RADON TRANSFORM ON REAL SYMMETRIC VARIETIES: KERNEL AND COKERNEL

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1. Introduction

Our concern is with

$$Y = G/H$$

a semisimple irreducible real symmetric variety (space).[1]

Our concern is with

$$L^2(Y)$$

the space of square integrable function on Y with respect to a G-invariant measure. This Hilbert space has a natural splitting

$$L^2(Y) = L^2_{\mathrm{mc}}(Y) \oplus L^2_{\mathrm{mc}}(Y)^\perp$$

into most-continuous part and its orthocomplement. There is a Schwartz space $\mathcal{S}(Y) \subset L^2(Y)$ of rapidly decaying functions. With $\mathcal{S}_{\mathrm{mc}}(Y) = L^2_{\mathrm{mc}}(Y) \cap S(Y)$ and $\mathcal{S}_{\mathrm{mc}}(Y)^\perp = L^2_{\mathrm{mc}}(Y)^\perp \cap S(Y)$ one has

$$\mathcal{S}(Y) = \mathcal{S}_{\mathrm{mc}}(Y) \oplus \mathcal{S}_{\mathrm{mc}}(Y)^\perp .$$

Our concern is with the parameter space of generic real horocycles

$$\Xi = G/(M \cap H)N$$

where MAN is a minimal $\sigma\theta$-stable[2] parabolic subgroup of G.

Write $BC^\infty(\Xi)$ for the space of bounded smooth functions on Ξ. In this paper we verify the following facts:

- The map

$$\mathcal{R} : \mathcal{S}(Y) \to BC^\infty(\Xi), f \mapsto \left(gM_H N \mapsto \int_N f(gnH)\ dn \right)$$

  is well defined. (We call $\mathcal{R}$ the (minimal) Radon transform)
- $\mathcal{R}|_{S_{\mathrm{mc}}(Y)^\perp} = 0$.
- $\mathcal{R}|_{S_{\mathrm{mc}}(Y)}$ is injective.

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[1] This means G is a connected real semisimple Lie group, H is the fixed point group of an involutive automorphism σ of G such that there is no σ-stable normal subgroup $H \subset L \subset G$ with $\dim H < \dim L < \dim G$.

[2] θ is a Cartan involution commuting with σ.

2. Real symmetric varieties

2.1. Notation

The objective of this section is to introduce notation and to recall some facts regarding real symmetric varieties.

Let $G_{\mathbb{C}}$ be a simply connected linear algebraic group whose Lie algebra $\mathfrak{g}_{\mathbb{C}}$ we assume to be semi-simple. We fix a real form G of $G_{\mathbb{C}}$: this means that G is the fixed point set of an involutive automorphism σ of $G_{\mathbb{C}}$ and that $\mathfrak{g}$, the Lie algebra of G, yields $\mathfrak{g}_{\mathbb{C}}$ after complexifying.

Let now τ be a second involutive automorphism of $G_{\mathbb{C}}$ which we request to commute with σ. In particular, τ stabilizes G. We write

$$H_{\mathbb{C}} := G_{\mathbb{C}}^{\tau} \quad \text{and} \quad H := G^{\tau}$$

for the corresponding fixed point groups of τ in G, resp. $G_{\mathbb{C}}$. We note that $H_{\mathbb{C}}$ is always connected, but H usually is not; the basic example of $(G_{\mathbb{C}}, G) = (\mathrm{Sl}(2,\mathbb{C}), \mathrm{Sl}(2,\mathbb{R}))$ and $(H_{\mathbb{C}}, H) = (\mathrm{SO}(1,1;\mathbb{C}), \mathrm{SO}(1,1;\mathbb{R})$ already illistrates the situation.

With G and H we form the object of our concern

$$Y = G/H\,;$$

we refer to Y as a *real (semi-simple) symmetric variety (or space)*. Henceforth we will denote by $y_o = H$ the standard base point in Y.

At this point it is useful to introduce infinitesimal notation. Lie groups will always be denoted by upper case Latin letters, e.g. G, H, K etc., and the corresponding Lie algebras by lower case German letters, eg. $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ etc. It is convenient to use the same symbol τ for the derived automorphism $d\tau(\mathbf{1})$ of $\mathfrak{g}$. Let us denote by $\mathfrak{q}$ the -1-eigenspace of τ on $\mathfrak{g}$. Note that $\mathfrak{q}$ is an H-module which naturally identifies with the tangent space $T_{y_o}Y$ at the base point.

From now we will request that Y is irreducible, i.e. we assume that the only τ-invariant ideals in $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$. In practice this means that G is simple except for the group case $G/H = H \times H/H \simeq H$.

We recall that maximal compact subgroups $K < G$ are in one-to-one correspondences with Cartan involutions $\theta : G \to G$. The correspondence is given by $K = G^{\theta}$. We form the Riemann symmetric space

$$X = G/K$$

of the non-compact type and denote by $x_o = K$ the standard base point. As before we write θ for the derived involution on $\mathfrak{g}$. We let $\mathfrak{p} \subset \mathfrak{g}$ be the -1-eigenspace θ and note that the K-module $\mathfrak{p}$ identifies with $T_{x_o}X$.

According to Berger, we may (and will) assume that K is τ-invariant. This implies that both $\mathfrak{h}$ and $\mathfrak{q}$ are τ-stable. Let us fix a maximal abelian subspace

$$\mathfrak{a} \subset \mathfrak{q} \cap \mathfrak{p} .$$

We wish to point out that $\mathfrak{a}$ is unique modulo conjugation by $H \cap K$, see [3], Lemma 7.1.5. Set $A = \exp(\mathfrak{a})$.

Our next concern is the centralizer $Z_G(A)$ of A. We first remark that there is a natural splitting

$$Z_G(A) = A \times M .$$

The Lie algebra of M is given by

$$\mathfrak{m} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{a}^\perp$$

where $\mathfrak{a}^\perp$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form κ of $\mathfrak{g}$. We write M_{ns} for the noncompact semisimple part of M and note that

$$M_{ns} \subset H \tag{2.1}$$

(cf. [3], Lemma 7.1.4). Set

$$M_H = M \cap H = Z_H(A) .$$

Let $\mathfrak{m} = \mathfrak{m}_h + \mathfrak{m}_q$ is the splitting of $\mathfrak{m}$ into $+1$ and -1-eigenspace and note that $\mathfrak{m}_h$ is the Lie algebra of M_H. Then (2.1) implies that $\mathfrak{m}_q \subset \mathfrak{k}$ and $M_Q = \{m \in M \mid \tau(m) = m^{-1}\}$ is compact. Moreover:

$$M = M_H M_Q \quad \text{where} \quad M_Q \subset K \quad \text{and} \quad M_H \cap M_Q \quad \text{discrete} .$$

We turn our attention to the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^*$, let

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid (\forall Y \in \mathfrak{a})\ [Y, X] = \alpha(Y)X\}$$

and set

$$\Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq \{0\}\} .$$

It is a fact, due to Matsuki and Rossmann, that Σ is a (possibly reduced) root system, cf. [3], Prop. 7.2.1. Hence we may fix a positive system $\Sigma^+ \subset \Sigma$ and define a corresponding nilpotent subalgebra

$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha .$$

Set $N := \exp(\mathfrak{n})$. Note that $\tau(\mathfrak{n}) = \theta(\mathfrak{n})$. We record the decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \tau(\mathfrak{n}) .$$

We shift our focus to the real flag manifold of G associated to A and Σ^+. We define

$$P_{\min} := MAN$$

and note that $P_{\min}$ is a minimal $\theta\tau$-stable parabolic subgroup of G.

The open H-orbit decomposition on the flag manifold $G/P_{\min}$ is essential in the theory of H-spherical representations of G. In order to describe this decomposition we have to collect some facts on Weyl groups first.

Let us denote by $\mathcal{W}$ the Weyl group of the root system Σ. The Weyl group admits an analytic realization:

$$\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a}) .$$

The group $\mathcal{W}$ features a natural subgroup

$$\mathcal{W}_H := N_{H\cap K}(\mathfrak{a})/Z_{H\cap K}(\mathfrak{a}) .$$

Knowing $\mathcal{W}$ and $\mathcal{W}_H$, we can quote the Matsuki-Rossmann decomposition of G into open $H \times P_{\min}$-cosets:

$$G \doteq \amalg_{w\in W_H\backslash\mathcal{W}} HwP_{\min} , \tag{2.2}$$

where $\doteq$ means equality up to a finite union of strictly lower dimensional $H \times P_{\min}$-orbits.

2.2. **Horocycles**

This small section is devoted to horocycles on the symmetric variety Y. By a *(generic) horocycle* on Y we understand an orbit of a conjugate of N of maximal dimension (i.e. $\dim N$). The entity of all horocycles will be denoted by $\mathrm{Hor}(Y)$. We remark that G acts naturally on it.

Next our concern is with an appropriate parameter space for $\mathrm{Hor}(Y)$. We introduce the G-manifold

$$\Xi = G/M_H N$$

and regard the map

$$E : \Xi \to \mathrm{Hor}(Y), \quad \xi = gM_H N \mapsto E(\xi) = gN \cdot y_o .$$

As $N \cap H = \{\mathbf{1}\}$, the map is defined. Moreover, E is G-equivariant and one establishes as in [?] or [?] that E is a bijection.

3. Schwartz space and the definition of the Radon transform

3.1. Schwartz space

To begin with we shall define the Schwartz space on Y. We recall that

$$G = KAH \tag{3.1}$$

and often refer to (3.1) as the polar decomposition of G (with respect to H and K). Accordingly every $g \in G$ can be written as $g = k_g a_g h_g$ with $k_g \in K$ etc. It is important to notice that a_g is unique modulo $\mathcal{W}_H$. Therefore the prescription

$$\|gH\| := |\log a_g| \qquad (g \in G)$$

is well defined for $|\cdot|$ the Killing norm on $\mathfrak{p}$. An alternative, and often useful, description of $\|\cdot\|$ is as follows

$$\|y\| = \frac{1}{4}\left|\log\left[y\tau(y)^{-1}\theta(y\tau(y)^{-1})^{-1}\right]\right| \qquad (y \in Y)\,. \tag{3.2}$$

For $u \in \mathcal{U}(\mathfrak{g})$ we write L_u for the corresponding differential operator on Y, i.e. for $u \in \mathfrak{g}$

$$(L_u f)(y) = \frac{d}{dt}\Big|_{t=0} f(\exp(-tu)y)\,,$$

whenever f is a differentiable function at y. With these preliminaries one defines

$$\mathcal{S}(Y) = \{f \in C^\infty(Y) \mid \forall u \in \mathcal{U}(\mathfrak{g})\ \forall n \in \mathbb{N} \quad \sup_{y \in Y} \Theta(y)(1 + \|y\|)^n |(L_u f)(y)| < \infty\}$$

where $\Theta(gH) = \phi_0(g\tau(g)^{-1})^{-1/2}$ and ϕ_0 Harish-Chandra's basic spherical function.

It is not to hard to see that $\mathcal{S}(Y)$ with the obvious family of defining seminorms is a Fréchet space. Moreover $\mathcal{S}(Y)$ is G-invariant and G acts smoothly on it. We note that $\mathcal{S}(Y) \subset L^2(Y)$ is a dense subspace.

We move from Y to Ξ and recall that

$$G = KAM_H N \tag{3.3}$$

and remark that the A-component $a(g)$ of an element $g \in G$ is unique. For our analysis discussion of Ξ a less sophisticated space will suffice:

$$BC^\infty(\Xi),$$

the G-Fréchet module of smooth and bounded functions.

3.2. **Definition of the Radon transform**

We state the result.

Theorem 3.1. *Let $f \in \mathcal{S}(Y)$. Then the following assertions hold:*

(i) *The integral $\int_N f(nH)\, dn$ is absolutely convergent.*

(ii) *The prescription*

$$gM_H N \mapsto \int_N f(gnH)\, dn$$

defines a function in $BC^\infty(\Xi)$.

It follows from the theorem that the map

$$\mathcal{R} : \mathcal{S}(Y) \to BC^\infty(\Xi), \quad f \mapsto \mathcal{R}(f);\ \mathcal{R}(f)(gM_H N) = \int_N f(gnH)\, dn$$

is defined and G-equivariant. We refer to $\mathcal{R}$ as the *(most-continuous) Radon transform* of the symmetric space Y.

The proof of the theorem is a familiar and rather standard exercise in technical matters (see [4], Thm. 7.2.1 for a similar result and in particular p. 232 - 233). Thus we will confine ourselves with a sketch based on the main example.

Proof. Let us confine ourselves to the basic case of $Y = G/H = \mathrm{Sl}(2, \mathbb{R})/\mathrm{SO}(1,1)$ with A the diagonal group and $N = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix}$.

(i) For $x \in \mathbb{R}$ and $n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ we have to determine $a_x \in A$ such that $n_x \in K a_x H$. We use (3.2) and start:

$$z_x := n_x \tau(n_x)^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 - x^2 & -x \\ x & 1 \end{pmatrix}$$

and hence

$$\begin{aligned} y_x := z_x \theta(z_x)^{-1} &= \begin{pmatrix} 1 - x^2 & -x \\ x & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 - x^2 & x \\ -x & 1 \end{pmatrix} \\ &= \begin{pmatrix} (1 - x^2)^2 + x^2 & * \\ * & 1 + x^2 \end{pmatrix}. \end{aligned}$$

For $|x|$ large we have $\log|y_x| = |\log y_x|$. Furthermore up to an irrelevant constant

$$|y_x| = [\mathrm{tr}(y_x y_x)]^{\frac{1}{2}} \geq \frac{1}{2}[(1-x^2)^2 + x^2 + 1 + x^2]$$
$$\geq \frac{1}{2}[x^4+1]$$

Therefore, for $|x|$ large

$$\|n_x\| \geq \frac{1}{4}\log(x^4/2 + 1/2)$$

From Harish-Chandra's basic estimates of ϕ_0 and our computation of z_x we further get that

$$\Theta(n_x) \geq |x|\,.$$

Therefore for $f \in \mathcal{S}(Y)$ we obtain that $x \mapsto |f(n_x H)|$ growths slower than $\frac{1}{|x|\cdot|\log x|^N}$ for any fixed $N > 0$ and $|x|$ large. This shows (i).
(ii) Let $f \in \mathcal{S}(Y)$ and set $F := \mathcal{R}(f)$. From the proof of (i) we know that F is smooth. It remains to see that F is bounded. From (3.3) we see that it is enough to show that $F|_A$ is bounded. We do this by direct computation. For $t > 0$ we set

$$a_t = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}.$$

Then

$$a_t n_x = \begin{pmatrix} t & tx \\ 0 & 1/t \end{pmatrix}$$

and thus

$$z_{t,x} := a_t n_x \tau(a_t n_x)^{-1} = \begin{pmatrix} t & tx \\ 0 & 1/t \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ -tx & 1/t \end{pmatrix}$$
$$= \begin{pmatrix} t^2(1-x^2) & -x \\ x & 1/t^2 \end{pmatrix}.$$

With that we get

$$y_{t,x} = z_{t,x}\theta(z_{t,x})^{-1} = \begin{pmatrix} t^4(1-x^2)^2 + x^2 & * \\ * & 1/t^4 + x^2 \end{pmatrix}.$$

For $t \geq 1$ we conclude that

$$\|a_t n_x\| \gtrsim \log\left(\begin{cases} c_1 t^4 & \text{for } |x| \leq 1/2, \\ c_2 t^4 x^4 - c_3 & \text{for } |x| \geq 1/2\,. \end{cases}\right)$$

and for $|t| < 1$ one has

$$\|a_t n_x\| \geq \log |x| \,.$$

From that we obtain (ii). □

4. The kernel of the Radon transform: discrete spectrum

In this section we show that the discrete spectrum of $L^2(Y)$ lies in the kernel of $\mathcal{R}$. In fact we shoe even more: namely $\mathcal{R}_{\mathcal{S}_{\rm mc}(Y)^\perp} = 0$.

Recall our minimal $\theta\tau$-stable parabolic subgroup

$$P_{\rm min} = MAN \,.$$

In the sequel we use the symbol Q for a $\theta\tau$-stable parabolic which contains $P_{\rm min}$. There are only finitely many. We write

$$Q = M_Q A_Q N_Q$$

for its standard factorization and observe:

- $M_Q \supset M$,
- $A_Q \subset A$,
- $N_Q \subset N$.

Next we let

$$L^2(Y) = \bigoplus_{Q \supset P_{\rm min}} L^2(Y)_Q$$

where $L^2(Y)_Q$ stands for the part corresponding to representations which are induced off from Q by discrete series of $M_Q/M_Q \cap H$.

Let $\mathcal{S}(Y)_Q = L^2(Y)_Q \cap \mathcal{S}(Y)$. We observe that:

$$\mathcal{S}(Y)_Q \subset L^2(Y)_Q \quad \text{is dense} \,. \tag{4.1}$$

Indeed this can be deduced from the fact that $\mathcal{S}(Y)_Q$ is stable under convolution with function from $L^1(G)$ of appropriate rapid decay.

For the extreme choices of Q there is a special terminology:

$$L^2(Y)_{\rm disc} := L^2(Y)_G \quad \text{and} \quad L^2(Y)_{\rm mc} := L^2(Y)_{P_{\rm min}}$$

and one refers to the *discrete* and *most continuous* part of the square-intergrable spectrum. Likewise we declare $\mathcal{S}(Y)_{\rm disc}$ and $\mathcal{S}(Y)_{\rm mc}$.

Theorem 4.1. $\mathcal{R}(\mathcal{S}(Y)_{\rm disc}) = \{0\}$.

Proof. The proof is the same as for the group, see [4], Th. 7.2.2 for a useful exposition.

Let $f \in \mathcal{S}(Y)_{\rm disc}$. We have to show that $\mathcal{R}(f) = 0$. By standard density arguments we may assume that f belongs to a single discrete series representation and that f is K-finite. Let

$$V = \mathcal{U}(\mathfrak{g}_\mathbb{C})f$$

be the corresponding Harish-Chandra module and set $T := \mathcal{R}|_V$. Then T factors over the Jacquet module $j(V) = V/\mathfrak{n}V$. We recall that $j(V)$ is an admissible finitely generated $\mathfrak{m} + \mathfrak{a}$-module. Hence

$$\dim \mathcal{U}(\mathfrak{a})T(f) < \infty\,.$$

Consequently

$$T(f)(aM_HN) = \sum_\mu a^\mu p_\mu(\log a) \qquad (a \in A)$$

where μ runs over a finite subset in $\mathfrak{a}_\mathbb{C}^*$ and p_μ is a polynomial (see [4], 8.A.2.10). From $T(f) \in BC^\infty(\Xi)$ we thus conclude that $T(f) = 0$ as was to be shown. □

As a consequence of the previous theorem we obtain the main result of this subsection.

Theorem 4.2. *Let $Q \supsetneq P_{\min}$. Then $\mathcal{R}(\mathcal{S}(Y)_Q) = \{0\}$.*

Proof. If $Q = G$, then this part of the previous theorem. The general case will be reduced to that. So suppose that $P_{\min} \subsetneq Q \subsetneq G$. We first observe that

$$N = N_Q \rtimes N^Q$$

with $\{\mathbf{1}\} \neq N^Q \subset M_Q$. Accordingly we have

$$\mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2 \tag{4.2}$$

with $\mathcal{R}_1 = \int_{N_Q}$ and $\mathcal{R}_2 = \int_{N^Q}$. Let now $f \in \mathcal{S}(Y)_Q$. Without loss of generality we may assume that f corresponds to a wave packet which is associated to a single discrete series σ of $M_Q/M_Q \cap H$. By the previous theorem we conclude that $\mathcal{R}_2(f) = 0$ and this completes the proof. □

5. Restriction of the Radon transform to the most-continuous spectrum

The objective of this section is to show that $\mathcal{R}$ is faithful on the most-continuous spectrum.

We recall a few facts on the spectrum of $L^2(\Xi)$ and the most-continuous spectrum on Y and start with the "horocyclic picture". The homogeneous space Ξ carries a G-invariant measure. Consequently left shifts by G in the argument of a function on Ξ yields a unitary representation , say L, of G on $L^2(\Xi)$; in symbols

$$(L(g)f)(\xi) = f(g^{-1}\cdot\xi) \qquad (f \in L^2(\Xi), g \in G, \xi \in \Xi)\,.$$

It is important to note that the G-action on Ξ admits a commutating action of A from the right

$$\xi \cdot a = gaMN \qquad (\xi = gM_HN \in \Xi, a \in A);$$

this is because A normalizes M_HN. Therefore the description

$$(R(a)f)(\xi) = a^\rho \cdot f(\xi \cdot a) \qquad (f \in L^2(\Xi), a \in A, \xi \in \Xi)$$

defines a unitary representation $(R, L^2(\Xi))$ of A which commutes with the G-representation L.

For each $\lambda \in \mathfrak{a}_\mathbb{C}^*$ let us set

$$L^2(\Xi)_\lambda := \{f : G \to \mathbb{C} \mid \begin{array}{l} \bullet\ f \text{ measurable}, \\ \bullet\ f(\cdot man) = a^{-\rho-\lambda} f(\cdot)\ \forall man \in P_{\min}, \\ \bullet\ \int_K |f(k)|^2 \, dk < \infty\} \end{array}$$

Likewise we write $C^\infty(\Xi)_\lambda$ for the smooth elements of $L^2(\Xi)_\lambda$. The disintegration of $L^2(\Xi)$ is then given by

$$L^2(\Xi) = \int_{i\mathfrak{a}^*}^\oplus L^2(\Xi)_\lambda \, d\lambda\,.$$

One obtains the inclusion

$$\mathcal{S}(\Xi) \subset \int_{i\mathfrak{a}^*}^\oplus C^\infty(\Xi)_\lambda \, d\lambda\,.$$

The decomposition of $L^2(\Xi)_\lambda$ into irreducible G-modules is now very simple. We observe that M acts on Ξ from the right and hence induces a unitary representation on $L^2(\Xi)_\lambda$ by

$$(R_\lambda(m)f)(\xi) = f(\xi \cdot m) \qquad (m \in M, f \in L^2(\Xi)_\lambda, \xi \in \Xi)\,.$$

Note that R_λ is trivial on M_H, that M_H is an (infinetisimal) factor of M and that M/M_H is compact. Thus the M_H-spherical unitary dual $\widehat{M/M_H}$ of M is discrete and each $\sigma \in \widehat{M/M_H}$ gives rise to a module

$$L^2(\Xi)_{\sigma,\lambda}$$

which consists of of those elements of $L^2(\Xi)_\lambda$ which transform under R_λ as σ . Consequently

$$L^2(\Xi)_\lambda = \bigoplus_{\sigma\in\widehat{M/M_H}} L^2(\Xi)_{\sigma,\lambda}\,.$$

In the next step we recall the Plancherel decomposition for the most continuous spectrum (cf. [1]).

Some generalities upfront. For a representation π of a group L on some topological vector space V we denote by π^* the dual representation on the (strong) topological dual V^* of V.

Let $\sigma \in \widehat{M/M_H}$ and V_σ a unitary representation module for σ. For thechnical reasons it is now more convenient for us to work with the opposite parabolic $\overline{P_{\min}} = MA\overline{N}$ with $\overline{N} = \theta(N)$.

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we define

$$\begin{aligned}\mathcal{H}_{\sigma,\lambda} := \{f : G \to V_\sigma \mid &\bullet\ f \text{ measurable},\\ &\bullet\ f(\cdot ma\overline{n}) = a^{\rho+\lambda}\sigma(m)^{-1}f(\cdot)\ \forall ma\overline{n} \in \overline{P_{\min}},\\ &\bullet\ \int_K \langle f(k), f(k)\rangle_\sigma\, dk < \infty\}\,.\end{aligned}$$

The group G acts on $\mathcal{H}_{\sigma,\lambda}$ by displacements from the left and the so-obtained Hilbert representation will be denoted by $\pi_{\sigma,\lambda}$. Sometimes it is useful to realize $\mathcal{H}_{\sigma,\lambda}$ as V_σ-valued functions on N; we speak of the non-compact realization then. Define a weight function on N by

$$w_\lambda(n) = a^{-2\operatorname{Re}\lambda}$$

where $a \in A$ is determined by $n \in Ka\overline{N}$. Then the map

$$\mathcal{H}_{\sigma,\lambda} \to L^2(N, w_\lambda(n)dn) \otimes V_\sigma, \quad f \mapsto f|_N$$

is an isometric isomorphism.

We remark that:

- $\pi_{\sigma,\lambda}$ is irreducible for generic λ.
- $\pi_{\sigma,\lambda}$ is unitary for $\lambda \in i\mathfrak{a}^*$.
- The dual representation of $\pi_{\sigma,\lambda}$ is canonically isomorphic to $\pi_{\sigma^*,-\lambda}$; the dual pairing is given by

$$\langle f, g\rangle := \int_N (f(n), g(n))_\sigma\, dn$$

for $f \in \mathcal{H}_{\sigma,\lambda}$, $g \in \mathcal{H}_{\sigma^*,-\lambda}$ and $(\,,\,)_\sigma$ the natural pairing between V_σ and V_σ^*.

- For increasing $\operatorname{Re}\lambda$ the decay rate of smooth vectors $\mathcal{H}_{\sigma,\lambda}^\infty$ (in the non-compact model) increases.

Next we wish to recall the H-fixed elements in the distribution module $(\mathcal{H}_{\sigma,\lambda}^\infty)^*$. We first set for each $w \in \mathcal{W}/\mathcal{W}_H$

$$V^*(\sigma, w) := (V_\sigma^*)^{wM_Hw^{-1}}$$

and then

$$V^*(\sigma) := \bigoplus_{w\in\mathcal{W}/\mathcal{W}_H} V^*(\sigma, w)\,.$$

For each w we denote by

$$V^*(\sigma) \to V^*(\sigma, w), \quad \eta \mapsto \eta_w\,.$$

the orthogonal projection. In the sequel we will use the terminolgy $\operatorname{Re}\lambda >> 0$ if

$$(\operatorname{Re}\lambda - \rho)(\alpha^\vee) > 0 \quad \forall \alpha \in \Sigma^+\,.$$

Then, for $\operatorname{Re}\lambda >> 0$ the description

$$j(\sigma^*, -\lambda)(\eta)(g) = \begin{cases} a^{\rho-\lambda}\sigma^*(m^{-1})\eta_w & \text{if } g = hwma\overline{n} \in HwMA\overline{N}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

defines a continuous H-fixed element in $\mathcal{H}_{\sigma^*,-\lambda}$. We may meromorphically continue $j(\sigma^*, \cdot)$ in the λ-variable and obtain, for generic values of λ the identity

$$j(\sigma^*, -\lambda)(V(\sigma^*)) = ((\mathcal{H}_{\sigma,\lambda}^\infty)^*)^H\,.$$

For a smooth vector $v \in \mathcal{H}_{\sigma,\lambda}$ and $\eta \in V(\sigma^*)$ we obtain a smooth function on $Y = G/H$ by setting

$$F_{v,\eta}(gH) = \langle \pi_{\sigma,\lambda}(g^{-1})v, j(\sigma^*, -\lambda)(\eta)\rangle\,.$$

The Plancherel theorem for $L^2(Y)_{\mathrm{mc}}$, see for instance [1], then asserts the existence of a meromorphic assignment

$$\mathfrak{a}_\mathbb{C}^* \to \mathrm{Gl}(V^*(\sigma)), \quad \lambda \mapsto C(\sigma, \lambda)$$

such that the map

$$\widehat{\bigoplus}_{\sigma\in\widehat{M/M_H}} \int_{i\mathfrak{a}_+^*}^{\oplus} \mathcal{H}_{\sigma,\lambda} \otimes V(\sigma^*)\; d\lambda \to L^2(Y)_{\mathrm{mc}}$$

which for smooth vectors on the left is defined by

$$\sum_\sigma (v_{\sigma,\lambda}\otimes\eta)_\lambda \mapsto \left(gH \mapsto \sum_\sigma \int_{i\mathfrak{a}_+^*} F_{v_{\sigma,\lambda},C(\sigma,\lambda)\eta}(gH)\ d\lambda\right)$$

extends to a unitary G-equivalence.

Theorem 5.1. *$\mathcal{R}$ restricted to $\mathcal{S}(Y)_{\rm mc}$ is injective.*

Proof. By the $G\times M$-equivariance of the Radon transform it is sufficient to prove injectivity for the restriction to

$$\int_{i\mathfrak{a}_+^*}^{\oplus} \mathcal{H}_{\sigma,\lambda}\otimes V(\sigma^*)\ d\lambda\,.$$

So let us fix σ. To begin with we first observe that $\mathcal{R}$ is defined stalkwise provided $\lambda \gg 0$ is large enough. In fact let $\lambda\gg 0$ and let $\phi\in C_c^\infty(N)\otimes V_\sigma \subset \mathcal{H}_{\sigma,\lambda}^\infty$ (we use the non-compact model now). Let $\eta\in V^*(\sigma)$ and note that $j(\sigma^*,-\lambda)\eta$ is a continuous function on N with polynomial growth. Accordingly

$$\mathcal{R}(F_{\phi,\eta})(aM_HN) = \int_N\int_N (\phi(ann'), j(\sigma^*,-\lambda)(\eta)(n'))_\sigma\ dn'\ dn$$

is defined for all $a\in A$. Thus in the large parameter range, $\mathcal{R}$ is a well defined integral operator. In particular it is faithful there, Naturally the faithfulness extends analytically by standard arguments. □

Remark 5.2. *It is not hard to show that*

$$\mathcal{R}\left(\mathcal{S}(Y)\cap\int_{i\mathfrak{a}_+^*}^{\oplus}\mathcal{H}_{\lambda,\sigma}\otimes C(\sigma,\lambda)^{-1}V^*(\sigma,w)\ d\lambda\right)\subset \int_{iw\mathcal{W}_H\mathfrak{a}_+^*} C^\infty(\Xi)_{\sigma,\lambda}\,d\lambda$$

for all $w\in\mathcal{W}$.

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