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for complex manifolds

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MPI/88-24

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My lecture at the Como Conference was a survey on the theory of elliptic genera as developed by Ochanine, Landweber, Stong and Witten. A good global reference are the Proceedings of the 1986 Princeton Conference [1]. In this contribution to the Proceedings of the Como Conference I shall not reproduce my lecture, but rather sketch a theory of elliptic genera of level N for compact complex manifolds which I presented in the last part of my course at the University of Bonn during the Wintersemester 1987/88. For a natural number $N > 1$ the elliptic genus of level N of a compact complex manifold M of dimension d is a modular form of weight d for the group $\Gamma_1(N)$. In the cusps of $\Gamma_1(N)$ the genus degenerates either to $\chi_y(M)/(1+y)^d$ where $-y$ is an N^{th}-root of unity different from 1 or to $\chi(M,K^{k/N})$ where K is the canonical line bundle and $0 < k < N$. Here $\chi_y(M) = \sum_{p=0}^{d} \chi^p(M)y^p$ with $\chi^p(M) = \chi(M,\Omega^p) = \sum_{q=0}^{d} (-1)^q h^{p,q}$ is the χ_y-genus introduced in [13] and $\chi(M,K^{k/N})$ is the genus with respect to the characteristic power series

$$\frac{x}{1-e^{-x}} \cdot e^{-(k/N)\cdot x}$$

which equals the holomorphic Euler number of M with coefficients in the line bundle L^k provided $K = L^N$ (see [13]).

For $N = 2$ the genus is expressible in Pontrjagin numbers and hence defined for an oriented differentiable manifold M. The only possible value of $-y$ is -1 and the genus degenerates in the two cusps to

$$\chi_1(M)/2^d = \text{sign}(M)/2^d \qquad (\dim_{\mathbb{R}} M = 2d)$$

or to

$$\chi(M,K^{1/2}) = \hat{A}(M) .$$

Only recently I realized that Witten in [19] studied also complex manifolds. His discussion includes the genus studied here, at least if one restricts attention to the cusps with specialization $\chi(M,K^{k/N})$.

In this report I shall also try to give an account of the rigidity theorem for complex manifolds with circle actions which for $N = 2$ are due to Taubes [18] and Witten with a new exposition by Bott [9]. These rigidity theorems hold if the first Chern class of M is divisible by N, i.e. if a holomorphic line bundle L with $L^N = K$ exists.

The results in this paper hold also for manifolds with a stable almost complex structure and for circle actions which preserve this structure. For simplicity we have formulated the results for complex manifolds only.

I would like to thank the students of my course Thomas Berger and Rainer Jung for writing notes. Many thanks to Nils-Peter Skoruppa who lectured several times in my course when I was away and with whom I had helpful discussion on modular forms. After my course I had intensive discussions with Michael Atiyah on the rigidity theorem in Oxford and also with Don Zagier at the Max-Planck-Institut.

1. In the following N is a fixed natural number >1, the "variable" x runs through the complex numbers, $\mathbb{H}$ is the upper half plane, $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$. For a lattice L in $\mathbb{C}$ we consider the elliptic function $g(x)$ with divisor $N\cdot 0 - N\cdot\alpha$ where $\alpha \in \mathbb{C}$ is an N-division point ($\alpha \notin L$, $N\alpha \in L$). The function g is uniquely determined by L and α (regarded as element of $\mathbb{C}/L$) if we demand that the power series of g in the origin begins with x^N. The function $f(x) = g(x)^{1/N}$ is uniquely defined if we request $f(x) = x$ + higher terms. The function f is elliptic with respect to a sublattice L' of L whose index in L equals the order of α as element of $\mathbb{C}/L$. For $\omega \in L$ the function $f(x+\omega)/f(x)$ is constant and equals an N-th root of unity. After multiplying L with a non-vanishing complex number we can assume that

$$(1) \qquad L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z}) \quad \text{and}$$

$$0 \neq \alpha = 2\pi i\left[\frac{k}{N}\tau + \frac{\ell}{N}\right] \quad \text{with} \quad 0 \leq k < N \quad \text{and} \quad 0 \leq \ell < N$$

To write down a product development for $f(x)$ in the case that L and α are as in (1) we introduce the entire function

$$(2) \qquad \Phi(x) = (1-e^{-x}) \prod_{n=1}^{\infty} (1-q^n e^{-x})(1-q^n e^{x})/(1-q^n)^2$$

which has zeros of order 1 in the points of L. The function $\Phi(x)$ equals the Weierstraß sigma-function for L up to a factor of the form $\exp(b_1 x + b_2 x^2)$. It can be proved easily that

$$(3) \qquad f(x) = e^{\frac{k}{N}x}\,\Phi(x)\Phi(-\alpha)/\Phi(x-\alpha)$$

Namely, it suffices to check

$$\Phi(x+2\pi i\tau) = -e^{-x}e^{-2\pi i\tau}\Phi(x).$$

For this replace in (2) the exponential e^x by λ and then substitute λ by λq to see that the factor $-\lambda^{-1}q^{-1}$ comes out. In fact, we have (for $\zeta = e^{2\pi i/N}$)

(4)
$$f(x+2\pi i) = \zeta^{k} f(x)$$
$$f(x+2\pi i\tau) = \zeta^{-\ell} f(x)$$

The function $f(x)$ as belonging to L and α (see (1)) degenerates for $q \longrightarrow 0$ to a function $f_\infty(x)$.

We have

(5)
$$f_\infty(x) = e^{(k/N)x} \cdot (1-e^{-x}) \quad \text{for} \quad k > 0$$
$$f_\infty(x) = (1-e^{-x})(1-e^{\alpha})/(1-e^{\alpha-x}) \quad \text{for} \quad k = 0.$$

For reasons which are apparent from the introduction we put $e^{\alpha} = -y$ for $k = 0$ and have in this case

$$f_\infty(x) = (1-e^{-x})(1+y)/(1+ye^{-x}) \quad \text{with} \quad -y = \zeta^{\ell} \neq 1.$$

The involution $x \longrightarrow -x+\alpha$ interchanges the zeros and poles of $f(x)$. Therefore,

$$f(x)f(-x+\alpha),$$

which is elliptic for L, is in fact a constant $\neq 0$. We write the constant as c^{-2}. Then c^{2N} depends only on L and the chosen n-division point as point of $\mathbb{C}/L$. If the lattice and α are normalized as in (1), then

$$f(x)(f-x+\alpha) = e^{(k/N)\alpha-\alpha}\Phi(-\alpha)^{2} = c^{-2}$$

and

(6)
$$c^{2N} = \Phi(-\alpha)^{-2N} q^{\frac{k(N-k)}{N}} \cdot \zeta^{-k\ell}$$

The coefficients of the power series developments of $f(x)/x$, $x/f(x)$ and $x\frac{f'(x)}{f(x)}$ determine each other. If one replaces in such a series x by λx, one obtains the corresponding function for the lattice $\lambda^{-1}L$ and the n-division point $\lambda^{-1}\alpha$. Therefore the coefficient of x^r in any of these series' as function of the pair L, α with $\alpha \in \mathbb{C}/L$ is homogeneous of degree $-r$. Also c^{2N} is such a function of L and α. It is homogeneous of degree $-2N$ and is related to Dedekind's η-function.

If the pair L,α is chosen as in (1), then the coefficients of $f(x)/x$ are indeed modular forms of weight r for the subgroup consisting of the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $SL_2(\mathbb{Z})$ which satisfy

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k \\ \ell \end{bmatrix} = \begin{bmatrix} k \\ \ell \end{bmatrix} \mod N.$$

Also c^{2N} is a modular form of weight $2N$ for this group. (It still has to be shown that these forms are holomorphic in the cusps. See the next section.)

2. The classification of pairs L,α where L is a lattice in $\mathbb{C}$ and $\alpha \in \mathbb{C}/L$ with $N\alpha = 0$ (but $\alpha \neq 0$), up to multiplication by some complex number $\lambda \neq 0$, leads to the introduction of the modular group

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N,\ a \equiv d \equiv 1 \bmod N \right\}$$

If we assume that the N-divison point has order N in $\mathbb{C}/L$, then the classes of pairs L,α are in one-to-one correspondence with the points of the modular curve $\mathbb{H}/\Gamma_1(N)$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acts on $\mathbb{H}$ by

$$\tau \longmapsto \frac{a\tau+b}{c\tau+d}$$

This follows from the fact that each pair L, α is equivalent to a pair of type (1) with $\alpha = 2\pi i/N$ (i.e. $k = 0$, $\ell = 1$). The coefficients of x^r in $\frac{x}{f(x)}$, $\frac{f(x)}{x}$, $x\frac{f'(x)}{f(x)}$ are modular forms of weight r for $\Gamma_1(N)$. It remains to show that such a coefficient is holomorphic in each cusp of $\mathbb{H}/\Gamma_1(N)$. Transforming $f(x)$ (taken for the lattice (1) with $\alpha = 2\pi i/N$) to a cusp gives a function $f(x)$ for some $\alpha = 2\pi i(\frac{k}{N}\tau + \frac{\ell}{N})$ and the same lattice. The formulas (2) and (3) show immediately that the q development of each coefficients of $f(x)$ has only non-negative powers of q, fractional if $k > 0$, but then a suitable root of q is the local uniformizing variable at the cusp.

The coefficients e_r of $x\frac{f'(x)}{f(x)}$ are the Eisenstein series. Their q-developments (for $k = 0$ and $\ell = 1$) for example can be read off from

$$(7) \qquad x\frac{f'(x)}{f(x)} = \sum_{n=0}^{\infty} \frac{xq^n e^{-x}}{1-q^n e^{-x}} - \sum_{n=1}^{\infty} \frac{xq^n e^x}{1-q^n e^x}$$

$$- \sum_{n=0}^{\infty} \frac{x\zeta q^n e^{-x}}{1-\zeta q^n e^{-x}} + \sum_{n=1}^{\infty} \frac{x\zeta^{-1} q^n e^x}{1-\zeta^{-1} q^n e^x}$$

Furthermore, c^{2N} (see (6)) is a modular form of weight $2N$. For more detailed formulae concerning these q-developments in the case $N = 2$ see [20]. For $N > 2$ and $N \neq 4$ the number of cusps of $\Gamma_1(N)$ equals

$$\frac{1}{2} \sum_{d/N} \varphi(d)\varphi(\tfrac{N}{d}),$$

where φ is Euler's function. Each cusp can be represented by several division points α as in (1).

3. Let M_d be a compact complex manifold. The Chern classes c_i of M_d are elements of the 2i-dimensional cohomology group $H^{2i}(M_d, \mathbb{Z})$. Let c be the total Chern class of M_d split up formally

(8) $$c = \sum_{i=0}^{d} c_i = (1+x_1)(1+x_2)\dots(1+x_d)$$

where $x_1, x_2, \dots, x_d$ can be regarded as 2-dimensional cohomology classes in some manifold fibred over M_d (see [13], § 13.3). Let $Q(x)$ be a fixed power series in the indeterminate x starting with 1 whose coefficients are in some commutative ring containing $\mathbb{Z}$. Then

(9) $$\varphi_Q(M_d) = Q(x_1)Q(x_2)\dots Q(x_d)[M_d]$$

is the genus of M_d with respect to the power series Q where in (9) the symmetric expression $Q(x_1)Q(x_2)\dots Q(x_d)$ is written in terms of the Chern classes in view of (8) and the 2d-dimensional component of this expression is evaluated on M_d (compare [13], § 10.2). We define the elliptic genus $\varphi_N(M_d)$ by using the power series

(10) $$Q(x) = \frac{x}{f(x)} = \frac{x\Phi(x-\alpha)}{\Phi(x)\Phi(-\alpha)}$$

where $\alpha = 2\pi i/N$ and $f(x)$ is taken for the pair L, α with $L = 2\pi i(\mathbb{Z}\tau+\mathbb{Z})$. We put again $\zeta = e^{2\pi i/N}$.

Theorem. The elliptic genus $\varphi_N(M_d)$ is a modular form of weight d for the group $\Gamma_1(N)$. If one represents a cusp of $\Gamma_1(N)$ by $2\pi i(\frac{k}{N}\tau + \frac{\ell}{N})$ with $0 \leq k < N$ and $0 \leq \ell < N$, then the value of $\varphi_N(M_d)$ in this cusp equals

$$\chi(M_d, K^{k/N}) \quad \text{if} \quad k > 0$$

and $\chi_y(M_d)/(1+y)^d$ if $k = 0$ and $-y = \zeta^{\ell}$.

The theorem follows from the remarks in section 2 and from (5) by recalling, that $\chi(M_d, K^{k/N})$ is the genus for the power series

$$\frac{x}{1-e^{-x}} \cdot e^{-(k/N)x}$$

and $\chi_y(M_d)/(1+y)^d$ is the genus for the power series

$$\frac{x}{1-e^{-x}}(1+ye^{-x})/(1+y) \ ,$$

see [13].

A genus can be defined also by a power series $Q(x)$ not beginning with 1 (we assume $Q(0) = a_0 \neq 0$). The definition is done by equation (9) again. Then $a_0^{-1}Q(a_0x)$ gives the same genus with a normalized power series (i.e. the constant term equals 1). We now define $\tilde{\varphi}_N(M_d)$ using the power series

$$\tilde{Q}(x) = \frac{x\Phi(x-\alpha)}{\Phi(x)}, \qquad \alpha = 2\pi i/N. \tag{11}$$

<u>Theorem.</u> <u>The elliptic genus</u> $\tilde{\varphi}_N(M_d)$ <u>is a modular function for</u> $\Gamma_1(N)$ <u>if</u> $d \equiv 0 \bmod 2N$. <u>We have</u>

$$\varphi_N(M_d) = \tilde{\varphi}_N(M_d)(\Phi(-\alpha))^{-d} = \tilde{\varphi}_N(M_d)\cdot c^d$$

The result follows from the preceding theorem and the consideration in section 1 and 2 which show that $\Phi(-\alpha)^{-d} = c^d$ is a modular form for $\Gamma_1(N)$ of weight d.

If d is not divisible by $2N$, then

$$\tilde{\varphi}_N(M_d)^{2N/(d,2N)}$$

is a modular function (where $(d,2N)$ is the greatest common divisor of d and $2N$).One simply applies the theorem to the $2N/(d,2N)$-th power of M_d.

The function $\tilde{\varphi}_N$ has poles in the cusps represented by (1) with $k > 0$. The order of the pole is given by (6). Let us restrict to the case that N is a prime. For $N = 2$ we have 2 cusps represented by $(k,\ell) = (0,1)$

and $(k,\ell) = (1,0)$. For N odd, we have $2 \cdot \frac{N-1}{2}$ cusps. There are $\frac{N-1}{2}$ cusps represented by $(k,\ell) = (0,\ell)$ and $1 \leq \ell \leq \frac{N-1}{2}$ and $\frac{N-1}{2}$ cusps represented by $(k,\ell) = (k,0)$ and $1 \leq k \leq \frac{N-1}{2}$. In the first kind of cusps the q development of $\tilde{\varphi}_N(M_d)$ begin with the constant term $\chi_y(M_d)$ with $y = -\zeta^{\ell}$, in the latter case it starts with

$$\chi(M_d, K^{k/N}) \cdot \tilde{q}^{-k(N-k)d/2N}$$

where $\tilde{q}$ is a local uniformizing parameter for this cusp of $\mathbb{H}/\Gamma_1(N)$. (We have $\tilde{q}^N = q$ in (6)).

4. For a complex vector bundle W of dimension r the exterior powers $\Lambda^i W$ and the symmetric powers $S^i W$ are well-defined vector bundles. Their Chern classes can be calculated from those of W. If $c_1, \dots, c_r$ are the Chern classes of W (where c_i is in the $(2i)$-dimensional cohomology of the base space) and if we write formally

$$c = 1 + c_1 + c_2 + \dots + c_r = (1+x_1)(1+x_2)\dots(1+x_r)$$

then the Chern character (in the rational cohomology of the base space) is given by

$$ch(W) = e^{x_1} + e^{x_2} + \dots + e^{x_r}$$

Over the rationals c and ch determine each other. For the exterior powers we write with some indeterminate t

$$\Lambda_t(W) = \sum_{i=0}^{r} \Lambda^i W \cdot t^i$$

and for the symmetric powers

$$S_t(W) = \sum_{i=0}^{\infty} S^i W \cdot t^i$$

Then we have for the Chern character

(12) $$\mathrm{ch}(\Lambda_t W) = \prod_{i=1}^{r} (1 + te^{x_i})$$

(13) $$\mathrm{ch}(S_t W) = \prod_{i=1}^{r} (1 - te^{x_i})^{-1}$$

formula (12) was often used in [13]. Formula (13) is, of course, a special case of the general method to calculate the Chern classes associated to given vector bundles by representations [7].

Following Witten's idea (see [19]) we write the elliptic genus $\tilde{\varphi}_N(M_d)$, or rather its q-development in the standard cusp, in the form

(14) $$\tilde{\varphi}_N(M_d) = \sum_{n=0}^{\infty} \chi_y(M_d, R_n) q^n .$$

Here, as before, $-y = \zeta = e^{2\pi i/N}$. Furthermore R_n is a virtual vector bundle associated to the complex tangent bundle of M_d by a virtual representation of $GL(d,\mathbb{C})$ (with coefficients in $\mathbb{Z}(\zeta)$).

For a vector bundle W the polynomial $\chi_y(M_d, W)$ is defined in [13]. We have, if T is the tangent bundle of M_d,

$$\chi_y(M_d, W) = \sum_{p=0}^{d} \chi(M_d, \Lambda^p T^* \otimes W) y^p$$

We now can specify the R_n in (14). Let us recall that $\tilde{\varphi}_N(M_d)$ is the genus belonging to the power series (11)

$$(15) \qquad \tilde{Q}(x) = \frac{x}{1-e^{-x}}(1+ye^{-x})\prod_{n=1}^{\infty}\frac{1+yq^n e^{-x}}{1-q^n e^{-x}}\cdot\frac{1+y^{-1}q^n e^{x}}{1-q^n e^{x}}$$

Therefore (by (12) and (13))

$$(16) \qquad \sum_{n=0}^{\infty} R_n q^n = \prod_{n=1}^{\infty} A_{yq^n} T^* \cdot \prod_{n=1}^{\infty} A_{y^{-1}q^n} T \cdot \prod_{n=1}^{\infty} S_{q^n}(T + T^*)$$

(with $-y = \zeta = e^{2\pi i/N}$).

We have

$$R_0 = 1, \quad R_1 = (1-\zeta)T^* + (1-\zeta^{-1})T$$

Modulo the ideal $(1-\zeta)$ of $\mathbb{Z}(\zeta)$, the elliptic genus $\tilde{\varphi}(M_d)$ equals the Euler-Poincare number $e(M_d)$.

According to Witten's philosophy (compare also [2] and [3]) if we had a χ_y-operator on the loop space $\mathcal{L}M_d$ of M_d, we could try to calculate (or define) its equivariant χ_y-genus for the natural S^1-action on $\mathcal{L}M_d$ with $q \in S^1$ $(q = e^{2\pi i\tau},\ \tau \in \mathbb{R})$ by the Atiyah-Bott-Singer ([4], [6]) fixed point theorem (fixed point set M_d (constant loops) in $\mathcal{L}M_d$). The result for the equivariant χ_y-genus $\chi_y(\mathcal{L}M_d, q)$ would be that it is the genus with respect to the power series

$$x \prod_{n=-\infty}^{\infty} \frac{1+yq^n e^{-x}}{1-q^n e^{-x}}$$

This does not make sense as a power series in q, but formal manipulations bring it to the form (15) provided $(-y)^d = 1$. Observe that (15) is a meromorphic function in the two variables x and q where $(x,q) \in \mathbb{C}^2$ and $|q| < 1$.

5. The genus $\tilde{\varphi}_N(M_d)$ has in the standard cusp a development whose coefficients are integral. They are elements of $\mathbb{Z}(\zeta)$. See formula (14). In the cusps (represented by (1) with $0 < k < N$) this is not so. The coefficients are of the form

$$\chi(M_d, K^{k/N} \otimes W_n)$$

where W_n is a virtual vector bundle associated to the tangent bundle by a virtual representation of $GL(d,\mathbb{C})$ with coefficients in $\mathbb{Z}(\zeta)$. The W_n can be calculated using (3). These coefficients are in general not integral. If, however, the first Chern class c_1 of M_d is divisible by N they are integral. This divisibility condition is equivalent to the existence of a holomorphic complex line bundle with $L^N = K$ and the coefficients

$$\chi(M_d, L^k \otimes W_n)$$

become "Riemann-Roch numbers" [13] which are integral

Theorem. If the first Chern class c_1 of the complex manifold M_d is divisible by N, then the coefficients of the q-developments of the genus $\tilde{\varphi}_N(M_d)$ in all cusps (given by (1)) are integral $(\in \mathbb{Z}(\zeta))$; for the elliptic genus $\varphi_N(M_d)$ the coefficients are integral in a cusp with $k > 0$, in the cusps with $k = 0$ they become integral after multiplication with $(1-\zeta)^d$.

6. Let M_d be a compact complex manifold together with an action of the circle S^1 on M_d by holomorphic maps. We write elements of the circle as $\lambda = e^{2\pi i z}$ where $z \in \mathbb{R}/\mathbb{Z}$. The group S^1 acts on the virtual bundles R_n (see (16)). It also acts on the "cohomology group"

(17)
$$H^q(M_d, \Lambda^p T^* \otimes R_n)$$

which is in fact a formal direct sum of cohomology groups $H^q(M_d, \Lambda^p T^* \otimes W)$ with coefficients in $\mathbb{Z}(\zeta)$. Since S^1 acts, we get from (17) (considered equivariantly) a character of S^1, i.e. a finite Laurent series in λ. Taking alternating sums over q in (17) gives us a character

$$\chi(M_d, \Lambda^p T^* \otimes R_n, \lambda)$$

and finally

$$\chi_y(M_d, R_n, \lambda) = \sum_{p=0}^{d} \chi(M_d, \Lambda^p T^* \otimes R_n, \lambda) y^p$$

and

(18) $$\tilde{\varphi}_N(M_d, \lambda) = \sum_{n=0}^{\infty} \chi_y(M_d, R_n, \lambda) q^n$$

It may be more convenient to return to our elliptic genus φ_N with characteristic power series $Q(x)$ (see (10)) and consider it equivariantly

(19) $$\varphi_N(M_d, \lambda) = \tilde{\varphi}_N(M_d, \lambda) \cdot \Phi(-\alpha)^{-d}$$

$$= \sum_{n=0}^{\infty} \chi_y(M_d, S_n, \lambda) q^n$$

where the S_n are virtual bundles (coefficients in $\mathbb{Q}(\zeta)$). We can calculate $\varphi_N(M_d, \lambda)$ using the Atiyah-Bott-Singer fixed point theorem (holomorphic Lefschetz theorem [6], p. 566). Before doing this some remarks concerning the fixed point set $M_d^{S^1}$ of the action are necessary.

The set $M_d^{S^1}$ is a smooth submanifold of M_d being a disjoint union of connected submanifolds of various dimensions. For each fixed point p, the circle acts in the tangent space T_p, hence integers $m_1,\ldots,m_d$ are defined such that $\lambda \in S^1$ acts by the diagonal matrix $(\lambda^{m_1},\lambda^{m_2},\ldots,\lambda^{m_d})$. For each $r \in \mathbb{Z}$ we consider those m_i which are equal to r. This leads to the eigenspace E_r of T_p. Of course, E_0 is the tangent space in p of the connected component of $M_d^{S_1}$ to which p belongs. The numbers $m_1,\ldots,m_d$ (well defined up to order) depend only on the component of $M_d^{S_1}$. Over each component we have eigenspace bundles, also denoted by E_r.

The characteristic power series of the elliptic genus φ_N is given in (10) in the form $Q(x) = x/f(x)$. For the fixed point theorem we need $1/f(x)$. We put

$$F(x) = 1/f(x) = \frac{\Phi(x-\alpha)}{\Phi(x)\Phi(-\alpha)} \tag{20}$$

We shall now give a formula for $\varphi_N(M_d,\lambda)$ using the holomorphic Lefschetz theorem writing it down in short hand form which will need some explanation

$$\varphi_N(M_d,\lambda) = (e_0\cdot F(x_1+2\pi i m_1 z)\ldots F(x_d+2\pi i m_d z))[M_d^{S^1}] \tag{21}$$

where e_0 is the product over these x_i for which $m_i = 0$. Recall $\lambda = e^{2\pi i z}$. Formula (21) has the following meaning. For each component of the fixed point set, e_0 is the Euler class (highest Chern class) of its tangent bundle E_0, the formal roots of the total Chern class of E_0 are the x_i with $m_i = 0$. The x_i with $m_i = r \neq 0$ are the formal roots of the total Chern class of the eigenspace bundle E_r over the component. Thus for E_0 one uses in the above product $xF(x) = Q(x)$ and for E_r $(r \neq 0)$ the function $F(x+2\pi i r z)$ which for $rz \notin \mathbb{Z}\tau+\mathbb{Z}$ has no pole for $x = 0$ (and we use a general z) and hence is a power series in x. Then one evaluates the expression in (21) on the component and adds over all components. Observe that the rotation numbers $(m_1, \dots, m_d)$ depend on the component and also the meaning of the x_i which are the formal roots of the total Chern class restricted to the component. According to (4) our function $F(x)$ has the property

$$(22) \qquad F(x+2\pi i) = F(x), \quad F(x+2\pi i\tau) = \zeta F(x)$$

where $\zeta = e^{2\pi i/N}$.

It follows immediately that $\varphi_N(M_d, \lambda)$ can be extended to an elliptic function in z (with $\lambda = e^{2\pi i z}$) for the lattice $\mathbb{Z}\cdot N\tau+\mathbb{Z}$. More precisely:

Let υ be an index for the connectedness components $(M_d^{S^1})_\upsilon$ of the fixed point set $M_d^{S^1}$. Then according to (21)

$$(23) \qquad \varphi_N(M_d, \lambda) = \sum_\upsilon \varphi_N(M_d, \lambda)_\upsilon$$

where $\varphi_N(M_d, \lambda)_\upsilon$ is an elliptic function for the lattice $\mathbb{Z}N\tau+\mathbb{Z}$ associated to $(M_d^{S^1})_\upsilon$. Indeed,

$$(24) \qquad \varphi_N(M_d, \lambda q)_\upsilon = \varphi_N(M_d, e^{2\pi i(z+\tau)})_\upsilon$$

$$= \zeta^{m_1+\dots+m_d} \varphi_N(M_d, \lambda)_\upsilon$$

The exponent $m_1+\dots+m_d$ depends on υ, even the residue class of the exponent mod N depends on υ in general.

<u>Definition:</u> The S^1-action on M_d is called N-balanced if for the components $(M_d^{S^1})_\upsilon$ of the fixed point set the residue class of $m_1+\dots+m_d$ modulo N does not depend on υ. If the action is N-balanced, the common residue class of $m_1+\dots+m_d$ is called the type of the action and denoted by t.

We have proved

Theorem. For an N-balanced S^1-action of type t on the complex manifold M_d, the equivariant elliptic genus $\varphi_N(M_d,\lambda)$ with $\lambda = e^{2\pi iz}$ is an elliptic function for the lattice $\mathbf{Z}\cdot N\tau + \mathbf{Z}$ which satisfies

$$\text{(25)} \qquad \varphi_N(M_d,\lambda q) = \varphi_N(M_d,e^{2\pi i(z+\tau)}) = \zeta^t\varphi_N(M_d,\lambda)$$

Remark Of course, φ_N can be regarded as a function of τ and z. In τ it is a modular form of weight d. In fact, φ_N is a meromorphic Jacobi form on $\Gamma_1(N)$ of weight d and index 0. (see [11]).

7. We now shall approach the rigidity theorems which under certain conditions state that the finite Laurent series' $\chi_y(M_d,R_n,\lambda)$ (see (16) and (18)) do not depend on λ. (Recall $-y = e^{2\pi i/N}$). This rigidity means that the elliptic function $\varphi_N(M_d,\lambda)$ of the preceeding theorem is a constant (see (19)), i.e. we have to show that it has no poles. The rigidity results were not included in my course at the University of Bonn. When Michael Atiyah came to Bonn in February 1988 he explained to me Bott's approach [9] and that it is rather close to our old paper [5] and we discussed it in Oxford in March. I did not study Taubes' paper [18] in detail, but rather looked in Bott's report [9]. Then I carried out the proof for the level N case during my visit in Cambridge (England) in March 1988 as a guest of Robinson college.

Let us consider $\varphi_N(M_d,\lambda)$ as a function of λ and q. It is meromorphic for $\lambda \in \mathbb{C}^*$ and $|q| < 1$. According to (21) it can have poles only for $mz \in \mathbf{Z}\tau+\mathbf{Z}$ where m is a rotation number $\neq 0$ occuring for one of the components of $M_d^{S^1}$. Of course, $mz \in \mathbf{Z}\tau + \mathbf{Z}$ means $\lambda^m = q^n$ where $n \in \mathbf{Z}$. We have

(26) $$\varphi_N(M_d,\lambda) = \sum_{n=0}^{\infty} c_n(\lambda)q^n$$

with $c_n(\lambda)$ a finite Laurent series (compare (18),(19)). The meromorphic function $\varphi_N(M_d,\lambda)$ can have poles only on the curves $\lambda^m = q^n$. If (λ,q) does not lie on such a curve, then the series (22) converges. If $\lambda_0 = e^{2\pi ik/r}$ is a primitive r-th root of unity, then (λ_0,q) lies precisely on the curves $\lambda^m = q^n$ with $m \equiv 0 \mod r$ and $n = 0$. But still

$$\varphi_N(M_d,\lambda_0) = \sum_{n=0}^{\infty} c_n(\lambda_0)q^n$$

converges for $|q| < 1$, because this q-development can be calculated from the fixed point set $M_d^{\lambda_0}$ of λ_0, namely

$$\varphi_N(M_d,\lambda_0) = (e_0F(x_1+2\pi im_1k/r)\ldots(F(x_d+2\pi im_dk/r))[M_d^{\lambda}]$$

where e_0 is now the product over those x_j for which $m_j \equiv 0 \mod r$ and where we interpret the formula in a similar way as in (21). If $|q| \leq R < 1$, then there is a neighborhood U of λ_0 in $\mathbb{C}^*$ such that no point (λ,q) with $\lambda \in U - \{\lambda_0\}$ and $|q| \leq R$ lies on one of the curves $\lambda^m = q^n$. We know that $\sum_{n=0}^{\infty} c_n(\lambda)q^n$ converges for $\lambda \in U$ and $|q| < R$.

It is not immediately clear that our elliptic function $\varphi_N(M_d,\lambda)$ is holomorphic in λ_0 and has there the value $\varphi_N(M_d,\lambda_0)$ calculated from the fixed point set of λ_0. But in fact $\varphi_N(M_d,\lambda)$ is holomorphic for $\lambda \in U$ and $|q| < R$ and (26) (for this range of λ and q) is the power series development of a holomorphic function in two variables with respect to one of the variables. <u>In particular, there is no pole for</u> $\lambda = \lambda_0$. We conclude this from the following lemma.

Lemma. Suppose $g(\lambda,q)$ is a meromorphic function in the two complex variables λ, q where $\lambda \in U \subset \mathbb{C}^*$ and $q \in D_R$ where D_R is the open disc around the origin of positive radius R. Assume that $\lambda_0 \in U$ and $h(\lambda,q) = (\lambda-\lambda_0)^m g(\lambda,q)$ (with $m \geq 0$, $m \in \mathbb{Z}$) is holomorphic in $U \times D_R$. If

$$g(\lambda,q) = \sum_{n=0}^{\infty} c_n(\lambda_0)q^n \tag{27}$$

in $(U - \{\lambda_0\}) \times D_R$ where $c_n(\lambda)$ is a finite Laurent series, then $g(\lambda,q)$ is holomorphic in $U \times D_R$ and (27) holds in $U \times D_R$.

Proof: The holomorphic function $g(\lambda,q)$ has a development in $U \times D_R$ of the form

$$h(\lambda,q) = \sum_{n=0}^{\infty} d_n(\lambda)q^n$$

where the functions $d_n(\lambda)$ are holomorphic in U. Since $c_n(\lambda) = (\lambda - \lambda_0)^{-m} d_n(\lambda)$ for $\lambda \neq \lambda_0$ and $\lim_{\lambda \to \lambda_0} c_n(\lambda) = c_n(\lambda_0)$ we see that the $d_n(\lambda)$ are divisible by $(\lambda - \lambda_0)^m$.

Actually, we did not need the above discussion of the fixed point set of λ_0 and the convergence of (26) for $\lambda = \lambda_0$.

An S^1-action is semi-free (i.e. the fixed point set of any $\lambda \in S^1$, $\lambda \neq 0$, equals $M_d^{S^1}$) if and only if all non-vanishing rotation numbers m equal ± 1. Therefore, for a semi-free action, $M_d^{S^1}$ can have poles only for $\lambda = q^n$ with $n \in \mathbb{Z}$.

Theorem. For an N-balanced semi-free S^1-action of type t on the complex manifold M_d, the equivariant elliptic genus $\varphi_N(M_d,\lambda)$ does not depend on λ. It equals the elliptic genus $\varphi_N(M_d)$. If $t \not\equiv 0 \bmod N$, then $\varphi_N(M_d) = 0$. (Compare [14] and [16].)

Proof: By the lemma, there is no pole for $\lambda = 1$. Because of (25) there are no poles for $\lambda = q^n$. The vanishing of $\varphi_N(M_d)$ follows also from (25).

8. Let M_d be a complex manifold with first Chern class $c_1 \in H^2(M_d,\mathbb{Z})$ divisible by N. The importance of this condition was already apparent in section 5. We choose a holomorphic line bundle L with $L^N = K$. Now suppose we have an S^1-action on M_d. Consider the N-fold covering $S^1 \longrightarrow S^1$ with $\mu \longmapsto \lambda = \mu^N$. Then μ acts on M_d and K through λ. This action can be lifted to L. If p is a fixed point of the given S^1-action with rotation numbers $m_1,m_2,\dots,m_d$, then μ acts in the fibre L_p by $\mu^{-(m_1+\dots+m_d)}$. However, if $\mu = \zeta = e^{2\pi i/N}$, then it operates trivially on M_d. The action of ζ in each fibre of L is by multiplication with ζ^{-t}, where t is a residue class mod N which does not depend on the base point of the fibre. (Assume that M_d is connected.) It follows that the action is N-balanced of type t (see the definition in section 6).

The condition $c_1 \equiv 0 \bmod N$ implies a stronger property than N-balanced. Let $G_m \subset S^1$ be the group of m^{th} roots of unity. The fixed point set of G_m is a submanifold of M_d which includes $M_d^{S^1}$ and is strictly larger if and only if there is a rotation number divisible by m. We denote the fixed point set of G_m by M_d^m. There is the map $S^1 \longrightarrow S^1$ with $\mu \longmapsto \lambda = \mu^N$ which we considered before. Hence any $\mu \in S^1$ with $\mu^{mN} = 1$ operates trivially on M_d^m, however it operates on every fibre L_p $(p \in M_d^m)$ by multiplication with some mN-th root of unity which only depends on the connected component of M_d^m which contains p. Since

μ acts on L_p (for $p \in M_d^{S^1}$) by $\mu^{-(m_1+\ldots+m_1)}$ where the m_j are the rotation numbers of the action in p, it follows that the residue class of $m_1+\ldots+m_d$ mod mN depends only on the connected components of M_d^m and not on the components of $M_d^{S^1}$ contained in them.

Let X be a connected component of M_d^m and $(M_d^{S^1})_\upsilon$ a component of $M_d^{S^1}$ contained in X with rotation numbers $m_1,\ldots,m_d$. Over X the tangent bundle T of M_d splits into vector bundles $\tilde{E}_k$ where $k = 0,1,\ldots,m-1$ and the action of G_m in $\tilde{E}_k$ is by multiplication with λ^k if $\lambda \in G_m$. Of course, $\tilde{E}_0$ is the tangent bundle of X. Over $(M_d^{S^1})_\upsilon$ we have

(28) $$\tilde{E}_k = \sum_{r \equiv k \bmod m} E_r \qquad \text{(see section 6)}$$

We write the rotation numbers in the following form

(29) $$m_i = r_i m + k_i \quad \text{where} \quad k_i = 0,1,\ldots,m-1$$

Since the integer $\sum k_i = \sum_{k=0}^{m-1} k \dim \tilde{E}_k$ depends only on X, we see that $m\cdot\sum r_i$ mod mN depends only on X. Hence, $\sum r_i$ mod N depends only on X and not on the components $(M_d^{S^1})_\upsilon$ contained in it. We put

(30) $$\sum r_i = t(m,X) \bmod N$$

Of course, $t(1,M_d)$ is the tpye t of the action (for connected M_d).

9. Let M_d be a compact complex manifold with $c_1 \equiv 0 \bmod N$. We assume that we have an S^1-action and wish to show that the elliptic function $\varphi_N(M_d, \lambda)$ has no poles. Let X be a connected component of M_d^m (see section 8). We define

$$(31) \qquad \varphi_N(X,\lambda) = \sum_{\upsilon} \varphi_N(M_d,\lambda)_{\upsilon}$$

where the summation is over those connected components $(M_d^{S^1})_{\upsilon}$ which are contained in X (see (23)). This is a short hand notation. Do not confuse (31) with the elliptic genus of X. Let $(M_d^{S^1})_{\upsilon}$ have the rotation numbers $m_1, \ldots, m_d$. According to (21) we have

$$(32) \qquad \varphi_N(M_d,\lambda)_{\upsilon} = (e_0 F(x_1+2\pi i m_1 z) \ldots F(x_d+2\pi i m_d z))[(M_d^{S^1})_{\upsilon}]$$

Let s be an integer and replace in (32) the variable z by $z + \frac{s}{m}\tau$ (in other words, replace λ by $\lambda \cdot q^{s/m}$), then $\varphi_N(M_d, \lambda q^{s/m})_{\upsilon}$ is again an elliptic function in z for the lattice $\mathbf{Z} \cdot N\tau + \mathbf{Z}$. It follows from (22), (29) and (30) that

$$(33) \qquad \varphi_N(M_d,\lambda q^{s/m})_{\upsilon} =$$

$$\zeta^{st(m,X)} \cdot (e_0 \prod_{j=1}^{d} F(x_j + 2\pi \cdot i m_j z + 2\pi i \frac{sk_j}{m} \tau)[M_d^{S^1})_{\upsilon}]$$

If we write down the q-development of the right hand side of (33) (with fractional powers of q) we see that $\varphi_N(X, \lambda q^{s/m})$ is of the form

$$(34) \qquad \varphi_N(X,\lambda q^{s/m}) = \sum_{n=0}^{\infty} \chi_y(X, S_n, \lambda) q^{n/m}$$

where the S_n are virtual equivariant bundles constructed from the bundles $\tilde{E}_k$ over X. For $m = 1$ we come back to (19).

The elliptic function (34) has no poles for $|\lambda| = 1$. We use again the lemma in section 7.

10. We are now able to prove the rigidity theorem.

<u>Theorem</u>. <u>Let</u> M_d <u>be a compact complex manifold with first Chern class</u> $c_1 \in H^2(M_d,\mathbb{Z})$ <u>divisible by</u> N. <u>Suppose an</u> S^1<u>-action on</u> M_d <u>is given</u>. <u>Then the equivariant elliptic genus</u> $\varphi_N(M_d,\lambda)$ <u>does not depend on</u> $\lambda \in S^1$. <u>It equals the elliptic genus</u> $\varphi_N(M_d) = \varphi_N(M_d,1)$. <u>If the type</u> t <u>of the action is</u> $\not\equiv 0 \bmod N$, <u>then</u> $\varphi_N(M_d) \equiv 0$.

<u>Proof.</u> Let m be a natural number ≥ 1 and $\lambda^{\pm m} = q^n$. Then λ is of the form $\lambda = \lambda_0 q^{s/m}$ where $\lambda_0^m = 1$ and $s = \pm n$. We have

$$\varphi_N(M_d,\lambda_0 q^{s/m}) = \sum_X \varphi_N(X,\lambda_0 q^{s/m})$$

where the summation is over all the connected components of M_d^m. Since the elliptic function (34) has no poles for λ_0, the result follows. The vanishing $\varphi_N(M_d) = 0$ for $t \not\equiv 0 \bmod N$ follows again from (25).

11. We want to point out some applications of the rigidity theorem.

If we develop in a cusp (1) with $k > 0$, we get a different version of the rigidity theorem (compare [19]). In particular, we get that $\chi(M_d,L^k,\lambda)$ does not depend on λ for $k = 1,\dots,N-1$, in fact $\chi(M_d,L^k) = 0$. This is a well-known result ([12],[15]). For $N = 2$ and $k = 1$ it corresponds to the theorem in [5] on the $\hat{A}$-genus.

The elliptic genus of level N is strictly multiplicative in fibre bundles with a manifold M_d with $c_1(M_d) \equiv 0 \bmod N$ as fibre and a compact connected Lie group G of automorphisms of M_d as structure group (compare [14] and [16]).

This we wish to apply, for example, to the compact irreducible hermitian symmetric spaces G/U studied in [7] § 16. There we gave a formula for the coefficient $\lambda(G/U)$ in

$$c_1(G/U) = \lambda(G/U) \cdot g$$

where g is a positive generator of the infinite cyclic group $H^2(G/U)$.

Take a system $w_1,\dots,w_d$ of positive complementary roots for G/U (see [7]). Here d is the complex dimension of G/U. The roots $w_1,\dots,w_d$ are linear forms in $x_1,\dots,x_\ell$ where $\ell = \text{rank}(U) = \text{rank}(G)$, the $x_1,\dots,x_\ell$ can be identified with a base of $H^1(T,\mathbf{Z})$ where T is the maximal torus of U. Without proof we state the following result which is equivalent to the strict multiplicativity of the elliptic genus for G/U-bundles.

Theorem. Let $F(x) = f(x)^{-1}$ be the elliptic function introduced for level N (see section 1). Let $w_1,\dots,w_d$ be positive complementary roots for the irreducible hermitian symmetric space G/U. Suppose $\lambda(G/U) \equiv 0 \bmod N$. Then

$$\sum_{\sigma\in W(G)/W(U)} F(\sigma(w_1))F(\sigma(w_2))\dots F(\sigma(w_d)) = \varphi_N(G/U). \tag{35}$$

Here $W(G)$, $W(U)$ are the Weyl groups. (An element $\sigma \in W(U)$ permutes $w_1,\dots,w_d$. Therefore the sum over the $W(U)$-cosets is well-defined.)

The formula (35) is an identity in the ℓ variables $x_1,\dots,x_\ell$. The sum is a constant, i.e. does not depend on these variables anymore.

The rigidity theorem in section 10 also gave a vanishing result. We give an example: Consider the Grassmannian

$$W(m,n) = U(m+n)/(U(m) \times U(n))$$

We use the notation of [7]. As a system of positive roots of $U(m+n)$, we take

$$\{-x_i + x_j \mid 1 \leq i < j \leq m + n\}.$$

The complementary roots w_r are given by $1 \leq i \leq m$ and $m + 1 \leq j \leq m + n$. Their sum equals

$$\sum w_r = -n \sum_{i=1}^{m} x_i + m \sum_{j=m+1}^{m+n} x_j = \tag{36}$$

$$-(m + n) \sum_{i=1}^{m} x_i + m \sum_{j=1}^{m+n} x_j$$

We put $-\sum_{i=1}^{m} x_i = g$ and $\sum_{j=1}^{m+n} x_j = \sigma_1$. Then g becomes the positive generator of $H^2(W(m,n),\mathbb{Z})$ whereas σ_1 vanishes if regarded as element of this cohomology group. Therefore,

$$\lambda(W(m,n)) = m + n.$$

We also see from (36) that $W(m,n)$ admits an N-balanced circle action of type m if $m + n \equiv 0 \bmod N$. We obtain

Proposition The elliptic genus $\varphi_N(W(m,n))$ vanishes if $m + n \equiv 0 \bmod N$ and $m \not\equiv 0 \bmod N$. For the complex projective spaces $P_n(\mathbb{C}) = W(n,1)$ we have

$$\varphi_N(P_n(\mathbb{C})) = 0 \quad \text{if} \quad n + 1 \equiv 0 \bmod N.$$

12. The elliptic function f defined in section 1 satisfies a differential equation

$$\left(\frac{f'}{f}\right)^N + a_1\left(\frac{f'}{f}\right)^{N-1} + \ldots + a_{N-1}\left(\frac{f'}{f}\right) + a_N \tag{37}$$

$$= \frac{1}{f^N} + a_{2N}f^N \quad \text{with} \quad a_{2N} = c^{2N} \quad \text{(see section 1)}$$

where the a_j are modular forms of weight j for $\Gamma_1(N)$, (if $\alpha = 2\pi i/N$ in (1)). The polynomial

$$P(\xi) = \xi^N + a_1\xi^{N-1} + \ldots + a_{N-1}\xi + a_N$$

has the following properties:

1) $a_{N-1} = 0$

2) If $P'(\xi) = 0$, but $\xi \neq 0$, then $P(\xi)^2 = 4a_{2N}$

The property 2) implies that the values at the critical points ξ with $\xi \neq 0$ are all equal up to sign. In this case, the polynomial might be called almost - Cebycev. Theodore J. Rivlin wrote to me that polynomials with essentially such properties occur in the literature under the name Zolotarev-polynomials. Also their relation to elliptic functions is known (see for example [10]). I plan to write a separate paper on these matters. For $N = 2$ the differential equation is of the form

$$(f')^2 = 1 - a_2f^2 + a_4f^4,$$

very well known for the elliptic genus of level 2 (see [14]).

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