ELLIPTIC HYPERGEOMETRIC TERMS

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Abstract. General structure of the multivariate plain and q-hypergeometric terms and univariate elliptic hypergeometric terms is described. Some explicit examples of the totally elliptic hypergeometric terms leading to multidimensional integrals on root systems, either computable or obeying non-trivial symmetry transformations, are presented.

Contents

1. Plain hypergeometric case

The definition of the hypergeometric series goes as far back as to Euler and, in a more general setting, to Pochhammer and Horn [1, 7].

Definition 1. *The formal series*

$$\sum_{m\in\mathbb{Z}^n} c(m) = \sum_{m\in\mathbb{Z}^n} c(m_1,\ldots,m_n)$$

is called plain hypergeometric series, if the ratios

$$\frac{c(m_1,\ldots,m_i+1,\ldots,m_n)}{c(m_1,\ldots,m_n)} = R_i(m_1,\ldots,m_n)$$

are rational functions of $m_1,\ldots,m_n$.

Suppose that given rational functions $R_i(m)$, called certificates, satisfy the consistency conditions

$$R_i(m_1,\ldots,m_k+1,\ldots,m_n)R_k(m) = R_k(m_1,\ldots,m_i+1,\ldots,m_n)R_i(m).$$

The general form of corresponding (admissible) plain hypergeometric series was determined by Ore and Sato (e.g., see the survey [7]).

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Theorem 1. *General admissible plain hypergeometric terms $c(m)$ have the form*

$$c(m) = \frac{R(m)}{\prod_{j=1}^{K} \Gamma(\epsilon_j(m) + a_j)} \prod_{k=1}^{n} z_k^{m_k},$$

where z_k, a_j are arbitrary complex parameters, $K = 0, 1, 2, \dots$, $\epsilon_j(m) = \sum_{k=1}^{n} \epsilon_{jk} m_k$, $\epsilon_{jk} \in \mathbb{Z}$, $R(m)$ is some rational function of $m_1, \dots, m_n$, and $\Gamma(x)$ is the standard Euler gamma function.

Using the inversion formula $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$, some of the Γ-functions can be put from the denominator of $c(m)$ to its numerator.

In a similar way one can treat hypergeometric integrals [18].

Definition 2. *The integrals*

$$\int_D \Delta(x) dx = \int_D \Delta(x_1, \dots, x_n) dx_1 \dots dx_n,$$

for some domain of integration $D \in \mathbb{C}^n$, are called plain hypergeometric integrals, if the ratios

$$\frac{\Delta(x_1 \dots, x_i + 1, \dots, x_n)}{\Delta(x_1, \dots, x_n)} = R_i(x_1, \dots, x_n)$$

are rational functions of $x_1, \dots, x_n$.

The general admissible plain hypergeometric terms $\Delta(x)$ have the form

$$\Delta(x) = \varphi(x) R(x) \frac{\prod_{j=1}^{K} \Gamma(\sum_{k=1}^{n} \mu_{jk} x_k + b_j)}{\prod_{j=1}^{M} \Gamma(\sum_{k=1}^{n} \epsilon_{jk} x_k + a_j)} \prod_{k=1}^{n} z_k^{x_k},$$

where z_k, a_j, b_j are arbitrary complex parameters, $K, M = 0, 1, 2, \dots$, $\mu_{jk}, \epsilon_{jk} \in \mathbb{Z}$, $R(x)$ is some rational function of $x_1, \dots, x_n$, and $\varphi(x) = \varphi(x_1, \dots, x_i + 1, \dots, x_n)$ is an arbitrary periodic function. The plain hypergeometric series can be obtained from integrals as sums of residues for particular sequences of poles of $\Delta(x)$.

The $\Gamma(x)$-function can be defined as a special meromorphic solution of the functional equation

$$\Gamma(x+1) = x\Gamma(x),$$

the general solution of which has the form $\varphi(x)\Gamma(x)$, where $\varphi(x+1) = \varphi(x)$ is an arbitrary periodic function.

For $n = 1$, definition 2 yields the Meijer function for the choices $\varphi = 1$, $R = 1$ and D one of the contours 1) $\{-\mathrm{i}\infty, +\mathrm{i}\infty\}$ separating sequences of equidistant poles going to the left and to the right of the complex plane, 2) $\{-\infty - \mathrm{i}A, -\infty + \mathrm{i}B\}$ encircling sequences of poles going to the left (for some choice of the positive constants A and B), 3) $\{+\infty - \mathrm{i}A, +\infty + \mathrm{i}B\}$ encircling sequences of poles going to the right (for some choice of the constants A and B).

It is worth to remark that a limiting form of the plain hypergeometric terms is determined by the system of partial differential equations

$$\frac{1}{\Delta(x)} \frac{\partial \Delta(x)}{\partial x_i} = R_i(x).$$

2. q-HYPERGEOMETRIC CASE

q-deformations of hypergeometric functions were introduced by Heine a long time ago [1].

Definition 3. *The formal series*

$$\sum_{m\in\mathbb{Z}^n} c(m) = \sum_{m\in\mathbb{Z}^n} c(m_1,\dots,m_n)$$

is called q-hypergeometric, if

$$\frac{c(m_1\dots,m_i+1,\dots,m_n)}{c(m_1,\dots,m_n)} = R_i(q^{m_1},\dots,q^{m_n})$$

are rational functions of $q^{m_1},\dots,q^{m_r}$, where q is an arbitrary complex parameter.

This is a natural extension of the previous definition and it leads [7] to the following theorem.

Theorem 2. *General admissible q-hypergeometric terms $c(m)$ have the form*

$$c(m) = R(q^m)\frac{\prod_{j=1}^K (a_j;q)_{\mu_j(m)}}{\prod_{j=1}^M (b_j;q)_{\epsilon_j(m)}}\prod_{k=1}^n x_k^{m_k},$$

where x_k, a_j, b_j are arbitrary complex parameters, $K, M = 0,1,2,\dots$, $\mu_j(m) = \sum_{k=1}^n \mu_{jk}m_k$ and $\epsilon_j(m) = \sum_{k=1}^n \epsilon_{jk}m_k$ with $\mu_{jk},\epsilon_{jk}\in\mathbb{Z}$, $R(q^m)$ is some rational function, and

$$(x;q)_n := \begin{cases} \prod_{j=0}^{n-1}(1-xq^j), & \text{for } n>0\\ \prod_{j=1}^{-n}(1-xq^{-j})^{-1}, & \text{for } n<0\end{cases}$$

is the q-shifted factorial (or the q-Pochhammer symbol).

Definition 4. *The integrals*

$$\int_D \Delta(x)dx = \int_D \Delta(x_1,\dots,x_n)dx_1\dots dx_n,$$

for some domain of integration $D\in\mathbb{C}^n$, are called q-hypergeometric, if the ratios

$$\frac{\Delta(x_1,\dots,x_i+1,\dots,x_n)}{\Delta(x_1,\dots,x_n)} = R_i(q^{x_1},\dots,q^{x_n})$$

are rational functions of $q^{x_1},\dots,q^{x_n}$.

Define q-gamma functions as special meromorphic solutions of the finite-difference equation

$$f(u+\omega_1) = (1-e^{2\pi \mathrm{i} u/\omega_2})f(u), \tag{2.1}$$

where $\omega_1,\omega_2\in\mathbb{C}$. Evidently, solutions of this equation are defined modulo multiplication by an arbitrary $\varphi(u+\omega_1)=\varphi(u)$ periodic function. Introducing the variables

$$q := e^{2\pi \mathrm{i}\omega_1/\omega_2}, \qquad z := e^{2\pi \mathrm{i} u/\omega_2},$$

this equation can be replaced by

$$\Gamma_q(qz) = (1-z)\Gamma_q(z).$$

For $|q|<1$ its particular solution, analytic near the point $z=0$, is determined by a simple iteration, which yields

$$\Gamma_q(z)=\frac{1}{(z;q)_\infty}=\prod_{j=0}^\infty\frac{1}{1-zq^j},\qquad \Gamma_q(0)=1,$$

which can be considered as a q-gamma function for $|q|<1$. More precisely, the Thomae-Jackson q-gamma function has the form

$$\Gamma(u;q)=(1-q)^{1-u}\frac{(q;q)_\infty}{(q^u;q)_\infty},\qquad \lim_{q\to1}\Gamma(u;q)=\Gamma(u).$$

It satisfies the equations

$$\Gamma(u+1;q)=\frac{1-q^u}{1-q}\Gamma(u;q),\qquad \Gamma(u-\frac{2\pi\mathrm{i}}{\log q};q)=(1-q)^{\frac{2\pi\mathrm{i}}{\log q}}\Gamma(u;q).$$

For $|q|<1$ the q-Pochhammer symbol can be written as

$$(t;q)_n=\frac{(t;q)_\infty}{(tq^n;q)_\infty},\qquad n\in\mathbb{Z}.$$

For $|q|=1$ the equation $f(qz)=(1-z)f(z)$ does not have meromorphic solutions for $z\in\mathbb{C}^*$. In this case it is necessary to consider equation (2.1) and search for its solutions meromorphic in $u\in\mathbb{C}$. The modified q-gamma function

$$\gamma(u;\omega_1,\omega_2)=\exp\left(-\int_{\mathbb{R}+\mathrm{i}0}\frac{e^{ux}}{(1-e^{\omega_1x})(1-e^{\omega_2x})}\frac{dx}{x}\right),\tag{2.2}$$

where the contour $\mathbb{R}+\mathrm{i}0$ passes along the real axis turning over the point $x=0$ from above in an infinitesimal way, solves (2.1) and remains meromorphic for $\omega_1,\omega_2>0$ when $|q|=1$. (2.2) is known as the Barnes-Shintani "double sign" function, the noncompact dilogarithm, or the hyperbolic gamma function (see the survey [14] for relevant references).

Let $\mathrm{Re}(\omega_1),\mathrm{Re}(\omega_2)>0$. Then the integral (2.2) is convergent for $0<\mathrm{Re}(u)<\mathrm{Re}(\omega_1+\omega_2)$. Under appropriate restrictions on u and $\omega_{1,2}$, it can be computed as a convergent sum of residues of the poles in the upper half-plane. For $\mathrm{Im}(\omega_1/\omega_2)>0$ (i.e., for $|q|<1$) this leads to the expression

$$\gamma(u;\omega_1,\omega_2)=\frac{(e^{2\pi\mathrm{i}u/\omega_1}\tilde{q};\tilde{q})_\infty}{(e^{2\pi\mathrm{i}u/\omega_2};q)_\infty},\tag{2.3}$$

which is continued analytically to the whole complex plane of u. Here

$$\tilde{q}:=e^{-2\pi\mathrm{i}\omega_2/\omega_1}$$

is a particular modular transform of q.

The function $\gamma(u;\omega_1,\omega_2)$ is symmetric in ω_1,ω_2. Therefore it satisfies the second equation

$$f(u+\omega_2)=(1-e^{2\pi\mathrm{i}u/\omega_1})f(u).\tag{2.4}$$

Suppose ω_1,ω_2 are real and linearly independent over $\mathbb{Z}$, i.e. $|q|=1$, but $q^n\neq1$. Then, equations (2.1) and (2.4) taken together define their solution $\gamma(u;\omega_1,\omega_2)$ uniquely up to multiplication by a constant. For $|q|<1$, there is a functional freedom in multiplication of $\gamma(u;\omega_1,\omega_2)$ by an arbitrary elliptic function

$$\varphi(u+\omega_1)=\varphi(u+\omega_2)=\varphi(u).$$

For $|q|<1$, the general admissible q-hypergeometric term $\Delta(x)$ (integral kernel) has the form

$$\Delta(x)=\varphi(x)R(q^x)\frac{\prod_{j=1}^M(w_jq^{\sum_{k=1}^n\epsilon_{jk}x_k};q)_\infty}{\prod_{j=1}^K(t_jq^{\sum_{k=1}^n\mu_{jk}x_k};q)_\infty}\prod_{k=1}^n z_k^{x_k},$$

where z_k,t_j,w_j are arbitrary complex parameters, $K,M=0,1,2,\ldots$, $\mu_{jk},\epsilon_{jk}\in\mathbb{Z}$, $R(q^x)$ is an arbitrary rational function of $q^{x_1},\ldots,q^{x_n}$, and $\varphi(x)=\varphi(x_1,\ldots,x_i+1,\ldots,x_n)$ is an arbitrary periodic function.

If we drop the factor $\varphi(x)$, then the function $\rho(q^x):=\Delta(x)$ satisfies the system of q-difference equations

$$\rho(\ldots qy_i\ldots)=R_i(y)\rho(y_1,\ldots,y_n),$$

where $R_i(y)$ are rational functions such that

$$R_i(y_1,\ldots,qy_k,\ldots,y_n)R_k(y)=R_k(y_1,\ldots,qy_i,\ldots,y_n)R_i(y).$$

Replacement of linear differences by q-shifts essentially simplifies formulae.

To construct $\Delta(x)$ for $|q|=1$ one needs the modified q-gamma function $\gamma(x;\omega_1,\omega_2)$. It is natural to require symmetry of $\Delta(x)$ in ω_1 and ω_2, i.e., to require fulfillment of the system of equations

$$\Delta(\ldots,x_i+\omega_k,\ldots)=R_i(e^{2\pi\mathrm{i}x_1/\omega_{k+1}},\ldots,e^{2\pi\mathrm{i}x_n/\omega_{k+1}};\omega_k,\omega_{k+1})\Delta(x),$$

where $i=1,\ldots,n$ and $k=1,2,\ \ \omega_{k+2}=\omega_k$. This yields uniquely up to multiplication by a constant

$$\Delta(x)=\exp\Big(\frac{\pi\mathrm{i}}{\omega_1\omega_2}\sum_{j,k=1}^n\mu_{jk}x_jx_k+\sum_{j=1}^n c_jx_j\Big)\prod_{j=1}^M\gamma\Big(\sum_{k=1}^n\epsilon_{jk}x_k+a_j;\omega_1,\omega_2\Big),$$

where $\epsilon_{jk},\mu_{jk}\in\mathbb{Z}$ and $a_j,c_j\in\mathbb{C}$.

3. Elliptic case

The key analytic object of the theory of elliptic functions is the theta series having a convenient multiplicative form due to the Jacobi triple product identity

$$\sum_{n\in\mathbb{Z}}p^{n(n-1)/2}(-x)^n=(p;p)_\infty\theta_p(x),$$

where

$$\theta_p(x)=(x;p)_\infty(px^{-1};p)_\infty,\quad (x;p)_\infty=\prod_{j=0}^\infty(1-xp^j)$$

for any $x\in\mathbb{C}^*$ and $p\in\mathbb{C},|p|<1$. This function obeys the symmetry properties

$$\theta_p(x^{-1})=\theta_p(px)=-x^{-1}\theta_p(x)$$

and the addition law

$$\theta_p(xw^{\pm1},yz^{\pm1})-\theta_p(xz^{\pm1},yw^{\pm1})=yw^{-1}\theta_p(xy^{\pm1},wz^{\pm1}),$$

where $x,y,w,z\in\mathbb{C}^*$. We use the conventions

$$\theta_p(x_1,\ldots,x_k)=\prod_{j=1}^k\theta_p(x_j),\qquad \theta_p(tx^{\pm1})=\theta_p(tx,tx^{-1}).$$

For arbitrary $q \in \mathbb{C}$ and $n \in \mathbb{Z}$, the elliptic shifted factorials are defined as

$$\theta_p(x;q)_n := \begin{cases} \prod_{j=0}^{n-1} \theta_p(xq^j), & \text{for } n > 0 \\ \prod_{j=1}^{-n} \theta_p(xq^{-j})^{-1}, & \text{for } n < 0 \end{cases}$$

and $\theta_p(x;q)_0 = 1$. For $p = 0$ we have $\theta_0(x) = 1 - x$ and $\theta_0(x;q)_n = (x;q)_n$.

For arbitrary $m \in \mathbb{Z}$, we have the quasiperiodicity relations

$$\theta_p(p^m x) = (-x)^{-m} p^{-\frac{m(m-1)}{2}} \theta_p(x),$$
$$\theta_p(p^m x;q)_k = (-x)^{-mk} q^{-\frac{mk(k-1)}{2}} p^{-\frac{km(m-1)}{2}} \theta_p(x;q)_k,$$
$$\theta_p(x;pq)_k = (-x)^{-\frac{k(k-1)}{2}} q^{-\frac{k(k-1)(2k-1)}{6}} p^{-\frac{k(k-1)(k-2)}{6}} \theta_p(x;q)_k.$$

Elliptic gamma functions are defined as special meromorphic solutions of the finite difference equation

$$f(u+\omega_1) = \theta_p(e^{2\pi \mathrm{i} u/\omega_2}) f(u), \tag{3.1}$$

which passes to (2.1) for $p \to 0$ and fixed u. It is not difficult to see that the double infinite product

$$\Gamma_{p,q}(z) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}, \tag{3.2}$$

where $|p|, |q| < 1$ and $z \in \mathbb{C}^*$, satisfies the equations

$$\Gamma_{p,q}(qz) = \theta_p(z) \Gamma_{p,q}(z), \quad \Gamma_{p,q}(pz) = \theta_q(z) \Gamma_{p,q}(z). \tag{3.3}$$

The second relation follows from the first one due to the symmetry in q and p. Thus, the function

$$f(u) = \Gamma_{p,q}(e^{2\pi \mathrm{i} u/\omega_2}), \quad q = e^{2\pi \mathrm{i} \omega_1/\omega_2}, \tag{3.4}$$

defines a solution of equation (3.1) for $|q| < 1$. For fixed z, $\Gamma_{0,q}(z) = 1/(z;q)_\infty$.

The standard elliptic gamma function (3.4) is directly related to the Barnes multiple gamma function of the third order [2]. Its special cases and different properties were investigated by Jackson, Baxter, Ruijsenaars, Felder, Varchenko, Rains, and the present author (see survey [14]).

Suppose three complex variables $\omega_{1,2,3}$ are linearly independent over $\mathbb{Z}$. Then the well known Jacobi theorem states that if a meromorphic $\varphi(u)$ satisfies the system of equations

$$\varphi(u+\omega_1) = \varphi(u+\omega_2) = \varphi(u+\omega_3) = \varphi(u),$$

then $\varphi(u) = const$. Define the bases

$$q = e^{2\pi \mathrm{i} \frac{\omega_1}{\omega_2}}, \quad p = e^{2\pi \mathrm{i} \frac{\omega_3}{\omega_2}}, \quad r = e^{2\pi \mathrm{i} \frac{\omega_3}{\omega_1}}$$

and their particular modular transformations

$$\tilde{q} = e^{-2\pi \mathrm{i} \frac{\omega_2}{\omega_1}}, \quad \tilde{p} = e^{-2\pi \mathrm{i} \frac{\omega_2}{\omega_3}}, \quad \tilde{r} = e^{-2\pi \mathrm{i} \frac{\omega_1}{\omega_3}}.$$

The incommensurability condition for ω_k takes now the form $p^n \neq q^m$, or $r^n \neq \tilde{q}^m$, etc., $n, m \in \mathbb{Z}$.

The elliptic gamma function (3.4) can be defined uniquely as the meromorphic solution of the system of three equations:

$$f(u+\omega_1) = \theta_p(e^{2\pi \mathrm{i} u/\omega_2}) f(u),$$
$$f(u+\omega_2) = f(u),$$
$$f(u+\omega_3) = \theta_q(e^{2\pi \mathrm{i} u/\omega_2}) f(u)$$

with the normalization condition $f(\sum_{k=1}^3 \omega_k/2) = 1$.

The modified elliptic gamma function has the form [11]

$$G(u;\omega) = \Gamma_{p,q}(e^{2\pi\mathrm{i}\frac{u}{\omega_2}})\Gamma_{\tilde{q},r}(re^{-2\pi\mathrm{i}\frac{u}{\omega_1}}). \tag{3.5}$$

It defines the unique solution of three equations:

$$f(u+\omega_2) = \theta_r(e^{2\pi\mathrm{i}u/\omega_1})f(u), \qquad f(u+\omega_3) = e^{-\pi\mathrm{i}B_{2,2}(u|\omega_1,\omega_2)}f(u)$$

and equation (3.1) with the normalization $f(\sum_{k=1}^3 \omega_k/2) = 1$. Here

$$B_{2,2}(u|\omega_1,\omega_2) = \frac{1}{\omega_1\omega_2}\left(u^2 - (\omega_1+\omega_2)u + \frac{\omega_1^2+\omega_2^2}{6} + \frac{\omega_1\omega_2}{2}\right) \tag{3.6}$$

is the second Bernoulli polynomial. The function

$$G(u;\omega) = e^{-\frac{\pi\mathrm{i}}{3}B_{3,3}(u|\omega)}\Gamma_{\tilde{r},\tilde{p}}(e^{-2\pi\mathrm{i}\frac{u}{\omega_3}}), \tag{3.7}$$

where $|\tilde{p}|, |\tilde{r}| < 1$,

$$\begin{aligned} B_{3,3}(u|\omega_1,\omega_2,\omega_3) = \frac{1}{\omega_1\omega_2\omega_3}\Bigg(& u^3 - \frac{3u^2}{2}\sum_{k=1}^3 \omega_k \\ &+ \frac{u}{2}\left(\sum_{k=1}^3 \omega_k^2 + 3\sum_{j<k}\omega_j\omega_k\right) - \frac{1}{4}\left(\sum_{k=1}^3\omega_k\right)\sum_{j<k}\omega_j\omega_k\Bigg) \end{aligned}$$

is the third Bernoulli polynomial, satisfies the same three equations and the normalization as function (3.5). Hence they coincide and their equality reflects one of the $SL(3;\mathbb{Z})$ modular group transformation laws [4]. In general it is expected that the elliptic hypergeometric integrals to be described below represent some automorphic forms in the cohomology class of $SL(3;\mathbb{Z})$.

Evidently, one deals in this picture with three elliptic curves with the modular parameters

$$\tau_1 = \frac{\omega_1}{\omega_2}, \qquad \tau_2 = \frac{\omega_3}{\omega_2}, \qquad \tau_3 = \frac{\omega_3}{\omega_1},$$

satisfying the constraint $\tau_3 = \tau_2/\tau_1$.

From expression (3.7) it follows that $G(u;\omega)$ is a meromorphic function of u for $\omega_1/\omega_2 > 0$, when $|q| = 1$. Therefore, not surprisingly, for $\mathrm{Im}(\omega_3/\omega_1), \mathrm{Im}(\omega_3/\omega_2) \to +\infty$ one has

$$\lim_{p,r\to 0} G(u;\omega) = \gamma(u;\omega_1,\omega_2).$$

For $|q| > 1$ a solution of (3.1) is given by $\Gamma_{p,q^{-1}}(q^{-1}e^{2\pi\mathrm{i}u/\omega_2})^{-1}$.

We use the conventions

$$\begin{aligned} &\Gamma_{p,q}(t_1,\ldots,t_k) := \Gamma_{p,q}(t_1)\cdots\Gamma_{p,q}(t_k), \\ &\Gamma_{p,q}(tz^{\pm 1}) := \Gamma_{p,q}(tz)\Gamma_{p,q}(tz^{-1}), \quad \Gamma_{p,q}(z^{\pm 2}) := \Gamma_{p,q}(z^2)\Gamma_{p,q}(z^{-2}). \end{aligned}$$

Some useful properties of $\Gamma_{p,q}(z)$ are:

$$\theta_p(x;q)_n = \frac{\Gamma_{p,q}(xq^n)}{\Gamma_{p,q}(x)},$$

the reflection equation $\Gamma_{p,q}(z)\Gamma_{p,q}(pq/z) = 1$, the duplication formula

$$\Gamma_{p,q}(z^2) = \Gamma_{p,q}(z,-z,q^{1/2}z,-q^{1/2}z,p^{1/2}z,-p^{1/2}z,(pq)^{1/2}z,-(pq)^{1/2}z),$$

and the limiting relation

$$\lim_{z\to 1}(1-z)\Gamma_{p,q}(z)=\frac{1}{(p;p)_\infty(q;q)_\infty}.$$

We skip consideration of elliptic hypergeometric series which are constructed similarly to the elliptic hypergeometric integrals which we explain now. Also, we stick to the multiplicative notation, skipping analysis of the additive finite difference equations for the integral kernels. Let us define the n-dimensional integrals

$$I(y_1,\ldots,y_m)=\int_{x\in D}\Delta(x_1,\ldots,x_n;y_1,\ldots,y_m)\prod_{j=1}^n\frac{dx_j}{x_j},$$

where $D\subset\mathbb{C}^n$ is some domain of integration and $\Delta(x_1,\ldots,x_n;y_1,\ldots,y_m)$ is a meromorphic function of x_j, y_k, where y_k denote the "external" parameters. The following definition was introduced in [11].

Definition 5. *The integral $I(y_1,\ldots,y_m;p,q)$ is called the elliptic hypergeometric integral if there are two distinguished complex parameters p and q such that the kernel $\Delta(x_1,\ldots,x_n;y_1,\ldots,y_m;p,q)$ (the elliptic hypergeometric term) satisfies the following system of linear first order q-difference equations in the integration variables x_j:*

$$\frac{\Delta(\ldots qx_j\ldots;y_1,\ldots,y_m;p,q)}{\Delta(x_1,\ldots,x_n;y_1,\ldots,y_m;p,q)}=h_j(x_1,\ldots,x_n;y_1,\ldots,y_m;q;p),$$

where the q-certificates h_j, $j=1,\ldots,n$, are some p-elliptic functions of the variables x_k.

Let us describe the explicit form of the univariate, $n=1$, elliptic hypergeometric term Δ. For this we recall that any meromorphic p-elliptic function $f(px)=f(x)$ can be represented as a ratio of theta functions

$$f_p(x)=z\prod_{k=1}^N\frac{\theta_p(t_kx)}{\theta_p(w_kx)},\qquad \prod_{k=1}^N t_k=\prod_{k=1}^N w_k,$$

where $z,t_1,\ldots,t_N,w_1,\ldots,w_N\in\mathbb{C}^*$ are arbitrary variables parametrizing the function's divisor. The integer $N=0,2,3,\ldots$ is called the order of the elliptic function, and, in association with the hypergeometric functions [14], the constraint on products of the parameters is called the balancing condition. Since $z=\theta_p(zx,px)/\theta_p(pzx,x)$, the parameter z can be obtained from the ratios of theta functions by a special choice of the parameters t_k and w_k without violation of the balancing condition. We can thus set $z=1$.

The equation $\Delta(qx)=f_p(x)\Delta(x)$ determining the elliptic hypergeometric terms has the general solution for $|q|<1$

$$\Delta(x)=\varphi(x)\prod_{k=1}^N\frac{\Gamma_{p,q}(t_kx)}{\Gamma_{p,q}(w_kx)},\qquad \varphi(x)=\prod_{k=1}^M\frac{\theta_q(a_kx)}{\theta_q(b_kx)},\quad \prod_{k=1}^M a_k=\prod_{k=1}^M b_k,$$

where $\varphi(qx)=\varphi(x)$ is an arbitrary q-elliptic function of order M. Since

$$\varphi(x)=\prod_{k=1}^M\frac{\Gamma_{p,q}(pa_kx,b_kx)}{\Gamma_{p,q}(a_kx,pb_kx)},$$

we see that $\varphi(x)$ can be obtained from the $\Gamma_{p,q}$-factors as a result of a special choice of the parameters t_k and w_k (such a reduction preserves the balancing condition). So, we can drop $\varphi(x)$ and get the general univariate elliptic hypergeometric term

$$\Delta(x;t_1,\ldots,t_N,w_1,\ldots,w_N;p,q)=\prod_{k=1}^N\frac{\Gamma_{p,q}(t_kx)}{\Gamma_{p,q}(w_kx)},\qquad \prod_{k=1}^N\frac{t_k}{w_k}=1.$$

Note that this function is symmetric in p and q. For incommensurate p and q, the pair of equations

$$\Delta(qx)=f_p(x)\Delta(x),\qquad \Delta(px)=f_q(x)\Delta(x)$$

determines the kernel $\Delta(x)$ up to a multiplicative constant (which may, of course, depend on the parameters t_k and w_k).

It is easy to see that for $|q|>1$, we have

$$\Delta(x;t_1,\ldots,t_N,w_1,\ldots,w_N;p,q)=\prod_{k=1}^N\frac{\Gamma_{p,q^{-1}}(q^{-1}w_kx)}{\Gamma_{p,q^{-1}}(q^{-1}t_kx)}$$

with the same balancing condition. The region $|q|=1$ is considered as in the previous section – it is necessary to pass to the additive form of the equations for integral kernels and express the corresponding elliptic hypergeometric terms in terms of $G(u;\omega)$-functions, which is skipped here.

Although there are some ideas and reasonable arguments how the general elliptic hypergeometric terms should look like in the multivariable setting, we prefer to state it as an open problem — the formulation of an elliptic (or, more generally, a theta-hypergeometric [11]) analogue of the Ore-Sato theorem.

4. Totally elliptic hypergeometric terms

The following definition was introduced by the author in 2001.

Definition 6. *A meromorphic function $f(x_1,\ldots,x_n;p)$ of $n+1$ indeterminates $x_j\in\mathbb{C}^*$ and $p\in\mathbb{C}$ with $|p|<1$ or $|p|>1$ is called totally p-elliptic if*

$$f(px_1,\ldots,x_n;p)=\ldots=f(x_1,\ldots,px_n;p)=f(x_1,\ldots,x_n;p),$$

and if its divisor forms a non-trivial manifold of maximal possible dimension.

The neat point of this definition consists in the demand of absence of constraints on the divisor of the elliptic functions except of those following from the p-ellipticity (i.e., positions of zeros and poles of f should not be prefixed). For instance, the Weierstraß function $\mathcal{P}(u)$ with the periods 1 and τ is not totally elliptic since the positions of its poles and zeros are fixed. Its linear fractional transform can be written as a p-elliptic function $f(z)=f(pz)=\theta_p(az,bz)/\theta_p(cz,dz)$, where $z=e^{2\pi\mathrm{i}u}$, $p=e^{2\pi\mathrm{i}\tau}$, $ab=cd$, but it is not p-elliptic for indeterminates a,b,c (if d is counted as a dependent variable). E.g., the function

$$f(x,y,w,z;p)=\frac{\theta_p(xw^{\pm1},yz^{\pm1})}{\theta_p(xz^{\pm1},yw^{\pm1})}$$

is totally p-elliptic. It was conjectured also that any totally elliptic function is automatically modular invariant.

Using the notion of total ellipticity and the results of [11, 13] it is natural to enrich the structure of elliptic hypergeometric integrals by the following definition.

Definition 7. *An elliptic hypergeometric integral*

$$I(y_1,\ldots,y_m,;p,q)=\int_{x\in D}\Delta(x_1,\ldots,x_n;y_1,\ldots,y_m;p,q)\prod_{j=1}^n\frac{dx_j}{x_j}$$

is called totally elliptic if all q-certificates $h_j(x_1,\ldots,x_n;y_1,\ldots,y_m;q;p)$, including the external y_k-variables certificates

$$h_{n+k}(x;y;q;p)=\frac{\Delta(x;\ldots qy_k\ldots;p,q)}{\Delta(x_1,\ldots,x_n;y_1,\ldots,y_m;p,q)},\quad k=1,\ldots,m,$$

are totally elliptic functions (i.e., they are p-elliptic in x_j,y_k, and q).

Some of the properties of such integrals are described in the following theorem.

Theorem 3 (Rains, Spiridonov, 2004)**.** *Define the meromorphic function*

$$\Delta(x_1,\ldots,x_n;p,q)=\prod_{a=1}^K\Gamma_{p,q}(x_1^{m_1^{(a)}}x_2^{m_2^{(a)}}\ldots x_n^{m_n^{(a)}})^{\epsilon(m^{(a)})},\tag{4.1}$$

where $m_j^{(a)}\in\mathbb{Z}$, $j=1,\ldots,n$, $a=1,\ldots,K$, and $\epsilon(m^{(a)})=\epsilon(m_1^{(a)},\ldots,m_n^{(a)})$ are arbitrary $\mathbb{Z}^n\to\mathbb{Z}$ maps with finite support. Suppose Δ is a totally elliptic hypergeometric term, i.e. all its q-certificates are p-elliptic functions of q and $x_1,\ldots,x_n$. Then these certificates are also modular invariant.

Proof. The q-certificates have the form

$$h_i(x;q;p)=\frac{\Delta(\ldots qx_i\ldots;p,q)}{\Delta(x_1,\ldots,x_n;p,q)}=\prod_{a=1}^K\theta_p(x^{m^{(a)}};q)_{m_i^{(a)}}^{\epsilon(m^{(a)})}.$$

The conditions for h_i to be elliptic in x_j are

$$1=\frac{h_i(\ldots px_j\ldots;q;p)}{h_i(x;q;p)}=\prod_{a=1}^K\Bigg(\Big[-\prod_{l=1}^n x_l^{m_l^{(a)}}\Big]^{-\epsilon(m^{(a)})m_i^{(a)}m_j^{(a)}}$$
$$\times\, q^{-\frac12\epsilon(m^{(a)})m_j^{(a)}m_i^{(a)}(m_i^{(a)}-1)}p^{-\frac12\epsilon(m^{(a)})m_i^{(a)}m_j^{(a)}(m_j^{(a)}-1)}\Bigg),$$

which yield the constraints

$$\sum_{a=1}^K\epsilon(m^{(a)})m_i^{(a)}m_j^{(a)}m_k^{(a)}=0,\tag{4.2}$$
$$\sum_{a=1}^K\epsilon(m^{(a)})m_i^{(a)}m_j^{(a)}=0\tag{4.3}$$

for $1\le i,j,k\le n$. The conditions of ellipticity in q have the form

$$1=\frac{h_i(x;pq;p)}{h_i(x;q;p)}=\prod_{a=1}^K\Bigg(\Big[-\prod_{l=1}^n x_l^{m_l^{(a)}}\Big]^{-\frac12\epsilon(m^{(a)})m_i^{(a)}(m_i^{(a)}-1)}$$
$$\times\, q^{-\frac16\epsilon(m^{(a)})m_i^{(a)}(m_i^{(a)}-1)(2m_i^{(a)}-1)}p^{-\frac16\epsilon(m^{(a)})m_i^{(a)}(m_i^{(a)}-1)(m_i^{(a)}-2)}\Bigg).$$

They add the constraint

$$\sum_{a=1}^{K} \epsilon(m^{(a)}) m_i^{(a)} = 0, \tag{4.4}$$

which guarantees that h_i has equal numbers of theta functions in the numerator and denominator.

Recall the notation $q = e^{2\pi \mathrm{i}\omega_1/\omega_2}$, $\tilde{q} = e^{-2\pi \mathrm{i}\omega_2/\omega_1}$. In terms of the variable $\tau = \omega_1/\omega_2$ the full modular group $SL(2,\mathbb{Z})$ is generated by the transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The function $\theta_q(x)$ is evidently invariant with respect to the first transformation and

$$\theta_{\tilde{q}}(e^{-2\pi \mathrm{i}u/\omega_1}) = e^{\pi \mathrm{i} B_{2,2}(u|\omega_1,\omega_2)} \theta_q(e^{2\pi \mathrm{i}u/\omega_2}),$$

where $B_{2,2}(u|\omega_1,\omega_2)$ is the second Bernoulli polynomial (3.6).

After denoting $x_l = e^{2\pi \mathrm{i}\gamma_l/\omega_2}$ and $p = e^{2\pi \mathrm{i}\omega_3/\omega_2}$, the conditions of modular invariance of $h_i(x;q;p)$ read

$$\begin{aligned} 1 &= \frac{h_i(e^{-2\pi \mathrm{i}\gamma/\omega_3}; e^{-2\pi \mathrm{i}\omega_1/\omega_3}; e^{-2\pi \mathrm{i}\omega_2/\omega_3})}{h_i(e^{2\pi \mathrm{i}\gamma/\omega_2}; e^{2\pi \mathrm{i}\omega_1/\omega_2}; e^{2\pi \mathrm{i}\omega_3/\omega_2})} \\ &= \exp\left[\pi \mathrm{i} \sum_{a=1}^{K} \epsilon(m^{(a)}) \sum_{l=0}^{m_i^{(a)}-1} B_{2,2}\Big(\omega_1 l + \sum_{j=1}^{n} \gamma_j m_j^{(a)} \Big| \omega_2, \omega_3\Big)\right], \end{aligned}$$

and they are automatically satisfied due to the constraints following from the total ellipticity. □

The classification of the totally elliptic hypergeometric terms and, in particular, of all solutions $m^{(a)}, \epsilon(m^{(a)})$ of the Diophantine equations (4.2), (4.3), (4.4) is an open problem.

The following nontrivial elliptic hypergeometric term with $n = 6$ and $K = 29$ was shown to be totally elliptic in [13]

$$\Delta(x;t_1,\ldots,t_6;p,q) = \frac{\prod_{j=1}^{6} \Gamma_{p,q}(t_j x^{\pm 1})}{\Gamma_{p,q}(x^{\pm 2}) \prod_{1\le i<j\le 6} \Gamma_{p,q}(t_i t_j)}, \qquad \prod_{j=1}^{6} t_j = pq,$$

or, after plugging in $t_6 = pq/\prod_{i=1}^{5} t_i$,

$$\Delta(x;t_1,\ldots,t_5;p,q) = \frac{\prod_{j=1}^{5} \Gamma_{p,q}(t_j x^{\pm 1}, t_j^{-1}\prod_{i=1}^{5} t_i)}{\Gamma_{p,q}(x^{\pm 2}, \prod_{i=1}^{5} t_i\, x^{\pm 1})) \prod_{1\le i<j\le 5} \Gamma_{p,q}(t_i t_j)}.$$

Theorem 4 (Spiridonov, 2000). *Elliptic beta integral. If $|t_j| < 1$, then*

$$\kappa \int_{\mathbb{T}} \Delta(x;t_1,\ldots,t_5;p,q) \frac{dx}{x} = 1, \qquad \kappa = \frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}}, \tag{4.5}$$

where $\mathbb{T}$ is the unit circle with positive orientation.

Proof. [13] The partial q-difference equation for the kernel

$$\begin{aligned} &\Delta(x;qt_1,t_2,\ldots,t_5;p,q) - \Delta(x;t_1,\ldots,t_5;p,q) \\ &\qquad = g(q^{-1}x)\Delta(q^{-1}x;,t_1,\ldots,t_5;p,q) - g(x)\Delta(x;t_1,\ldots,t_5;p,q), \end{aligned}$$

where

$$g(x) = \frac{\prod_{m=1}^{5} \theta_p(t_m x)}{\prod_{m=2}^{5} \theta_p(t_1 t_m)} \frac{\theta_p(t_1 \prod_{j=1}^{5} t_j)}{\theta_p(x^2, x\prod_{j=1}^{5} t_j)} \frac{t_1}{x},$$

and its partner obtained after permutation of p and q show that the integral of interest is a constant independent of the parameters t_j. Giving to t_j special values such that the integral is saturated by (i.e., its value is given by) the sum of residues of a fixed pair of poles, we find the needed constant. $\square$

For $p \to 0$ followed by the $t_5 \to 0$ and

$$t_1 = q^{\alpha-1/2},\ t_2 = -q^{\beta-1/2},\ t_3 = q^{1/2},\ t_4 = -q^{1/2}, \quad q \to 1$$

limit, equality (4.5) reduces to the classical Euler beta-integral evaluation

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0.$$

An elliptic analogue of the Gauß hypergeometric function ${}_2F_1(a,b,;c;z)$ has the form [11, 14]

$$V(t_1,\dots,t_8;p,q) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma_{p,q}(t_j x^{\pm 1})}{\Gamma_{p,q}(x^{\pm 2})}\frac{dx}{x}, \qquad \prod_{j=1}^8 t_j = (pq)^2, \tag{4.6}$$

where $|t_j| < 1$. For $t_7 t_8 = pq$ (and other similar restrictions), it reduces to the elliptic beta integral. This function obeys symmetry transformations attached to the exceptional root system E_7 and satisfies the elliptic hypergeometric equation, a second order difference equation with some elliptic coefficients (reducing in a special limit to the standard hypergeometric equation).

5. Multidimensional integrals

Let us introduce the meromorphic function

$$\Delta_n(z,t;p,q) = \prod_{1\le i<j\le n} \frac{1}{\Gamma_{p,q}(z_i^{\pm 1} z_j^{\pm 1})} \prod_{j=1}^n \frac{\prod_{i=1}^{2n+2m+4} \Gamma_{p,q}(t_i z_j^{\pm 1})}{\Gamma_{p,q}(z_j^{\pm 2})},$$

and associate with it the following multidimensional elliptic hypergeometric integral (type I integral for the BC_n root system)

$$I_n^{(m)}(t_1,\dots,t_{2n+2m+4}) = \kappa_n \int_{\mathbb{T}^n} \Delta_n(z,t;p,q) \prod_{j=1}^n \frac{dz_j}{z_j},$$

where $|t_j| < 1$,

$$\prod_{j=1}^{2n+2m+4} t_j = (pq)^{m+1}, \qquad \kappa_n = \frac{(p;p)_\infty^n (q;q)_\infty^n}{2^n n! (2\pi \mathrm{i})^n}.$$

Theorem 5. [3] *The integral* $I_n^{(m)}$ *satisfies the* $2\binom{2n+2m+4}{n+2}$*-set of* $(n+2)$*-term recurrences*

$$\sum_{i=1}^{n+2} \frac{t_i}{\prod_{j=1,\, j\ne i}^{n+2} \theta_p(t_i t_j^{\pm 1})} I_n^{(m)}(t_1,\dots,qt_i\dots,t_{2n+2m+4}) = 0, \tag{5.1}$$

where $\prod_{j=1}^{2n+2m+4} t_j = (pq)^m p$.

Theorem 6. [10] *If $t_1 \cdots t_{2m+2n+4} = (pq)^{m+1} q$, then we have the following $2\binom{2n+2m+4}{m+2}$-set of $(m+2)$-term recurrences for $I_n^{(m)}$:*

$$\sum_{1\le k\le m+2} \frac{\prod_{m+3\le i\le 2n+2m+4} \theta_p(t_i t_k/q)}{t_k \prod_{1\le i\le m+2; i\ne k} \theta_p(t_i/t_k)} I_n^{(m)}(t_1,\ldots,q^{-1}t_k\ldots,t_{2n+2m+4}) = 0.$$

Using these two sets of recurrences, for each variable t_k it is possible to define the fundamental matrix function $M_{ij} \propto I_n^{(m)}(\ \ldots, p^{-1}t_i,\ldots,q^{-1}t_j,\ldots)$, satisfying the two $\binom{n+m}{n} \times \binom{n+m}{n}$-matrix equations

$$M(qt_k) = A(t_k)M(t_k), \qquad M(pt_k) = M(t_k)B(t_k),$$

for some matrices $A(t_k)$ with p-elliptic entries and $B(t_k)$ with q-elliptic entries. Taken together with the condition of meromorphicity of M in each variable t_i, this set of equations determines the matrix M uniquely up to multiplication by a constant [10].

It can be easily checked that

$$\prod_{1\le r<s\le 2n+2m+4} \Gamma_{p,q}(t_r t_s)\, I_m^{(n)}\left(\frac{\sqrt{pq}}{t_1},\ldots,\frac{\sqrt{pq}}{t_{2n+2m+4}}\right)$$

satisfies the same recurrences as $I_n^{(m)}(t_1,\ldots,t_{2n+2m+4})$ (the recurrences are just swapped after replacing $I_n^{(m)}$ by this expression). But there is only one solution of the above system of equations, hence, these two expressions are proportional to each other and the constant of proportionality is found by residue calculus.

Theorem 7. [9] *The integrals $I_n^{(m)}$ satisfy the relation*

$$I_n^{(m)}(t_1,\ldots,t_{2n+2m+4}) = \prod_{1\le r<s\le 2n+2m+4} \Gamma_{p,q}(t_r t_s)\, I_m^{(n)}\left(\frac{\sqrt{pq}}{t_1},\ldots,\frac{\sqrt{pq}}{t_{2n+2m+4}}\right). \tag{5.2}$$

The main new results of the present paper are described in the following two theorems. They represent a natural extension of some of the results of [13] and [10].

Theorem 8. *The ratio*

$$\rho(z,y;t;p,q) = \prod_{1\le r<s\le 2n+2m+4} \Gamma_{p,q}(t_r t_s)^{-1} \frac{\Delta_n(z;t;p,q)}{\Delta_m(y/\sqrt{pq};\sqrt{pq}/t;p,q)}$$

is a totally elliptic hypergeometric term.

Proof. Explicitly,

$$\rho(z,y;t;p,q) = \prod_{1\le r<s\le 2n+2m+4} \frac{1}{\Gamma_{p,q}(t_r t_s)} \frac{\prod_{1\le i<j\le m} \Gamma_{p,q}\left(\frac{y_i y_j}{pq}, \frac{pq}{y_i y_j}, \frac{y_i}{y_j}, \frac{y_j}{y_i}\right)}{\prod_{1\le i<j\le n} \Gamma_{p,q}(z_i^{\pm 1} z_j^{\pm 1})}$$
$$\times \frac{\prod_{j=1}^m \Gamma_{p,q}\left(\frac{y_j^2}{pq}, \frac{pq}{y_j^2}\right)}{\prod_{j=1}^n \Gamma_{p,q}(z_j^{\pm 2})} \prod_{r=1}^{2n+2m+4} \frac{\prod_{j=1}^n \Gamma_{p,q}(t_r z_j^{\pm 1})}{\prod_{j=1}^m \Gamma_{p,q}\left(\frac{y_j}{t_r}, \frac{pq}{t_r y_j}\right)}.$$

This term contains elliptic gamma functions with non-removable integer powers of pq in their arguments showing that ansatz (4.1) is not the most general one. After substitution $y_j \to \sqrt{pq}y_j$, we see that the ρ-function becomes equal to a ratio of the kernels of integrals in (5.2).

The z_k-variable q-certificate is

$$\begin{aligned} h_k^{(z)}(z;t;q;p) &= \frac{\rho(\dots qz_k \dots, y; t; p, q)}{\rho(z, y; t; p, q)} \\ &= \frac{\theta_p(q^{-2}z_k^{-2}, q^{-1}z_k^{-2})}{\theta_p(qz_k^2, z_k^2)} \prod_{i=1,\neq k}^{n} \frac{\theta_p(q^{-1}z_k^{-1}z_i^{\pm 1})}{\theta_p(z_k z_i^{\pm 1})} \prod_{r=1}^{2n+2m+4} \frac{\theta_p(t_r z_k)}{\theta_p(q^{-1}t_r z_k^{-1})}. \end{aligned}$$

The equality $h_k^{(z)}(\dots pz_l \dots) = h_k^{(z)}(z;t;q;p)$ for $l \neq k$ is automatically true, and for $l = k$ it requires the balancing condition

$$\frac{h_k^{(z)}(\dots pz_k \dots)}{h_k^{(z)}(z;t;q;p)} = \frac{(pq)^{2m+2}}{\prod_{r=1}^{2n+2m+4} t_r^2} = 1.$$

The equality $h_k^{(z)}(\dots pt_i, \dots, p^{-1}t_j \dots) = h_k^{(z)}(z;t;q;p)$ is valid automatically. Most non-trivial is the ellipticity in q since the balancing condition contains the multiplier q^{m+1}. Assuming that $t_1 = (pq)^{m+1}/\prod_{r=2}^{2n+2m+4} t_r$, we find

$$\begin{aligned} \frac{h_k^{(z)}(z; p^{m+1}t_1, t_2 \dots; pq; p)}{h_k^{(z)}(z;t;q;p)} &= \frac{1}{(pqz_k)^4} \prod_{i=1,\neq k}^{n} \frac{1}{(pqz_k)^2} \prod_{r=2}^{2n+2m+4} \frac{-pqz_k}{t_r} \\ &\times \frac{(-q^{-1}t_1 z_k^{-1})^m p^{m(m-1)/2}}{(-t_1 z_k)^{m+1} p^{m(m+1)/2}} = 1. \end{aligned}$$

Taken together, these conditions guarantee that $h_k^{(z)}$ is invariant with respect to all transformations $z_l \to p^{n_l} z_l$, $t_r \to p^{m_r} t_r$, $q \to p^N q$ such that $n_l, m_r, N \in \mathbb{Z}$ and $\sum_{r=1}^{2n+2m+4} m_r = N(m+1)$.

The y_k-variables q-certificates are equivalent to $h_k^{(z)}$. Therefore it remains to consider the t_j-variables q-certificates

$$\begin{aligned} h_{ik}^{(t)}(z;t;q;p) &= \frac{\rho(\dots qt_i, \dots, q^{-1}t_k \dots)}{\rho(z, y; t; p, q)} \\ &= \prod_{l=1,\neq i,k}^{2n+2m+4} \frac{\theta_p(q^{-1}t_k t_l)}{\theta_p(t_i t_l)} \prod_{j=1}^{n} \frac{\theta_p(t_i z_j^{\pm 1})}{\theta_p(q^{-1}t_k z_j^{\pm 1})} \prod_{j=1}^{m} \frac{\theta_p(q^{-1}t_i^{-1}y_j, pt_i^{-1}y_j^{-1})}{\theta_p(pqt_k^{-1}y_j^{-1}, t_k^{-1}y_j)}. \end{aligned}$$

The ellipticity in z_j and y_l is easy to check. The invariance of $h_{ik}^{(t)}$ with respect to the transformations $t_a \to pt_a, t_b \to p^{-1}t_b$ for various choices of the indices a and b is also verified without difficulties. Finally, assuming that $t_s = (pq)^{m+1}/\prod_{r=1,\neq s}^{2n+2m+4} t_r$, where $s \neq i, k$, we find

$$\begin{aligned} \frac{h_{ik}^{(t)}(\dots p^{m+1}t_s \dots; pq; p)}{h_{ik}^{(t)}(z;t;q;p)} &= \frac{(-t_i t_s)^{m+1} p^{m(m+1)/2}}{(-q^{-1}t_k t_s)^m p^{m(m-1)/2}} \\ &\times \prod_{l=1,\neq i,k,s}^{2n+2m+4} \frac{-t_k t_l}{pq} \prod_{j=1}^{n} \frac{p^2q^2}{t_k^2} \prod_{j=1}^{m} \frac{1}{t_i t_k} = 1, \end{aligned}$$

and a similar (though even more complicated) picture holds for $s = i$ or $s = k$, which completes the proof. □

As mentioned already, sums of residues of particular sequences of the integral kernel poles form elliptic hypergeometric series. Under special constraints, exact computation or symmetry transformation formulae for integrals reduce to particular totally elliptic functions identities [14]. An analogue of the total ellipticity requirement for elliptic beta integrals was (implicitly) found in [13]. The value of the above theorem consists in the first extension of the latter property to an integral transformation formula. In general, this "ratio of integral kernels" criterion provides a powerful technical tool for conjecturing new relations between integrals.

Indeed, an analogous result can be established for the following type I elliptic hypergeometric integral for the A_n-root system [11]

$$I_n^{(m)}(s_1,\ldots,s_{n+m+2};t_1,\ldots,t_{n+m+2})=\mu_n\int_{\mathbb{T}^n}\Delta_n(z;s,t;p,q)\prod_{j=1}^n\frac{dz_j}{z_j},$$

where

$$\Delta_n(z;s,t;p,q)=\prod_{1\le j<k\le n+1}\frac{1}{\Gamma_{p,q}(z_jz_k^{-1},z_j^{-1}z_k)}\prod_{j=1}^{n+1}\prod_{l=1}^{n+m+2}\Gamma_{p,q}(s_lz_j,t_lz_j^{-1}),$$

and $|t_j|,|s_j|<1$,

$$\prod_{j=1}^{n+1}z_j=1,\qquad \prod_{l=1}^{n+m+2}s_lt_l=(pq)^{m+1},\qquad \mu_n=\frac{(p;p)_\infty^n(q;q)_\infty^n}{(n+1)!(2\pi\mathrm{i})^n}.$$

We denote $T=\prod_{j=1}^{n+m+2}t_j$, $S=\prod_{j=1}^{n+m+2}s_j$, so that $ST=(pq)^{m+1}$, and let all $|t_k|$, $|s_k|$, $|T^{\frac{1}{m+1}}/t_k|$, $|S^{\frac{1}{m+1}}/s_k|<1$. Then [9]

$$\begin{aligned}I_n^{(m)}(s_1,\ldots,s_{n+m+2};t_1,\ldots,t_{n+m+2})&=\prod_{j,k=1}^{n+m+2}\Gamma_{p,q}(t_js_k)\\ &\times I_m^{(n)}\left(\frac{S^{\frac{1}{m+1}}}{s_1},\ldots,\frac{S^{\frac{1}{m+1}}}{s_{n+m+2}};\frac{T^{\frac{1}{m+1}}}{t_1},\ldots,\frac{T^{\frac{1}{m+1}}}{t_{n+m+2}}\right).\end{aligned}\tag{5.3}$$

Theorem 9. *The ratio*

$$\rho(z,y;s,t;p,q)=\frac{\Delta_n(z;s,t;p,q)}{\Delta_m(y;s^{-1},pqt^{-1};p,q)}\prod_{k,r=1}^{n+m+2}\frac{1}{\Gamma_{p,q}(s_kt_r)},$$

where $\prod_{j=1}^n z_j=1$, $\prod_{j=1}^m y_j=S$, *is a totally elliptic hypergeometric term.*

Proof. Explicitly,

$$\begin{aligned}\rho(z,y;s,t;p,q)=&\prod_{1\le k,r\le n+m+2}\frac{1}{\Gamma_{p,q}(s_kt_r)}\prod_{r=1}^{n+m+2}\frac{\prod_{j=1}^{n+1}\Gamma_{p,q}(s_rz_j,t_rz_j^{-1})}{\prod_{j=1}^{m+1}\Gamma_{p,q}(s_r^{-1}y_j,pqt_r^{-1}y_j^{-1})}\\ &\times\frac{\prod_{1\le i<j\le m+1}\Gamma_{p,q}\left(y_i^{-1}y_j,y_iy_j^{-1}\right)}{\prod_{1\le i<j\le n+1}\Gamma_{p,q}(z_i^{-1}z_j,z_iz_j^{-1})}.\end{aligned}$$

After substitution $y_j\to S^{1/(m+1)}y_j$, we see that ρ becomes equal to the ratio of kernels of the integrals occurring in relation (5.3).

The z_k-variables q-certificates have the form

$$h_k^{(z)}(z;s,t;q;p) = \frac{\rho(\ldots qz_k\ldots,q^{-1}z_{n+1},y;t;p,q)}{\rho(z,y;t;p,q)} = \frac{\theta_p(q^{-2}z_k^{-1}z_{n+1},q^{-1}z_k^{-1}z_{n+1})}{\theta_p(qz_kz_{n+1}^{-1},z_kz_{n+1}^{-1})}$$
$$\times \prod_{i=1,\neq k}^{n} \frac{\theta_p(q^{-1}z_iz_k^{-1},q^{-1}z_i^{-1}z_{n+1})}{\theta_p(z_i^{-1}z_k,z_i,z_{n+1}^{-1})} \prod_{r=1}^{n+m+2} \frac{\theta_p(s_rz_k,t_rz_{n+1}^{-1})}{\theta_p(q^{-1}s_rz_{n+1},q^{-1}t_rz_k^{-1})}.$$

Direct computations yield

$$\frac{h_k^{(z)}(\ldots pz_l,\ldots,p^{-1}z_{n+1}\ldots)}{h_k^{(z)}(z;s,t;q;p)} = \frac{(pq)^{m+1}}{\prod_{r=1}^{n+m+2}s_rt_r} = 1, \quad l\neq k,$$
$$\frac{h_k^{(z)}(\ldots pz_k,\ldots,p^{-1}z_{n+1}\ldots)}{h_k^{(z)}(z;s,t;q;p)} = \frac{(pq)^{2(m+1)}}{\prod_{r=1}^{n+m+2}s_r^2t_r^2} = 1,$$
$$\frac{h_k^{(z)}(\ldots ps_a,\ldots,p^{-1}s_b\ldots)}{h_k^{(z)}(z;s,t;q;p)} = \frac{h_k^{(z)}(\ldots ps_a,\ldots,p^{-1}t_b\ldots)}{h_k^{(z)}(z;s,t;q;p)} = 1. \qquad (5.4)$$

The most complicated is the verification of the equality

$$\frac{h_k^{(z)}(\ldots p^{m+1}t_a\ldots;pq;p)}{h_k^{(z)}(z;s,t;q;p)} = \frac{(pq)^{m+1}}{\prod_{r=1}^{n+m+2}s_rt_r} = 1.$$

The y_k-variables q-certificates are equivalent to $h_k^{(z)}$. Therefore we can pass to the t_j,s_k-variables q-certificates (reduced by their permutational symmetry)

$$h_{il}^{(t)}(z,y;s,t;q;p) = \frac{\rho(\ldots qt_i,\ldots,q^{-1}t_l\ldots)}{\rho(z,y;s,t;p,q)}$$
$$= \prod_{k=1}^{n+m+2}\frac{\theta_p(q^{-1}s_kt_l)}{\theta_p(s_kt_i)} \prod_{j=1}^{n+1}\frac{\theta_p(t_iz_j^{-1})}{\theta_p(q^{-1}t_lz_j^{-1})} \prod_{j=1}^{m+1}\frac{\theta_p(pt_i^{-1}y_j^{-1})}{\theta_p(pqt_l^{-1}y_j^{-1})}.$$

This function is automatically invariant with respect to the replacements $t_a\to pt_a$ for any a and fixed other variables, $q\to pq$ (without touching other parameters) as well as $s_a\to ps_a,\ s_b\to p^{-1}s_b$.

The mixing q-certificates have the form

$$h_{ijl}^{(y,s,t)}(z,y;s,t;q;p) = \frac{\rho(\ldots qy_i,\ldots,qs_j,\ldots q^{-1}t_l,\ldots)}{\rho(z,y;s,t;p,q)}$$
$$= \frac{\prod_{r=1,\neq j}^{n+m+2}\theta_p(q^{-1}s_rt_l)}{\prod_{r=1,\neq l}^{n+m+2}\theta_p(s_jt_r)} \prod_{k=1}^{n+1}\frac{\theta_p(s_jz_k)}{\theta_p(q^{-1}t_lz_k^{-1})} \prod_{k=1,\neq i}^{m+1}\frac{\theta_p(q^{-1}s_j^{-1}y_k)}{\theta_p(pqt_l^{-1}y_k^{-1})}$$
$$\times \frac{\prod_{r=1,\neq l}^{n+m+2}\theta_p(pt_r^{-1}y_i^{-1})}{\prod_{r=1,\neq j}^{n+m+2}\theta_p(s_r^{-1}y_i)} \prod_{k=1,\neq i}^{m+1}\frac{\theta_p(y_iy_k^{-1})}{\theta_p(q^{-1}y_i^{-1}y_k)}.$$

The invariance of this function with respect to the transformations described for other certificates is verified in a direct way. E.g., the most complicated case is

$$\frac{h_{ijl}^{(y,s,t)}(\ldots p^{m+1}t_l\ldots;pq;p)}{h_{ijl}^{(y,s,t)}(z,y;s,t;q;p)} = \frac{\prod_{k=1}^{n+1}(-q^{-1}t_l z_k^{-1})^m p^{m(m-1)/2}}{\prod_{r=1,\neq j}^{n+m+2}(-q^{-1}s_r t_l)^m p^{m(m-1)/2}}$$
$$\times \prod_{k=1,\neq i}^{m+1} \frac{(-y_k)^{m+1}p^{m(m+1)/2}}{(pq)^{m+1}s_j t_l^{-m}} \prod_{k=1,\neq i}^{m+1} \frac{-pqy_i}{y_k} = 1.$$

To summarize, all q-certificates have the symmetries

$$z_k \to p^{n_k}z_k, \quad y_j \to p^{m_j}y_j, \quad s_l \to p^{\alpha_l}s_l, \quad t_l \to p^{\beta_l}t_l, \quad q \to p^N q,$$

where $n_k, m_j, \alpha_l, \beta_l, N \in \mathbb{Z}$ and $\sum_{k=1}^{n+1} n_k = 0$, $\sum_{j=1}^{m+1} m_j = \sum_{l=1}^{n+m+2} \alpha_l$, and $\sum_{l=1}^{n+m+2}(\alpha_l+\beta_l) = N(m+1)$. This completes the proof of the theorem. □

In the univariate case, $n = 1$, both type I A_n and BC_n-integrals coincide with the $V(t_1,\ldots,t_8;p,q)$-function described in the previous section. The corresponding symmetry transformations are mere consequences of the E_7-root system transformation found in [11]. The simplest $p \to 0$ limit of the arbitrary rank integrals reduces them to the Gustafson integrals [8].

In joint work with Vartanov [15, 16] we gave a large list of known simple transformation formulae for elliptic hypergeometric integrals related to the Weyl groups (including the cases of computable integrals). There are about seven proven different multiple elliptic beta integrals associated to root systems and a similar number of proven symmetry transformations for higher order elliptic hypergeometric functions. Additionally, there are about fifteen conjectures for new elliptic beta integrals and a similar number of new conjectured transformation formulae for integrals with a bigger number of parameters. In particular, there are interesting examples of elliptic hypergeometric terms having fractional powers of pq in the elliptic gamma function arguments. In such cases the total ellipticity condition should be modified — it is necessary to consider dilations by appropriate powers of q (or p). Using the preceding two theorems as a starting point, for all these relations we have checked validity of the following conjecture.

Conjecture. [16]. *The condition of total ellipticity for elliptic hypergeometric terms is necessary for the existence of exact evaluation formulae for elliptic beta integrals or of nontrivial Weyl group symmetry transformations for higher order elliptic hypergeometric functions (emerging in the "ratio of integral kernels" spirit, as described above).*

Most of the new relations for integrals were inspired by the Seiberg dualities for $\mathcal{N} = 1$ four dimensional supersymmetric field theories, where elliptic hypergeometric integrals play the role of superconformal (topological) indices with an appropriate matrix integral representation. For further details, see [16]. A few more identities are conjectured in recent interesting papers by Gadde et al [5, 6] as a consequence of known dualities for $\mathcal{N} = 2$ four dimensional superconformal field theories. Here it should be stressed that there are actually infinitely many symmetry relations for elliptic hypergeometric integrals generated in a recursive way from some canonical exact formulae, see [12] for a tree of identities following from the univariate elliptic beta integral and its root system generalizations following from the integral transforms of [17]. Therefore, many identities are, in fact, relatively simple consequences of some universal relations, whose enumeration would be of most interest.

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