac{(n-2)^2}{n^2}(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R}). \tag{52}$$

But it is clear from (50) that (52) is equivalent to

$$-n + 2(n-1)\bar{R} - \frac{n-2}{n}|h|^2 \geq \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R})}, \tag{53}$$

therefore the right hand side of (51) is non-negative. Because M^n is compact and the operator $\Box$ is self-adjoint, we have $\int_{M^n} \Box(nH)dv = 0$. Thus either

$$|h|^2 = n\bar{R}, \tag{54}$$

that is, $|h|^2 = nH^2$, M^n is a totally umbilical hypersurface; or

$$|h|^2 = \frac{n}{(n-2)(n\bar{R}-2)}[n(n-1)\bar{R}^2 - 4(n-1)\bar{R} + n]. \tag{55}$$

In the latter case, equalities hold in (30) and Lemma 4.1, and it follows that M^n has at most two distinct constant principal curvatures. We conclude that M^n is totally umbilical from the compactness of M^n. This completes the proof of Corollary 1.

Corollary 2. Let M^n be an n-dimensional compact hypersurface with constant normalized scalar curvature R and $R+1 \geq 0$ in H^{n+1}. If M has non-negative sectional curvature, then M is a totally umbilical hypersurface.

Proof. Because M^n is compact and the operator $\square$ is self-adjoint, form (25), we have

$$\int_{M^n} [|\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2] = 0. \tag{56}$$

If M^n has constant normalized scalar curvature R and $R \geq -1$, from Lemma 4.1, we have $|\nabla h|^2 \geq n^2|\nabla H|^2$. So if M has non-negative sectional curvature, form (56) we have $|\nabla h|^2 = n^2|\nabla H|^2$ and $R_{ijij} = 0$, when $k_i \neq k_j$ on M^n. Since $R_{ijij} = -1 + k_i k_j$, then either M^n is totally umbilical, or M^n has two different principal curvatures, in the latter case, M^n is still totally umbilical from the compactness of M^n. This completes the proof of Corollary 2.

Acknowledgements

This paper was written during the author's stay at the Max-Planck-Institut für Mathematik in Bonn. The author would like to exprss his thanks to the institute for its hospitality and support.

References

[1] H. Alencar and M. P. do Carmo, Hypersurfaces with constant mean curvature in hyperbolic space forms, Proc. Amer. Math. Soc. 120(1994), 1223-1229.

[2] B.Y. Chen, Totally mean curvature and submanifolds of finite type, World Scientific, Singapore, 1984.

[3] S.Y. Cheng and S.T. Yau, Hypersurfaces with constant scalar curvature, Math. Ann., 225(1977), 195-204.

[4] Z. H. Hou, Hypersurfaces in sphere with constant mean curvature, Proc. Amer. Math. Soc., 125(1997), 1193-1196.

[5] H.B. Lawson, Jr., Local rigidity theorems for minimal hypersurfaces, Ann. of Math. (2), 89(1969), 187-197.

[6] H. Li, Hypersurfaces with constant scalar curvature in space forms, Math. Ann. 305(1996), 665-672.

[7] J.M. Morvan and B.Q. Wu, Hypersurfaces with constant mean curvature in hyperbolic space form, Deom. Dedicata, 59(1996), 197-222.

[8] K. Nomizu and B. Smyth, A formula for Simon's type and hypersurfaces, J. Diff. Geom., 3(1969), 367-377.

[9] M. Okumuru, Hypersurfaces and a pinching problem on the second fundamental tensor, Amer. J. Math., 96(1974), 207-213.

[10] H. Omori, Isometric immersions of Riemannian manifolds, J. Math. Soc. Japan, 19(1967), 205-214.

[11] P.J. Ryan, Hypersurfaces with parallel Ricci tensor, Osaka J. Math. 8(1971), 251-259.

[12] J. Simons, Minimal varieties in Riemannian manifolds, Ann. of Math. (2) 88(1968), 62-105.

[13] R. Walter, Compact hypersurfaces with a constant higher mean curvature function, Math. Ann., 270(1985), 125-145.

[14] B.Q. Wu, Hypersurface with constant mean curvature in H^{n+1}, The Math. Heritage of C. F. Gauss, World Scientific, Singapore, 1991, 862-871.

[15] S.T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure and Appl. Math., 28(1975), 201-228.

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