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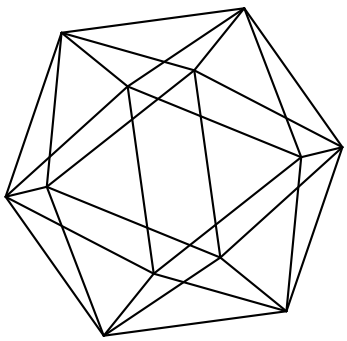
by

Andy Wand



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MAPPING CLASS GROUP RELATIONS, STEIN FILLINGS, AND PLANAR OPEN BOOK DECOMPOSITIONS

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Abstract. The aim of this paper is to use mapping class group relations to approach the 'geography' problem for Stein fillings of a contact 3-manifold. In particular, we adapt a formula of Endo and Nagami so as to calculate the signature of such fillings as a sum of the signatures of basic relations in the monodromy of a related open book decomposition. We combine this with a theorem of Wendl to show that for any Stein filling of a contact structure supported by a planar open book decomposition, the sum of the signature and Euler characteristic depends only on the contact manifold. This gives a simple obstruction to planarity, which we interpret in terms of existence of certain configurations of curves in a factorization of the monodromy.

1. Introduction

In recent years, a large body of work has brought to light surprising connections between open book decompositions, contact manifolds, Lefschetz fibrations, and symplectic and Stein manifolds. Giroux [9] has demonstrated a 1-1 correspondence between stabilization classes of open book decompositions and contact 3-manifolds up to isotopy of the contact structure, and further shown that such a manifold has a Stein filling if and only if the monodromy of some open book decomposition associated to it through this correspondence has a factorization into positive Dehn twists. Work of Giroux and others (in particular Loi and Piergallini [1] and Akbulut and Ozbagci[12]) has further shown that such a factorization defines a Lefschetz fibration of a 4-manifold filling of the contact manifold, which in turn defines a Stein structure on the filling and thus an induced contact structure on the boundary. In the case of a factorization into twists along homologically non-trivial curves, this induced structure agrees with the original structure. Conversely, any Stein filling induces such a Lefschetz fibration and open book decomposition.

Via the above framework, one may translate questions concerning Stein fillings of a given contact manifold into questions concerning positive factorizations of the set of monodromies of its open book decompositions. It is however generally quite difficult to understand how the sets of possible factorizations of stabilization-equivalent open book decompositions are related.

In this paper, we are concerned with *curve configurations* in a given mapping class φ, by which we mean any subword of a positive factorization of φ into Dehn twists. In particular, we are motivated by the idea of using an understanding of possible curve configurations in the monodromy of a given open book decomposition

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to understand properties of its stabilization class (such as support genus, see e.g. [7]) and also the set of Stein fillings of the supported contact manifold.

In Section 4, we adapt Endo and Nagmi's [5] notion of the *signature* of a relation (itself a generalization of the Meyer cocycle [14]) to the setting of contact structures and Stein fillings. This gives a simple method of calculating the effect of changing a factorization of the monodromy of an open book decomposition on the Euler characteristic and signature of the associated filling. As an application, we find that well-known presentations of the mapping class group restrict the 'geography' of Stein fillings of a contact 3-manifold.

This technique also gives more stringent restrictions on the set of Stein fillings associated to a contact structure via a *given* supporting open book decomposition. These restrictions, however, are not in general preserved by stabilization of the open book, and as such are not in general properties of the contact structure itself. Indeed, in [21] we constructed examples of positive open book decompositions for which stabilization increases the set of related Stein fillings. If, however, contact (M, ξ) is supported by *planar* open book decomposition (Σ, φ), then the situation is somewhat more restrictive, due to a recent result of Wendl [22] which in effect says that we do not have to stabilize (Σ, φ); that the set of fillings related to (Σ, φ) is exactly the set of fillings of (M, ξ). This result allows us to use the above restrictions to demonstrate new obstructions to a contact structure being supported by a planar open book through existence of particular curve configurations in any supporting positive open book. These obstructions are of a substantially different flavor than the known obstructions due to Etnyre [6] and Ozsvath, Stipsicz and Szabo [18].

As a final comment, the dependence of these results on Wendl's theorem means that this approach, as is, has no hope of giving obstructions to support genus greater than zero. Furthermore, our above-mentioned earlier examples (constructed in [21]) are of genus 2, so there can be no analogue of Wendl's theorem for genus 2 or higher. The case of genus 1 remains unknown.

The organization of the paper is as follows. Sections 2 and 3 give basic definitions concerning mapping class groups, Lefschetz fibrations, and open book decompositions. In Section 4 we recall Endo and Nagami's signature of a relation, adapting their concept for a more general setting. In Section 5, we combine this with Wendl's result to give necessary conditions on Stein fillings of planar contact structures, which we interpret in terms of existence of certain curve configurations in Section 6.

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2. Mapping class groups and relators

Let $\Sigma = \Sigma_{g,b}$ be a compact, orientable surface of genus g with b boundary components. The (restricted) mapping class group of Σ, denoted Γ_Σ, is the group of isotopy classes of orientation preserving diffeomorphisms of Σ which restrict to the identity on $\partial\Sigma$. If $b = 0$, i.e. Σ is closed, we write simply $\Sigma = \Sigma_g$. We denote by $Dehn^+(\Sigma)$ the subset of mapping classes which admit factorizations into positive Dehn twists, and by $Fac^+(\varphi)$ the set of such factorizations.

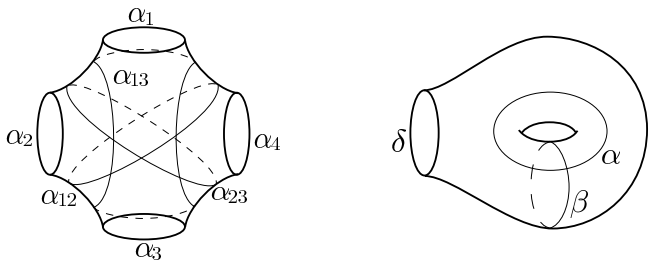
Denoting by $\mathcal{F}$ the free group generated by isotopy classes of simple closed curves on Σ, there is a natural homomorphism $g : \mathcal{F} \to \Gamma_\Sigma$ sending a curve α to the positive

Dehn twist τ_α about α. It is a classical theorem of Dehn that g is surjective. We call each element r of $Ker(g)$ a *relator* in the generators of Γ_Σ.

We have the following presentation of Γ_Σ, due to Gervais [8] and Luo [13]:

Theorem 2.1. *For a compact oriented surface Σ, the mapping class group Γ_Σ has the following presentation:*
generators: $\{\tau_\alpha : \alpha$ *a simple closed curve in* $\Sigma\}$.
relators:

(I) τ_α *for* α *the isotopy class of a null homotopic loop.*
(II) $\tau_\alpha\tau_\beta\tau_\alpha^{-1}\tau_\beta^{-1}$ *for* $|\alpha \cap \beta| = 0$.
(III)$\tau_{\tau_\alpha\beta}\tau_\alpha\tau_\beta^{-1}\tau_\alpha^{-1}$ *(the braid relation)*
(IV)$\tau_{\alpha_1}^{-1}\tau_{\alpha_2}^{-1}\tau_{\alpha_3}^{-1}\tau_{\alpha_4}^{-1}\tau_{\alpha_{12}}\tau_{\alpha_{23}}\tau_{\alpha_{13}}$ *for curves as in Figure 1(a) (the lantern relation)*
(V) $\tau_\delta(\tau_\alpha\tau_\beta)^{-6}$ *for curves as in Figure 1(b) (the 2-chain relation)*



(a) (b)

FIGURE 1. Curves involved in the lantern and 2-chain relations

Suppose $r = \lambda_1^{-1}\lambda_2$ is a relator, and λ a word in Γ_Σ which can be written $\lambda = \lambda_3\lambda_1\lambda_4$ (where each λ_i is a positive word). Then we say $\lambda' = \lambda_3\lambda_2\lambda_4$ is an *r-substitution* of λ. If r is a braid relator, i.e. of type (III) above, an r substitution is often referred to as a *Hurwitz move*. We will unless otherwise specified be considering words only up to the relations of type (I), (II), and (III), as these preserve most of the information we will be interested in. Note that, with this convention, a r-substitution may always be viewed as concatenation of words. In particular, setting $r' = \lambda_4^{-1} \circ r \circ \lambda_4$, we may write the above r-substitution as $\lambda = \lambda_3\lambda_1\lambda_4 \overset{r'}{\rightsquigarrow} \lambda_3\lambda_1\lambda_4 r' = \lambda_3\lambda_2\lambda_4$.

It follows then that any r-substitution of a word λ takes the form $\lambda(\lambda_{1_1}^{-1}\ \lambda_{1_2})\dots(\lambda_{n_1}^{-1}\lambda_{n_2})$, where each $r_i := \lambda_{i_1}^{-1}\lambda_{i_2}$ is a relator of type (IV) or (V). We write $r = \prod r_i$.

As an example, consider the '3-chain' relator $r = \tau_{\alpha_1}^{-1}\tau_{\alpha_2}^{-1}(\tau_{\alpha_{23}}\tau_\alpha\tau_\beta)^4$, where curves are indicated in Figure 2 (the general definition of an n-chain relation, due to Wajnryb [19], is given in Section 6). We may decompose r as r_1r_2, where r_1 is the lantern relator $\tau_{\alpha_1}^{-1}\tau_{\alpha_2}^{-1}\tau_{\alpha_3}^{-1}\tau_{\alpha_4}^{-1}\tau_{\alpha_{12}}\tau_{\alpha_{23}}\tau_{\alpha_{13}}$ and r_2 the 2-chain $\tau_\delta^{-1}(\tau_\alpha\tau_\beta)^6$. We recover r by pasting the supporting surfaces along a common subsurface (here a pair of pants) so that $\alpha_{12} \simeq \delta$, and $\alpha_3 \simeq \alpha_4 \simeq \beta$, and performing a sequence of Hurwitz moves and cancelations.

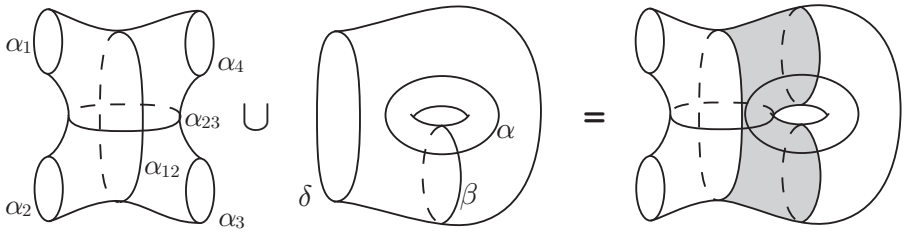


FIGURE 2. Decomposition of the 3-chain relator

3. Lefschetz fibrations and open book decompositions

Let X and B be compact oriented smooth manifolds of dimension 4 and 2 respectively, possibly with boundary. A *Lefschetz fibration* $f : X \to B$ is then a smooth surjective map which is a locally trivial fibration outside of finitely many critical values $\{b_i\} \in int(B)$, where each singular fiber $f^{-1}(b_i)$ has a unique critical point, at which f can be modeled in some choice of complex coordinates by $f(z1, z2) = z_1^2 + z_2^2$. If $b' \in B$ is near a critical value b_i, then there is a simple closed curve C in $f^{-1}(b')$, called a *vanishing cycle*, such that the singular fiber $f^{-1}(b_i)$ can be identified with $f^{-1}(b')$ after collapsing C to a point. The boundary of a regular neighborhood of a singular fiber is a surface bundle over the circle with monodromy a right-handed Dehn twist along the corresponding vanishing cycle. Once we fix an identification of Σ with the fiber over a base point of B, the topology of the Lefschetz fibration is determined by its monodromy representation $\Psi : \pi_1(B - \{\text{critical values}\}) \to \Gamma_\Sigma$. If the base is $B = D^2$ the monodromy along $\partial D^2 = S^1$ is given by the product of right-handed Dehn twists corresponding to the singular fibers, and called the *total monodromy* of the fibration. A Lefschetz fibration over S^2 can be decomposed into two Lefschetz fibrations over D^2, one of which is trivial; consequently, a Lefschetz fibration over S^2 is determined by a relator in the mapping class group. Conversely, given a product of right-handed Dehn twists in the mapping class group, we can construct the corresponding Lefschetz fibration over D^2, and if the given product of right-handed Dehn twists is isotopic to the identity (and $g \geq 2$), then the fibration extends uniquely over S^2. The monodromy representation also provides a handlebody decomposition of a Lefschetz fibration over D^2: we attach 2-handles to $\Sigma \times D^2$ along the vanishing cycles with framing -1 relative to the framing that the circle inherits from the fiber. (For more detail, see e.g. [10]).

For this paper, the base B will be either S^2 or D^2. Having specified the base, we may specify a Lefshetz fibration over B by the data of the diffeomorphism type of a generic fiber $\Sigma = f^{-1}(b)$, and a word $\lambda = \tau_{\alpha_n} \cdots \tau_{\alpha_1}$ in Γ_Σ given as a composition of positive Dehn twists, with the condition that λ be a factorization of the identity in Γ_Σ if the base is S^2. We denote the resulting 4-manifold by $X_{(\Sigma,\lambda)}$, which is unique up to Hurwitz equivalence (i.e. under relations of type (III) from the previous section) and global conjugation of λ.

An *open book decomposition* is a pair (Σ, φ), where $\varphi \in \Gamma_\Sigma$. From the mapping torus $(\Sigma \times [0, 1])/\sim$, where $(\varphi(x), 0) \sim (x, 1)$ for $x \in \Sigma$, we obtain a closed 3-manifold M_Σ by gluing solid tori to the boundary so as to identify (y, t) with (y, t') for $y \in \partial\Sigma$. For a closed 3-manifold M, a celebrated result of Giroux [9] gives a

1-1 correspondence between open book decompositions of M up to a stabilization operation and contact structures on M up to isotopy.

In the case that a Lefschetz fibration over D^2 has fiber $\Sigma = \Sigma_{g,b}$, $b \neq 0$, the boundary $M = \partial X_{\Sigma,\lambda}$ has a natural open book decomposition (Σ, φ), where $\lambda \in Fac^+(\varphi)$. Conversely, given an open book decomposition (Σ, φ), a positive factorization λ obviously determines a Lefschetz fibration $X_{\Sigma,\lambda}$, which by Eliashberg [3] determines a Stein structure on $X_{\Sigma,\lambda}$. Even more, if each vanishing cycle is homologically non-trivial in the fiber, then the contact structure induced on M by the Stein filling given by the Lefschetz fibration agrees with the contact structure supported by the open book decomposition (Σ, φ) through the Giroux correspondence (full details may be found in [1],[17]).

4. Signature and Euler characteristic of a relation

In [5], Endo and Nagami introduce the concept of the *signature* of a relation in a mapping class group, generalizing a formula of Meyer for the signature of a surface bundle over a surface to the case of a Lefschetz fibration over S^2 with closed fiber. A particularly useful aspect of this approach is that it allows one to calculate the signature of a Lefschetz fibration as the sum of basic relations in the monodromy.

Definition 4.1. (Endo and Nagami [5]) Let $\Sigma = \Sigma_g$, $g : \mathcal{F} \to \Gamma_\Sigma$ the homomorphism defined in Section 2, and $\tau_g : \Gamma_\Sigma \times \Gamma_\Sigma \to \mathbb{Z}$ the signature cocycle of Meyer. Then there is an explicit homomorphism $c_g : ker(g) \to \mathbb{Z}$ inducing the evaluation map $H_2(\Gamma_\Sigma) \to \mathbb{Z}$ for the cohomology class of τ_g. For a relator $r \in ker(g)$, the *signature* of r is $I(r) := -c_g(r) - s(r)$, where $s(r)$ is the total exponent of the Dehn twists in r.

The following calculations may be found in [5]:

Lemma 4.2. *Let r_b, r_l and r_c be the braid, lantern and 2-chain relators (III, IV and V of Theorem 2.1), and I, s as in Definition 4.1. Then*

- $I(r_b) = s(r_b) = 0$
- $I(r_l) = 1,\ s(r_l) = -1$
- $I(r_c) = -7,\ s(r_c) = 11$

Now, following section 2, for r any relator, we write $r = \prod r_i$, where each r_i is either a lantern or a 2-chain relator. Clearly, we have $s(r) = \sum s(r_i)$. Furthermore, it follows from Definition 4.1 that $I(r) = \sum I(r_i)$. Thus, for example, for r the 3-chain relation from the example at the end of Section 2, we have $r = r_1 r_2$ where r_1 is a lantern, r_2 (the inverse of) a 2-chain relator. Thus $I(r) = -1 + 7 = 6$, while $s(r) = 1 - 11 = -10$.

It is straightforward to see that if a Lefschetz fibration is modified by a r-substitution in the associated mapping class factorization, then the change in the Euler characteristic of the 4-manifold is exactly $s(r)$; i.e. if λ' is an r-substitution of λ, then $s(r) = e(X_{\Sigma,\lambda'}) - e(X_{\Sigma,\lambda})$. That the analogous statement holds for $I(r)$ in the case of a closed Lefschetz fibration over S^2 is the content of Theorem 4.3 of [5]:

Theorem 4.3. *[Endo and Nagami] Let $X_{\Sigma_g,\lambda}, X_{\Sigma_g,\lambda'}$ be Lefschetz fibrations over S^2, where λ' is a r-substitution of λ. Then*

$$\sigma(X_{\Sigma_g,\lambda'}) - \sigma(X_{\Sigma_g,\lambda}) = I(r).$$

For the purposes of this paper, we require a version of Theorem 4.3 which covers the case of Lefschetz fibrations with open book decomposition boundary. We have:

Theorem 4.4. *Let $\Sigma = \Sigma_{g,b}$ be a surface with boundary, and $X_{\Sigma,\lambda}, X_{\Sigma,\lambda'}$ Lefschetz fibrations over D^2, where λ' is a r-substitution of λ. Then*

$$\sigma(X_{\Sigma,\lambda'}) - \sigma(X_{\Sigma,\lambda}) = I(r). \ *$$

Our approach is to embed $X_{\Sigma,\lambda}$ and $X_{\Sigma,\lambda'}$ into Lefschetz fibrations which satisfy the hypotheses of Theorem 4.3, and then, using Novikov additivity and Wall's formula for non-additivity of the signature, show that the signature equality $*$ holds at each step as we remove what is necessary to recover our original fibrations. We require the following application of Wall non-additivity:

Lemma 4.5. *Suppose X_i, X_i' are compact 4-manifolds such that $\partial X_i = \partial X_i'$, $i = 1, 2$. Let $X = X_1 \cup_f X_2$, $X' = X_1' \cup_f X_2'$ be the result of gluing along a common submanifold N of the boundaries via an orientation reversing diffeomorphism f. Then*

$$\sigma(X) - \sigma(X_1) - \sigma(X_2) = \sigma(X') - \sigma(X_1') - \sigma(X_2').$$

Proof. By Wall's formula for additivity of the signature in this situation [20], we have

$$\sigma(X) - \sigma(X_1) - \sigma(X_2) = -\sigma(V; A, B, C)$$

where the 'correction term' $\sigma(V; A, B, C)$ depends only on the inclusions of ∂N in $\partial X_1 - N, N$ and $\partial X_2 - N$. In particular, the calculation for $\sigma(X') - \sigma(X_1') - \sigma(X_2')$ gives the same correction term, from which the result follows. □

Proof. (of Theorem 4.4)

To set things up, let $\Sigma' \cong \Sigma_{0,b+1}$, $\Sigma'' := \Sigma \cup_{\partial(\Sigma)} \Sigma'$, and $\hat{\Sigma}$ the closed surface obtained by filling in the remaining boundary component (Figure 3). It is well-known that any mapping class on $\Sigma_{1,g}$ has a factorization such that all negative twists are about the boundary component δ, so in particular there is n such that $\tau_\delta^n \circ \lambda^{-1}$ has positive factorization λ_1. The word $\lambda_1 \circ \lambda$ is thus a factorization of the identity element in $\Gamma_{\hat{\Sigma}}$, so $X_{(\hat{\Sigma},\lambda_1\circ\lambda)}$ gives a Lefschetz fibration over S^2 (a similar construction was used in [2], where one may also find a proof of the above 'well-known' fact). Now, $\lambda = \lambda'$ as elements of Γ_Σ, so the above goes through identically for λ'. By Theorem 4.3 we have

$$\sigma(X_{(\hat{\Sigma},\lambda_1\circ\lambda')}) - \sigma(X_{(\hat{\Sigma},\lambda_1\circ\lambda)}) = I(r).$$

It remains to check that the equality $*$ holds as we remove these new bits to recover the signatures $\sigma(X_{\Sigma_g,\lambda})$ and $\sigma(X_{\Sigma_g,\lambda'})$ for the Lefschetz fibrations over D^2. The Lefschetz fibration $X_{(\hat{\Sigma},\lambda_1\circ\lambda)}$ over D^2 is obtained from the (closed) fibration over S^2 by deleting a piece with zero signature, so by Novikov additivity the equality $*$ holds for these. Similarly, using Lemma 4.5, removing a disc from the fiber (corresponding to a 4-ball in the corresponding 4 manifold) to recover the Lefschetz fibrations with fiber Σ'' gives

$$\sigma(X_{(\hat{\Sigma},\lambda_1\circ\lambda')}) - \sigma(X_{(\Sigma'',\lambda_1\circ\lambda')}) = \sigma(X_{(\hat{\Sigma},\lambda_1\circ\lambda)}) - \sigma(X_{(\Sigma'',\lambda_1\circ\lambda)})$$

and so

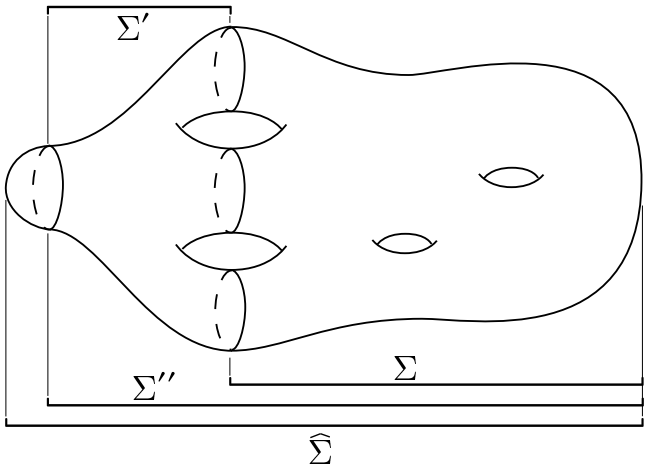


FIGURE 3. Surfaces involved in proof of Theorem 4.4

$$\sigma(X_{(\Sigma'',\lambda_1\circ\lambda')}) - \sigma(X_{(\Sigma'',\lambda_1\circ\lambda)}) = \sigma(X_{(\hat{\Sigma},\lambda_1\circ\lambda')}) - \sigma(X_{(\hat{\Sigma},\lambda_1\circ\lambda)}) = I(r).$$

Then $X_{\Sigma'',\lambda_1\circ\lambda}$ is a fiber sum of X_{Σ'',λ_1} and $X_{\Sigma'',\lambda}$, so

$$\sigma(X_{(\Sigma'',\lambda')}) - \sigma(X_{(\Sigma'',\lambda)}) = I(r).$$

Finally, another application of Lemma 4.5 gives the equality for Σ, as desired. □

5. Applications

Let (Σ, φ) be a positive open book decomposition supporting (M, ξ). Now, while any $\lambda \in Fac^+(\varphi)$ determines a Stein filling $X_{\Sigma,\lambda}$ of (M, ξ), it is not the case that *any* Stein filling X can be given as $X_{\Sigma,\lambda}$ for some $\lambda \in Fac^+(\varphi)$ (see Section 5 of [21] for an explicit counterexample). It does however follow easily from Giroux's theorem that there is some (Σ', φ') obtainable by some number of positive stabilizations on (Σ, φ) such that this holds; i.e for any Stein filling X of (M, ξ), there is $\lambda \in Fac^+(\varphi')$ such that $X = X_{\Sigma,\lambda}$. Note then that by Theorem 4.4, and the calculations of Lemma 4.2, we find that, if X' is any other Stein filling of (M, ξ), then $e(X) + \sigma(X) \equiv e(X') + \sigma(X')(\text{mod}4)$.

Corollary 5.1. *If X, X' are Stein fillings of contact (M, ξ), then $e(X) + \sigma(X) \equiv e(X') + \sigma(X')(mod4)$.*

If, however, (M, ξ) is supported by *planar* (Σ, φ), then the situation is somewhat more restrictive, due to a recent result of Wendl which in effect says that we do not have to stabilize (Σ, φ). In particular, in [22], Wendl has shown that if planar (Σ, φ) supports (M, ξ), and (X, ω) is any minimal strong symplectic filling of (M, ξ), then (after possibly enlarging X by a trivial symplectic cobordism preserving the contact boundary), (X, ω) admits a Lefschetz fibration whose boundary is (Σ, φ). It follows that (X, ω) is symplectically deformation equivalent to Stein $X_{\Sigma,\lambda}$ for $\lambda \in Fac^+(\varphi)$.

To summarize using our notation,

Theorem 5.2 (Wendl)**.** *Suppose that X is a minimal strong symplectic filling of (M, ξ), and that the latter admits a planar open book decomposition (Σ, φ). Then there is $\lambda \in Fac^+(\varphi)$ such that $X = X_{\Sigma,\lambda}$.*

To see that this gives a restriction, we have the following direct corollary of Theorem 2.1:

Corollary 5.3. *Let $\Sigma = \Sigma_{0,b}$ be a planar surface, $\varphi \in \Gamma_\Sigma$, and $\lambda_1, \lambda_2 \in Fac^+\varphi$. Then λ_1, λ_2 are related by an r-substitution where $r = \prod r_i$ is a concatenation of lantern relations.*

Now, it follows that, for each i, either $I(r_i) = 1$ and $s(r_i) = -1$, or $I(r_i) = -1$ and $s(r_i) = 1$. In particular, we have:

Lemma 5.4. *Let $\Sigma = \Sigma_{0,b}$ be a planar surface, $\varphi \in \Gamma_\Sigma$, and $\lambda_1, \lambda_2 \in Fac^+(\varphi)$. Then $e(X_{\Sigma,\lambda_1}) + \sigma(X_{\Sigma,\lambda_1}) = e(X_{\Sigma,\lambda_2}) + \sigma(X_{\Sigma,\lambda_2})$.*

And thus:

Corollary 5.5. *If X_1, X_2 are Stein fillings of planar (M, ξ), then $e(X_1)+\sigma(X_2) = e(X_1) + \sigma(X_2)$.*

Note that, using Wendl's result (see the discussion preceding Theorem 5.2), we may replace 'Stein' in Corollary 5.5 with 'minimal strong symplectic'. In fact, a recent strengthening of Theorem 5.2 due to Niederkrüger and Wendl [15], and brought to our attention by Chris Wendl, extends the result to the more general case of minimal *weak* symplectic fillings. Of course, for the purposes of Corollary 5.5, we may drop the minimality condition, as the blowup process preserves $e + \sigma$ (for definitions and details, see e.g. [10]).

6. Curve configurations as obstructions to planarity

We begin with some terminology:

Definition 6.1. Let $\varphi \in Dehn^+(\Sigma)$. We say φ *contains* the positive word λ if there is a positive word λ' such that $\lambda' \circ \lambda \in Fac^+(\varphi)$.

Definition 6.2. Suppose (Σ, φ) is an open book decomposition satisfying $\varphi \in Dehn^+(\Sigma)$, and let $\Sigma' \cong \Sigma_{g,n}$ be a subsurface of Σ such that φ contains the multicurve $\partial\Sigma'$. Then we say φ *bounds* $\Sigma_{g,n}$.

From the discussion following Lemma 4.2, along with Corollary 5.5, it follows immediately that if positve φ bounds $\Sigma_{1,2}$, then (by performing a substitution with the 3-chain relator) we see that the supported contact manifold admits Stein fillings X, X' such that $e(X) + \sigma(X) = e(X') + \sigma(X') + 4$. By Corollary 5.5, (M, ξ) is therefore not planar.

To generalize this obstruction, we begin with the following well-known (see e.g. [19]) generalization of the 2- and 3-chain relations introduced in section 2. Let $\alpha_1, \ldots, \alpha_n$ be a chain of curves; i.e. such that each pair α_i, α_{i+1} have a single point of intersection, while curves of non-consecutive index are disjoint. Note that for n even, a regular neighborhood of such a chain is a surface of genus $n/2$ with 1 boundary component, which we denote δ, while if n is odd, a regular neighborhood is a surface of genus $(n-1)/2$ with 2 boundary components, which we denote δ_1 and δ_2.

Lemma 6.3. *(see, e.g. [19]) For even n, $\tau_\delta^{-1}(\tau_{\alpha_1}\dots\tau_{\alpha_n})^{2n+2}$ is a relator, while for odd n, $\tau_{\delta_1}^{-1}\tau_{\delta_2}^{-1}(\tau_{\alpha_1}\dots\tau_{\alpha_n})^{n+1}$ is a relator.*

Using the n-chain relation, we see that if φ bounds either $\Sigma_{g,1}$ or $\Sigma_{g,2}$, $g>1$, then φ bounds $\Sigma_{g',2}$ for each $2\le g'\le g$. In particular, φ bounds $\Sigma_{1,2}$, and so any of these is an obstruction to planarity.

We may play a similar game to obtain relators (and thus obstructions to planarity) if φ bounds $\Sigma_{1,n}$ for $2\le n\le 9$ (see [11] for an explicit construction, from which one may easily calculate I and s for each relator). Note that the non-existence of elliptic fibrations with more than 9 disjoint sections means that there is no such relator for $n>9$.

In the genus 2 case, Onaran [16] has given relators for $\Sigma_{2,n}$, $n\le 8$. It is straightforward to check that each of these gives an obstruction to planarity; i.e. if φ bounds $\Sigma_{2,n}$ for $n\le 8$, then the contact manifold supported by (Σ,φ) is not planar.

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