NIELSEN NUMBER IS A KNOT INVARIANT

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Abstract. We show that the Nielsen number is a knot invariant via representation variety

1. Introduction

We briefly describe the few basic notions of Nielsen fixed point theory(see [2]). We assume X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, i.e. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(Fix(\tilde{f})) \subset Fix(f)$ is called *the fixed point class of f determined by the lifting class* $[\tilde{f}]$.Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X. A fixed point class is called *essential* if its index is nonzero. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$.The Nielsen number is always finite. $N(f)$ is a homotopy invariant. In the category of compact, connected polyhedra, the Nielsen number of a map is, apart from certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as f.
Let us consider a braid representative of a knot and induced map of corresponding representation variety(see section 2).We prove in section 3 that the Nielsen number of induced map is a invariant under Markov moves and so is a knot invariant.
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2. Knot invariants via representations spaces

We recall firstly the Lin's construction in [6] for the intersection number of the representation spaces corresponding to a braid representative of a knot K in S^3. Let (S^3, D^3_+, D^3_-, S^2) be a Heegaard decomposition of S^3 with genus 0, where

$$S^3 = D^3_+ \cup_{S^2} D^3_-, \quad \partial D^3_+ = \partial D^3_- = D^3_+ \cap D^3_- = S^2.$$

Suppose that a knot $K \subset S^3$ is in general position with respect to this Heegaard decomposition. So $K \cap S^2 = \{x_1, \cdots, x_n, y_1, \cdots, y_n\}$, $K \cap D^3_\pm$ is a collection of unknotted,

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unlinked arcs $\{\gamma_1^{\pm}, \cdots, \gamma_n^{\pm}\} \subset D^3_{\pm}$, where $\partial\gamma_i^- = \{x_i, y_i\}$ and $\{\gamma_1^+, \cdots, \gamma_n^+\} = K \cap D^3_+$ becomes a braid of n strands inside D^3_+. Denote by β a corresponding word in the braid group B_n. For the top end points x_i of γ_i^+, the bottom end points of $\{\gamma_1^+, \cdots, \gamma_n^+\}$ give a permutation of $\{y_1, \cdots, y_n\}$ which generates a map

$$\pi : B_n \to S_n,$$

where $\pi(\beta)$ is the permutation of $\{y_1, \cdots, y_n\}$ in the symmetric group of n letters. Let $K = \overline{\beta}$ be the closure of β. It is well-known that there is a correspondence between a knot and a braid β with $\pi(\beta)$ is a complete cycle of the n letters (see [1]).

There is a corresponding Heegaard decomposition for the complement of a K,

$$S^3 \setminus K = (D^3_+ \setminus K) \cup_{(S^2\setminus K)} (D^3_- \setminus K),$$

$$D^3_{\pm} \setminus K = D^3_{\pm} \setminus (D^3_{\pm} \cap K), \quad S^2 \setminus K = S^2 \setminus (S^2 \cap K).$$

Thus by Seifert-van Kampen theorem we have following diagramm

$$\begin{array}{ccc} \pi_1(S^2 \setminus K) & \to & \pi_1(D^3_+ \setminus K) \\ \downarrow & & \downarrow \\ \pi_1(D^3_- \setminus K) & \to & \pi_1(S^3 \setminus K), \end{array}$$

and a corresponding diagramm of representation spaces

$$\begin{array}{ccc} \mathcal{R}(S^2 \setminus K) & \leftarrow & \mathcal{R}(D^3_+ \setminus K) \\ \uparrow & & \uparrow \\ \mathcal{R}(D^3_- \setminus K) & \leftarrow & \mathcal{R}(S^3 \setminus K), \end{array} \tag{1}$$

where $\mathcal{R}(X) = Hom(\pi_1(X), SU(2))/SU(2)$ for $X = S^2 \setminus K, D^3_{\pm} \setminus K, S^3 \setminus K$.

In [7], Magnus used the trace free matrices to represent the generators of a free group to show that the faithfulness of a representation of braid groups in the automorphism groups of the rings generated by the character functions on free groups. This is original idea to have representations with trace free along all meridians which Lin worked in [6] to define the knot invariant. It has been carried out by M. Heusener and J. Kroll in [3] for the representation of knot groups with the trace of the meridian fixed (not necessary zero). Let $\mathcal{R}(S^2 \setminus K)^{[i]}$ be the space of $SU(2)$ representations $\rho : \pi_1(S^2 \setminus K) \to SU(2)$ such that

$$\rho([m_{x_i}]) \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho([m_{y_i}]) \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{2}$$

where $m_{x_i}, m_{y_i}, i = 1, 2, \cdots, n$ are the meridian circles around x_i, y_i respectively. Note that $\pi_1(S^2 \setminus K)$ is generated by $m_{x_i}, m_{y_i}, i = 1, 2, \cdots, n$ and one relation $\prod_{i=1}^n m_{x_i} = \prod_{i=1}^n m_{y_i}$. Corresponding to (1), we have

$$\begin{array}{ccc} \mathcal{R}(S^2 \setminus K)^{[i]} & \leftarrow & \mathcal{R}(D^3_+ \setminus K)^{[i]} \\ \uparrow & & \uparrow \\ \mathcal{R}(D^3_- \setminus K)^{[i]} & \leftarrow & \mathcal{R}(S^3 \setminus K)^{[i]}. \end{array} \tag{3}$$

The conjugacy class in $SU(2)$ is completely determined by its trace. So the condition (2) can be reformulated for $\rho \in \mathcal{R}(X)^{[i]}$,

$$trace\rho([m_{x_i}]) = trace\rho([m_{y_i}]) = 0. \tag{4}$$

The space $\mathcal{R}(S^2\setminus K)^{[i]}$ can be identified with the space of $2n$ matrices $X_1\cdots,X_n,Y_1,\cdots,Y_n$ in $SU(2)$ satisfying

$$trace(X_i) = trace(Y_i) = 0, \quad \text{for } i = 1,\cdots,n, \tag{5}$$

$$X_1\cdot X_2\cdots X_n = Y_1\cdot Y_2\cdots Y_n. \tag{6}$$

Let Q_n be the space $\{(X_1,\cdots,X_n)\in SU(2)^n|\ trace(X_i)=0, i=1,\cdots,n\}$. Let $\mathcal{R}^*(S^2\setminus K)^{[i]}$ be the subset of $\mathcal{R}(S^2\setminus K)^{[i]}$ consisting of irreducible representations. Note that $\mathcal{R}^*(S^2\setminus K)^{[i]} = (H_n\setminus S_n)/SU(2)$ in Lin's notation [6], where

$$H_n = \{(X_1,\cdots,X_n,Y_1,\cdots,Y_n)\in Q_n\times Q_n|\ \ X_1\cdots X_n = Y_1\cdots Y_n\},$$

S_n is the subspace of H_n consisting of all the reducible points. Here $H_n\setminus S_n$ is the total space of a $SU(2)$-fiber bundle over $\mathcal{R}^*(S^2\setminus K)^{[i]}$.

Given $\beta\in B_n$, we denote by Γ_β the graph of β in $Q_n\times Q_n$, i.e.

$$\Gamma_\beta = \{(X_1,\cdots,X_n,\beta(X_1),\cdots,\beta(X_n))\in Q_n\times Q_n\}.$$

As an automorphism of the free group $Z[m_{x_1}]*Z[m_{x_2}]*\cdots*Z[m_{x_n}]$, this element $\beta\in B_n$ preserves the word $[m_{x_1}]\cdots[m_{x_n}]$. Thus we have $X_1\cdots X_n = \beta(X_1)\cdots\beta(X_n)$, or in other words Γ_β is a subspace of H_n. In fact, for $\overline{\beta} = K$, this subspace Γ_β coincides with the subspace of representations $\rho:\pi_1(S^2\setminus K)\to SU(2)$ in H_n which can be extended to $\pi_1(D^3_+\setminus K)$, $\Gamma_\beta = Hom(\pi_1(D^3_+\setminus K), SU(2))^{[i]}$. Hence the space $\mathcal{R}^*(D^3_+\setminus K)^{[i]} = \Gamma_{\beta,irre}/SU(2)$ is the irreducible $SU(2)$ representations with traceless condition over $D^3_+\setminus K$.

In the special case $\beta = id$, then Γ_{id} represents the diagonal in $Q_n\times Q_n$,

$$\Gamma_{id} = \{(X_1,\cdots,X_n,X_1,\cdots,X_n)\in Q_n\times Q_n\}.$$

Since $K\cap D^3_-$ represents the trivial braid, this space $\Gamma_{id}\subset H_n$ can be identified with the subspace of representations in $Hom(\pi_1(S^2\setminus K), SU(2))^{[i]}$ which can be extended to $\pi_1(D^3_-\setminus K)$, i.e. $\Gamma_{id} = Hom(\pi_1(D^3_-\setminus K), SU(2))^{[i]}$. By Seifert, Van-Kampen Theorem, the intersection $\Gamma_\beta\cap\Gamma_{id}$ is the same as the space of representations of $\pi_1(S^3\setminus K)$ satisfying the monodromy condition $[i]$ (see (1)),

$$\Gamma_\beta\cap\Gamma_{id} = Hom(\pi_1(S^3\setminus K), SU(2))^{[i]}.$$

Given $\beta\in B_n$ with $\overline{\beta} = K$, there is an induced diffeomorphism (still denoted by β) from Q_n to itself. Such a diffeomorphism also induces a diffeomorphism $f_\beta:\mathcal{R}^*(S^2\setminus K)^{[i]}\to\mathcal{R}^*(S^2\setminus K)^{[i]}$of the representation variety.

Note that $\overline{\Gamma}_\beta = (\Gamma_\beta\setminus(\Gamma_\beta\cap S_n))/SU(2)$ is the image of the "diagonal" $\overline{\Gamma}_{id}$ under diffeomorphism f_β. By Seifert- van Kampen theorem (3), it is clear that the fixed point set of f_β is

$$Fix(f_\beta|_{\mathcal{R}^*(S^2\setminus K)^{[i]}}) = \overline{\Gamma}_\beta\cap\overline{\Gamma}_{id} = \mathcal{R}^*(S^3\setminus K)^{[i]}.$$

The oriented submanifolds $\overline{\Gamma}_\beta = \mathcal{R}^*(D^3_+\setminus K)^{[i]}, \overline{\Gamma}_{id} = \mathcal{R}^*(D^3_-\setminus K)^{[i]}$ intersects each other in a compact subspace of $\mathcal{R}^*(S^2\setminus K)^{[i]}$ from Lemma 1.6 in [6]. Hence we can perturb f_β, or in another words perturb $\mathcal{R}^*(D^3_+\setminus K)^{[i]}$ to $\hat{\mathcal{R}}^*(D^3_+\setminus K)^{[i]}$ by a compactly support isotopy so that $\hat{\mathcal{R}}^*(D^3_+\setminus K)^{[i]}$ intersects $\mathcal{R}^*(D^3_-\setminus K)^{[i]}$ transversally at a finite number of intersection points. Denote the perturbed diffeomorphism by $\hat{f}_\beta$. So its fixed points are all nondegenerated.

The Casson-Lin invariant of a knot $K = \overline{\beta}$ is given by counting the algebraic intersection number of $\hat{\mathcal{R}}^*(D^3_+ \setminus K)^{[i]}$ and $\mathcal{R}^*(D^3_- \setminus K)^{[i]}$, or the algebraic number of $\mathrm{Fix}(\hat{f}_\beta)$,

$$\lambda_{CL}(K) = \lambda_{CL}(\beta) = Algebraic(\#\,\mathrm{Fix}(\hat{f}_\beta)) = Algebraic(\#(\hat{\mathcal{R}}^*(D^3_+ \setminus K)^{[i]} \cap \mathcal{R}^*(D^3_- \setminus K)^{[i]})).$$

The results proved by Lin in [6] show that the Casson-Lin invariant $\lambda_{CL}(K) = \lambda_{CL}(\beta)$ is independent of its braid representatives, i.e. $\lambda_{CL}(\beta)$ is invariant under the Markov moves of type I and type II on β and is one half of the classical signature of the knot K.

3. Nielsen number is a knot invariant

In this article we propose to count fixed points of f_β in a Nielsen way - using the classical Nielsen numbers of f_β. Nielsen counting of fixed points is a counting in the presence of the fundamental group. In order to get an invariant of knots from braids, we have to verify that Nielsen number $N(f_\beta)$ is invariant under Markov moves. A Markov move of type I changes $\sigma \in B_n$ to $\xi^{-1}\sigma\xi \in B_n$ for any $\xi \in B_n$, and the Markov move of type II changes $\sigma \in B_n$ to $\sigma_n^{\pm}\sigma \in B_{n+1}$, or the inverses of these operations. It is well-known that two braids β_1 and β_2 has isotopic closure if and only if β_1 can be changed to β_2 by a sequence of finitely many Markov moves [1].

Theorem 1. *If $\overline{\beta_1} = \overline{\beta_2} = K$ as a knot, $\beta_1 \in B_n, \beta_2 \in B_m$, then*

$$N(f_{\beta_1}) = N(f_{\beta_2}).$$

So the Nielsen number $N(f_\beta)$ is a knot invariant.

Proof: We only need to show that for $\beta \in B_n$ with $\overline{\beta}$ being a knot K, the Markov moves of type I and type II on β provide either a cojugacy or a isotopy of f_β. Hence from the commutativity and the invariance property under isotopy of the Nielsen numbers, we get that $N(f_\beta)$ is an invariant of knot $K = \overline{\beta}$.

Suppose we have the Markov move of type I: change β to $\xi^{-1}\beta\xi$ for some $\xi \in B_n$. The element ξ in B_n induces a diffeomorphism $\xi : Q_n \to Q_n$ is orientation preserving as observed by Lin in [6]. Note that B_n is generated by $\sigma_1, \cdots, \sigma_{n-1}$. For any $\sigma_i^{\pm}$, the induced diffeomorphism $\sigma_i^{\pm} \times \sigma_i^{\pm} : Q_n \times Q_n \to Q_n \times Q_n$ is an orientation preserving diffeomorphism. So ξ is also a orientation preserving diffeomorphism since orientation preserving properties are invariant under the composition operation. Hence there is a homeomorphism

$$\xi \times \xi : Q_n \times Q_n \to Q_n \times Q_n,$$

which commutes with the $SU(2)$-action and

$$\xi \times \xi(\mathcal{R}^*(S^2 \setminus K)^{[i]}) = \mathcal{R}^*(S^2 \setminus K)^{[i]} \quad \text{(changing variables by } \xi \times \xi\text{)},$$

$$\xi \times \xi(\mathcal{R}^*(D^3_- \setminus K)^{[i]}) = \mathcal{R}^*(D^3_- \setminus K)^{[i]} \quad \text{(in new coordinate } \xi(X_1), \cdots, \xi(X_n)\text{)},$$

$$\xi \times \xi(\mathcal{R}^*(D^3_+ \setminus K)^{[i]}) = \mathcal{R}^*(D^3_+ \setminus K)^{[i]} \quad \text{(in new coordinate } \xi(X_1), \cdots, \xi(X_n)\text{)},$$

as oriented manifolds. Let $g_\xi : \mathcal{R}^*(S^2 \setminus K)^{[i]} \to \mathcal{R}^*(S^2 \setminus K)^{[i]}$ be the induced homeomorphism, induced from $\xi \times \xi$ as coordinate changes . Hence we get a conjugacy relation

$$g_\xi^{-1} \circ f_\beta \circ g_\xi = f_{\xi^{-1}\beta\xi},$$

from changing variables via g_ξ. Note that $Fix(f_{\xi^{-1}\beta\xi})$ is identified with $Fix(f_\beta)$ under g_ξ. Thus the Markov move of type I preserves the conjugacy class of f_β Therefore by commutativity of the Nielsen number (see [4]) we have,

$$N(f_{\xi^{-1}\beta\xi}) = N(g_\xi^{-1} \circ f_\beta \circ g_\xi) = N(f_\beta). \tag{7}$$

It is clear that the argument goes through for the inverse operation of Markov move of type I.

Suppose we have the Markov move of type II: change β to $\sigma_n\beta \in B_{n+1}$. Recall that $\sigma_n(x_i) = x_i, 1 \le i \le n-1, \sigma_n(x_n) = x_n x_{n+1} x_n^{-1}$ and $\sigma_n(x_{n+1}) = x_n$. We need to identify the Nielsen number from the construction in $\hat{H}_n$ into the one from $\hat{H}_{n+1}$. Following Lin [6], there is an imbedding $q : Q_n \times Q_n \to Q_{n+1} \times Q_{n+1}$ given by

$$q(X_1, \cdots, X_n, Y_1, \cdots, Y_n) = (X_1, \cdots, X_n, Y_n, Y_1, \cdots, Y_n, Y_n).$$

Such an imbedding commutes with the $SU(2)$-action and $q(H_n) \subset H_{n+1}$, and induces an imbedding

$$\hat{q} : \hat{H}_n(= \mathcal{R}^*(S^2 \setminus \overline{\beta})^{[i]}) \to \hat{H}_{n+1}(= \mathcal{R}^*(S^2 \setminus \overline{\sigma_n\beta})^{[i]}).$$

Note that the symplectic structure of $\hat{H}_{n+1}$ restricted on $\hat{q}(\hat{H}_n)$ is the symplectic structure on $\hat{H}_n$. Under this imbedding, we have $\hat{q}(f_\beta) : \hat{H}_{n+1} \to \hat{H}_{n+1}$ is given by

$$(X_1, \cdots, X_n, X_1, \cdots, X_n) \mapsto (X_1, \cdots, X_n, \beta(X_n), \beta(X_1), \cdots, \beta(X_n), \beta(X_n)). \tag{8}$$

The image of $\hat{q}(f_\beta)$ is invariant under the operation of σ_n. Also the corresponding diffeomorphism $f_{\sigma_n\beta}$ is given by

$$\begin{aligned} &f_{\sigma_n\beta}(X_1, \cdots, X_n, X_{n+1}, X_1, \cdots, X_n, X_{n+1}) \\ &= (X_1, \cdots, X_{n+1}, \beta(X_1), \cdots, \beta(X_{n-1}), \beta(X_n)X_{n+1}\beta(X_n)^{-1}, \beta(X_n)). \end{aligned} \tag{9}$$

Thus we have

$$\hat{q}(\mathcal{R}^*(D^3_- \setminus \overline{\beta})^{[i]}) \subset \mathcal{R}^*(D^3_- \setminus \overline{\sigma_n\beta})^{[i]}, \quad \hat{q}(\mathcal{R}^*(D^3_+ \setminus \overline{\beta})^{[i]}) \subset \mathcal{R}^*(D^3_+ \setminus \overline{\sigma_n\beta})^{[i]}.$$

The fixed points of $f_{\sigma_n\beta}$ are elements

$$\beta(X_i) = X_i, 1 \le i \le n_1; \quad \beta(X_n)X_{n+1}\beta(X_n)^{-1} = X_n, \quad \beta(X_n) = X_{n+1},$$

which is equivalent to $\beta(X_i) = X_i, 1 \le i \le n$, i.e.

$$\mathrm{Fix}(f_{\sigma_n\beta}) = \ Fix(\hat{q}(f_\beta)) = \mathrm{Fix}(f_\beta).$$

Then there is a (Hamiltonian) isotopy $\psi_t : \hat{H}_{n+1}(= (H_{n+1} \setminus S_{n+1})/SU(2)) \to \hat{H}_{n+1}$ between $\psi_{t_0} = \hat{q}(f_\beta)$ by (8) and $\psi_{t_1} = f_{\sigma_n\beta}$ (9)(see [5, 6] for the explicit constructions). So we have

$$N(f_{\sigma_n\beta}) = N(\hat{g}(f_\beta)) = N(f_\beta). \tag{10}$$

The first equality is from the invariance property of Nielsen numbers under the isotopy ψ_t and the second from the natural identification. We can similarly prove that

$$N(f_{\sigma_n^{-1}\beta}) = N(f_\beta).$$

Remark 2. It is known for a long time, that the problem of computation of Nielsen numbers is a very difficult problem. By this reason, we strongly believe that the Nielsen number $N(f_\beta)$ is a new knots invariant, which cannot be reduced to the known knots invariants, as it happened in the case of the Casson-Lin invariant of knots.

Example 3. "Pillowcase". For $n = 2$ the irreducible variety $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ is a 2-sphere with four cone points deleted(see [6]). So, in this case the space $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ is non-simply-connected and the Nielsen number $N(f_\beta)$ is not trivial(not 0 or 1) for general β.

It is a important problem to find a non-simply-connected examples of the space $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ for $n > 2$.

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