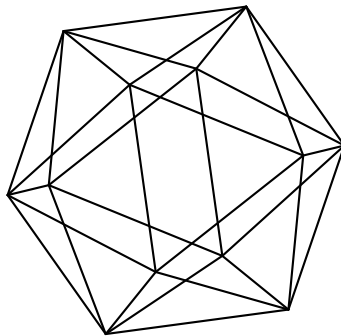


# Max-Planck-Institut für Mathematik Bonn

On semiconjugate rational functions

by

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## Abstract

We classify rational solutions of the functional equation  $A \circ X = X \circ B$  in terms of groups acting properly discontinuously on  $\mathbb{C}$  or  $\mathbb{C}\mathbb{P}^1$ , generalizing classical results of Julia, Fatou, and Ritt about commuting rational functions. We also give a description of rational solutions of the more general functional equation  $A \circ C = D \circ B$  under the condition that the algebraic curve  $A(x) - D(y) = 0$  is irreducible.

## 1 Introduction

The problem of description of commuting rational functions, that is of rational solutions of the functional equation

$$f_1 \circ f_2 = f_2 \circ f_1, \tag{1}$$

was considered for the first time in the early twenties of the past century in the papers of Fatou, Julia, and Ritt [6], [8], [15]. In all these papers it was assumed that the functions  $f_1$  and  $f_2$  have no iterate in common, that is

$$f_1^{\circ n} \neq f_2^{\circ m}$$

for all  $n, m \in \mathbb{N}$ . Fatou and Julia used in their papers dynamical methods requiring an additional assumption that the Julia set of  $f_1$  or  $f_2$  does not coincide with the whole complex plane. They showed that in this case up to a conjugacy  $f_1, f_2$  are either powers or Chebyshev polynomials. In contrast, Ritt used a method of algebraic-topological character which was free of any assumptions about the Julia set. Accordingly, the list of commuting functions obtained by Ritt is longer and includes in particular rational functions arising from multiplication theorems of doubly periodic meromorphic functions. Below we put a citation from the Ritt paper which contains a formulation of his main result as well as some additional remarks.

“If the rational functions  $\Phi(z)$  and  $\Psi(z)$ , each of degree greater than unity, are permutable, and if no iterate of  $\Phi(z)$  is identical with any iterate of  $\Psi(z)$ , there exists a periodical meromorphic function  $f(z)$ , and four numbers  $a, b, c$  and  $d$  such that

$$f(az + b) = \Phi[f(z)], \quad f(cz + d) = \Psi[f(z)].$$

The possibilities for  $f(z)$  are: any linear function of  $e^z$ ,  $\cos z$ ,  $\wp(z)$ ; in the lemniscatic case ( $g_3 = 0$ ),  $\wp^2$ ; in the equianharmonic case ( $g_2 = 0$ ),  $\wp'z$  and  $\wp^3z$ . There are, essentially, the only periodic meromorphic functions which have rational multiplication theorems.

The multipliers  $a$  and  $c$  must be such that if  $\omega$  is any period of  $f(z)$ ,  $a\omega$  and  $c\omega$  are also periods of  $f(z)$ .

If  $p$  represents the order of  $f(z)$ , that is, the number of times  $f(z)$  assumes any given value in a primitive period strip or in a primitive period parallelogram, the products

$$b(1 - e^{2\pi i/p}), \quad d(1 - e^{2\pi i/p})$$

must be periods of  $f(z)$ .

Finally,

$$(a - 1)d - (c - 1)b$$

must be a period of  $f(z)$ .

The condition that  $\Phi(z)$  and  $\Psi(z)$  have no iterate in common, can be replaced by one which is certainly not stronger, and which satisfied, for instance, if there does not exist a rational function  $\sigma(z)$ , of degree greater than unity, such that

$$\Phi(z) = \varphi[\sigma(z)], \quad \Psi(z) = \psi[\sigma(z)],$$

where  $\varphi(z)$  and  $\psi(z)$  are rational.

The existence of the periodic function  $f(z)$  is demonstrated by a method which is almost entirely algebraic. It would be interesting to know whether a proof can also be effected by the use of the Poincaré functions employed by Julia."

Note that an affirmative answer to the last question was given more than sixty years later by Eremenko in the paper [4], where the result of Ritt was reproved with the use of the modern methods of the iteration theory.

The problem of description of commuting rational functions is a part of the more general problem of description of semi-conjugate rational functions, that is of rational solutions of the equation

$$A \circ X = X \circ B, \tag{2}$$

where it is assumed that the degree of  $X$  is greater than one. We also will assume that the degree of  $A$  and  $B$  is greater than one since otherwise any solution reduces to a solution of the form

$$\varepsilon^s z \circ x^s R(x^n) = x^s R(x^n) \circ \varepsilon z,$$

where  $R$  is a rational function and  $\varepsilon^n = 1$  (see [7], [11]).

Since for any solution of equation (2) the equality

$$A^{\circ n} \circ X = X \circ B^{\circ n}$$

holds for any  $n \in \mathbb{N}$ , equation (2) is of a great interest in the complex dynamics (see e.g. [3] and references therein). Nevertheless, essentially nothing

is known about its solutions except the case where all the functions  $A, X, B$  are polynomials. For this case a complete description of solutions of (2) was obtained in [10] as a corollary of the decomposition theory of polynomials. This theory was developed by Ritt in the paper [16] and includes in particular an explicit description of polynomial solutions of the more general than (2) functional equation

$$A \circ C = D \circ B. \quad (3)$$

The Ritt theory may be extended to a decomposition theory of Laurent polynomials ([13], [19]). However, the methods of [16], [13], [19] are not applicable to a description of arbitrary rational solutions of (2) or (3).

The main goal of this paper is to provide a “geometric” description of solutions of equation (2). Similarly to the papers about commuting rational functions, we mostly will work under an additional assumption about solutions  $A, X, B$  of (2). Namely, we will assume that each of the functions  $A, B, X$  is of degree greater than one, and there exists no rational function  $W$  of degree greater than one such that

$$B = \tilde{B} \circ W, \quad X = \tilde{X} \circ W, \quad (4)$$

for some rational functions  $\tilde{X}, \tilde{B}$ . We will call such solutions primitive. Notice that this assumption was also mentioned by Ritt in the above extract concerning commuting rational function. However, in distinction with equation (1) any solution of equation (2) may be reduced either to a primitive solution or to a solution with  $\deg X = 1$  by a simple iterative process. Indeed, it is easy to see that if a triple  $A, X, B$  is a non-primitive solution of (2) such that (4) holds, then the triple  $A, \tilde{X}, W \circ \tilde{B}$  is also a solution of (2). This new solution is not necessary primitive, however  $\deg \tilde{X} < \deg X$ . Therefore, after a finite number of similar transformations we will arrive either to a primitive solution or to a solution with  $\deg X = 1$ .

Let us introduce now a general construction which permits to describe all primitive solutions of (2). Let  $R$  be a simply connected Riemann surface,  $\Gamma$  a group acting properly discontinuously on  $R$ , and  $\theta_\Gamma$  the projection map

$$\theta_\Gamma : R \rightarrow R/\Gamma.$$

Further, assume that  $\tau : R \rightarrow R$  is a holomorphic map which maps any orbit of  $\Gamma$  to an orbit of  $\Gamma$ , or equivalently a holomorphic map such that for any  $\sigma \in \Gamma$  the equality

$$\tau \circ \sigma = \varphi(\sigma) \circ \tau \quad (5)$$

holds for some homomorphism  $\varphi : \Gamma \rightarrow \Gamma$ . Clearly,  $\tau$  is such a map if and only if  $\tau$  descends to a holomorphic map  $A : R/\Gamma \rightarrow R/\Gamma$  such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\tau} & R \\ \downarrow \theta_\Gamma & & \downarrow \theta_\Gamma \\ R/\Gamma & \xrightarrow{A} & R/\Gamma \end{array} \quad (6)$$

is commutative. Finally, assume that  $\Gamma'$  is a subgroup of  $\Gamma$  such that the homomorphism  $\varphi$  in (5) satisfies the condition  $\varphi(\Gamma') \subseteq \Gamma'$ .

Since the inclusion  $\Gamma' \subseteq \Gamma$  implies that there exists a holomorphic function  $X : R/\Gamma' \rightarrow R/\Gamma$  such that  $\theta_\Gamma = X \circ \theta_{\Gamma'}$ , starting from a collection  $R, \Gamma', \Gamma, \tau$  as above we obtain the commutative diagram

$$\begin{array}{ccccccc}
R & \xrightarrow{\tau} & R & \xrightarrow{id} & R & \xrightarrow{\tau} & R \\
\downarrow \theta_{\Gamma'} & & \downarrow \theta_{\Gamma'} & & \downarrow \theta_\Gamma & & \downarrow \theta_\Gamma \\
R/\Gamma' & \xrightarrow{B} & R/\Gamma' & \xrightarrow{X} & R/\Gamma & \xrightarrow{A} & R/\Gamma
\end{array} \tag{7}$$

implying the equalities

$$A \circ X \circ \theta_{\Gamma'} = \theta_\Gamma \circ \tau, \quad X \circ B \circ \theta_{\Gamma'} = \theta_\Gamma \circ \tau,$$

which in their turn yield the equality

$$A \circ X \circ \theta_{\Gamma'} = X \circ B \circ \theta_{\Gamma'}. \tag{8}$$

Clearly, if  $R/\Gamma' = R/\Gamma = \mathbb{CP}^1$ , then  $A, X, B$  are rational functions and equality (8) implies that the triple  $A, X, B$  is a solution of (2). Roughly speaking, our main result states that all rational primitive solutions of (2) may be obtained in this way either for  $R = \mathbb{CP}^1$  or for  $R = \mathbb{C}$ . More precisely, say that a solution is *spherical* if it is obtained from a collection  $R, \Gamma', \Gamma, \tau$  as above, where  $R = \mathbb{CP}^1$ ,  $\tau$  is a rational function, and  $\Gamma', \Gamma$  are finite subgroups of the group  $Aut(\mathbb{CP}^1)$ . The second type of solutions called *Euclidean* is defined by the condition that  $R = \mathbb{C}$ ,  $\tau$  is a linear function, and  $\Gamma', \Gamma$  are subgroups of the group  $Aut(\mathbb{C})$  which contain translations by elements of some lattice  $L$  of rank two in  $\mathbb{C}$ . It is well known that in both cases  $R/\Gamma' = R/\Gamma = \mathbb{CP}^1$  and therefore diagram (7) produces a solution of (2). In this notation our main result may be formulated as follows.

**Theorem 1.1.** *Every primitive solution of the equation  $A \circ X = X \circ B$  is either Euclidean or spherical.*

Recall that in the spherical case the automorphism group  $Aut(\mathbb{CP}^1)$  consists of Möbius transformations. Furthermore, any non-trivial finite subgroup  $\Gamma$  of  $Aut(\mathbb{CP}^1)$  is cyclic, dihedral, tetrahedral, octahedral, or icosahedral. The corresponding functions  $\theta_\Gamma$  were calculated for the first time by Klein in [9]. Notice that since  $X$  is a compositional left factor of  $\theta_\Gamma$ , the Klein classification implies in particular that, up to the change  $X \rightarrow \alpha \circ X \circ \beta$ , where  $\alpha, \beta$  are Möbius transformations, apart from the series  $z^n$ ,  $T_n(z)$ , and  $\frac{1}{2} \left( z^n + \frac{1}{z^n} \right)$ , there exists only a finite number of rational functions  $X$  appearing as a component of a spherical solution  $A, X, B$  (see Corollary 5.1 below). Notice also that in the spherical case diagram (6) by itself induces a solution of (2). Clearly, such a solution is obtained from the above construction for  $\Gamma' = \{e\}$ .

In the Euclidean case the corresponding automorphism group  $Aut(\mathbb{C})$  consists of linear functions. Furthermore, for any group  $\Gamma \subset Aut(\mathbb{C})$  containing



translations by elements of some lattice  $L$  of rank two in  $\mathbb{C}$  there exists an integer  $n$  equal 2,3,4, or 6, satisfying  $\varepsilon_n L = L$ , where  $\varepsilon_n = \exp(2\pi/n)$ , such that  $\Gamma$  is generated by translations by elements of  $L$ , and the transformation  $z \rightarrow \varepsilon_n z$ . Accordingly, the function  $\theta_\Gamma$  is either the Weierstrass function  $\wp(z)$  corresponding to  $L$ , or may be written in terms of the Weierstrass function as  $\wp'(z)$ ,  $\wp^2(z)$ ,  $\wp'^2(z)$ . Note that for any pair  $\Gamma' \subset \Gamma$  of such groups and any integer  $m \geq 2$  we obtain an Euclidean solution of (2) setting  $\tau(z) = mz$ .

A natural generalization of the above construction leads to functional equation (3). Namely, let  $R$  be a simply connected Riemann surface,  $\Gamma_1, \Gamma_2$  two groups acting properly discontinuously on  $R$ , and  $\tau : R \rightarrow R$  a holomorphic function such that for any  $\sigma \in \Gamma_1$  the equality

$$\tau \circ \sigma = \varphi(\sigma) \circ \tau \tag{9}$$

holds for some homomorphism  $\varphi : \Gamma_1 \rightarrow \Gamma_2$ . Furthermore, let  $\Gamma'_1$  be a subgroup of  $\Gamma_1$ , and  $\Gamma'_2$  a subgroup of  $\Gamma_2$  such that  $\varphi(\Gamma'_1) \subset \Gamma'_2$ . Then there exist holomorphic functions  $A, B, C, D$  such that the following diagram

$$\begin{array}{ccccc}
 R & \xrightarrow{\tau} & & R & \\
 \theta_{\Gamma_1} \searrow & & & & \swarrow \theta_{\Gamma_2} \\
 & R/\Gamma_1 \xrightarrow{A} R/\Gamma_2 & & & \\
 id \uparrow & C \uparrow & D \uparrow & & id \uparrow \\
 & R/\Gamma'_1 \xrightarrow{B} R/\Gamma'_2 & & & \\
 \theta_{\Gamma'_1} \swarrow & & & & \searrow \theta_{\Gamma'_2} \\
 R & \xrightarrow{\tau} & & R & 
 \end{array}$$

is commutative. Therefore, if  $R/\Gamma_1 = R/\Gamma_2 = R/\Gamma'_1 = R/\Gamma'_2 = \mathbb{C}\mathbb{P}^1$  we obtain a solution (3). We will call solutions of (3) obtained in this way *geometric*. Notice that in distinction with the definitions of spherical and Euclidean solutions the Riemann surface  $R$  appearing in this definition may be a unit disk.

In order to formulate our main result about solutions of (3) we need a convenient modification of the notion of primitive solution for equation (3). Namely, as above we will assume that each of the functions  $A, B, C, D$  in (3) is of degree greater than one and there exists no rational function  $W$  of degree greater than one such that

$$C = \tilde{C} \circ W, \quad B = \tilde{B} \circ W, \tag{10}$$

for some rational functions  $\tilde{C}, \tilde{B}$ . Additionally, we will assume that the algebraic curve  $A(x) - D(y) = 0$  is irreducible (notice that if  $C = D$ , then the first condition implies the second, see Lemma 3.2 below). We will call such solutions of (3) *good*. Under this notation our main result about solutions of (3) is the following statement.

**Theorem 1.2.** *Every good solution of the equation  $A \circ C = D \circ B$  is geometric.*

Notice that one can obtain a geometric solution of (3) starting from any group  $\Omega$ , acting properly discontinuously on  $R$ , and two its subgroups  $\Omega_1, \Omega_2$ , assuming that the corresponding Riemann surfaces are isomorphic to  $\mathbb{CP}^1$ , setting  $\Gamma_2 = \Omega, \Gamma_1 = \Omega_1, \Gamma'_2 = \Omega_2, \Gamma'_1 = \Omega_1 \cap \Omega_2$ , and  $\tau = id$ . For  $R = \mathbb{C}$  this particular case of the above construction was described in [1], while for  $R = \mathbb{CP}^1$  similar examples were given in [5].

The paper is organized as follows. In the second section we recall some basic definitions and results related to orbifolds whose underlying space is a Riemann surface and reformulate our results in terms of orbifolds. In the third section we recall some general properties of the functional equation

$$f \circ p = g \circ q \tag{11}$$

on Riemann surfaces basing on the fiber product approach. In the fourth section we introduce the concept of quasi-covering map between orbifolds and relate it with functional equation (11). In the fifth section we prove Theorem 1.1 and Theorem 1.2. We also deduce the Ritt-Eremenko theorem from Theorem 1.1.

In the sixth section we establish some additional properties of spherical primitive solutions of (2). In particular, we show that for such a solution the homomorphism  $\varphi$  in (5) is an automorphism. Since  $\Gamma$  is finite, this implies that some iteration of the function  $\tau$  is  $\Gamma$ -equivariant (that is the corresponding homomorphism  $\varphi$  in (5) satisfies the condition  $\varphi(\sigma) = \sigma$ ). This property of spherical solutions is quite important since as it was shown in [2] the problem of description of  $\Gamma$ -equivariant functions for finite subgroups of  $Aut(\mathbb{CP}^1)$  reduces to the classical problem of description of homogeneous  $\Gamma$ -invariant polynomials solved by Klein. In conclusion, we give several explicit examples of spherical primitive solutions of (2).

Notice that in contrast to the papers [6], [8], [4] our approach does not use any dynamical methods at all but as in [15],[16] relies on algebraic-topological ideas.

## 2 Orbifolds

In this section we recall some basic definitions and results related to orbifolds whose underlying space is a Riemann surface (see [17] and [12], Appendix E) and reformulate our main results in terms of orbifolds.

### 2.1 Orbifolds on Riemann surfaces

A pair  $\mathcal{O} = (R, \nu)$  consisting of a Riemann surface  $R$  and a ramification function  $\nu : R \rightarrow \mathbb{N}$  which takes the value  $\nu(z) = 1$  except at isolated set of points is called an orbifold. The Euler characteristic of an orbifold  $\mathcal{O} = (R, \nu)$  is defined by the formula

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left( \frac{1}{\nu(z)} - 1 \right),$$

where  $\chi(R)$  is the Euler characteristic of  $R$ .

If  $R_1, R_2$  are Riemann surfaces provided with ramification functions  $\nu_1, \nu_2$ , and  $f : R_1 \rightarrow R_2$  is a holomorphic branched covering map, then  $f$  is said to be a *covering map*  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  between orbifolds  $\mathcal{O}_1 = (R_1, \nu_1)$  and  $\mathcal{O}_2 = (R_2, \nu_2)$  if for any  $z \in R_1$  the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \quad (12)$$

holds, where  $\deg_z f$  denotes the local degree of  $f$  at the point  $z$ . If  $f$  is such a map and  $R_1, R_2$  are compact, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2), \quad (13)$$

where  $d$  is the degree of  $f$ . If for any  $z \in R_1$  instead of equality (12) a weaker condition that  $\nu_2(f(z))$  divides  $\nu_1(z) \deg_z f$  holds, then  $f$  is said to be a *holomorphic map*  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  between orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$ .

A universal cover of an orbifold  $\mathcal{O}$  is a covering map between orbifolds  $\tilde{\theta}_{\mathcal{O}} : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$  such that  $\tilde{R}$  is simply connected and  $\tilde{\nu}(z) \equiv 1$ . If  $\tilde{\theta}_{\mathcal{O}}$  is such a map, then there exists a group  $\tilde{\Gamma}_{\mathcal{O}}$  of conformal automorphisms of  $\tilde{R}$  such that the equality  $\tilde{\theta}_{\mathcal{O}}(z_1) = \tilde{\theta}_{\mathcal{O}}(z_2)$  holds for  $z_1, z_2 \in \tilde{R}$  if and only if  $z_1 = \sigma(z_2)$  for some  $\sigma \in \tilde{\Gamma}_{\mathcal{O}}$ . A universal cover of  $\mathcal{O}$  exists and is unique up to a conformal isomorphism of  $\tilde{R}$ , unless  $\mathcal{O}$  is the Riemann sphere with one ramified point, or  $\mathcal{O}$  is the Riemann sphere with two ramified points for which  $\nu(z_1) \neq \nu(z_2)$ .

If  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a covering map between orbifolds, then for any choice of  $\tilde{\theta}_{\mathcal{O}_1}$  and  $\tilde{\theta}_{\mathcal{O}_2}$  there exists a conformal isomorphism  $\tau : \tilde{\mathcal{O}}_1 \rightarrow \tilde{\mathcal{O}}_2$  such that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{O}}_1 & \xrightarrow{\tau} & \tilde{\mathcal{O}}_2 \\ \downarrow \tilde{\theta}_{\mathcal{O}_1} & & \downarrow \tilde{\theta}_{\mathcal{O}_2} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \quad (14)$$

is commutative. More generally, the following statement holds.

**Proposition 2.1.** *Let  $f : R_1 \rightarrow R_2$  be a holomorphic branched covering map. Then  $f$  lifts to a holomorphic map  $\tau : \tilde{\mathcal{O}}_1 \rightarrow \tilde{\mathcal{O}}_2$  such that the diagram (14) is commutative if and only if  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a holomorphic map between orbifolds.*

*Proof.* If diagram (14) is commutative, then for any  $z \in \tilde{R}_1$  the equality

$$\deg_z (f \circ \tilde{\theta}_{\mathcal{O}_1}) = \deg_z (\tilde{\theta}_{\mathcal{O}_2} \circ \tau)$$

holds. It follows now from

$$\deg_z (f \circ \tilde{\theta}_{\mathcal{O}_1}) = \deg_z \tilde{\theta}_{\mathcal{O}_1} \deg_{\tilde{\theta}_{\mathcal{O}_1}(z)} f = \nu_1(\tilde{\theta}_{\mathcal{O}_1}(z)) \deg_{\tilde{\theta}_{\mathcal{O}_1}(z)} f,$$

and

$$\deg_z (\tilde{\theta}_{\mathcal{O}_2} \circ \tau) = \deg_z \tau \deg_{\tau(z)} \tilde{\theta}_{\mathcal{O}_2} = \deg_z \tau \nu_2(\tilde{\theta}_{\mathcal{O}_2}(\tau(z))) = \deg_z \tau \nu_2(f(\tilde{\theta}_{\mathcal{O}_1}(z))),$$

that

$$\nu_2(f(z)) \mid \nu_1(z) \deg_z f \quad (15)$$

holds for any  $z \in R_1$ .

Assume now that (15) holds and denote by  $\tau$  the complete analytic continuation of the germ  $\tilde{\theta}_{\mathcal{O}_2}^{-1} \circ f \circ \tilde{\theta}_{\mathcal{O}_1}$ , where  $\tilde{\theta}_{\mathcal{O}_2}^{-1}$  is a branch of the function inverse to  $\tilde{\theta}_{\mathcal{O}_2}$ . Clearly, the local multiplicity of the map  $f \circ \tilde{\theta}_{\mathcal{O}_1}$  at a point  $z \in \tilde{\mathcal{O}}_1$  equals

$$\deg_z \tilde{\theta}_{\mathcal{O}_1} \deg_{\tilde{\theta}_{\mathcal{O}_1}(z)} f = \nu(\tilde{\theta}_{\mathcal{O}_1}(z)) \deg_{\tilde{\theta}_{\mathcal{O}_1}(z)} f.$$

On the other hand, the order of the permutation of branches of the function inverse to  $\tilde{\theta}_{\mathcal{O}_2}$  induced by the analytic continuation along a small loop around the point  $(f \circ \tilde{\theta}_{\mathcal{O}_1})(z)$  is equal to  $\nu_2((f \circ \tilde{\theta}_{\mathcal{O}_1})(z))$ . Therefore, (15) implies that the function  $\tau$  has no ramification points. Since  $\mathcal{R}$  is simply connected, this implies that  $\tau$  is single valued. Finally, clearly diagram (14) is commutative.  $\square$

Notice that the function  $\tau$  in (14) is defined by  $f$  and  $\tilde{\theta}_{\mathcal{O}_1}, \tilde{\theta}_{\mathcal{O}_2}$  up to a transformation  $\tau \rightarrow g \circ \tau$ , where  $g \in \tilde{\Gamma}_{\mathcal{O}_2}$ .

It is easy to see that in its turn a holomorphic map  $\tau : \tilde{\mathcal{O}}_1 \rightarrow \tilde{\mathcal{O}}_2$  descends to a holomorphic map between orbifolds  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  such that diagram (14) is commutative if and only if  $\tau$  maps any orbit of  $\tilde{\Gamma}_{\mathcal{O}_1}$  to an orbit of  $\tilde{\Gamma}_{\mathcal{O}_2}$ , or equivalently if there exists a homomorphism  $\varphi : \tilde{\Gamma}_{\mathcal{O}_1} \rightarrow \tilde{\Gamma}_{\mathcal{O}_2}$  such that for any  $\sigma \in \tilde{\Gamma}_{\mathcal{O}_1}$  the equality

$$\tau \circ \sigma = \varphi(\sigma) \circ \tau \quad (16)$$

holds. We will denote the set of all such maps by  $\mathcal{E}(\tilde{\Gamma}_{\mathcal{O}_1}, \tilde{\Gamma}_{\mathcal{O}_2})$ .

**Proposition 2.2.** *Let  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a holomorphic map between orbifolds with compact supports. Then*

$$\chi(\mathcal{O}_1) \leq \chi(\mathcal{O}_2) \deg f \quad (17)$$

and the equality holds if and only if  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a covering map between orbifolds.

*Proof.* Denote by  $S_1$  (resp.  $S_2$ ) the set of ramified points of  $\mathcal{O}_1$  (resp. of  $\mathcal{O}_2$ ) and by  $\mathcal{C}(f)$  the set of critical values  $f$ . Set

$$S = S_2 \cup f(S_1) \cup \mathcal{C}(f), \quad \hat{R}_2 = R_2 \setminus S, \quad \hat{R}_1 = f^{-1}\{\hat{R}_2\}.$$

Since  $f : \hat{R}_1 \rightarrow \hat{R}_2$  is a covering map between surfaces, we have:

$$\chi(\hat{R}_1) = d\chi(\hat{R}_2), \quad (18)$$

where  $d = \deg f$ . Furthermore, the inequality

$$\frac{\nu_2(f(z))}{\nu_1(z)} \leq \deg_z f$$

implies the inequality

$$\sum_{\substack{x \in R_1 \\ f(x)=f(z)}} \frac{1}{\nu_1(x)} \leq \frac{d}{\nu_2(f(z))}, \quad (19)$$

where the equality holds if and only if (12) holds for any  $x \in f^{-1}\{z\}$ .

Since removing a point from a surface reduces the Euler characteristic by one we have:

$$\chi(\mathcal{O}_1) = \chi(R_1) + \sum_{\substack{x \in R_1 \\ f(x) \in S}} \left( \frac{1}{\nu_1(x)} - 1 \right) = \chi(\widehat{R}_1) + \sum_{\substack{x \in R_1 \\ f(x) \in S}} \frac{1}{\nu_1(x)}. \quad (20)$$

It follows now from (18), (19), and (20) that

$$\chi(\mathcal{O}_1) \leq d\chi(\widehat{R}_2) + \sum_{z \in S} \frac{d}{\nu_2(z)} = d\chi(\mathcal{O}_2),$$

where the equality holds if and only if (12) holds for any  $z \in f^{-1}\{S\}$ . Since the definition of  $S$  implies that (12) is satisfied also for  $z \notin f^{-1}\{S\}$ , we conclude that the equality in (17) holds if and only if  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a covering map between orbifolds.  $\square$

**Corollary 2.1.** *Let  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  be a holomorphic map of degree greater than one between orbifolds with compact supports such that  $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = l$ . Then  $l \geq 0$ .*  $\square$

## 2.2 Orbifolds with $\chi(\mathcal{O}) \geq 0$ and main theorems

Orbifolds  $\mathcal{O}$  with  $\chi(\mathcal{O}) \geq 0$  and  $R = \mathbb{CP}^1$  are closely related to groups acting properly discontinuously on  $\mathbb{C}$  or  $\mathbb{CP}^1$ . Namely, if  $\chi(\mathcal{O}) = 0$ , then  $\widetilde{\mathcal{O}} = \mathbb{C}$  and the collection of ramification indices of  $\mathcal{O}$  is either  $(2, 2, 2, 2)$ , or one of the following triples  $(3, 3, 3)$ ,  $(2, 4, 4)$ ,  $(2, 3, 6)$ . Correspondingly, there exists a lattice  $L$  of rank two in  $\mathbb{C}$  and an integer  $n$  equal 2 or 3, 4, 6, satisfying  $\varepsilon_n L = L$ , where  $\varepsilon_n = \exp(2\pi/n)$ , such that the group  $\widetilde{\Gamma}_{\mathcal{O}}$  is generated by translations of  $\mathbb{C}$  by elements of  $L$  and the transformation  $z \rightarrow \varepsilon_n z$ . Notice that for the collection of ramification indices  $(2, 2, 2, 2)$  the complex structure of  $\mathbb{C}/L$  may be arbitrary and the function  $\widetilde{\theta}_{\mathcal{O}}$  coincides with the corresponding Weierstrass function  $\wp(z)$ . On the other hand, for the collections  $(2, 4, 4)$ ,  $(2, 3, 6)$ , and  $(3, 3, 3)$  this structure is rigid and arises from the tiling of  $\mathbb{C}$  by squares, equilateral triangles, or alternately colored equilateral triangles, respectively. Accordingly, the functions  $\widetilde{\theta}_{\mathcal{O}}$  may be written in terms of the corresponding Weierstrass functions as  $\wp^2(z)$ ,  $\wp'^2(z)$ , and  $\wp'(z)$ .

Further, if  $\chi(\mathcal{O}) > 0$  and  $\mathcal{O}$  is neither non-ramified sphere nor one of two orbifolds without the universal cover, then  $\widetilde{\mathcal{O}} = \mathbb{CP}^1$  and the collection of ramification indices of  $\mathcal{O}$  is either  $(n, n)$ , or  $(2, 2, n)$  for some  $n \geq 2$ , or one of the

following triples  $(2, 3, 3)$ ,  $(2, 3, 4)$ ,  $(2, 3, 5)$ . The corresponding groups  $\tilde{\Gamma}_{\mathcal{O}}$  are finite subgroups of the automorphism group of the Riemann sphere, namely, the cyclic, dihedral, tetrahedral, octahedral, and icosahedral respectively. Accordingly, the functions  $\tilde{\theta}_{\mathcal{O}}$  are rational functions of degree  $n$ ,  $2n$ ,  $12$ ,  $24$ , and  $60$  (see e.g. [9]).

The above classification implies that the definition of a spherical solution of (2) given in introduction may be reformulated as follows: a solution  $A, X, B$  of (2) is spherical if and only if there exist orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$  of positive Euler characteristic whose underlying surface is the Riemann sphere and a holomorphic function  $\tau \in \mathcal{E}(\tilde{\Gamma}_{\mathcal{O}_1}, \tilde{\Gamma}_{\mathcal{O}_1}) \cap \mathcal{E}(\tilde{\Gamma}_{\mathcal{O}_2}, \tilde{\Gamma}_{\mathcal{O}_2})$  such that the diagram

$$\begin{array}{ccccccc}
\tilde{\mathcal{O}}_1 & \xrightarrow{\tau} & \tilde{\mathcal{O}}_1 & \xrightarrow{id} & \tilde{\mathcal{O}}_2 & \xrightarrow{\tau} & \tilde{\mathcal{O}}_2 \\
\downarrow \tilde{\theta}_{\mathcal{O}_1} & & \downarrow \tilde{\theta}_{\mathcal{O}_1} & & \downarrow \tilde{\theta}_{\mathcal{O}_2} & & \downarrow \tilde{\theta}_{\mathcal{O}_2} \\
\mathcal{O}_1 & \xrightarrow{B} & \mathcal{O}_1 & \xrightarrow{X} & \mathcal{O}_2 & \xrightarrow{A} & \mathcal{O}_2,
\end{array} \tag{21}$$

is commutative.

Similarly, a solution  $A, X, B$  of (2) is Euclidean if and only if there exist orbifolds  $\mathcal{O}_1, \mathcal{O}_2$  of zero Euler characteristic whose underlying surface is the Riemann sphere and a holomorphic function  $\tau \in \mathcal{E}(\tilde{\Gamma}_{\mathcal{O}_1}, \tilde{\Gamma}_{\mathcal{O}_1}) \cap \mathcal{E}(\tilde{\Gamma}_{\mathcal{O}_2}, \tilde{\Gamma}_{\mathcal{O}_2})$  such that the diagram (21) is commutative. Indeed, since  $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = 0$  the rational functions  $A, X, B$  are covering maps between corresponding orbifolds by Proposition 2.2. In particular,  $\tau$  is an automorphism of  $\mathbb{C}$  and therefore is a linear function in correspondence with the definition given in the introduction.

Further, a solution  $A, B, C, D$  of (3) is geometric if and only if there exist orbifolds  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}'_1, \mathcal{O}'_2$  whose underlying surface is the Riemann sphere, and a holomorphic function  $\tau \in \mathcal{E}(\tilde{\Gamma}_{\mathcal{O}_1}, \tilde{\Gamma}_{\mathcal{O}_2}) \cap \mathcal{E}(\tilde{\Gamma}_{\mathcal{O}'_1}, \tilde{\Gamma}_{\mathcal{O}'_2})$  such that the diagram

$$\begin{array}{ccccc}
\tilde{\mathcal{O}}_1 & \xrightarrow{\tau} & & & \tilde{\mathcal{O}}_2 \\
& \searrow \tilde{\theta}_{\mathcal{O}_1} & & & \swarrow \tilde{\theta}_{\mathcal{O}_2} \\
& & \mathcal{O}_1 & \xrightarrow{A} & \mathcal{O}_2 \\
& & \uparrow C & & \uparrow D \\
& & \mathcal{O}'_1 & \xrightarrow{B} & \mathcal{O}'_2 \\
& \swarrow \tilde{\theta}_{\mathcal{O}'_1} & & & \searrow \tilde{\theta}_{\mathcal{O}'_2} \\
\tilde{\mathcal{O}}'_1 & \xrightarrow{\tau} & & & \tilde{\mathcal{O}}'_2 \\
& \uparrow id & & & \uparrow id
\end{array}$$

is commutative.

Finally, the Ritt-Eremenko theorem in the above notation is equivalent to the following statement: rational functions  $A$  and  $X$  with no iteration in common commute if and only if there exists an orbifold  $\mathcal{O} = (R, \nu)$  with zero Euler characteristic such that  $R$  is either  $\mathbb{C} \setminus \{0\}$ , or  $\mathbb{C}$ , or  $\mathbb{C}\mathbb{P}^1$  (for any such an

orbifold  $\tilde{R} = \mathbb{C}$ ) and linear functions  $\tau_1, \tau_2 \in \mathcal{E}(\tilde{\Gamma}_\mathcal{O}, \tilde{\Gamma}_\mathcal{O})$  such that the diagram

$$\begin{array}{ccccccc}
\tilde{\mathcal{O}} & \xrightarrow{\tau_1} & \tilde{\mathcal{O}} & \xrightarrow{\tau_2} & \tilde{\mathcal{O}} & \xrightarrow{\tau_1} & \tilde{\mathcal{O}} \\
\downarrow \tilde{\theta}_\mathcal{O} & & \downarrow \tilde{\theta}_\mathcal{O} & & \downarrow \tilde{\theta}_\mathcal{O} & & \downarrow \tilde{\theta}_\mathcal{O} \\
\mathcal{O} & \xrightarrow{A} & \mathcal{O} & \xrightarrow{X} & \mathcal{O} & \xrightarrow{A} & \mathcal{O},
\end{array} \tag{22}$$

is commutative and the equality

$$\tilde{\theta}_\mathcal{O} \circ \tau_2 \circ \tau_1 = \tilde{\theta}_\mathcal{O} \circ \tau_1 \circ \tau_2 \tag{23}$$

holds. From this point of view, the list of functions  $f$  given by Ritt coincides with the list of functions  $\theta_\mathcal{O}$  for all possible orbifolds  $\mathcal{O}$  as above, while the requirements imposed on numbers  $a, b, c, d$  reduce to the requirements that  $\tau_1, \tau_2 \in \mathcal{E}(\tilde{\Gamma}_\mathcal{O}, \tilde{\Gamma}_\mathcal{O})$  and equality (23) holds.

Point out that in distinction with Theorem 1.1 in the formulation of the Ritt-Eremenko theorem only one orbifold  $\mathcal{O}$  appears. On the other hand, the underlying surface  $R$  of  $\mathcal{O}$  is not necessary  $\mathbb{CP}^1$  but may be also  $\mathbb{C} \setminus \{0\}$  or  $\mathbb{C}$ . However, up to a conjugacy the only pairs of commuting rational functions obtained from orbifolds whose underlying surface is  $\mathbb{C} \setminus \{0\}$  or  $\mathbb{C}$  are the pairs  $z^n, z^m$  or  $T_n, T_m$  correspondingly.

### 3 Equation $f \circ p = g \circ q$ and fiber products

In this section we recall, mostly without proofs, some general results related to the functional equation

$$h = f \circ p = g \circ q, \tag{24}$$

where  $h : R \rightarrow \mathbb{CP}^1$ ,  $p : R \rightarrow C_1$ ,  $f : C_1 \rightarrow \mathbb{CP}^1$ ,  $q : R \rightarrow C_2$ ,  $g : C_2 \rightarrow \mathbb{CP}^1$  are holomorphic functions on compact Riemann surfaces. For more details we refer the reader to [13], Section 2 and 3.

Let  $h : R \rightarrow \mathbb{CP}^1$  be a holomorphic function on a compact Riemann surface  $R$  and  $\mathcal{C}(h) = \{z_1, z_2, \dots, z_r\}$  the set of critical values of  $h$ . Fix a point  $z_0 \in \mathbb{CP}^1 \setminus \mathcal{C}(h)$  and small loops  $\gamma_i$  around  $z_i$ ,  $1 \leq i \leq r$ , such that  $\gamma_1 \gamma_2 \dots \gamma_r = 1$  in  $\pi_1(\mathbb{CP}^1 \setminus \mathcal{C}(h), z_0)$ . Denote by  $\delta_i$ ,  $1 \leq i \leq r$ , a permutation of points of  $h^{-1}\{z_0\}$  induced by the lifting of  $\gamma_i$ ,  $1 \leq i \leq r$ , by  $h$ , and by  $G_h$  the permutation group generated by  $\delta_i$ ,  $1 \leq i \leq r$ . The group  $G_h$  is called the monodromy group of  $h$ . Clearly,

$$\delta_1 \delta_2 \dots \delta_r = 1 \tag{25}$$

in  $G_h$ .

Recall that the group  $G_h$  is related to compositional properties of the function  $h$  as follows. If the function  $h$  can be decomposed into a composition  $h = f \circ p$  of holomorphic functions  $p : R \rightarrow C_1$  and  $f : C_1 \rightarrow \mathbb{CP}^1$ , where  $\deg f = d$ , then the group  $G_h$  has an imprimitivity system consisting of  $d$  blocks  $\mathcal{A}_i = p^{-1}\{t_i\}$ ,  $1 \leq i \leq d$ , where  $\{t_1, t_2, \dots, t_d\} = f^{-1}\{z_0\}$ , and the permutation group induced by the action of  $G_h$  on these blocks is permutation isomorphic to

the group  $G_f$ . Furthermore, any imprimitivity system of  $G_h$  arises from a decomposition of  $h$  and decompositions  $h = f \circ p$  and  $h = g \circ q$ , where  $q : R \rightarrow C_2$ ,  $g : C_2 \rightarrow \mathbb{CP}^1$ , lead to the same imprimitivity system if and only there exists an isomorphism  $\mu : C_2 \rightarrow C_1$  such that

$$f = g \circ \mu^{-1}, \quad p = \mu \circ q.$$

In the last case the decompositions  $h = f \circ p$  and  $h = g \circ q$  are called equivalent. Abusing of notation, usually we will mean by a decomposition a corresponding equivalence class of decompositions.

We will say that two holomorphic functions  $p : R \rightarrow C_1$  and  $q : R \rightarrow C_2$  have no non-trivial compositional common right factor, if the equalities

$$p = \tilde{p} \circ w, \quad q = \tilde{q} \circ w,$$

where  $w : R \rightarrow \tilde{R}$ ,  $\tilde{p} : \tilde{R} \rightarrow C_1$ ,  $\tilde{q} : \tilde{R} \rightarrow C_2$  are holomorphic functions, imply that  $\deg w = 1$ .

**Theorem 3.1.** *For any two fixed holomorphic functions  $f : C_1 \rightarrow \mathbb{CP}^1$  and  $g : C_2 \rightarrow \mathbb{CP}^1$  there exist holomorphic functions  $h_j : R_j \rightarrow \mathbb{CP}^1$  (components of the fiber product of  $f$  and  $g$ ) and  $p_j : R_j \rightarrow C_1$ ,  $q_j : R_j \rightarrow C_2$  such that the following conditions are satisfied:*

- $h_j = f \circ p_j = g \circ q_j$ ,
- $\sum_j \deg h_j = \deg f \deg g$ ,
- for any solution  $h, p, q$  of (24) there exist an index  $j$  and a holomorphic function  $w : R \rightarrow R_j$  such that  $h = h_j \circ w$ ,  $p = p_j \circ w$ ,  $q = q_j \circ w$ .  $\square$

Recall briefly the construction of the functions  $h_j$ . Set  $n = \deg f$ ,  $m = \deg g$ , and denote by  $S = \{z_1, z_2, \dots, z_r\}$  the union of  $\mathcal{C}(f)$  and  $\mathcal{C}(g)$ . As above, fix a point  $z_0$  from  $\mathbb{CP}^1 \setminus S$ , small loops  $\gamma_i$  around  $z_i$ ,  $1 \leq i \leq r$ , such that  $\gamma_1 \gamma_2 \dots \gamma_r = 1$  in  $\pi_1(\mathbb{CP}^1 \setminus S, z_0)$ , and for  $i$ ,  $1 \leq i \leq r$ , denote by  $\alpha_i \in S_n$  (resp.  $\beta_i \in S_m$ ) a permutation of points of  $f^{-1}\{z_0\}$  (resp. of  $g^{-1}\{z_0\}$ ) induced by the lifting of  $\gamma_i$  by  $f$  (resp.  $g$ ). Clearly, the permutations  $\alpha_i$  (resp.  $\beta_i$ ),  $1 \leq i \leq r$ , generate the monodromy group of  $f$  (resp. of  $g$ ) and

$$\alpha_1 \alpha_2 \dots \alpha_r = 1, \quad \beta_1 \beta_2 \dots \beta_r = 1. \quad (26)$$

Define now permutations  $\delta_1, \delta_2, \dots, \delta_r \in S_{nm}$  as follows: consider the set of  $mn$  elements  $c_{j_1, j_2}$ ,  $1 \leq j_1 \leq n$ ,  $1 \leq j_2 \leq m$ , and set  $(c_{j_1, j_2})^{\delta_i} = c_{j'_1, j'_2}$ , where

$$j'_1 = j_1^{\alpha_i}, \quad j'_2 = j_2^{\beta_i}, \quad 1 \leq i \leq r.$$

It is convenient to consider  $c_{j_1, j_2}$ ,  $1 \leq j_1 \leq n$ ,  $1 \leq j_2 \leq m$ , as elements of a  $n \times m$  matrix  $M$ . Then the action of the permutation  $\delta_i$ ,  $1 \leq i \leq r$ , reduces to the permutation of rows of  $M$  in accordance with the permutation  $\alpha_i$  and the permutation of columns of  $M$  in accordance with the permutation  $\beta_i$ .



In general, the permutation group generated by  $\delta_i$ ,  $1 \leq i \leq r$ , is not transitive on the set  $c_{j_1, j_2}$ ,  $1 \leq j_1 \leq n$ ,  $1 \leq j_2 \leq m$ . However, on each transitivity set  $U_j$  the induced permutations  $\delta_i(j)$ ,  $1 \leq i \leq r$ , satisfy the equality

$$\delta_1(j)\delta_2(j)\dots\delta_r(j) = 1.$$

By the Riemann existence theorem this implies that there exist compact Riemann surfaces  $R_j$  and holomorphic functions  $h_j : R_j \rightarrow \mathbb{CP}^1$  non ramified outside of  $S$  such that the permutations  $\delta_i(j)$ ,  $1 \leq i \leq r$ , are induced by the lifting of  $\gamma_i$  by  $h_j$ . Moreover, it is easy to see by construction that the intersections of the transitivity set  $U_j$  with the rows of  $M$  form an imprimitivity system  $\Omega_f(j)$  for the group generated by  $\delta_i(j)$ ,  $1 \leq i \leq r$ , such that the permutations of blocks of  $\Omega_f(j)$  induced by  $\delta_i(j)$ ,  $1 \leq i \leq r$ , coincide with  $\alpha_i$ . Similarly, the intersections of  $U_j$  with the columns of  $M$  form an imprimitivity system  $\Omega_g(j)$  such that the permutations of blocks of  $\Omega_g(j)$  induced by  $\delta_i(j)$ ,  $1 \leq i \leq r$ , coincide with  $\beta_i$ . Therefore,  $h_j = f \circ p_j = g \circ q_j$  for some functions  $p_j$  and  $q_j$ .

Clearly,

$$\deg_z h_j = \deg_z p_j \deg_{p_j(z)} f = \deg_z q_j \deg_{q_j(z)} g. \quad (27)$$

On the other hand, the definition of the permutations  $\delta_i$ ,  $1 \leq i \leq r$ , yields that for any  $z \in R_j$  the equality

$$\deg_z h_j = \text{LCM}(\deg_{p_j(z)} f, \deg_{q_j(z)} g) \quad (28)$$

holds. Therefore, Theorem 3.1 implies the following statement.

**Lemma 3.1.** *If  $h, f, p, g, q$  is a solution of (24) such that  $p$  and  $q$  have no non-trivial common compositional right factor, then for any  $z \in R$  the local degrees  $\deg_z p$  and  $\deg_z q$  are coprime.*  $\square$

Below we will study a class of solutions of equation (24) which is defined as follows. Let  $h, f, p, g, q$  be a solution of (24), and  $\mathcal{A}$  and  $\mathcal{B}$  imprimitivity systems of the group  $G_h$  corresponding to the decompositions  $h = f \circ p$  and  $h = g \circ q$  respectively. We say that the solution  $h, f, p, g, q$  is *good* if any block of  $\mathcal{A}$  intersects with any block of  $\mathcal{B}$  and this intersection consists of a unique element. Clearly, this is equivalent to the requirement that the fiber product of  $f$  and  $g$  has a unique component and  $p$  and  $q$  have no non-trivial common compositional right factor. Furthermore, the above construction implies easily the following statement.

**Lemma 3.2.** *A solution  $h, f, p, g, q$  of (24) is good whenever any two of the following three conditions are satisfied:*

- the fiber product of  $f$  and  $g$  has a unique component,
- $p$  and  $q$  have no non-trivial common compositional right factor,
- $\deg f = \deg q, \quad \deg g = \deg p.$   $\square$

If  $f$  and  $g$  are rational functions on the Riemann sphere, then the fiber product of  $f$  and  $g$  has a unique component if and only if the algebraic curve  $f(x) - g(y) = 0$  is irreducible (see e.g. [13], Proposition 2.4). Therefore, it follows from Lemma 3.2 that the definitions of good solutions for equations (3) and (24) are compatible. Besides, Lemma 3.2 implies that a solution (2) is primitive if and only if, considered as a solution of (3), it is a good solution.

Finally, let us mention the following property of good solutions of (24).

**Lemma 3.3.** *Let  $h, f, p, g, q$  be a good solution of (24),  $z_1$  a point from the set  $g^{-1}\{z_0\}$ , and  $\sigma \in G_h$  a permutation which maps the set  $q^{-1}\{z_1\}$  to itself. Then the permutation induced by  $\sigma$  on  $q^{-1}\{z_1\}$  has the same cyclic structure as the permutation induced by  $\sigma$  on blocks  $\mathcal{A}_i = p^{-1}\{t_i\}$ ,  $1 \leq i \leq d$ , where  $\{t_1, t_2, \dots, t_d\} = f^{-1}\{z_0\}$ .*

*Proof.* Since the set  $q^{-1}\{z_1\}$  is a block, it follows from the definition of a good solution that there is a natural one-to-one correspondence between the elements of  $q^{-1}\{z_1\}$  and the blocks  $\mathcal{A}_i$ ,  $1 \leq i \leq d$ . Furthermore, the action of any  $\sigma \in G_h$  which maps  $q^{-1}\{z_1\}$  to itself obviously respects this correspondence.  $\square$

## 4 Equation $f \circ p = g \circ q$ and quasi-covering maps

With each holomorphic function  $f : R_1 \rightarrow R_2$  between compact Riemann surfaces one can associate naturally two orbifolds  $\mathcal{O}_1^f = (R_1, \nu_1^f)$  and  $\mathcal{O}_2^f = (R_2, \nu_2^f)$ , where  $\nu_2^f(z)$  is equal to the least common multiple of local degrees of  $f$  at the points of the preimage  $f^{-1}\{z\}$  and  $\nu_1^f(z)$  is equal to  $\nu_2^f(f(z))/\deg_z f$ . Clearly, by construction,  $f$  is a covering map between orbifolds  $f : \mathcal{O}_1^f \rightarrow \mathcal{O}_2^f$ .

**Lemma 4.1.** *Orbifolds  $\mathcal{O}_1^f$  and  $\mathcal{O}_2^f$  have a universal cover.*

*Proof.* Indeed, equality (25) implies that  $f$  may not have only one critical value, and that if  $f$  has two critical values, then, by transitivity of  $G_h$ , the corresponding permutations have the same order equal to  $\deg f$ . Therefore,  $\mathcal{O}_2^f$  has a universal cover.

Let  $\tilde{\theta} : \tilde{\mathcal{O}} \rightarrow \mathcal{O}_2^f$  be a universal cover of  $\mathcal{O}_2^f$  and  $\hat{\theta}$  the complete analytic continuation of the germ  $(f^{-1} \circ \tilde{\theta}) : \tilde{\mathcal{O}} \rightarrow \mathcal{O}_1^f$ , where  $f^{-1}$  is a branch of the algebraic function inverse to  $f$ . It is easy to see that the equalities

$$\deg_z \tilde{\theta} = \nu_2^f(\tilde{\theta}(z)), \quad z \in \tilde{\mathcal{O}}, \quad (29)$$

and

$$\nu_2^f(f(z)) = \nu_1^f(z) \deg_z f, \quad z \in \mathcal{O}_1^f, \quad (30)$$

imply that  $\hat{\theta}$  is single valued. Moreover, since  $f \circ \hat{\theta} = \tilde{\theta}$  and hence

$$\deg_z \hat{\theta} \deg_{\hat{\theta}(z)} f = \deg_z \tilde{\theta},$$

equalities (29) and (30) yield the equality  $\nu_1^f(\hat{\theta}(z)) = \deg_z \hat{\theta}$  implying that  $\hat{\theta} : \tilde{\mathcal{O}} \rightarrow \mathcal{O}_1^f$  is a universal cover of  $\mathcal{O}_1^f$ .  $\square$

Let  $R_1, R_2$  be Riemann surfaces provided with ramification functions  $\nu_1, \nu_2$ , and  $f : R_1 \rightarrow R_2$  a holomorphic branched covering map. Say that  $f$  is a *quasi-covering map between orbifolds*  $\mathcal{O}_1 = (R_1, \nu_1)$  and  $\mathcal{O}_2 = (R_2, \nu_2)$  if for any  $z \in R_1$  the equality

$$\nu_2(f(z)) = \nu_1(z) \text{GCD}(\deg_z f, \nu_2(f(z))) \quad (31)$$

holds. Clearly, any covering map  $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  between orbifolds is a quasi-covering map and any quasi-covering map is a holomorphic map. Notice also that a quasi-covering map between orbifolds maps any ramified point of  $\mathcal{O}_1$  to a ramified point of  $\mathcal{O}_2$ .

**Theorem 4.1.** *Let  $h, f, p, g, q$  be a good solution of equation (24). Then  $g$  is a quasi-covering map between orbifolds  $g : \mathcal{O}_2^g \rightarrow \mathcal{O}_2^f$ , while  $p$  is a quasi-covering map between orbifolds  $p : \mathcal{O}_1^q \rightarrow \mathcal{O}_1^f$ .*

*Proof.* Let  $z \in C_2$  be a point,  $\rho \subset C_2$  a small free loop around  $z$ , and  $z_1 \in \rho$  a point such that  $g(z_1) = z_0$  is a regular value of  $h$ . Then the permutation of points of  $h^{-1}\{z_0\}$  corresponding to the analytic continuation of  $h^{-1}$  along the curve  $g(\rho) \subset \mathbb{C}\mathbb{P}^1$  induces a permutation  $\sigma_1$  of points of  $q^{-1}\{z_1\}$  as well as a permutation  $\sigma_2$  of points of  $f^{-1}\{z_0\}$ . Furthermore, by Lemma 3.3 the permutations  $\sigma_1$  and  $\sigma_2$  have the same cyclic structure. In particular, they have the same order equal to  $\nu_2^q(z)$ .

On the other hand, since  $g$  is modeled locally as  $w \rightarrow w^{\deg_z g}$ , the permutation  $\sigma_2$  has the same cyclic structure as the permutation  $\sigma_3^{\deg_z g}$ , where  $\sigma_3$  is the permutation of points  $f^{-1}\{z_0\}$  induced by the analytic continuation of  $h^{-1}$  along a small free loop  $\tilde{\rho}$  around  $g(z)$ . Since the order of  $\sigma_3$  is equal to  $\nu_2^f(g(z))$ , this yields that the order of  $\sigma_2$  is equal to  $\nu_2^f(g(z))/\text{GCD}(\deg_z g, \nu_2^f(g(z)))$  implying the equality

$$\nu_2^f(g(z)) = \nu_2^q(z) \text{GCD}(\deg_z g, \nu_2^f(g(z))). \quad (32)$$

In order to prove the second part of the theorem substitute  $z = q(z)$  in (32). We obtain the equality

$$\nu_2^f((f \circ p)(z)) = \nu_2^q(q(z)) \text{GCD}(\deg_{q(z)} g, \nu_2^f((f \circ p)(z)))$$

implying the equality

$$\nu_1^f(p(z)) \deg_{p(z)} f = \nu_1^q(z) \deg_z q \text{GCD}(\deg_{q(z)} g, \nu_1^f(p(z)) \deg_{p(z)} f). \quad (33)$$

Since for any integer numbers  $a, b, c$  the equality  $\text{GCD}(ac, bc) = c \text{GCD}(a, b)$  holds, it follows from (27) that

$$\begin{aligned} & \deg_z q \text{GCD}(\deg_{q(z)} g, \nu_1^f(p(z)) \deg_{p(z)} f) = \\ & \text{GCD}(\deg_{q(z)} g \deg_z q, \nu_1^f(p(z)) \deg_{p(z)} f \deg_z q) = \end{aligned}$$

$$\begin{aligned} \text{GCD} \left( \deg_{p(z)} f \deg_z p, \nu_1^f(p(z)) \deg_{p(z)} f \deg_z q \right) = \\ \deg_{p(z)} f \text{GCD} \left( \deg_z p, \nu_1^f(p(z)) \deg_z q \right). \end{aligned}$$

Now (33) implies that

$$\nu_1^f(p(z)) = \nu_1^q(z) \text{GCD} \left( \deg_z p, \nu_1^f(p(z)) \deg_z q \right).$$

Finally, since by Lemma 3.1 the equality  $\text{GCD}(\deg_z p, \deg_z q) = 1$  holds, we arrive to the equality

$$\nu_1^f(p(z)) = \nu_1^q(z) \text{GCD} \left( \deg_z p, \nu_1^f(p(z)) \right). \quad \square$$

## 5 Proofs of the main theorems

*Proof of Theorem 1.1.* Since a primitive solution  $A, X, B$  of (2) considered as a solution of (24) for

$$f = q = X, \quad p = B, \quad g = A$$

is good, it follows from Theorem 4.1 that  $A : \mathcal{O}_2^X \rightarrow \mathcal{O}_1^X$  and  $B : \mathcal{O}_1^X \rightarrow \mathcal{O}_1^X$  are quasi-covering maps between orbifolds. Furthermore, by Corollary 2.1 the inequalities  $\chi(\mathcal{O}_1) \geq 0, \chi(\mathcal{O}_2) \geq 0$  hold. Therefore, setting  $\mathcal{O}_1 = \mathcal{O}_1^X, \mathcal{O}_2 = \mathcal{O}_2^X$  we obtain the commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{O}}_1 & \xrightarrow{\alpha} & \widetilde{\mathcal{O}}_2 & & \\ \downarrow \widetilde{\theta}_{\mathcal{O}_1} & & \downarrow \widetilde{\theta}_{\mathcal{O}_2} & & \\ \mathcal{O}_1 & \xrightarrow{B} & \mathcal{O}_1 & \xrightarrow{X} & \mathcal{O}_2 \xrightarrow{A} \mathcal{O}_2, \end{array} \quad (34)$$

where  $X$  is a covering map between orbifolds and  $\alpha$  is an isomorphism. Since changing if necessary  $\widetilde{\theta}_{\mathcal{O}_1}$  without loss of generality we may assume that  $\alpha = id$ , it follows from Proposition 2.1 that there exist  $\tau \in \mathcal{E}(\widetilde{\Gamma}_{\mathcal{O}_1}, \widetilde{\Gamma}_{\mathcal{O}_1})$  and  $\widetilde{\tau} \in \mathcal{E}(\widetilde{\Gamma}_{\mathcal{O}_2}, \widetilde{\Gamma}_{\mathcal{O}_2})$  such that the diagram

$$\begin{array}{ccccccc} \widetilde{\mathcal{O}}_1 & \xrightarrow{\tau} & \widetilde{\mathcal{O}}_1 & \xrightarrow{id} & \widetilde{\mathcal{O}}_2 & \xrightarrow{\widetilde{\tau}} & \widetilde{\mathcal{O}}_2 \\ \downarrow \widetilde{\theta}_{\mathcal{O}_1} & & \downarrow \widetilde{\theta}_{\mathcal{O}_1} & & \downarrow \widetilde{\theta}_{\mathcal{O}_2} & & \downarrow \widetilde{\theta}_{\mathcal{O}_2} \\ \mathcal{O}_1 & \xrightarrow{B} & \mathcal{O}_1 & \xrightarrow{X} & \mathcal{O}_2 & \xrightarrow{A} & \mathcal{O}_2 \end{array} \quad (35)$$

is commutative. Finally, since the equality

$$A \circ X \circ \widetilde{\theta}_{\mathcal{O}_1} = X \circ B \circ \widetilde{\theta}_{\mathcal{O}_1}$$

implies the equality

$$\widetilde{\theta}_{\mathcal{O}_2} \circ \tau = \widetilde{\theta}_{\mathcal{O}_2} \circ \widetilde{\tau}$$

there exists  $\sigma \in \widetilde{\Gamma}_{\mathcal{O}_2}$  such that  $\widetilde{\tau} = \sigma \circ \tau$ . Therefore, (35) still holds for  $\widetilde{\tau} = \tau$ .  $\square$

**Corollary 5.1.** *Up to the change  $X \rightarrow \alpha \circ X \circ \beta$ , where  $\alpha, \beta$  are Möbius transformations, apart from the series  $z^n$ ,  $T_n(z)$ , and  $\frac{1}{2}(z^n + \frac{1}{z^n})$ , there exists only a finite number of rational functions  $X$  appearing as a component of a spherical solution  $A, X, B$ .*

*Proof.* Indeed, it follows from Theorem 1.1 that if  $A, X, B$  is a spherical primitive solution of (2), then  $X$  is a compositional left factor of the function  $\tilde{\theta}_\Gamma$ , where  $\Gamma$  is a finite subgroup of  $Aut(\mathbb{CP}^1)$ . Therefore, since there exist only a finite number of subgroups of  $Aut(\mathbb{CP}^1)$  distinct from cyclic and dihedral, the statement follows from the observation that any compositional left factor of the rational functions  $\tilde{\theta}_{\mathbb{Z}_n} = z^n$  has the form  $z^m \circ \alpha$ , where  $\alpha$  is a Möbius transformation, while any compositional left factor of the rational functions  $\tilde{\theta}_{D_{2n}} = \frac{1}{2}(z^n + \frac{1}{z^n})$  has the form  $\frac{1}{2}(z^m + \frac{1}{z^m}) \circ \alpha$  or  $T_m \circ \alpha$  (see e.g. Appendix of [5]).  $\square$

*Proof of Theorem 1.2.* Setting

$$\mathcal{O}'_1 = \mathcal{O}_1^C, \quad \mathcal{O}_1 = \mathcal{O}_2^C, \quad \mathcal{O}'_2 = \mathcal{O}_1^D, \quad \mathcal{O}_2 = \mathcal{O}_2^D$$

and arguing as above we obtain the commutative diagram

$$\begin{array}{ccccc}
 \tilde{\mathcal{O}}_1 & \xrightarrow{\tilde{\tau}} & & & \tilde{\mathcal{O}}_2 \\
 & \searrow \tilde{\theta}_{\mathcal{O}_1} & & & \swarrow \tilde{\theta}_{\mathcal{O}_2} \\
 & & \mathcal{O}_1 & \xrightarrow{A} & \mathcal{O}_2 \\
 & & \uparrow C & & \uparrow D \\
 id & & \mathcal{O}'_1 & \xrightarrow{B} & \mathcal{O}'_2 & id \\
 & \swarrow \tilde{\theta}_{\mathcal{O}'_1} & & & \nwarrow \tilde{\theta}_{\mathcal{O}'_2} \\
 \tilde{\mathcal{O}}'_1 & \xrightarrow{\tau} & & & \tilde{\mathcal{O}}'_2
 \end{array}$$

where  $\tau \in \mathcal{E}(\tilde{\Gamma}_{\mathcal{O}_1}, \tilde{\Gamma}_{\mathcal{O}_2})$  and  $\tilde{\tau} \in \mathcal{E}(\tilde{\Gamma}_{\mathcal{O}'_1}, \tilde{\Gamma}_{\mathcal{O}'_2})$ . Furthermore, it follows from

$$A \circ C \circ \tilde{\theta}_{\mathcal{O}'_1} = B \circ D \circ \tilde{\theta}_{\mathcal{O}_1}$$

that

$$\tilde{\theta}_{\mathcal{O}_2} \circ \tilde{\tau} = \tilde{\theta}_{\mathcal{O}_2} \circ \tau. \quad \square$$

*Proof of the Ritt-Eremenko theorem.* Since pairs of commuting rational functions in the Ritt-Eremenko theorem, obtained from orbifolds whose underlying surface is  $\mathbb{C} \setminus \{0\}$  or  $\mathbb{C}$ , reduce correspondingly to the pairs  $z^n, z^m$  or  $T_n, T_m$ , it is enough to prove the following two statements. First, that if (21) holds for  $A = B$ , where  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are orbifolds of zero Euler characteristic whose underlying surface is the Riemann sphere, then  $\mathcal{O}_1 = \mathcal{O}_2$ ; in this case  $\tilde{\theta}_{\mathcal{O}_2} = \tilde{\theta}_{\mathcal{O}_1} \circ \alpha$  for some linear function  $\alpha$  and diagram (21) reduces to diagram (22), where

$$\tilde{\theta}_{\mathcal{O}} = \tilde{\theta}_{\mathcal{O}_1}, \quad \tau_1 = \tau, \quad \tau_2 = \alpha, \quad \tau_3 = \alpha^{-1} \circ \tau \circ \alpha.$$

Second, that if (21) holds for  $A = B$ , where  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are orbifolds of positive Euler characteristic whose underlying surface is the Riemann sphere, then up to a conjugacy the pair  $A, X$  is either the pair  $z^n, z^m$  or the pair  $T_n, T_m$ .

In order to prove the first statement observe that if  $\mathcal{O}_1 = (\mathbb{CP}^1, \nu_1)$  and  $\mathcal{O}_2 = (\mathbb{CP}^1, \nu_2)$  are orbifolds, and  $A$  is a rational function such that  $A : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  is a covering map between orbifolds, then for all  $x \in \mathbb{CP}^1$  the equalities

$$\nu_1(x) = c_x \nu_1^A(x), \quad \nu_2(f(x)) = c_x \nu_2^A(f(x))$$

hold, where  $c_x$  are some natural numbers satisfying the equality  $c_{x_1} = c_{x_2}$  whenever  $f(x_1) = f(x_2)$ . Furthermore, it is easy to see that if  $\mathcal{O} = (\mathbb{CP}^1, \nu)$  is an orbifold of zero Euler characteristic, then for any orbifold  $\tilde{\mathcal{O}} = (\mathbb{CP}^1, \tilde{\nu})$  such that  $\tilde{\nu}(x) = c_x \nu(x)$ , the Euler characteristic of  $\tilde{\mathcal{O}}$  is negative unless all  $c_x$  are equal to one. Therefore, if (21) holds for  $A = B$ , where  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are orbifolds of zero Euler characteristic, then

$$\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_1^A = \mathcal{O}_2^A$$

since the equality  $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = 0$  implies that both maps  $A : \mathcal{O}_1 \rightarrow \mathcal{O}_1$  and  $A : \mathcal{O}_2 \rightarrow \mathcal{O}_2$  are covering maps between orbifolds.

Let us prove now the second statement. Recall that by construction ramified points of  $\mathcal{O}_2 = \mathcal{O}_2^X$  coincide with critical values of  $X$ . Assume first that the collection of ramification indices of  $\mathcal{O}_2 = \mathcal{O}_2^X$  is  $(d, d)$ . Then up to the change  $X \rightarrow \alpha \circ X \circ \beta$ , where  $\alpha, \beta$  are Möbius transformations,  $X = z^n$ , where  $n|d$ . By symmetry between  $X$  and  $A$ , the function  $X$  is a quasi-covering map between orbifolds  $\mathcal{O}_2^A$  and  $\mathcal{O}_2^A$  of positive Euler characteristic implying easily that either the collection of ramification indices of  $\mathcal{O}_2^A$  is  $(m, m)$ , where  $\text{GCD}(m, n) = 1$ , or  $n = 2$  and the collection of ramification indices of  $\mathcal{O}_2^A$  is  $(2, 2, m)$ , where  $m$  is odd; in the first case pairs of commuting rational functions obtained from diagram (21) reduce to pairs of powers while in the second case to pairs  $T_2, T_m$ .

Further, if the collection of ramification indices of  $\mathcal{O}_2 = \mathcal{O}_2^X$  is  $(2, 2, d)$ , then without loss of generality we may assume that either  $X = \frac{1}{2} \left( z^n + \frac{1}{z^n} \right)$ , or  $X = T_n$ , where  $n|d$  (see the proof of Corollary 5.1). Moreover, as above  $X$  is a quasi-covering map between orbifolds  $\mathcal{O}_2^A$  and  $\mathcal{O}_2^A$  of positive Euler characteristic, and by symmetry we can assume that the collection of ramification indices of  $\mathcal{O}_2^A$  is distinct from  $(d, d)$ . It is not hard to check that  $\frac{1}{2} \left( z^n + \frac{1}{z^n} \right)$  may not be such a quasi-covering while  $T_n$  is such a quasi-covering if and only if the collection of ramification indices of  $\mathcal{O}_2^A$  is  $(2, 2, m)$ , where  $\text{GCD}(n, m) = 1$ . This implies that if the collection of ramification indices of  $\mathcal{O}_2 = \mathcal{O}_2^X$  is  $(2, 2, d)$ , then pairs of commuting rational functions obtained from diagram (21) reduce to pairs of Chebyshev polynomials.

Finally, assuming that the collection of ramification indices of  $\mathcal{O}_2 = \mathcal{O}_2^X$  is either  $(2, 3, 3)$ , or  $(2, 3, 4)$ , or  $(2, 3, 5)$  (by symmetry we can assume that the same is true for  $\mathcal{O}_2^A$ ), it is easy to obtain a contradiction between the condition that  $X$  is a covering map between orbifolds  $\mathcal{O}_1^X$  and  $\mathcal{O}_2^X$  and the condition that  $X$  is a quasi-covering map between orbifolds  $\mathcal{O}_2^A$  and  $\mathcal{O}_2^A$ . Indeed, formula (31) implies that if  $f$  is a quasi-covering map between orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , then  $\mathcal{O}_1$

is defined in a unique way by  $\mathcal{O}_2$  and  $f$ . Therefore,  $\mathcal{O}_2^A$  may not coincide with  $\mathcal{O}_2^X$  for otherwise  $\mathcal{O}_1^X$  would coincide with  $\mathcal{O}_2^X$  in contradiction with (13). In particular, switching if necessary between  $A$  and  $X$  without loss of generality we may assume that the collection of ramification indices of  $\mathcal{O}_2^A$  is either  $(2, 3, 4)$ , or  $(2, 3, 5)$ . It is easy to see that in the first (resp. in the second) case the condition that  $X$  is a quasi-covering map between orbifolds  $\mathcal{O}_2^A$  and  $\mathcal{O}_2^A$  implies that there exists a point  $z \in \mathbb{CP}^1$  where the local multiplicity of  $X$  is divisible by 4 (resp. by 5). On the other hand, the condition that  $X$  is a covering map between orbifolds  $\mathcal{O}_1^X$  and  $\mathcal{O}_2^X$  with  $\mathcal{O}_2^X \neq \mathcal{O}_2^A$  implies that every local multiplicity of  $X$  in the first (resp. in the second) case is equal to 2, 3, or 5 (resp. to 2, 3 or 4).  $\square$

## 6 Further properties of spherical solutions and examples

In this section we give a characterization of quasi-covering maps  $f : \mathcal{O} \rightarrow \mathcal{O}$ , where  $\mathcal{O} = (R, \nu)$  is an orbifold with  $R = \mathbb{CP}^1$  and  $\chi(\mathcal{O}) > 0$ , and give several explicit examples of primitive spherical solutions of (2).

**Lemma 6.1.** *Let  $\mathcal{O} = (R, \nu)$  be an orbifold of positive Euler characteristic with  $R = \mathbb{CP}^1$ , and  $S$  the set of ramified points of  $\mathcal{O}$ . Then for any quasi-covering map between orbifolds  $f : \mathcal{O} \rightarrow \mathcal{O}$  and any  $s \in S$  the equality  $\nu(f(z)) = \nu(z)$  holds. Equivalently, for any  $s \in S$  the numbers  $\deg_z f$  and  $\nu(f(z))$  are coprime.*

*Proof.* Assume for example that the collection of ramification indices of  $\mathcal{O}$  is  $(2, 3, 5)$  and denote the points  $z \in \mathbb{CP}^1$ , where  $\nu(z)$  equals 2, 3, and 5, correspondingly by  $z_1$ ,  $z_2$ , and  $z_3$ . In this notation it follows directly from (31) that

$$f(z_3) = z_3, \quad f(z_2) = z_2, \quad f(z_1) = z_1 \quad (36)$$

and therefore the lemma is true. If the ramification collection of  $\mathcal{O}$  is  $(2, 3, 3)$ , then, keeping the notation introduced above, we see that either (36) or

$$f(z_3) = z_2, \quad f(z_2) = z_3, \quad f(z_1) = z_1 \quad (37)$$

holds implying that the lemma is true also in this case. Similarly, the lemma is true for orbifolds with ramification collections  $(n, n)$  for any  $n \geq 2$ , and  $(2, 2, n)$  for  $n$  equals 2 or odd.

If the collection of ramification indices of  $\mathcal{O}$  is  $(2, 3, 4)$ , then as above we conclude that either (36) or

$$f(z_3) = z_3, \quad f(z_2) = z_2, \quad f(z_1) = z_3 \quad (38)$$

holds. Let us show that (38) is impossible. Assume the inverse. Then (31) implies that

$$\text{GCD}(\deg_{z_1} f, 4) = 2, \quad \text{GCD}(\deg_{z_3} f, 4) = 1,$$

while at points of  $f^{-1}\{z_3\}$  distinct from  $z_1$ ,  $z_3$  the local multiplicity of  $f$  is divisible by 4. Therefore,  $\deg f$  is odd. On the other hand, since  $S \cap f^{-1}\{z_1\} = \emptyset$ ,

equality (31) implies that the local multiplicity of  $f$  at any point of  $f^{-1}\{z_1\}$  is even implying that  $\deg f$  is even, and the contradiction obtained proves the lemma in this case.

Assume finally that the collection of ramification indices of  $\mathcal{O}$  is  $(2, 2, n)$ , where  $n > 2$  is even. In order to prove the lemma in this case we only must show that  $f\{z_1, z_2\} \subseteq \{z_1, z_2\}$ . Suppose say that  $f^{-1}\{z_1\} \cap \{z_1, z_2\} = \emptyset$ . Then  $\deg f$  must be even implying that either

$$f^{-1}\{z_2\} \cap \{z_1, z_2\} = \{z_1, z_2\}, \quad \text{or} \quad f^{-1}\{z_2\} \cap \{z_1, z_2\} = \emptyset,$$

since otherwise  $\deg f$  would be odd. Furthermore, the second case is impossible. Indeed, otherwise (31) implies that

$$\text{GCD}(\deg_{z_1} f, n) = n/2, \quad \text{GCD}(\deg_{z_2} f, n) = n/2, \quad \text{GCD}(\deg_{z_3} f, n) = 1,$$

while at points of  $f^{-1}\{z_3\}$  distinct from  $z_1, z_2, z_3$  the local multiplicity of  $f$  is divisible by  $n$ . Since  $n$  is even the first two equalities imply that either both numbers  $\deg_{z_1} f, \deg_{z_2} f$  are even (if  $n/2$  is even), or they both are odd (if  $n/2$  is odd). In both cases we obtain a contradiction with the evenness of  $\deg f$ . This finishes the proof.  $\square$

**Theorem 6.1.** *Let  $\mathcal{O} = (R, \nu)$  be an orbifold of positive Euler characteristic with  $R = \mathbb{C}\mathbb{P}^1$ , and  $f : \mathcal{O} \rightarrow \mathcal{O}$  a holomorphic map between orbifolds. Then the following conditions are equivalent:*

- the map  $f$  is a quasi-covering map between orbifolds,
- the homomorphism  $\varphi$  of  $\tilde{\Gamma}_{\mathcal{O}}$  defined by the equality

$$\tau \circ \sigma = \varphi(\sigma) \circ \tau, \quad \sigma \in \tilde{\Gamma}_{\mathcal{O}}, \tag{39}$$

is an automorphism of  $\tilde{\Gamma}_{\mathcal{O}}$ ,

- the triple  $\tau, f, \tilde{\theta}_{\mathcal{O}}$  is a good solution of the equation

$$\tilde{\theta}_{\mathcal{O}} \circ \tau = f \circ \tilde{\theta}_{\mathcal{O}}. \tag{40}$$

*Proof.*  $2 \Leftrightarrow 3$  By Lemma 3.2 we only must show that the homomorphism  $\varphi$  is an automorphism if and only if the functions  $\tau$  and  $\tilde{\theta}_{\mathcal{O}}$  have no non-trivial common compositional right factor.

Assume that the group  $\text{Ker } \varphi$  is non-trivial. Since  $\text{Ker } \varphi$  is contained in a finite subgroup of the automorphism group of the sphere,  $\text{Ker } \varphi$  itself is such a subgroup and therefore coincides with  $\tilde{\Gamma}_{\hat{\mathcal{O}}}$  for some orbifold  $\hat{\mathcal{O}}$  on the Riemann sphere with  $\chi(\hat{\mathcal{O}}) > 0$ . Furthermore, equality (39) implies that  $\tau$  is a rational function in  $\tilde{\theta}_{\hat{\mathcal{O}}}$ . On the other hand, the inclusion  $\tilde{\Gamma}_{\hat{\mathcal{O}}} \subseteq \tilde{\Gamma}_{\mathcal{O}}$  implies that  $\tilde{\theta}_{\mathcal{O}}$  also is a rational function in  $\tilde{\theta}_{\hat{\mathcal{O}}}$ . Therefore,  $\tau$  and  $\tilde{\theta}_{\mathcal{O}}$  have the non-trivial common compositional right factor  $\tilde{\theta}_{\hat{\mathcal{O}}}$ .



In other direction, assume that  $\tau$  and  $\tilde{\theta}_\mathcal{O}$  have a non-trivial common compositional right factor. Since the equality  $\nu(\tilde{\theta}_\mathcal{O}(z)) = \deg_z \tilde{\theta}_\mathcal{O}$  characterizes universal coverings, it is easy to see that any compositional right factor of  $\tilde{\theta}_\mathcal{O}$  has the form  $\tilde{\theta}_{\tilde{\mathcal{O}}}$  for some orbifold  $\tilde{\mathcal{O}}$  on the Riemann sphere with  $\chi(\tilde{\mathcal{O}}) > 0$ . It follows now from the equality  $\tau = \tilde{\tau} \circ \tilde{\theta}_{\tilde{\mathcal{O}}}$ , where  $\tilde{\tau}$  is a rational function, that  $\tau$  is invariant with respect to the corresponding group  $\tilde{\Gamma}_{\tilde{\mathcal{O}}}$  implying that  $\text{Ker } \varphi$  contains the group  $\tilde{\Gamma}_{\tilde{\mathcal{O}}}$  and therefore is non-trivial.

1  $\Rightarrow$  3 As above it is enough to show that if  $f : \mathcal{O} \rightarrow \mathcal{O}$  is a quasi-covering map between orbifolds, then the functions  $\tau$  and  $\tilde{\theta}_\mathcal{O}$  have no non-trivial common compositional right factor. Assume the inverse. Then equality (40) implies the equality

$$\tilde{\theta}_\mathcal{O} \circ q = f \circ p, \quad (41)$$

where  $q$  and  $p$  are some compositional left factors of  $\tau$  and  $\tilde{\theta}_\mathcal{O}$  correspondingly, and  $\deg q < \deg \tau$ ,  $\deg p < \deg \tilde{\theta}_\mathcal{O}$ . Denote the rational function defined by any part of equality (41) by  $h$ . It follows from the equality  $h = \tilde{\theta}_\mathcal{O} \circ q$  that for any  $z \in S$ , where  $S$  as above denotes the set of ramified points of  $\mathcal{O}$ , all local multiplicities of  $h$  at the points of  $h^{-1}\{z\}$  are divisible by  $\nu(z)$ . Since, on the other hand,  $h = f \circ p$ , and for any  $s \in S$  the equalities

$$\nu(f(z)) = \nu(z), \quad \text{GCD}(\deg_z f, \nu(f(z))) = 1$$

hold by Lemma 6.1, this implies that for any  $z \in S$  all local multiplicities of  $p$  at the points of  $p^{-1}\{z\}$  are divisible by  $\nu(z)$ . Therefore, since  $p$  is a compositional left factor of  $\tilde{\theta}_\mathcal{O}$  these local multiplicities are actually equal to  $\nu(z)$ . However, in this case  $p$  must be a universal cover of  $\mathcal{O}$  in contradiction with the uniqueness of a universal covering map.

3  $\Rightarrow$  1 Follows from Theorem 4.1 applied to equality (40) taking into account that  $\mathcal{O}_{\tilde{\theta}_\mathcal{O}}^2 = \mathcal{O}$ .  $\square$

*Remark.* Since the function  $\tau$  in (39) is defined by  $f$  up to a transformation  $\tau \rightarrow g \circ \tau$ , where  $g \in \tilde{\Gamma}_\mathcal{O}$ , the corresponding automorphism  $\varphi$  in (39) is defined up to a transformation  $\varphi \rightarrow g \circ \varphi \circ g^{-1}$ . This means that if the automorphism  $\varphi$  is inner, then without loss of generality we may assume that  $\varphi(\sigma) = \sigma$  or in other words that the function  $\tau$  is  $\tilde{\Gamma}_\mathcal{O}$ -equivariant. For example, since any automorphism of  $S_4$  is inner, quasi-covering maps in the octahedral case are in a one-to-one correspondence with  $\tilde{\Gamma}_\mathcal{O}$ -equivariant functions. More generally, even if the automorphism  $\varphi$  is not inner, it follows from the finiteness of the group  $\tilde{\Gamma}_\mathcal{O}$  that an appropriate iteration of the function  $\tau$  will be  $\tilde{\Gamma}_\mathcal{O}$ -equivariant. This property of  $\tau$  may be used for further study of primitive spherical solutions of (2) since as it is shown in [2] the problem of description of  $\tilde{\Gamma}_\mathcal{O}$ -equivariant functions, where  $\mathcal{O} = (R, \nu)$  is an orbifold with  $R = \mathbb{C}\mathbb{P}^1$  and  $\chi(\mathcal{O}) > 0$ , reduces to the classical problem of description of homogeneous  $\tilde{\Gamma}_\mathcal{O}$ -invariant polynomials solved by Klein (see also [18] for implicit calculations and examples).

In conclusion let us give several explicit examples of spherical primitive solutions of equation (2). Let  $\mathcal{O}$  be an orbifold with the collection of rami-

fication indices  $(n, n)$ . Then without loss of generality we may assume that  $\nu(0) = \nu(\infty) = n$  and  $\tilde{\theta}_{\mathcal{O}} = z^n$ . Furthermore, it is easy to see that any quasi-covering map between orbifolds  $f: \mathcal{O} \rightarrow \mathcal{O}$  has the form  $f = z^r R^n(z)$ , where  $R$  is an arbitrary rational function and  $r$  is an integer such that  $\text{GCD}(r, n) = 1$ . For the corresponding function  $\tau$  the equality

$$\tau = z^r R(z^n) \quad (42)$$

holds, and solutions provided by Theorem 1.1 take the form

$$z^r R^n(z) \circ z^m = z^m \circ z^r R^{n/m}(z^m),$$

where  $m$  is a divisor of  $n$ .

Let now  $\mathcal{O}$  be an orbifold with the collection of ramification indices  $(2, 2, n)$ . We may assume that  $\nu(1) = \nu(-1) = 2$ ,  $\nu(\infty) = n$ . Then the transformations

$$\alpha: z \rightarrow e^{2\pi i/n} z, \quad \beta: z \rightarrow \frac{1}{z},$$

generate  $\tilde{\Gamma}_{\mathcal{O}}$ , and

$$\tilde{\theta}_{\mathcal{O}} = \frac{1}{2} \left( z^n + \frac{1}{z^n} \right). \quad (43)$$

Clearly, for the function  $\tau(z) = z^m$ , where  $\text{GCD}(m, n) = 1$ , equality (39) holds for some automorphism  $\varphi$ . The corresponding function  $f$  is the  $m$ th Chebyshev polynomial  $T_m$ , and taking  $X$  equal to  $\frac{1}{2} \left( z^d + \frac{1}{z^d} \right)$ , where  $d|n$ , we obtain the following well known series of primitive solutions of (2)

$$T_m \circ \frac{1}{2} \left( z^d + \frac{1}{z^d} \right) = \frac{1}{2} \left( z^d + \frac{1}{z^d} \right) \circ z^m. \quad (44)$$

On the other hand, taking  $X$  equal to another compositional factor  $T_d$ ,  $d|n$ , of  $\tilde{\theta}_{\mathcal{O}}$  we obtain the series of solutions

$$T_m \circ T_d = T_d \circ T_m.$$

The simplest case where equality (39) holds for some automorphism  $\varphi$  but the function  $\tau(z)$  does not reduce to a power is the one where  $n = 2$  and

$$\tau(z) = \frac{z^2 - 2}{z - 2z^3}.$$

In this case we obtain the solution

$$\frac{(64z^3 - 64z^2 - 23z + 24)}{(4z - 5)^2} \circ \frac{1}{2} \left( z^2 + \frac{1}{z^2} \right) = \frac{1}{2} \left( z^2 + \frac{1}{z^2} \right) \circ \left( \frac{z^2 - 2}{z - 2z^3} \right).$$

Finally, the simplest example corresponding to the orbifold with the collection of ramification indices  $(2, 3, 3)$  is the following one:

$$\left( \frac{(4x - 1)^3}{27x} \right) \circ \left( \frac{z^3(8 - z^3)^3}{64(z^3 + 1)^3} \right) = \left( \frac{z^3(8 - z^3)^3}{64(z^3 + 1)^3} \right) \circ \left( \frac{2 - z^3}{3z} \right).$$

More examples of rational solutions of equation (2) will be presented in the forthcoming paper [14].

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## References

- [1] A. Bogatyrev, *Rational functions admitting double decomposition*, arXiv:1012.4577.
- [2] P. Doyle, C. McMullen, *Solving the quintic by iteration*, Acta Math. 163 (1989), no. 3-4, 151-180.
- [3] A. Eremenko, *Invariant curves and semiconjugacies of rational functions*, arXiv:1110.6552.
- [4] A. Eremenko, *Some functional equations connected with the iteration of rational functions* (Russian), Algebra i Analiz 1 (1989), 102-116; translation in Leningrad Math. J. 1 (1990), 905-919.
- [5] M. Muzychuk, F. Pakovich, *Jordan-Hölder theorem for imprimitivity systems and maximal decompositions of rational functions*, Proc. Lond. Math. Soc. (3) 102 (2011), no. 1, 1-24.
- [6] P. Fatou, *Sur l'iteration analytique et les substitutions permutables*, J. Math. Pures Appl. (9), 2, 1923, 343-384.
- [7] G. Hällström, *Über Halbvertauschbarkeit zwischen linearen und allgemeineren rationalen Funktionen*, Math. Japon. 4, 1957, 107-112.
- [8] G. Julia, *Mémoire sur la permutabilité des fractions rationnelles*, Ann. Sci. E'cole Norm. Sup. 39 (3) (1922), 131-215.
- [9] F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*, Dover Publications, Inc., New York, N.Y., 1956. xvi+289 pp,
- [10] H. Inou, *Extending local analytic conjugacies*, Trans. Amer. Math. Soc. 363 (2011), no. 1, 331-343,
- [11] A. Masuda, M. Zieve, *Rational functions with linear relations*, Proc. Amer. Math. Soc. 136 (2008), no. 4, 1403-1408.
- [12] J. Milnor, *Dynamics in one complex variable*, Princeton Annals in Mathematics 160. Princeton, NJ: Princeton University Press (2006).
- [13] F. Pakovich, *Prime and composite Laurent polynomials*, Bull. Sci. Math., 133 (2009), 693-732.

- [14] F. Pakovich, A. Zvonkin, *Dessins d'enfants and functional equations*, in preparation.
- [15] J. F. Ritt. *Permutable rational functions*, Trans. Amer. Math. Soc. 25 (1923), 399-448.
- [16] J. F. Ritt. *Prime and composite polynomials*, Trans. Amer. Math. Soc., 23, 1922, 51-66.
- [17] W. Thurston, *The geometry and topology of 3-manifolds*, mimeographed notes, Princeton, 1979.
- [18] S. Ushiki, *Julia sets with polyhedral symmetries*, Dynamical systems and related topics (Nagoya, 1990), 515-538, Adv. Ser. Dynam. Systems, 9, World Sci. Publ., River Edge, NJ, 1991.
- [19] M. Zieve, *Decompositions of Laurent polynomials*, arXiv:0710.1902.