CLOSED 1-FORMS WITH AT MOST ONE ZERO

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ABSTRACT. We prove that in any nonzero cohomology class $\xi \in H^1(M; \mathbf{R})$ there always exists a closed 1-form having at most one zero.

1. Statement of the result

Let M be a closed connected smooth manifold. By Hopf's theorem, there exists a nowhere zero tangent vector field on M if and only if $\chi(M) = 0$. If $\chi(M) \neq 0$ one may find a tangent vector field on M vanishing at a single point $p \in M$. A Riemannian metric on M determines a one-to-one correspondence between vectors and covectors; therefore on any closed connected manifold M there exists a smooth 1-form ω vanishing at most at one point $p \in M$. The question we address in this note is *whether the 1-form ω which is nonzero on $M - \{p\}$ can be chosen to be closed, $d\omega = 0$?*

The Novikov theory [8] gives bounds from below on the number of distinct zeros which have closed 1-forms ω lying in a prescribed cohomology class $\xi \in H^1(M; \mathbf{R})$. However the Novikov theory imposes an additional requirement that *all zeros of ω are non-degenerate in the sense of Morse.* The number of zeros is then at least the sum $\sum_j b_j(\xi)$ of the Novikov numbers $b_j(\xi)$.

If ω is a closed 1-form representing *the zero cohomology class* then $\omega = df$ where $f : M \to \mathbf{R}$ is a smooth function; in this case ω must have at least $\operatorname{cat}(M)$ geometrically distinct zeros, according to the classical Lusternik-Schnirelman theory [1].

Our goal in this paper is to show that in general, with the exception of two situations mentioned above, *there are no obstructions for constructing closed 1-forms possessing a single zero.* We prove the following statement:

Theorem 1. *Let M be a closed connected n-dimensional smooth manifold, and let $\xi \in H^1(M; \mathbf{R})$ be a nonzero real cohomology class. Then there exists a smooth closed 1-form ω in the class ξ having at most one zero.*

This result suggests that "the Lusternik-Schnirelman theory for closed 1-forms" (see [3, 4] and Chapter 10 of [5]) has a new character which is quite distinct from both the classical Lusternik-Schnirelman theory of functions and the Novikov theory of closed 1-forms.

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Theorem 1 was proven in [3] under an additional assumption that the class ξ is integral, $\xi \in H^1(M;\mathbf{Z})$. See also [5], Theorem 10.1. This essentially covers all rank 1 cohomology classes $\xi \in H^1(M;\mathbf{R})$ since any such class is a multiple of an integral class.

Theorem 1 has interesting implications in the theory of symplectic intersections, compare [9], [2]. Y. Eliashberg and M. Gromov mention in [2] that a statement in the spirit of Theorem 1 was made by Yu. Chekanov at a seminar talk in 1996. No written account of his work is available.

Let us mention briefly a similar question. We know that if $\chi(M) = 0$ then there exists a nowhere zero 1-form ω on M. Given $\chi(M) = 0$, one may ask if it is possible to find a nowhere zero 1-form ω on M which is closed $d\omega = 0$? The answer is negative in general. For example, vanishing of the Novikov numbers $b_j(\xi) = 0$ is a necessary condition for the class ξ to be representable by a closed 1-form without zeros. The full list of necessary and sufficient conditions (in the case $\dim M > 5$) is given by the theorem of Latour [6].

2. Preliminaries

Here we recall some basic terminology. We refer to [5] for more detail.

A smooth 1-form ω is a smooth section $x \mapsto \omega_x$, $x \in M$ of the cotangent bundle $T^*(M) \to M$. A *zero* of ω is a point $p \in M$ such that $\omega_p = 0$.

If ω is a closed 1-form on M, i.e. $d\omega = 0$, then in any simply connected domain $U \subset M$ there exists a smooth function $f : U \to R$ such that $\omega|_U = df$. Zeros of ω are precisely the critical points of f. A zero $p \in M$, $\omega_p = 0$ is said to be *Morse type* iff p is a Morse type critical point for f.

The *homomorphism of periods*

$$\mathrm{Per}_\xi : H_1(M) \to \mathbf{R} \tag{1}$$

is defined by

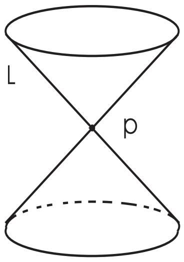
$$\mathrm{Per}_\xi([\gamma]) = \int_\gamma \omega \in \mathbf{R}. \tag{2}$$

Here $\xi = [\omega] \in H^1(M;\mathbf{R})$ is the de Rham cohomology class of ω and γ is a closed loop in M; the symbol $[\gamma] \in H_1(M)$ denotes the homology class of γ.

The image of the homomorphism of periods (1) is a finitely generated free abelian subgroup of $\mathbf{R}$; it is called the *group of periods*. Its rank is denoted $\mathrm{rk}(\xi)$ – the *rank of the cohomology class* $\xi \in H^1(M;\mathbf{R})$.

A closed 1-form ω with Morse zeros determines a *singular foliation* $\omega = 0$ on M. It is a decomposition of M into leaves: two points $p, q \in M$ belong to the same leaf if there exists a path $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$ and $\omega(\dot\gamma(t)) = 0$ for all t. Locally, in a simply connected domain $U \subset M$, we have $\omega|_U = df$, where $f : U \to \mathbf{R}$; each connected component of the level set $f^{-1}(c)$ lies in a single leaf. If U is small enough and does not contain the zeros of ω, one may find coordinates $x_1, \dots, x_n$ in U such that $f \equiv x_1$; hence the leaves in U are the sets $x_1 = c$. Near such points the singular foliation

$\omega = 0$ is a usual foliation. On the contrary, if U is a small neighborhood of a zero $p \in M$ of ω having Morse index $0 \le k \le n$, then there are coordinates $x_1, \dots, x_n$ in U such that $x_i(p) = 0$ and the leaves of $\omega = 0$ in U are the level sets $-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2 = c$. The leaf L with $c = 0$ contains the zero p. It has a *singularity* at p: a neighborhood of p in L is homeomorphic to a cone over the product $S^{k-1} \times S^{n-k-1}$. There are finitely many *singular leaves*, i.e. the leaves containing the zeros of ω.



We are particularly interested in the singular leaves containing the zeros of ω having Morse indices 1 and $n-1$. Removing such a zero p *locally* disconnects the leaf L. However globally the complement $L - p$ may or may not be connected.

The singular foliation $\omega = 0$ is *co-oriented*: the normal bundle to any leaf at any nonsingular point has a specified orientation.

We shall use the notion of a weakly complete closed 1-form introduced by G. Levitt [7]. A closed 1-form ω is called *weakly complete* if it has Morse type zeros and for any smooth path $\sigma : [0,1] \to M^*$ with $\int_\sigma \omega = 0$ the endpoints $\sigma(0)$ and $\sigma(1)$ lie in the same leaf of the foliation $\omega = 0$ on M^*. Here M^* denotes $M - \{p_1, \dots, p_m\}$ where p_j are the zeros of ω.

A weakly complete closed 1-form with $\xi = [\omega] \neq 0$ has no zeros with Morse indices 0 and n. According to Levitt [7], *any nonzero real cohomology class* $\xi \in H^1(M; \mathbf{R})$ *can be represented by a weakly complete closed 1-form.*

The plan of our proof of Theorem 1 is as follows. We start with a weakly complete closed 1-form ω lying in the prescribed cohomology class $\xi \in H^1(M; \mathbf{R})$, $\xi \neq 0$. We show that assuming $\mathrm{rk}(\xi) > 1$ all leaves of the singular foliation $\omega = 0$ are dense (see §3). We perturb ω such that the resulting closed 1-form ω' has a single singular leaf (see §4). After that we apply the technique of Takens [10] allowing us to collide the zeros in a single (highly degenerate) zero. We first prove Theorem 1 assuming that $n = \dim M > 2$; the special case $n = 2$ is treated separately later.

3. Density of the leaves

In this section we show that *if ω is weakly complete and* $\mathrm{rk}(\xi) > 1$ *then the leaves of* $\omega = 0$ *are dense.*

Note that in general the assumption $\mathrm{rk}(\xi) > 1$ alone does not imply that the leaves are dense, see the examples in §9.3 of [5].

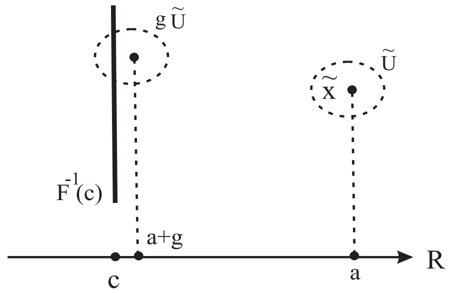
Let ω be a weakly complete closed 1-form in class ξ. Consider the covering $\pi : \tilde{M} \to M$ corresponding to the kernel of the homomorphism of periods $\mathrm{Per}_\xi : H_1(M) \to \mathbf{R}$, where $\xi = [\omega] \in H^1(M;\mathbf{R})$. Let $H \subset \mathbf{R}$ be the group of periods. The rank of H equals $\mathrm{rk}(\xi)$; since we assume that $\mathrm{rk}(\xi) > 1$, the group H is dense in $\mathbf{R}$. The group of periods H acts on the covering space $\tilde{M}$ as the group of covering transformations. We have $\pi^*\omega = dF$ where $F : \tilde{M} \to \mathbf{R}$ is a smooth function. The leaves of the singular foliation $\omega = 0$ are the images under the projection π of the level sets $F^{-1}(c)$; this property follows from the weak completeness of ω, see [7], Proposition II.1. For any $g \in H$ and $x \in \tilde{M}$ one has

$$F(gx) - F(x) \,=\, g \,\in\, \mathbf{R}. \tag{3}$$

Let $L = \pi(F^{-1}(c))$ be a leaf and let $x \in M$ be an arbitrary point. Our goal is to show that x lies in the closure $\bar{L}$ of L. Let $U \subset M$ be a neighborhood of x. We want to show that U intersects L. We shall assume that U is "small" in the following sense: $\xi|_U = 0$.

Consider a lift $\tilde{x} \in \tilde{M}$, $\pi(\tilde{x}) = x$. Let $\tilde{U}$ be a neighborhood of $\tilde{x}$ which is mapped by π homeomorphically onto U. We claim that *the set of values* $F(\tilde{U}) \subset \mathbf{R}$ *contains an interval* $(a-\epsilon, a+\epsilon)$ *where* $a = F(\tilde{x})$ *and* $\epsilon > 0$.

This claim is obvious if $\tilde{x}$ is not a critical point of F since in this case one may choose the coordinates $x_1, \dots, x_n$ around $\tilde{x}$ such that $F(x) = a + x_1$. In the case when $\tilde{x}$ is a critical point of F, one may choose the coordinates $x_1, \dots, x_n$ near the point $\tilde{x} \in \tilde{M}$ such that $F(x)$ is given by $a \pm x_1^2 \pm x_2^2 + \dots + \pm x_n^2$ and our claim follows since we know that the Morse index is distinct from 0 and n.



Because of the density of the group of translations $H \subset \mathbf{R}$ one may find $g \in H$ such that the real number $F(g\tilde{x}) = F(\tilde{x}) + g = a + g$ lies in the interval $(c - \epsilon, c + \epsilon)$. Then we obtain

$$c \,\in\, (a+g-\epsilon,\, a+g+\epsilon) \,\subset\, g + F(\tilde{U}) \,=\, F(g\tilde{U}). \tag{4}$$

Hence we see that the sets $F^{-1}(c)$ and $g\tilde{U}$ have a nonempty intersection. Therefore the neighborhood $U = \pi(g\tilde{U})$ intersects the leaf $L = \pi(F^{-1}(c))$ as claimed.

An obvious modification of the above argument proves a slightly more precise statement:

Given a point $x \in M$ and a leaf $L \subset M$ of the singular foliation $\omega = 0$, there exist two sequences of points $x_k \in L$ and $y_k \in L$ such that

$$x_k \to x \quad \text{and} \quad y_k \to x, \tag{5}$$

and, moreover,

$$\int_x^{x_k} \omega > 0, \quad \text{while} \quad \int_x^{y_k} \omega < 0. \tag{6}$$

The integrals in (6) are calculated along an arbitrary path lying in a small neighborhood of x.

This can also be expressed by saying that the leaf L approaches x from both the positive and the negative sides.

4. Modification

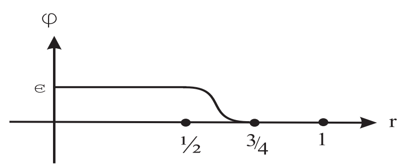
Our next goal is to replace ω by a Morse closed 1-form ω' which has the property that all its zeros lie on the same singular leaf of the singular foliation $\omega' = 0$. In this section we assume that $n = \dim M > 2$.

Let ω be a weakly complete Morse closed 1-form in class ξ where $\mathrm{rk}(\xi) > 1$. Let $p_1, \dots, p_m \in M$ be the zeros of ω. For each p_j choose a small neighborhood $U_j \ni p_j$ and local coordinates $x_1, \dots, x_n$ in U_j such that $x_i(p_j) = 0$ for $i = 1, \dots, n$ and

$$\omega|_{U_j} = df_j, \quad \text{where} \quad f_j = -x_1^2 - \dots - x_{m_j}^2 + x_{m_j+1}^2 + \dots + x_n^2. \tag{7}$$

Here m_j denotes the Morse index of p_j. We assume that the ball $\sum_{i=1}^n x_i^2 \le 1$ is contained in U_j and that $U_j \cap U_{j'} = \emptyset$ for $j \neq j'$. Denote by W_j the open ball $\sum_{i=1}^n x_i^2 < 1$.

Let $\phi : [0, 1] \to [0, 1]$ be a smooth function with the following properties: (a) $\phi \equiv 0$ on $[3/4, 1]$; (b) $\phi \equiv \epsilon > 0$ on $[0, 1/2]$; (c) $-1 < \phi' \le 0$. Such a



function exists assuming that $\epsilon > 0$ is small enough. (a), (b), (c) imply that

$$\phi'(r) > -2r, \quad \text{for} \quad r > 0. \tag{8}$$

We replace the closed 1-form ω by

$$\omega' = \omega - \sum_{j=1}^m \mu_j \cdot dg_j \tag{9}$$

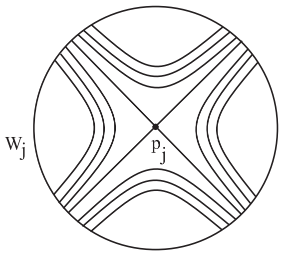
where $g_j : M \to \mathbf{R}$ is a smooth function with support in U_j. In the coordinates $x_1, \dots, x_n$ of U_j (see above) the function g_j is given by $g_j(x) = \phi(||x||)$. The parameters $\mu_j \in [-1, 1]$ appearing in (9) are specified later.

One has $\omega \equiv \omega'$ on $M - \cup_j U_j$ and near the zeros of ω. Let us show that ω' has no additional zeros. We have $\omega'|_{U_j} = d(f_j - \mu_j g_j)$ (where f_j is defined in (7)) and

$$\frac{\partial}{\partial x_i}(f_j - \mu_j g_j) = \pm 2x_i - \mu_j \phi'(||x||)\frac{x_i}{||x||} \tag{10}$$

If this partial derivative vanishes and $x_i \neq 0$ then $\phi'(r) = \pm 2r\mu_j^{-1}$ which may happen only for $r = ||x|| = 0$ according to (8).

We now show how to choose the parameters μ_j so that the closed 1-form ω' given by (9) has a unique singular leaf. Let L be a fixed nonsingular leaf of $\omega = 0$. Since L is dense in M (see §3) for any $j = 1, \dots, m$ the intersection $L \cap U_j$ contains infinitely many connected components approaching p_j from below and from above and the function f_j is constant on each of them. We



say that a subset $T_c \subset L \cap W_j$ is a *level set* if $T_c = f_j^{-1}(c) \cap W_j$ for some $c \in \mathbf{R}$. Note that $f_j(p_j) = 0$. The level set $c = 0$ contains the zero p_j; it is homeomorphic to the cone over the product $S^{m_j - 1} \times S^{n - m_j - 1}$. Each level set T_c with $c < 0$ is diffeomorphic to $S^{m_j - 1} \times D^{n - m_j}$ and each level set T_c with $c > 0$ is diffeomorphic to $D^{m_j} \times S^{n - m_j - 1}$. Recall that m_j denotes the Morse index of p_j.

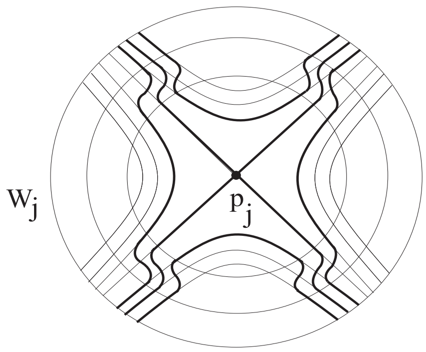
Let $\mathcal{V}_j = f_j(L \cap W_j) \subset \mathbf{R}$ denote the set of values of f_j on different level sets belonging to the leaf L. The zero 0 does not lie in $\mathcal{V}_j$ since we assume that the leaf L is nonsingular. However, according to the result proven in §3, the zero $0 \in \mathbf{R}$ is a limit point of $\mathcal{V}_j$ and, moreover, the closure of either of the sets $\mathcal{V}_j \cap (0, \infty)$ and $\mathcal{V}_j \cap (-\infty, 0)$ contains $0 \in \mathbf{R}$.

For the modification ω' (given by (9)) one has $\omega'|_{U_j} = dh_j$ where $h_j = f_j - \mu_j g_j$. The level sets T'_c for h_j are defined as $h_j^{-1}(c) \cap W_j$. Clearly T'_c is given by the equation

$$f_j(x) = \mu_j \phi(||x||) + c, \quad x \in W_j.$$

Hence for $||x|| \geq 3/4$ this is the same as T_c; for $||x|| \leq 1/2$ the level set T'_c coincides with $T_{c + \mu_j \epsilon}$. In the ring $1/2 \leq ||x|| \leq 3/4$ the level set T'_c is homeomorphic to a cylinder.

The following figure illustrates the distinction between the level sets T_c and T'_c in the case $\mu_j > 0$.



Case $\mu_j > 0$.

Examine the changes which undergoes the leaf L when we replace ω by ω'. Here we view L with the *leaf topology*; it is the topology induced on L from the covering $\tilde{M}$ using an arbitrary lift $L \to \tilde{M}$. First, let us assume that: (1) the Morse index m_j satisfies $m_j < n-1$; (2) the coefficient $\mu_j > 0$ is positive; (3) the number $-\epsilon\mu_j$ lies in the set $\mathcal{V}_j$. Then the complement

$$L - \bigcup_{\substack{c \in \mathcal{V}_j \\ -\epsilon\mu_j < c < 0}} T_c$$

is connected and it lies in a single leaf L' of the singular foliation $\omega' = 0$. We see that the new leaf L' is obtained from L by infinitely many surgeries. Namely, each level set $T_c \subset L$, where $c \in \mathcal{V}_j$ satisfies $-\epsilon\mu_j < c < 0$, is removed and replaced by a copy of $D^{m_j} \times S^{n-m_j-1}$; besides, the set $T_c \subset L$ where $c = -\epsilon\mu_j$, is removed and gets replaces by a cone over the product $S^{m_j-1} \times S^{n-m_j-1}$. Hence the new leaf L' contains the zero p_j.

Let us now show how one may modify the above construction in the case $m_j = n-1$. Since $n > 2$ we have in this case $n - m_j - 1 < n-2$; hence removing the sphere S^{n-m_j-1} from the leaf L does not disconnect L. We shall assume that the coefficient μ_j is *negative* and that the number $-\epsilon\mu_j$ lies in $\mathcal{V}_j \subset \mathbf{R}$. The complement

$$L - \bigcup_{\substack{c \in \mathcal{V}_j \\ 0 < c < -\epsilon\mu_j}} T_c$$

is connected and it lies in a single leaf L' of the singular foliation $\omega' = 0$. Clearly, L' is obtained from L by removing the level sets T_c where $c \in \mathcal{V}_j$ satisfies $0 < c < -\epsilon\mu_j$ (each such T_c is diffeomorphic to $D^{m_j} \times S^{n-m_j-1}$) and by replacing them by copies of $S^{m_j-1} \times D^{n-m_j}$. In addition, the set

$T_c \subset L$ where $c = -\epsilon\mu_j$, is removed and is replaces by a cone over the product $S^{m_j-1} \times S^{n-m_j-1}$.

We see that L' is a leaf of the singular foliation $\omega' = 0$ containing all the zeros $p_1, \dots, p_m$.

5. Proof of Theorem 1

Below we assume that $\mathrm{rk}(\xi) > 1$. The case $\mathrm{rk}(\xi) = 1$ is covered by Theorem 2.1 from [3].

The results of the preceding sections allow to complete the proof of Theorem 1 in the case $n = \dim M > 2$. Indeed, we showed in §4 how to construct a Morse closed 1-form ω' lying in the prescribed cohomology class ξ such that all zeros of ω' are Morse and belong to the same singular leaf L' of the singular foliation $\omega' = 0$. Now we may apply the colliding technique of F. Takens [10], pages 203–206. Namely, we may find a piecewise smooth tree $\Gamma \subset L'$ containing all the zeros of ω'. Let $U \subset M$ be a small neighborhood of Γ which is diffeomorphic to $\mathbf{R}^n$. We may find a continuous map $\Psi : M \to M$ with the following properties:

$\Psi(\Gamma)$ *is a single point* $p \in \Gamma$*;*

$\Psi|_{M-\Gamma}$ *is a diffeomorphism onto* $M - p$*;*

$\Psi(U) = U$;

Ψ *is the identity map on the complement of a small neighborhood* $V \subset M$ *of* Γ *where the closure* $\overline{V}$ *is contained in* U*.*

Consider a smooth function $f : U \to \mathbf{R}$ such that $df = \omega'|_U$; it exists and is unique up to a constant. The function $g = f \circ \Psi^{-1} : U \to \mathbf{R}$ is well-defined (since $f|_\Gamma$ is constant). g is continuous by the universal property of the quotient topology. Moreover, g is smooth on $M - p$. Applying Theorem 2.7 from [10], we see that we can replace g by a smooth function $h : U \to \mathbf{R}$ having a single critical point at p and such that $h = f$ on $U - \overline{V}$.

Let ω'' be a closed 1-form on M given by

$$\omega''|_{M-\overline{V}} = \omega'|_{M-\overline{V}} \quad \text{and} \quad \omega''|_U = dh. \tag{11}$$

Clearly ω'' is a smooth closed 1-form on M having no zeros in $M - \{p\}$. Moreover, ω'' lies in the cohomology class $\xi = [\omega']$ (since any loop in M is homologous to a loop in $M - \overline{V}$).

Now we prove Theorem 1 the case $n = 2$. We shall replace the construction of §4 (which requires $n > 2$) by a direct construction. The final argument using the Takens' technique [10] remains the same.

Let M be a closed surface and let $\xi \in H^1(M; \mathbf{R})$ be a nonzero cohomology class. We can split M into a connected sum

$$M = M_1 \sharp M_2 \sharp \dots \sharp M_k$$

where each M_j is a torus or a Klein bottle and such that the cohomology class $\xi_j = \xi|_{M_j} \in H^1(M_j; \mathbf{R})$ is nonzero. Let ω_j be a closed 1-form on M_j lying in the class ξ_j and having no zeros; obviously such a form exists. §9.3.2 of [5] describes the construction of connected sum of closed 1-forms on surfaces. Each connecting tube contributes two zeros. In fact there are three different ways of forming the connected sum, they are denoted by A, B, C on Figure 9.8 in [5]. In the type C connected sum the zeros lie on the same singular leaf. Hence by using the type C connected sum operation we get a closed 1-form ω on M having $2k-2$ zeros which all lie on the same singular leaf of the singular foliation $\omega = 0$. The colliding argument based on the technique of Takens [10] applies as in the case $n > 2$ and produces a closed 1-form with at most one zero lying in class ξ.

References

[1] B. Dubrovin, S. Novikov and A. Fomenko, *Modern Geometry; Methods of the homology theory*, 1984

[2] Y. Eliashberg, M. Gromov, *Lagrangian Intersection Theory, finite-dimensional approach*, In: "Geometry of Differential Equations" , A. Khovanskii, A. Varchenko, V. Vassiliev, editors, AMS, volume 186, 1998.

[3] M. Farber, *Zeros of closed 1-forms, homoclinic orbits and Lusternik–Schnirelman theory*, "Topological Methods in Nonlinear Analysis", **19**(2002), 123 - 152.

[4] M. Farber, *Lusternik–Schnirelman Theory and Dynamics*, in: "Lusternik - Schnirelmann Category and related topics", Contemporary Mathematics, **316**(2002), 95 - 111.

[5] M. Farber, *Topology of closed one-forms*, Mathematical Surveys and Monographs, Vol 108, American Mathematical Society, 2004.

[6] F. Latour, *Existence de 1-formes fermées non singulières dans une classe de cohomologie de De Rham*, Publ. IHES, **80**, 1994.

[7] G. Levitt, *1-formes fermées singulières et groupe fondamental*, Invent. Math., **88**(1987), 635 - 667.

[8] S. P. Novikov, *The Hamiltonian formalism and a multi-valued analogue of Morse theory*, Russian Math. Surveys, **37:5**(1982), pp. 1–56.

[9] J.-Cl. Sikorav, *Un problem de disjonction par isotopie symplectique dans un fibré cotangent*, Ann. Sci. École Norm. Sup. (4)**19**(1986), 543-552.

[10] F. Takens, *The minimal number of critical points of a function on a compact manifold and the Lusternik - Schnirelman category*, Invent. Math., **6**(1968), pp. 197 - 244.

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