Canonical maps of pointed nodal curves

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Abstract

In 1983 Knudsen [4] has proved that the three-canonical map of a pointed Deligne-Mumford stable curve (see [2]) is an embedding and the double-canonical map is base point free. Here we discuss the same question for the canonical map. The answer may be formulated almost purely topologically in terms of the dual graph with the exception for hyperelliptic case. These results were announced in [6] for ll-curves with the reference to the future author's paper "Algebraic geometry of ll-curves". It turned out however, that the theorem is valid for any pointed nodal curve. So the title has been changed. (An ll-curve in [6] is a Deligne-Mumford stable curve having only rational irreducible components and trivalent dual graph.)

1 Introduction

In this paper we shall study singular curves over an algebraicly closed field k, having only nodal singularities. Consider such a curve X, its irreducible components X_v, $v \in V$, and singular points p_e, $e \in E_{node}$. Then the *dual modular graph* $\Gamma(X)$ has V as the set of vertices and E_{node} as the set of edges; two vertices v_1 and v_2 are connected by an edge e if and only if the two branches of the singular point p_e belong to the components X_{v_1} and X_{v_2}. Thus the intersection points of two distinct components correspond to edges with distinct ends while nodal singularities of one irreducible component X_v correspond to loops starting and ending at the same vertex v in $\Gamma(X)$.

A pointed nodal curve is a nodal curve with a finite (maybe empty) set of marked nonsingular points $p_e,\ e \in E_{par}$. To such points we shall also attach edges in the dual modular graph $\Gamma(X)$, such an edge e having only one end v, where $p_e \in X_e$. We shall call such edges *parabolic edges*, or *half-edges*. So the dual modular graph of a pointed nodal curve has V as the set of vertices and $E = E_{node} \cup E_{par}$ as the set of edges; we don't want to make much difference between usual and parabolic edges. Note that the incidence relation for edges and vertices may be described in the uniform way: a vertex v is incident to an edge e if $p_e \in X_v$. We may also consider marked nodal curve as a non-compact curve, obtained from the compact one by deleting the marked points $p_e,\ e \in E_{par}$. According to such point of view a pointed nodal curve with empty set of marked points will be sometimes called *compact*, and the corresponding modular graph without parabolic edges will be called *closed modular graph*.

The modular graph $\Gamma(X)$ is non-oriented; each edge can of course possess two possible orientations. Let us denote the set of oriented edges of $\Gamma(X)$ by $\overrightarrow{E}$; the *star of outcoming edges* — by $S_-(v)$ (it is simply the set of oriented edges incident to the vertex v and pointing from v), the natural *source map* — by $s\colon \overrightarrow{E}_1 \to V$, so that $S_-(v) = s^{-1}(v)$. Consider the subset $\overrightarrow{E}_1 = \cup_{v \in V} S_-(v)$ of the set of oriented edges $\overrightarrow{E}$: $\overrightarrow{E}_1$ is the set of all outcoming edges (i.e., oriented edges, pointing from some vertex of Γ). If the modular graph $\Gamma(X)$ is closed, $\overrightarrow{E}_1 = \overrightarrow{E}$.

The normalization X^ν of the curve X is a disjoint union of nonsingular curves

$$X^\nu = \coprod_{v \in V} X_v^\nu, \tag{1.1}$$

the normalization map $\nu\colon X^\nu \to X$ is simply the collection of the normalization maps of the components $\nu_v\colon X_v^\nu \to X_v,\ v \in V$. There is a natural one-to-one correspondence between the preimages $\nu^{-1}p_e$ of the points $p_e,\ e \in E$, located on the component X_v, and the oriented edges of the star $S_-(v)$, we shall denote the point of X_v^ν, corresponding to the oriented edge $\overrightarrow{e} \in S_-(v)$, by $P_{\overrightarrow{e}}$. These considerations lead to the formal definition of modular graph given in the next section; it is slightly different from the usual definition (see for example [5]), because we use in it all oriented edges, not only flags.

The sheaf of regular differentials Ω_X^1 on X is locally free; the restriction of its lift $\nu^*\Omega_X^1$ to the component X_v^ν is the sheaf of differentials of the third kind on X_v^ν with poles only at the points $P_{\overrightarrow{e}},\ \overrightarrow{e} \in S_-(v)$:

$$(\nu^*\Omega_X^1)\big|_{X_v^\nu} = \Omega^1_{X_v^\nu}\left(\sum_{\overrightarrow{e} \in S_-(v)} P_{\overrightarrow{e}}\right). \tag{1.2}$$

Thus a regular differential on X is simply a collection of differentials of the third kind $\{\omega_v\}$ on each X_v^ν with poles only at the points $P_{\overrightarrow{e}}$, $\overrightarrow{e} \in S_-(v)$. Each singular point p_e, $e \in E_{node}$ imposes one constraint on the differentials $\{\omega_v\}$: if e connects vertices v_1 and v_2, then

$$res_{P_{\overrightarrow{e}}}\omega_{v_1} + res_{P_{\overleftarrow{e}}}\omega_{v_2} = 0. \tag{1.3}$$

(Here $\overrightarrow{e}$ and $\overleftarrow{e}$ mean two orientations on the edge e, so that $s(\overrightarrow{e}) = v_1$ and $s(\overleftarrow{e}) = v_2$.) Thus the collection of residues at the branches of the nodal points is skew-symmetric, moreover, it may be extended to a skew-symmetric function on the whole $\overrightarrow{E}$ by attaching $-res_{P_{\overleftarrow{e}}}\omega_v$ to the oriented edge $\overrightarrow{e} \notin \overrightarrow{E}_1$, directed to the marked point p_v. The main residue theorem for each component X_v^ν gives the cocycle condition

$$\sum_{\overrightarrow{e} \in S_-(v)} res_{P_{\overrightarrow{e}}}\omega_v = 0. \tag{1.4}$$

The map, attaching to the differential $\omega \in H^0(X, \Omega_X^1)$ the collection of residues at the marked points is the analogue of the period map for pointed nodal curves, the target space of this map will be discussed in the next section.

The m-canonical map is $\varphi_m \colon X \to \mathbb{P}(H^0(X, (\Omega_X^1)^{\otimes m})^*)$. In [4] it is proved that for a Deligne-Mumford stable curve X φ_m is base point free for $m \geq 2$ and is an embedding for $m \geq 3$. The following theorem presents the conditions for the double-canonical map to be an embedding.

Theorem 1.1 *The double-canonical map of a Deligne-Mumford stable curve X is an embedding if*
1) the modular graph $\Gamma(X)$ has no univalent vertices v with $g(X_v) = 1$;
2) the modular graph $\Gamma(X)$ has no trivalent vertices v with a loop and $g(X_v) = 0$;
3) X is not one of the four following compact (i.e. without marked points) genus 2 curves:

a) nonsingular genus 2 curve;
b) elliptic curve with one node;
c) rational curve with two nodes;
d) two nonsingular rational curves intersecting at three points.

On each irreducible component of type 1) and 2) and on each curve of type 3) φ_2 is a degree 2 map to a projective line.

The behavior of the canonical map depends on the following characteristic of the modular graph $\Gamma(X)$. Let Γ be an arbitrary modular graph, not necessarily connected; denote by $R(\Gamma)$ the set of its closed connected components (i.e. connected components without parabolic edges).

Definition 1.1 The **thickness** $th(\Gamma)$ of a modular graph Γ is the minimal number of edges of Γ required to be removed to increase $|R(\Gamma)|$.

Evidently $th(\Gamma)$ does not exceed the minimal valence of its vertices, since removing all the edges from $S(v)$ we obtain a new closed connected component, namely the vertex v. For a connected non-closed modular graph, $th(\Gamma)$ does not exceed the number of parabolic edges $|E_{par}|$: the removing of all the parabolic edges increases $|R(\Gamma)|$ from 0 to 1. On the following pictures we present examples of graphs of thickness 1, 2 and 3.

Figure 1: Thickness 1 modular graphs

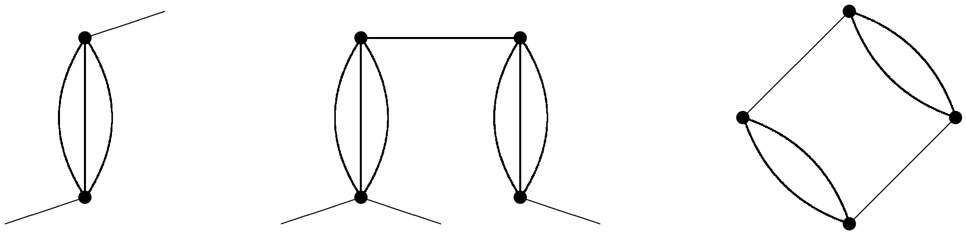

Figure 2: Thickness 2 modular graphs

We shall denote the dimension of the space of regular differentials on a nodal curve X by $g_\gamma(X)$; it is expressed in terms of the genera of irreducible

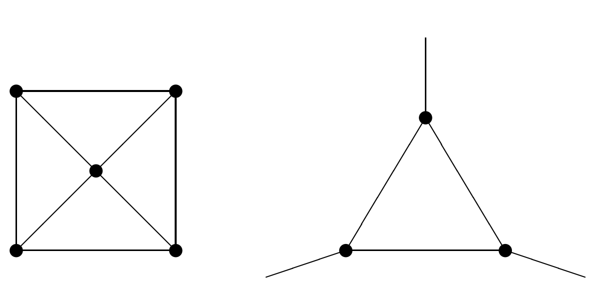

Figure 3: Thickness 3 modular graphs

components of X and the characteristics of the dual modular graph as follows:

$$g_\gamma = \sum_{v\in V} g(X_v^\nu) + |E| - |V| + |R|. \tag{1.5}$$

For compact (i.e. without marked points) nodal curves the arithmetic genus of X coincides with g_γ; of course, marking of some points does not change the arithmetic genus but may change g_γ.

The following statement shows that almost all nodal curves with $th(\Gamma(X)) \geq 3$ are Deligne-Mumford stable.

Proposition 1.1 *If $th(\Gamma(X)) \geq 3$ and $g_\gamma(X) > 1$, the curve X is Deligne-Mumford stable.*

Note that there are only three connected nodal curves with $th(\Gamma(X)) \geq 3$ and $g_\gamma(X) \leq 1$; these are the projective line ($g_\gamma = 0$), a nonsingular elliptic curve ($g_\gamma = 1$), and the nodal rational curve with one node ($g_\gamma = 1$).

Now we can state the main theorems.

Theorem 1.2 *The canonical map of a pointed nodal curve X having no connected components with exactly one marked point, is base point free iff $th(\Gamma(X)) \geq 2$.*

Note that we eliminate from consideration connected components having exactly one marked point for the following reasons. Formally the thickness of dual modular graph of such a curve is one, but the canonical map coincides with the canonical map of its closure (i.e., the same curve without any marked points).

In the same way in the study of the canonical map of a connected nodal curve X with exactly two marked points p_1 and p_2 we may replace X by a compact nodal curve $\tilde{X}$ without marked points, obtained from X by gluing together the points p_1 and p_2 into additional node $\tilde{p}$ with branches p_1 and p_2. (See Corollary 4.1). So the canonical map ignores a single marked point and glues together a single pair of marked points into one node. A more detailed description of the canonical map for the thickness 1 and 2 cases can be found in Propositions 4.3 and 4.4. And the main case is the case when the thickness of the dual modular graph is at least three.

Theorem 1.3 *The canonical map of a pointed nodal curve X is*
1) an embedding to $\mathbb{P}^{g_\gamma-1}$ iff $th(\Gamma(X)) \geq 3$, $g_\gamma(X) > 1$ and X is not hyperelliptic (see 2)); the image is a curve of degree

$$d_\gamma = 2(g_\gamma - R) - |E_{par}|; \tag{1.6}$$

2) a double covering of the rational degree $g_\gamma - 1$ Veronese curve in $\mathbb{P}^{g_\gamma-1}$ for the following two types of curves, which are called **hyperelliptic***:*

a) an irreducible curve X with X^ν rational, elliptic or hyperelliptic with an involution $i\colon X^\nu \to X^\nu$ transposing the branches of $n \geq \max(2-g(X^\nu), 0)$ singular points: $i(P_{\overleftarrow{e}}) = P_{\overrightarrow{e}}$ for all n edges (that are loops) of $\Gamma(X)$.

b) a reducible curve $X = X_1 \cup X_2$ having two nonsingular rational irreducible components X_1 and X_2 intersecting at $n \geq 3$ points and a projective isomorphism $i\colon X_1 \to X_2$ transposing the branches of singular points: $i(P_{\overleftarrow{e}}) = P_{\overrightarrow{e}}$ for all n edges of $\Gamma(X)$.

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Figure 4: Modular graphs dual to hyperelliptic nodal curves

2 Modular graphs

It will be convenient to define a modular graph Γ by the following data: $(V, \overrightarrow{E}, i, \overrightarrow{E}_-, s)$, where:

(1) V is the set of *vertices* of Γ;

(2) $\overrightarrow{E}$ is the set of *oriented edges* of the modular graph Γ;

(3) $i\colon \overrightarrow{E} \to \overrightarrow{E}$ is an *orientation-changing involution*, (that is fixed point free).

(4) $\overrightarrow{E}_-$ is the set of *outcoming oriented edges* of the modular graph Γ; it is claimed that $\overrightarrow{E} = \overrightarrow{E}_- \cup i(\overrightarrow{E}_-)$, so that each edge is either incoming or outcoming;

(5) $s\colon \overrightarrow{E}_- \to V$ is a surjective *source map*, assighning to every outcoming edge from $\overrightarrow{E}_-$ its source vertex.

Note that if $\overrightarrow{E} = \overrightarrow{E}_-$, then Γ is an ordinary graph without parabolic edges, so that each edge has two ends. We shall call such graphs *closed graphs*. In the general case $\overrightarrow{E}_-$ may be interpreted as the set of germs of outcoming edges, and $\overrightarrow{E}_+ = i(\overrightarrow{E}_-)$ as the set of germs of *incoming edges*, and the map $t = s \circ i\colon \overrightarrow{E}_+ \to V$ assighnes the target vertex to each germ of incoming edge. For every vertex $v \in V$ of the modular graph Γ the star of germs of outcoming edges is

$$S_-(v) = s^{-1}(v) \qquad (2.1)$$

and

$$S_+(v) = t^{-1}(v), \tag{2.2}$$

is the star of germs of incoming edges; evidently $i(S_-(v)) = S_+(v)$.

We define the *incidence coefficient* of a vertex v and an oriented edge $\overrightarrow{e}$ as

$$inc(\overrightarrow{e}, v) = \begin{cases} 1 & , \text{ if } \quad v = t(\overrightarrow{e}) \\ -1 & , \text{ if } \quad v = s(\overrightarrow{e}) \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

We define the *set of non-oriented edges* of a modular graph Γ as the quotient set $E = \overrightarrow{E}/i$ with the natural projection $\pi\colon \overrightarrow{E} \to E$. The number of elements in $S_-(v)$ or $S_+(v)$) is called the *valence* of the vertex v.

A morphism f from a modular graph $\Gamma' = (V', \overrightarrow{E}', i', \overrightarrow{E}'_-, s')$ to a modular graph $\Gamma = (V, \overrightarrow{E}, i, \overrightarrow{E}_-, s)$ is a pair of maps $f_V\colon V' \to V$ and $f_E\colon \overrightarrow{E}' \to \overrightarrow{E}$, so that $f_E(\overrightarrow{E}'_-) \subseteq \overrightarrow{E}_-$, $f_V s' = s f_E$ and $f_E i = i' f_E$. A modular graph $\Gamma' = (V', \overrightarrow{E}', i', \overrightarrow{E}'_-, s')$ is called a *subgraph* of a graph $\Gamma = (V, \overrightarrow{E}, i, \overrightarrow{E}_-, s)$, if $V' \subseteq V$, $\overrightarrow{E}' \subseteq \overrightarrow{E}$, $\overrightarrow{E}'_- \subseteq \overrightarrow{E}_-$, and the maps s' and i' are the restrictions of the maps s and i on $\overrightarrow{E}_-$ and $\overrightarrow{E}$. Thus a subgraph Γ' of a modular graph Γ is defined by two subsets $V' \subseteq V$ and $\overrightarrow{E}' \subseteq \overrightarrow{E}$. A subgraph $\Gamma' = (V', \overrightarrow{E}')$ of a modular graph Γ will be called *induced*, if the complementary subsets $V'' = V \setminus V'$ and $\overrightarrow{E}'' = \overrightarrow{E} \setminus \overrightarrow{E}'$; define some subgraph $\Gamma'' = (V'', \overrightarrow{E}'')$ of the modular graph Γ. In this case we shall say that Γ is divided into two induced subgraphs Γ' and Γ''. This is equivalent to

$$\overrightarrow{E}' \supset \overrightarrow{E} \setminus (s^{-1}(V'') \cup i(s^{-1}(V''))). \tag{2.4}$$

If the last inclusion is the equality, we say that the induced subgraph Γ' is a *closed subgraph* of Γ; the complementary subgraph Γ'' is then called an *open subgraph* of Γ. Thus the division of Γ into closed and open subgraphs is defined by a partitions of the set of vertices $V = V' \cup V''$, $\quad V' \cap V'' = \emptyset$, the corresponding partition of the set of oriented edges is given by

$$\overrightarrow{E}'' = (s^{-1}(V'') \cup i(s^{-1}(V''))); \qquad \overrightarrow{E}' = \overrightarrow{E} \setminus \overrightarrow{E}''. \tag{2.5}$$

A morphism f of a modular graph Γ' to a modular graph Γ is called a *star morphism*, if the restriction of the map f_E to the star of germs of outcoming edges for any vertex $v' \in V'$ is one-to-one correspondence

$$f_E\colon S_-(v') \xrightarrow{\sim} S_-(f_V(v)). \tag{2.6}$$

Note that the inclusion morphism of a subgraph Γ' into a modular graph Γ is a star morphism if and only if Γ' is an open subgraph of Γ.

Next we shall define homology and cohomology groups of a modular graph, different from the standard topological ones. We shall call them *"star"* homology and cohomology and denote by ${}_sH$. Fix an abelian group A and an orientation on the set of edges E, i. e. a map

$$\sigma\colon E \to \overrightarrow{E}, \tag{2.7}$$

assighning to every edge one of its two possible orientations, so that $\pi\circ\sigma = Id_E$. The standard proof of independence of all our constructions on the choice of σ will be omitted.

The coboundary map δ from the group of 1-cochains A^E to the group of 2-cochains A^V

$$\delta\colon A^E \to A^V, \tag{2.8}$$

is defined by

$$\delta(\{a_e\})_v = \sum_{\overrightarrow{e}\in S_-(v)} inc(\sigma(\pi(\overrightarrow{e})), v)\cdot a_{\pi(\overrightarrow{e})}; \tag{2.9}$$

then

$${}_sH^1(\Gamma, A) = \operatorname{Ker}\delta, \qquad {}_sH^2(\Gamma, A) = \operatorname{Coker}\delta. \tag{2.10}$$

Similarly the boundary operator

$$d\colon A^V \to A^E \tag{2.11}$$

is given by

$$d(v) = \sum_{\overrightarrow{e}\in S_-(v)} inc(\sigma(\pi(\overrightarrow{e})), v)\cdot \pi(\overrightarrow{e}); \tag{2.12}$$

and then

$${}_sH_1(\Gamma, A) = \operatorname{Coker} d, \qquad {}_sH_2(\Gamma, A) = \operatorname{Ker} d. \tag{2.13}$$

It is easy to verify that the star homology and cohomology behave properly with respect to star morphisms:

Proposition 2.1 *Let*

$$f\colon \Gamma' \to \Gamma \tag{2.14}$$

be a star morphism. Then there are natural homomorphisms

$$f^*\colon {}_sH^i(\Gamma, A) \to {}_sH^i(\Gamma', A) \tag{2.15}$$

and

$$f_*\colon {}_sH_i(\Gamma', A) \to {}_sH_i(\Gamma, A). \tag{2.16}$$

We shall also use the following simple statement.

Proposition 2.2 *Let $f\colon \Gamma' \to \Gamma$ be a star morphism, and let the map $f_V\colon V' \to V$ be surjective. Then the homomorphism $f^*\colon {}_sH^1(\Gamma, A) \to {}_sH^1(\Gamma', A)$ is injective.*

Note that for closed modular graphs the star cohomology are the simplicial homology of Γ, considered as a one dimensional simplicial complex (replacing 2 by 0 and vise versa). But for modular graphs with parabolic edges our constructions can't be described in such topological terms. These definitions and notations have the following motivations. Over the field of complex numbers the nodal curves are the degenerations of Riemann surfaces that can be obtained in the following way. Fix on a Riemann surface Σ several non-intersecting non-contractible circles $C_e,\ e \in E$, dividing the surface into parts $T_v,\ v \in V$, each homeomorphic to a Riemann surface of lower genus with $k_v > 1$ holes. Contracting each circle to a point we obtain (at least topologicaly) nodal curve X_Σ, whose dual modular graph is $\Gamma = \Gamma(X_\Sigma)$ having V as the set of vertices and E as the set of edges. Fix an orientation on the surface Σ. Then the choice of orientation on every circle C_e is equivalent to the choice of orientation on the corresponding edge e of the modular graph Γ. A linear combination of vertices $\sum a_v \cdot v$ of the modular graph Γ, which we called a 2-chain, corresponds to the usual 2-chain $\sum a_v \cdot T_v$ on the surface Σ, and its boundary in the homology of the surface Σ is calculated by the formula for the boundary operator for the star homology (2.12) of the modular graph Γ. In this way we obtain the injection

$${}_sH_1(\Gamma, A) \hookrightarrow H_1(\Sigma, A) \tag{2.17}$$

and the isomorphism

$${}_sH_2(\Gamma, A) \cong H_2(\Sigma, A). \tag{2.18}$$

The collection of intersection indices of an arbitrary cycle from $H_1(\Sigma, A)$ with the circles C_e is a 1-cocycle from ${}_sH^1(\Gamma, A)$, which gives the homomorphism

$$H_1(\Sigma, A) \to {}_sH^1(\Gamma, A). \tag{2.19}$$

If all the parts T_v are spheres (with holes) we obtain the exact sequence

$$0 \to {}_sH_1(\Gamma, A) \to H_1(\Sigma, A) \to {}_sH^1(\Gamma, A) \to 0. \tag{2.20}$$

So the star homology and cohomology of Γ come from respectively A and B cycles on the Riemann surface Σ.

Recall that $R(\Gamma)$ is the set of closed connected components of the modular graph Γ. It is easy to check the following statement.

Proposition 2.3

$$ {}_sH^2(\Gamma, A) \cong A^{R(\Gamma)}. \tag{2.21}$$

Corollary 2.1

$$ {}_sh^1(\Gamma) = |E| - |V| + |R|. \tag{2.22}$$

Take any closed (not necessarily induced) subgraph $\Gamma' = (V', E')$ of the modular graph Γ; put $V'' = V \setminus V'$, $E'' = E \setminus E'$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A^{E'} & \longrightarrow & A^{E} & \longrightarrow & A^{E''} & \to & 0 \\ & & \downarrow \delta' & & \downarrow \delta & & \downarrow \delta'' & & \\ 0 & \to & A^{V'} & \longrightarrow & A^{V} & \longrightarrow & A^{V''} & \to & 0, \end{array} \tag{2.23}$$

whose vertical arrows are given by the formula (2.9) (changing E and V to E' and V' or E'' and V''). Using the snake lemma, we obtain the exact sequence of a pair:

$$\begin{aligned} 0 \to {}_sH^1(\Gamma', A) \to {}_sH^1(\Gamma, A) \to {}_sH^1(\Gamma, \Gamma', A) \to \\ \to {}_sH^2(\Gamma', A) \to {}_sH^2(\Gamma, A) \to {}_sH^2(\Gamma, \Gamma', A) \to 0. \end{aligned} \tag{2.24}$$

If the closed subgraph Γ' is induced, the right column of (2.23) gives the star cohomology of the modular graph $\Gamma'' = (V'', E'')$. Using Proposition 2.3, we obtain the following statement:

Proposition 2.4 *Let Γ'' be an open subgraph of the modular graph Γ; Γ' be the complementary closed subgraph,*

$$ i\colon \Gamma'' \to \Gamma \tag{2.25}$$

— the inclusion map. Then the natural map $i^*\colon {}_sH^1(\Gamma, A) \to {}_sH^1(\Gamma'', A)$ can be included into the exact sequence

$$0 \to {}_sH^1(\Gamma', A) \to {}_sH^1(\Gamma, A) \xrightarrow{i^*} {}_sH^1(\Gamma'', A) \to \\ \to A^{R(\Gamma')} \to A^{R(\Gamma)} \to A^{R(\Gamma'')} \to 0. \quad (2.26)$$

Corollary 2.2 *Suppose that the assumptions of the Proposition 2.4 are fulfilled and every closed connected component of the induced closed subgraph Γ' is contained in a closed connected component of the modular graph Γ, and each closed connected component of the modular graph Γ contains no more than one closed connected component of the subgraph Γ'. Then there is the exact sequence*

$$0 \to {}_sH^1(\Gamma', A) \to {}_sH^1(\Gamma, A) \xrightarrow{i^*} {}_sH^1(\Gamma'', A) \to 0 \quad (2.27)$$

Corollary 2.3 *Suppose that the assumptions of the Proposition 2.4 are fulfilled and the graphs Γ and Γ' are connected. If the modular graph Γ contains parabolic edges, suppose that Γ' also contains parabolic edges. Then there is the exact sequence*

$$0 \to {}_sH^1(\Gamma', A) \to {}_sH^1(\Gamma, A) \xrightarrow{i^*} {}_sH^1(\Gamma'', A) \to 0 \quad (2.28)$$

Let us see, what happens to the cohomology of a modular graph after deleting some subset of edges $E'' \subset E$. Then the modular graph $\Gamma' = (V, E \backslash E'')$ is a closed (not necessarily induced) subgraph of the modular graph Γ, and using the exact sequence of a pair (2.24) it is easy to compare the cohomology of Γ and Γ'. In this case ${}_sH^1(\Gamma, \Gamma', A) = A^{E''}$ and ${}_sH^2(\Gamma, \Gamma', A) = 0$, so we obtain the exact sequence

$$0 \to {}_sH^1(\Gamma', A) \to {}_sH^1(\Gamma, A) \to A^{E''} \to A^{R'} \to A^R \to 0. \quad (2.29)$$

We have proved the following statement.

Proposition 2.5 *Suppose that the modular graph Γ' is obtained from the modular graph Γ by deleting some subset of edges $E'' \subset E$. Then there is the exact sequence*

$$0 \to {}_sH^1(\Gamma', A) \to {}_sH^1(\Gamma, A) \to A^{E''} \to A^{|R'|-|R|} \to 0. \tag{2.30}$$

Note that $|R'| - |R|$ is the increase of the number of closed connected components after deleting edges E''. By definition of thickness $th(\Gamma)$ of a modular graph Γ, $|R'| - |R| = 0$ if$|E''| < th(\Gamma)$. So we get the following corollary.

Corollary 2.4 *If $|E''| < th(\Gamma)$ then the edges from E'' impose independent constraints on ${}_sH^1(\Gamma, A)$.*

Let's consider the important particular case, when we delete all the parabolic edges of Γ, i.e. $E'' = E_{par}$.

Proposition 2.6 *There is the exact sequence*

$$0 \to {}_sH^1(\Gamma_{closed}, A) \to {}_sH^1(\Gamma, A) \to A^{E_{par}} \to A^{R_{par}} \to 0, \tag{2.31}$$

were the last homomorphism is given by the formula

$$\sum_{e \in E_{par}} a_e \cdot e \mapsto \sum_{r \in R_{par}} (\sum_{e \in E^r_{par}} a_e) \cdot r, \tag{2.32}$$

where R_{par} is the set of connected components of the modular graph Γ, containing parabolic edges, and E^r_{par} is the set of parabolic edges of the component $r \in R_{par}$.

3 Differentials

Consider a pointed nodal curve X having only nodal double singularities, let $\Gamma(X)$ be the dual modular graph, V its set of vertices, E its set of edges, R its set of closed connected components. Any section of the sheaf of regular differentials Ω^1_X on X is a collection $\{\omega_v\}$ of sections of the restrictions of the lifts $\nu^*\Omega^1_X$ to each component X^ν_v, that are simply sheaves of differentials of the third kind on X^ν_v with poles only at the points $P_{\overrightarrow{e}}$, $\overrightarrow{e} \in S_-(v)$:

$$(\nu^*\Omega^1_X)\big|_{X^\nu_v} = \Omega^1_{X^\nu_v}(\sum_{\overrightarrow{e} \in S_-(v)} P_{\overrightarrow{e}}). \tag{3.1}$$

These differentials satisfy the compatibility equations at each singular point p_e of intersection of the components X_{v_1} and X_{v_2}:

$$res_{P_{\overrightarrow{e}}}\omega_{v_1} + res_{P_{\overleftarrow{e}}}\omega_{v_2} = 0. \tag{3.2}$$

(Here $\overrightarrow{e}$ and $\overleftarrow{e}$ mean two orientations on the edge e, so that $s(\overrightarrow{e}) = v_1$ and $s(\overleftarrow{e}) = v_2$) Fix any orientation σ (see (2.7)) on the edges of the modular graph $\Gamma(X)$. Then we have a map

$$H^0(X, \Omega^1_X) \to k^E, \tag{3.3}$$

assighning to every regular differential $\omega \in H^0(X, \Omega^1_X)$ the collection of its residues at all the points p_e:

$$\omega \mapsto \{res_{P_{\sigma(e)}}\omega_{s(\sigma(e))},\ e \in E\}. \tag{3.4}$$

The equation (3.2) means that this map does not depend on the choice of the orientation σ, and the main residue theorem implies that the image is the cocycle. Thus we have a map

$$H^0(X, \Omega^1_X) \to\ {}_sH^1(\Gamma(X), k). \tag{3.5}$$

The constructed map is surjective, since on every nonsingular curve there exists a differential of the third kind with any prescribed collection of values of residues (with zero sum) at given points (see [1]). The kernel of this map consists of regular differentials on each X^ν_v.

Proposition 3.1 *There is the exact sequence*

$$0 \to \bigoplus_{v \in V} H^0(X^\nu_v, \Omega^1_{X^\nu_v}) \to H^0(X, \Omega^1_X) \to\ {}_sH^1(\Gamma(X), k) \to 0. \tag{3.6}$$

Corollary 3.1

$$\dim H^0(X, \Omega^1_X) = \sum_{v \in V} g(X^\nu_v) + |E| - |V| + |R|. \tag{3.7}$$

We shall denote this dimension by $g_\gamma(X)$.

Now using Proposition 3.6 we may rewrite the exact sequences of the previous section substituting $H^0(X, \Omega^1_X)$ for ${}_sH^1(\Gamma(X), k)$. To do that it is sometimes useful to consider the marked points of the curve as punctures, so that a pointed curve X with marked points p_e, $e \in E_{par}$ is considered as a non-compact curve, obtained from the compact curve X_{closed} by deleting the marked points. Then the modular graph $\Gamma(X)$ is closed if and only if X is compact. Combining (3.6) and (2.31) we obtain the exact sequence:

$$0 \to H^0(X_{closed}, \Omega^1_{X_{closed}}) \to H^0(X, \Omega^1_X) \to k^{E_{par}} \to k \to 0, \tag{3.8}$$

Here the left homomorphism a regular differential on X to the collection of its residues at the marked points, the middle one is the residue map and the right one is given by the equation (2.32). This is simply the main residue theorem for pointed nodal curves.

Proposition 3.2 *1) The sum of the residues of a regular differential at all marked points on a pointed curve with double nodal singularities equals zero. 2) On a pointed curve with double nodal singularities there exists a regular differential with any prescribed collection of residues at the marked points (with zero sum).*

Consider a closed induced subgraph Γ' of the modular graph $\Gamma(X)$, and let Γ'' be the complementary open subgraph. Consider the corresponding curves

$$X' = \bigcup_{v \in V'} X_v \qquad \text{and} \qquad X'' = \bigcup_{v \in V''} X_v. \tag{3.9}$$

The curves X' and X'' inherit corresponding marked points of X and X'' additionally acquires new marked points, namely, the branches of the singular points of intersection with X' (see (2.5)). Then $\Gamma' = \Gamma(X')$ and $\Gamma'' = \Gamma(X'')$. (Interpreting marked points as punctures, we may say that we simply tear of the components X' from X'', so that the intersection points go with X', leaving punctures on their former places at X''.)

Now combining (3.6) and (2.26) we obtain the following exact sequence:

$$\begin{aligned} 0 \to H^0(X', \Omega^1_{X'}) \xrightarrow{u} H^0(X, \Omega^1_X) \xrightarrow{q} H^0(X'', \Omega^1_{X''}) \\ \to k^{R'} \to k^R \to k^{R''} \to 0. \end{aligned} \tag{3.10}$$

Here R' and R'' are the sets of closed components of Γ' and Γ'', the inclusion u extends regular differentials from $H^0(X', \Omega^1_{X'})$ to X by zero on X'', and q is the restriction of regular differentials from $H^0(X, \Omega^1_X)$ to X''.

Using Corollary 2.2, we obtain the following

Proposition 3.3 *Let X' be a closed subset of the curve X, consisting of some irreducible components of X, put $X'' = X \setminus X'$. Suppose that each compact connected component of the curve X' is contained in a compact connected component of the curve X and each compact connected component of the curve X contains at most one compact connected component of the curve X'. Then every regular differential form on X'' can be extended to a regular differential form on X.*

Corollary 3.2 *Suppose that X' is a connected closed subset of a compact connected curve X consisting of some irreducible components of X, and $X'' = X \setminus X'$. Then every regular differential form on X'' can be extended to a regular differential form on X.*

Next let us consider a subgraph Γ' of $\Gamma(X)$, obtained by by deleting some subset of edges $E'' \subset E$, and let X' be the pointed nodal curve, corresponding to Γ'. Then there is the exact sequence

$$0 \to H^0(X', \Omega^1_{X'}) \to H^0(X, \Omega^1_X) \to k^{E''} \to k^{|R'|-|R|} \to 0. \tag{3.11}$$

Corollary 3.3 *Suppose that X is a connected curve, and let $E'' \subset E$ be a subset of edges of $\Gamma(X)$, such that $|R(\Gamma(X))|$ does not increase after deleting all the edges E''. Then there exists a regular differential on X with any prescribed collection of residues at the points p_e, $e \in E''$.*

4 The canonical map

Consider the canonical map

$$\varphi_X \colon X \to \mathbb{P}(H^0(X, \Omega^1_X)^*) = \mathbb{P}^{g_\gamma - 1}, \tag{4.1}$$

where $g_\gamma = h^0(X, \Omega^1_X)$.

First let us explain the role of parabolic points, by comparing the canonical maps for X and for its closure X_{closed}, obtained by "forgetting" all marked parabolic points. The exact sequence (3.8) immediately implies the following statement.

Proposition 4.1 *1) The canonical map of the closure X_{closed} of a pointed nodal curve X with $m > 1$ parabolic points is the composition of the canonical map of the pointed nodal curve X and a projection from the $(m-2)$-dimensional projective space, spanned by the points $\varphi_X(p_e)$, $e \in E_{par}$.*

2) The canonical map of a pointed nodal curve X with one parabolic point coincides with the canonical map of its closure X_{closed}.

Now suppose that there are $2m$ parabolic points on a nodal curve X, corresponding to $2m$ parabolic edges

$$e_1, \ldots, e_m,\ e'_1, \ldots, e'_m \tag{4.2}$$

of the modular graph $\Gamma(X)$. Consider the nodal curve $\tilde{X}$ obtained by gluing each pair of points p_{e_l}, $p_{e'_l}$ of the curve X into one nodal singular point $p_{\tilde{e}_l}$ of the curve $\tilde{X}$, $l = 1, \ldots m$. Suppose for simplicity, that the curve X had no closed connected components (such components would not change after the gluing), and the curve $\tilde{X}$ is connected (otherwise all the arguments may be repeated for each connected component of $\tilde{X}$). The gluing map $\pi \colon X \to \tilde{X}$ induces the star morphism of the graphs $\pi \colon \Gamma(X) \to \Gamma(\tilde{X})$. Combining Propositions 3.6 and 2.2, we obtain the inclusion $\pi^* \colon {}_sH^1(\Gamma(\tilde{X}), k) \hookrightarrow {}_sH^1(\Gamma(X), k)$, which is simply the lifting of differential forms from $\tilde{X}$ to X. Comparing the dimensions according to (2.22), we obtain the exact sequence

$$0 \to H^0(\tilde{X}, \Omega^1_{\tilde{X}}) \to H^0(X, \Omega^1_X) \to k^d \to 0, \tag{4.3}$$

$$\text{where } d = \begin{cases} m-1, & \text{if the curve } \tilde{X} \text{ is compact} \\ m, & \text{if the curve } \tilde{X} \text{ is not compact} \end{cases} \tag{4.4}$$

Proposition 4.2 *The canonical map of a pointed nodal curve $\tilde{X}$ obtained from a nodal curve X by pairwise gluing of m pairs of parabolic points of the curve X, is the composition of the canonical map of the curve X and, for $d > 0$, a projection from an $(d-1)$-dimensional projective space, where d is given by (4.4).*

Evidently, d can happen to be zero only if $m = 1$ and the curve $\tilde{X}$ is compact; in this case (4.3) is an isomorphism.

Corollary 4.1 *The canonical map of a pointed nodal curve X with two parabolic points is the composition of the "gluing map" $\pi \colon X \to \tilde{X}$, (which is an isomorphism outside these two points and maps each of them to the same nodal singular point) and the canonical map of the curve $\tilde{X}$.*

Suppose now that in the described situation the two parabolic points were located on different connected components X_1 and X_2 of the curve X. Then

$H^0(X,\Omega^1_X) = H^0(X_1,\Omega^1_{X_1}) \bigoplus H^0(X_2,\Omega^1_{X_2})$, so that the canonical map of the curve X is a pair of the canonical maps of the curves X_1 and X_2 into distinct projective subspaces in $\mathbb{P}(H^0(X,\Omega^1_X)^*)$. But as we have seen, this map is the canonical map of the compact connected curve $\tilde{X}$, which is, therefore, not defined at the point $p_{\tilde{e}_1}$. Of course, this is precisely the case, when the modular graph $\Gamma(\tilde{X})$ has thickness 1, since the removing of the edge $\tilde{e}_1$ breaks $\Gamma(\tilde{X})$ into two parts.

Proposition 4.3 *1) Let X be a connected compact nodal curve, and let $th(\Gamma(X)) = 1$. Then the canonical sheaf Ω^1_X is not ample.*
2) Suppose that the curve X is the union of two connected compact curves X_1 and X_2, having only one common nodal singular point p_e. Then p_e is a base point of the canonical map φ_X, and the restrictions of φ_X to X_1 and X_2 are the canonical maps of X_1 and X_2 into distinct projective subspaces in $\mathbb{P}(H^0(X,\Omega^1_X)^*)$.

Thus we have proved the necessity of the condition $th(\Gamma(X)) \geq 2$ of the first part of the Theorem 1.3.

Next let us consider the case, when the dual modular graph of a compact nodal curve X has $th(\Gamma(X)) = 2$. Suppose that the edges e_1 and e_2 connect two closed subgraphs $\Gamma(X_1)$ and $\Gamma(X_2)$, corresponding to two curves, X_1 and X_2, intersecting at two nodal points p_{e_1} and p_{e_2} of the curve $X = X_1 \cup X_2$. Consider the noncompact curves $X_i^0 = X_i \setminus \{p_{e_1}, p_{e_2}\}$, $i = 1, 2$ (in the other words, we mark the nonsingular points p_{e_1} and p_{e_2} on the curves X_1 and X_2), so that the graphs $\Gamma(X_1^0)$ and $\Gamma(X_2^0)$ are the open subgraphs of $\Gamma(X)$, and $\Gamma(X_2)$ $\Gamma(X_1)$ are the complementary closed subgraphs. The non-connected curve $X_1^0 \coprod X_2^0$ contains two pairs of parabolic points to be pairwise glued together to obtain X, so by the Proposition 4.2 the canonical map of X is the composition of the canonical map of $X_1^0 \coprod X_2^0$ and a projection from a point. But the canonical map of $X_1^0 \coprod X_2^0$, is of course, simply the pair of the canonical maps of the curves X_1^0 and X_2^0. The canonical map of each X_i^0, according to the Corollary 4.1, glues the points p_{e_1} and p_{e_2} together on each X_i^0, and therefore, the same happens for φ_X. Thus we have proved the necessity of the condition $th(\Gamma(X)) \geq 3$ of Theorem 1.3.

Proposition 4.4 *1) The canonical map of a connected compact nodal curve having $th(\Gamma(X)) = 2$ is not an embedding.*
2) Suppose that in the assumptions of the part 1 the edges e_1 and e_2 divide the modular graph $\Gamma(X)$ into two subgraphs, so that X is the union of two

connected compact curves intersecting at two nodal points p_{e_1} and p_{e_2} (which are nonsingular on each X_i). Then the restrictions of the canonical map φ_X to each X_i, $i = 1, 2$ are the canonical maps of the pointed curves X_i^0, obtained from the curves X_i^0 by marking the points p_{e_1} and p_{e_2} on each of them. The maps $\varphi_{X_1^0}$ and $\varphi_{X_2^0}$ take the curves X_1 and X_2 into two projective subspaces in $\mathbb{P}(H^0(X, \Omega_X^1)^*)$ with one common point M, and if $\varphi_{X_1^0}$ and $\varphi_{X_2^0}$ are defined at the points p_{e_1} and p_{e_2}, then $\varphi_X(p_{e_1}) = \varphi_X(p_{e_1}) = M$.

For thickness 2 non-compact curves all the considerations are quite similar, but we would rather show how to reduce all the problem to the compact case (see Proposition 4.2 below).

First of all let us introduce a very simple inductive argument that we shall use in the proof of the Theorem 1.3. Consider some open subgraph Γ'' of the dual modular graph $\Gamma(X)$ of some pointed nodal curve X and the corresponding pointed nodal curve $X'' = \cup_{v \in V''} X_v$. Suppose that the complementary closed subgraph Γ' is connected. Then the Corollary 3.2 implies the following lemma.

Lemma 4.1 *Let Γ'' be an open subgraph of the dual modular graph $\Gamma(X)$ of some pointed nodal curve X and suppose that the complementary closed subgraph Γ'' is connected. Then the restriction of the canonical map φ_X of the curve X to X'' coincides with the canonical map $\varphi_{X''}$ of the curve X'':*

$$\varphi_X|_{X''} = \varphi_{X''}. \tag{4.5}$$

Consider a non-compact pointed nodal curve X with m parabolic points. The cases $m = 1, 2$ are described in Proposition 4.1 and Corollary 4.1, so we shall assume $m \geq 3$. Consider the one-point compactification $\hat{\Gamma}$ of the modular graph $\Gamma(X)$, obtained by adding to V one additional vertex incident to all the parabolic edges of $\Gamma(X)$. It is easy to see that

$$th(\hat{\Gamma}) = th(\Gamma(X)). \tag{4.6}$$

To construct the corresponding curve $\hat{X}$ we simply take any rational component X_{v_0}, so that $\hat{X} = X \cup X_{v_0}$ and $\Gamma(\hat{X}) = \hat{\Gamma}$; our very simple arguments are independent on the concrete choice of the point in $M_{0,m}$ for X_{v_0}. We shall also call such a curve $\hat{X}$ a one-component compactification of X. The modular graph $\Gamma(X)$ is an open subgraph of $\hat{\Gamma}$, and the complementary closed subgraph is the vertex v_0. Using the exact sequence 3.10 we obtain the following statement.

Lemma 4.2 *Let X be a pointed nodal curve, $\hat{X}$ — its one-component compactification. Then*

$$H^0(X, \Omega^1_X) = H^0(\hat{X}, \Omega^1_{\hat{X}}) \tag{4.7}$$

and the restriction of the canonical map $\varphi_{\hat{X}}$ of the curve $\hat{X}$ to X coincides with the canonical map φ_X of the curve X.

Now we are ready to prove the first part of the Theorem 1.3. Due to the previous lemma it is sufficient to prove the theorem only for a compact nodal curve X. Suppose that $th(\Gamma(X)) \geq 2$, x — arbitrary point of X. To prove that the point x is not a base point it is sufficient to find a regular differential ω on X which is nonzero at the point x.

Firs of all, notice that since $th(\Gamma(X)) \geq 2$, the modular graph $\Gamma(X)$ remains connected after removing any edge $e \in E$. According to Corollary 3.3 it means that there exist a regular differential with any residue at the point p_e, so a singular point can't be a base point.

Now suppose that x is a nonsingular point of some irreducible component X_v of X. If X_v is non rational, for ω we may take any regular differential form on X_v^ν, nonvanishing at x, and extend it by zero to all the other components of X. So we may assume that X_v is rational. If there is a loop e at the vertex v, we may choose coordinate z on the projective line X_v^ν, so that the branches of the corresponding node are 0 and ∞, then for such ω we may take $\frac{dz}{z}$ on X_v^ν and extend it by zero to all the other components of X. So we may assume that v has no loops.

Now let us try to use the inductive argument of Lemma 4.1 for the open subgraph Γ'', having one vertex v_0 and edges $S(v_0)$. Since there are at least two edges in $S(v_0)$, and again, choosing coordinate z on the projective line X_v^ν, so that the corresponding points are 0 and ∞, we may take $\frac{dz}{z}$ on X_v^ν. According to Lemma 4.1, this differential can be extended to the whole X if the complementary modular graph Γ', obtained by removing v_0 (and $S(v_0)$), is connected.

So it is left to consider the case, when removing v_0 breaks $\Gamma(X)$ into several disjoint parts, let us name one connected component Γ_1, all the others — Γ_2. Then the edges, connecting Γ_1 with v_0 are the parabolic edges of Γ_1, so by Proposition 3.2 there exist a differential on the corresponding curve X_1

with any prescribed collection of residues. Again the condition $th(\Gamma(X)) \geq 2$ implies that there are at least two such edges, so we again can take $\frac{dz}{z}$ on X_v^ν, extend it to X_1 by the previous argument and extend it by zero to X_2.

Theorem 1.2 is proved.

5 Irreducible curves

In this section we shall prove Theorem 1.3 for irreducible nodal curves. We start with irreducible compact nodal curves. The dual modular graph of such a curve consists of a single vertex with several loops, corresponding to the self intersection points of the curve.

First of all, we need some more general inductive arguments. Suppose we have a compact nodal curve X, $th(\Gamma(X)) \geq 3$, whose canonical map φ_X is an embedding to $\mathbb{P}^{g_\gamma - 1}$ of degree

$$d_\gamma = 2(g_\gamma - 1), \tag{5.1}$$

where

$$g_\gamma = \sum_{v \in V} g(X_v^\nu) + |E| - |V| + 1. \tag{5.2}$$

(For connected compact curve $|R| = 1$, $|E_{par}| = 0$.) Choose two nonsingular points p_1 and p_2 on X and consider a new pointed nodal curve X^0, obtained from X by marking the points p_1 and p_2, and a new compact nodal curve $\tilde{X}$. obtained from X by gluing together the points p_1 and p_2 into additional node $\tilde{p}$ with branches p_1 and p_2. According to Proposition 4.1 and Corollary 4.1, the canonical maps φ^0 for X^0 and $\tilde{\varphi}$ for $\tilde{X}$ coincide, and the canonical map φ_X is the composition of $\varphi^0 = \tilde{\varphi}$ and the projection from the point $\varphi^0(p_1) = \varphi^0(p_2)$. Thus $\tilde{\varphi}$ is an isomorphism at least everywhere outside p_1 and p_2. Since

$$H^0(X^0, \Omega^1_{X^0}) = H^0(X, \Omega^1_X(p_1 + p_2)), \tag{5.3}$$

the image of φ^0 has degree $d_\gamma + 2$, hence both branches of the curve $\varphi^0(X)$ are nonsingular, and therefore $\varphi^0 = \tilde{\varphi}$ is an embedding. We have proved the following useful inductive statement.

Lemma 5.1 *Suppose that a compact connected nodal curve $\tilde{X}$ is obtained from a compact connected nodal curve X by gluing two nonsingular points into one additional node (so that the modular graph $\Gamma(X)$ is obtained from* *the modular graph* $\Gamma(\tilde{X})$ *by removing an edge). Suppose that* $th(\Gamma(X)) \geq 3$ *and the canonical map* φ_X *of* X *is an embedding. Then the canonical map* $\varphi_{\tilde{X}}$ *of* $\tilde{X}$ *is an embedding too.*

Take for example a nonsingular nonhyperelliptic curve ($g > 2$), then its canonical map is an embedding. Using the lemma we immediately prove the same for the corresponding nodal curves.

Corollary 5.1 *The canonical map of an irreducible compact nodal curve* X *with nonhyperelliptic normalization of genus greater than* 2 *is an embedding.*

Now take a hyperelliptic curve X ($g \geq 2$) with one node, such that its branches p_1 and p_2 are not transposed by the hyperelliptic involution $i\colon X^\nu \to X^\nu$; this means that $p_1 + p_2$ is not a zero divisor of any rational function on X^ν. The canonical map φ_X is given by the complete linear system $|\Omega^1_{X^\nu}(p_1 + p_2)|$, and to prove that it is an embedding it is sufficient to check that for any two points $a, b \in X^\nu$, $\{a, b\} \neq \{p_1, p_2\}$,

$$h^0(X^\nu, \Omega^1_{X^\nu}(p_1 + p_2 - a - b)) = h^0(X^\nu, \Omega^1_{X^\nu}(p_1 + p_2)) - 2. \tag{5.4}$$

By Serre duality this is equivalent to $h^0(X^\nu, \mathcal{O}_{X^\nu}(a + b - p_1 - p_2)) = 0$, what is, as we have seen, true. Then, using the inductive Lemma 5.1, we prove the first part of Theorem 1.3 for hyperelliptic curves (more pecisely, we should say "for a non-hyperelliptic iireducible nodal curve, whose normalization is hyperelliptic").

Proposition 5.1 *The canonical map of an irreducible compact nodal curve* X *with hyperelliptic normalization of genus greater than 1 is an embedding if the two branches of any node are not transposed by the hyperelliptic involution.*

In the same way we may prove such statements for elliptic and rational cases.

Proposition 5.2 *The canonical map of an irreducible compact nodal curve* X *with elliptic normalization and more than 1 node is an embedding if there is no elliptic involution transposing the branches of each node.*

Proposition 5.3 *The canonical map of an irreducible compact nodal curve* X *with rational normalization and more than 2 nodes is an embedding if there is no projective involution transposing the branches of each node.*

These two statements are proved by the same inductive argument, and for the base of induction we must take the elliptic curve with two nodes for Proposition 5.2 and the rational curve with three nodes for Proposition 5.3. These are plane quartics with two and, respectively, three nodes.

If the branches of all nodes are transposed by hyperelliptic, elliptic or rational involution i, it is easy to see that all the regular differentials on X are lifted from the rational curve X^ν/i with marked points, corresponding to the nodes of X, and so the canonical map of X is the composition of the double sheet factorization map and the canonical map of X^ν/i, which is just the Veronese map. (See, for example, [3] or [1].) This completes the proof of Theorem 1.3 for compact irreducible curves.

For non-compact curves we need some more general lemmas. Suppose, like in Lemma 5.1 we have a compact nodal curve X, $th(\Gamma(X)) \geq 3$, whose canonical map φ_X is an embedding to $\mathbb{P}^{g_\gamma - 1}$ of degree

$$d_\gamma(X) = 2(g_\gamma - 1), \tag{5.5}$$

where

$$g_\gamma(X) = \sum_{v\in V} g(X_v^\nu) + |E| - |V| + 1. \tag{5.6}$$

Marking three nonsingular points p_1, p_2, p_3 on X we obtain pointed nodal curve X^0. Then $g_\gamma(X^0) = g_\gamma(X) + 2$, $d_\gamma(X^0) = d_\gamma(X) + 2$, and according to Proposition 4.1, the canonical map of X is the composition of the canonical map of X^0 and projection from a line. Comparing the degrees we see that this line intersects $\varphi_{X^0}(X^0)$ exactly at three points $\varphi_{X^0}(p_i)$, $i = 1, 2, 3$ and these points are nonsingular on $\varphi_{X^0}(X^0)$, and since φ_X is an embedding φ_{X^0} is also an embedding. Of course the same happens when we add to X^0 some more marked points: every new point increases d_γ and g_γ to 1 and we can repeat the same argument. Thus we prove the following lemma.

Lemma 5.2 *1) Suppose that a pointed connected nodal curve X^0 is obtained from a compact connected nodal curve X by marking three nonsingular points (p_i), $i = 1, 2, 3$ (and so the modular graph $\Gamma(X^0)$ is obtained from the modular graph $\Gamma(\tilde{X})$ by adding 3 parabolic edges). Suppose that $th(\Gamma(X)) \geq 3$ and the canonical map φ_X of X is an embedding. Then the canonical map φ_{X^0} of X^0 is an embedding too. The images $\varphi_{X^0}(p_i)$, $i = 1, 2, 3$ lie on a projective line and they are the only intersection points of this line with the image $\varphi_{X^0}(X^0)$.*

2) Suppose that a pointed connected nodal curve X^0 is obtained from a pointed connected nodal curve X with at least three marked points by marking one more nonsingular point p, (and so the modular graph $\Gamma(X^0)$ is obtained from the modular graph $\Gamma(\tilde{X})$ by adding a parabolic edge). Suppose that $th(\Gamma(X)) \geq 3$ and the canonical map φ_X of X is an embedding. Then the canonical map φ_{X^0} of X^0 is an embedding too.

Corollary 5.2 *Suppose that a pointed connected nodal curve X^0 is obtained from a compact connected nodal curve X by marking $m \geq 3$ nonsingular points p_i, $i = 1, \dots, m$ (and so the modular graph $\Gamma(X)$ is obtained from the modular graph $\Gamma(\tilde{X})$ by adding m parabolic edges). Suppose that $th(\Gamma(X)) \geq 3$ and the canonical map φ_X of X is an embedding. Then the canonical map φ_{X^0} of X^0 is an embedding too. The images $\varphi_{X^0}(p_i)$, $i = 1, \dots m$ span an $(m-2)$-dimensional projective space $\mathbb{P}^{m-2}$; these points are in general position in that $\mathbb{P}^{m-2}$ and they are the only intersection points of that $\mathbb{P}^{m-2}$ with the image $\varphi_{X^0}(X^0)$.*

Next let us prove the analogue of Lemma 5.1.

Lemma 5.3 *Suppose that a connected pointed nodal curve $\tilde{X}$ with at least three marked points is obtained from a connected pointed nodal curve X by gluing two nonsingular nonmarked points into one additional node (and so the modular graph $\Gamma(X)$ is obtained from the modular graph $\Gamma(\tilde{X})$ by removing an edge). Suppose that $th(\Gamma(X)) \geq 3$ and the canonical map φ_X of X is an embedding. Then the canonical map $\varphi_{\tilde{X}}$ of $\tilde{X}$ is an embedding too.*

To prove the lemma let us take the two nonsingular points p_1 and p_2 on X that are glued together in $\tilde{X}$ and consider the new pointed nodal curve X^0, obtained from X by marking the points p_1 and p_2. Then $g_\gamma(\tilde{X}) = g_\gamma(X^0)+1 = g_\gamma(X)+2$ and $d_\gamma(\tilde{X}) = d_\gamma(X^0) = d_\gamma(X) + 2$, the canonical map $\varphi_{\tilde{X}}$ is the composition of the canonical embedding (Lemma 5.2) φ_{X^0} and a projection from a point. Hence $\varphi_{\tilde{X}}$ is isomorphic at least outside p_1 and p_2. The canonical embedding φ_X is the composition of $\varphi_{\tilde{X}}$ and the projection from the point $\varphi_{\tilde{X}}(p_1) = \varphi_{\tilde{X}}(p_2)$, decreasing degree by 2; so this point is a node and $\varphi_{\tilde{X}}$ is an embedding.

Now suppose we have a nonsingular irreducible pointed curve with at least three marked points, then the canonical map is evidently an embedding. Then Lemma 5.3 allows us also to add nodes.

Corollary 5.3 *The canonical map of a pointed irreducible nodal curve with at least three marked points is an embedding.*

6 Proof of the main theorem

We shall prove Theorem 1.3 by induction on number of irreducible components of the compact pointed curve X. Since there are special (namely, hyperelliptic) curves for $|V| = 2$, the general case is $|V| \geq 3$ so we need an individual study of the two- and three-conponent curves. To prove that a map of a reducible curve is an embedding, it is sufficient to verify two things. First, we must prove that the restriction of our map to each irreducible component is an embedding. Second, we must prove that our map distinguishes two points, located on distinct components of the curve X. Note that Corollary 3.3 implies that for $th(\Gamma(X)) \geq 3$ the canonical map distinguishes singular points, so we may restrict ourselves to the nonsingular ones. Moreover, it is easy to see that there is no problem with nonrational components and rational components having a self-intersection nodal point.

Lemma 6.1 *The canonical map of a compact nodal curve whose dual modular graph has thickness at least* 3 *distinguishes points located on different two components if at least one of these components is either nonrational or rational with a self-intersection node.*

To prove the lemma it is sufficient to construct a regular differential on X, distinguishing these two points. Such a regular differential exists if at least one of the components is not rational: we simply take a regular differential form on the normalization of nonrational component and extend it by zero to the others. The same trick goes when at least one rational component has a self-intersection node: we choose a coordinate z on this component in such a way that 0 and ∞ are the branches of the corresponding node and take $\frac{dz}{z}$ on this component and zero on the others.

Now it is quite easy to reduce the problem of distinguishing points on different components of a compact nodal curve to the case when all the components are rational. Suppose we have two nonsingular points located on the components X_i, $i = 1, 2$. We have seen that we may assume they are rational. Take any other component X_i according to Lemma 4.1 we may reduce the problem to that on the pointed nodal curve $X^0 = X \setminus X_i$ (which corresponds to the open subgraph $\Gamma(X) \setminus v_i$), because the restriction of the canonical map of X to X^0 coincides with the canonical map of X^0. Then we can pass to the one-component compactification (Lemma 4.2) of X^0 with the rational component X_i'.

Corollary 6.1 *Let Γ be a thickness 3 modular graph. Suppose that the canonical map of any compact nodal curve X' such that*

1) all the irreducible components of X' are rational and

2) $\Gamma(X') = \Gamma$

distinguishes points, located on different irreducible components. Then the canonical map of any compact nodal curve X such that $\Gamma(X) = \Gamma$ distinguishes points, located on different irreducible components.

Now we start with a compact nodal curve $X = X_1 \cup X_2$ having two irreducible components X_1 and X_2. The dual modular graph $\Gamma(X)$ has two vertices v_1 and v_2, connected by at least three edges, since $th(\Gamma(X) \geq 3$. Each vertex taken with all incident edges is an open subgraph, and the complementary closed subgraph (the other vertex) is connected. So the assumptions of Lemma 4.1 are fulfilled, and hence the restriction of the canonical map φ_X of X to each component X_i coincides with the canonical map φ_{X_i} of the component X_i, considered as a pointed irreducible nodal curve, where the marked points are the points of intersection with the other component. And according to Corollary 5.3 φ_{X_i} are embeddings.

Proposition 6.1 *The restriction of the canonical map to each component of a compact nodal curve with two irreducible components and thickness of the dual modular graph at least three is an embedding.*

Now it is left to find out when the canonical map distinguishes two points located on different components. Lemma 6.1 reduces the problem to the case, where both components are rational and have no self-intersection nodes.

Returning to the 2-component curve X we see that we are left with the case when X consists of two rational nonsingular curves, intersecting at $m \geq 3$ points. The restriction of the canonical map to each component is the Veronese map to the same projective space $\mathbb{P}^{m-2}$. It is well-known (see, for example, [1]), that two such Veronese curves having more than m points in common coincide. Thus the canonical map is either an embedding or has degree 2. In the last case the two components have the same canonical image. This provides the isomorphism transposing the branches of their intersection points; the existence of such an isomorphism implies that the images of the two components coincide, since a differential of the third kind on a rational curve is uniquely determined by the collection of its residues.

Now we have passed all the exceptional cases and may start proving that the canonical map for a compact connected nodal curve X with no less than 3 irreducible components and $th(\Gamma(X)) \geq 3$ is an embedding. The induction base is the case of 3-component curve that we shall discuss now.

Suppose we have such a curve X. Using Lemma 5.1 we may reduce the problem to that on the curve with less number of singular points by removing edges from the dual modular graph, until the thickness of the dual modular graph is still no less than 3. It is easy to verify that proceeding this way we shall at last obtain one of the two minimal 3-vertex thickness 3 graphs (see Figure 5).

Figure 5: Two minimal thickness 3 three-vertex graphs

The first modular graph remains connected after removing any one of its three vertices; by Lemma 4.1 this means that the restriction of the canonical map to each component coincides with the canonical map of the component, considered as a pointed irreducible nodal curve, where the marked points are the points of intersection with the other components. Then Lemma 5.3 provides that the canonical map of each component is an embedding. So it is left to prove that the canonical map distinguishes two points on different components. According to corollary 6.1 we may assume all the components to be rational, but then the canonical image of the curve is simply the plane conic with two secant lines.

The second 3-vertex modular graph ("double-theta" modular graph) is a bit more complicated, because removing the middle vertex disconnects it. So we prove that the canonical map is an embedding on the disjoint union of the rest two components using Lemma 4.1, but need some other arguments for the middle vertex. Note that if X_1 is nonrational, the canonical map for the compact 2-component curve $X_1 \cup X_2$ is already an embedding and then by Lemma 5.2 the same is true for the pointed curve $X_1 \cup X_2$, marked by three points of intersection with X_3. But this pointed curve corresponds to the open subgraph of the dual modular graph, and the complementary closed subgraph is connected (it is simply the vertex v_3) hence by Lemma 4.1 the restriction of the canonical map φ_X to $X_1 \cup X_2$ is an embedding. This completes the proof for non-rational X_2.

Now suppose that X_1 is rational; we must prove that the restriction of the canonical map to X_1 is an embedding and that the canonical map distinguishes a nonsingular point located on X_1 from a nonsingular point located on X_i, $i = 2, 3$. For the first purpose we may again reduce the problem to the pointed 2-component curve $X_1 \cup X_i$ (Lemma 4.1) and then pass to the one-component compactification (Lemma 4.2) with rational third component. Repeating the same for the other component, we restrict ourselves to the case when all the components are rational. For the second purpose Lemma 6.1 also restricts the problem to this case.

The canonical image of the nodal curve with rational components, having "double-theta" dual modular graph, may be presented as follows. Fix two triples of points on a Veronese degree 4 curve in $\mathbb{P}^4$ (all the points different) and take the two planes, spanned by each triple. Each plane has no other intersections with the curve, otherwise any fifth point located on the curve together with this plane would span a 3-dimensional hyperplane, intersecting our degree 4 curve at 5 different points, which is impossible. Moreover, if any secant or tangent line would intersect one of these panes, the plane and the line would span a 3-dimensional hyperplane, intersecting our degree 4 curve at 5 different points, which again is impossible. If these two planes were not in general position, they also would span a 3-dimensional hyperplane, intersecting our degree 4 curve at 6 different points, which is impossible too. So these two planes intersect at one point. The projection from this point maps our curve isomorphically to a degree 4 curve in $\mathbb{P}^3$, which is the canonical image of X_1. The two planes are mapped to the two lines, each transversally intersecting X_1 exactly at three points. These are the canonical images of X_2 and X_3. This completes the proof for the 3-component case.

Starting from the "double-theta" modular graph, we may prove the Theorem 1.3 for any compact nodal curve X, whose dual modular graph $\Gamma(X)$ has $m+1$ vertices $v_0, v_1, \ldots, v_m$, and $3m$ edges so that each v_i, $i = 1, \ldots, m$, is connected with v_0 by 3 edges ("star-theta" modular graph).

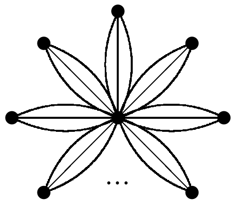

Figure 6: "Star-theta" graph — the only minimal thickness 3 "one-capital graph".

We have already proved it for $m = 2$; on each step we simply add three nonsingular marked points to the component X_0 and apply Lemma 5.2 and 4.1.

Now let us prove Theorem 1.3 for any compact nodal curve X having more than three vertices. Suppose that we have already proved the theorem for all compact nodal curves having less irreducible components than X. As above, it is sufficient to prove that the restriction of the canonical map to each component is an embedding and that the canonical map distinguishes pairs of points located on different components. Choose a component X_1, corresponding to the vertex v_1 of the dual modular graph. Suppose we find two other vertices v' and v'' connected by an edge, such that the induced closed subgraph defined by v' and v'' is connected. The complementary open subgraph corresponds to pointed nodal curve $X^0 = X \setminus (X_{v'} \cup X_{v''})$ (with marked points of its intersection with $X_{v'} \cup X_{v''}$), and the restriction of the canonical map φ_X to X^0 is the canonical map φ_{X^0} of X^0 (Lemma 4.1). But the canonical map φ_{X^0} coincides with the restriction to X^0 of the canonical map of its one-component compactification (Lemma 4.2) $\hat{X}^0$, which is a compact nodal curve having one irreducible component less than X. The thickness of the dual modular graph $\Gamma(\hat{X}^0)$ may only increase, whence by the inductive assumption the canonical map of $\hat{X}^0$ is an embedding, and therefore the restriction of it to X_0 is an embedding too.

So the only problem is whether we can always find such v' and v''. Evidently, we can't do this only when all the vertices are connected only with v_1. (This is the case of a country, where all the towns are connected only with the capital.) But in this case we can do the removing edges procedure (using Lemma 5.1) until we obtain the "star-theta" modular graph, for which the theorem is already proved.

Thus we have proved that the restriction of the canonical map to each component of X is an embedding.

Suppose we have two nonsingular points on the two components X_1 and X_2. As we have already seen the problem of distinguishing points on different components may be reduced to the case when all the components are rational. Next we proceed as above. Suppose we find two other vertices v' and v'', connected by an edge, so that the induced closed subgraph defined by v' and v'' is connected. The complementary open subgraph corresponds to a pointed nodal curve $X^0 = X \setminus (X_{v'} \cup X_{v''})$ (with marked points of intersection of it with $X_{v'} \cup X_{v''}$), and the restriction of the canonical map φ_X to X^0 is the canonical map φ_{X^0} of X^0 (Lemma 4.1). But the canonical map φ_{X^0} coincides with the restriction to X^0 of the canonical map of its one-component compactification (Lemma 4.2) $\hat{X}^0$, which is a compact nodal curve having one irreducible component less than X. The thickness of the dual modular graph $\Gamma(\hat{X}^0)$ may only increase, so by the inductive assumption the canonical map of $\hat{X}^0$ is an embedding, and therefore it distinguishes the points on the components X_1 and X_2.

When can't we find such v' and v''? It can happen only when all the other vertices are connected only with v_1 and v_2. (This is the case of a country with two capitals, so that every other town is connected only with the capitals.) It is not very hard to verify that in this case the modular graph $\Gamma(X)$ contains a thickness 3 closed subgraph of one of the following nine types (see Figure 7):

It is sufficient to find regular differentials on the corresponding curves, distinguishing our points. Then the desired differential on X is obtained by extending these differentials by zero to all the other components. So we may simply present the canonical embeddings of all the corresponding curves, and that is trivial. Here we shall present only the most interesting curve, the one corresponding to the first modular graph on Figure 7. The six-valent vertex corresponds to the a Veronese degree 4 curve in $\mathbb{P}^4$, the three vertices different from v_1 and v_2 correspond to its three secant lines. If any two of these lines intersect, they define a plane having four common points with the Veronese curve; any fifth point located on the curve together with this plane would span a 3-dimensional hyperplane, intersecting our degree 4 curve at 5 different points, which is impossible. Any two of these lines span projective 3-subspace which is intersected by the third line at one point, and there is a unique line in the projective 3-space, passing through a given point and

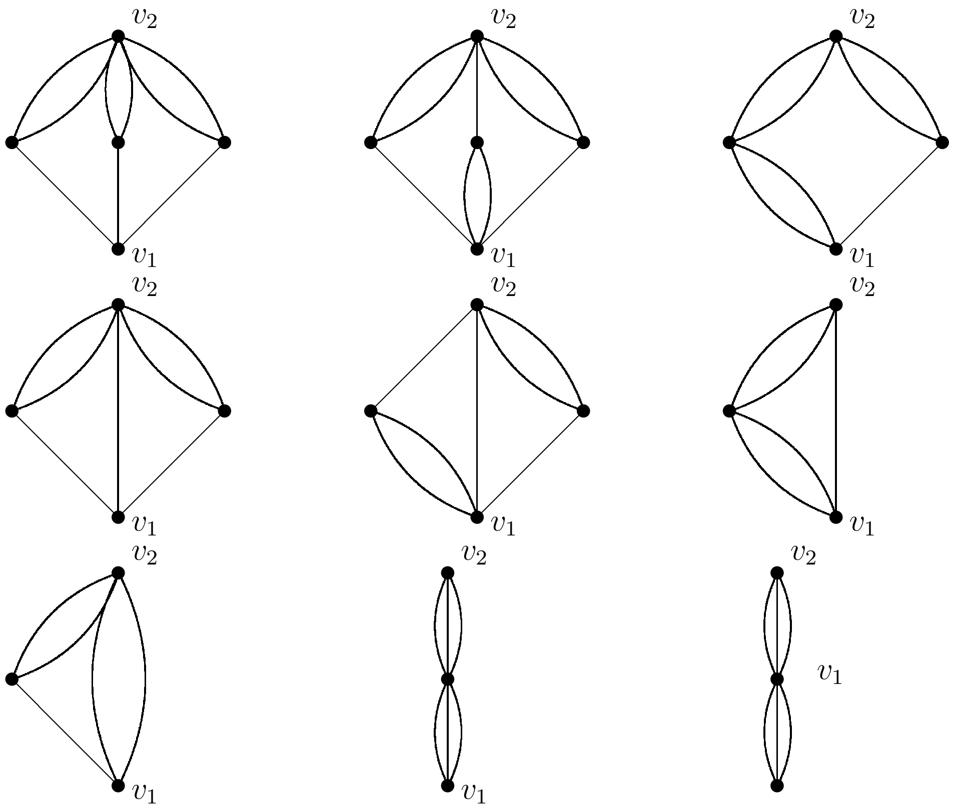

Figure 7: Nine minimal "two-capital" thickness 3 graphs.

intersecting two given lines in general position. This is the last component of our curve.

Now let us say few words about Theorem 1.1 which is much more simple. First of all, a regular quadratic differential on X is the collection of meromorphic differentials on X_v, $v \in V$, having poles of order at most 2 at the points $P_{\overrightarrow{e}}$:

$$(\nu^*(\Omega^1_X)^{\otimes 2})\big|_{X^\nu_v} = (\Omega^1_{X^\nu_v})^{\otimes 2}\left(\sum_{\overrightarrow{e}\in S_-(v)} 2P_{\overrightarrow{e}}\right). \tag{6.1}$$

The compatibility conditions are now symmetric:

$$bires_{P_{\overrightarrow{e}}}\omega_{v_1} = bires_{P_{\overleftarrow{e}}}\omega_{v_2}. \tag{6.2}$$

(see [7] for the definition of the biresidue of a quadratic differential). Thus we obtain the map

$$H^0(X, (\Omega^1_X)^{\otimes 2}) \to k^E, \tag{6.3}$$

and it is not hard to verify that for a Deligne-Mumford stable nodal curve this map is surjective. This means that the restriction of the double-canonical map of X to each component X_v is the double-canonical map of X (considered as a pointed nodal curve, corresponding to the open subgraph of $\Gamma(X)$, induced by the vertex v). The only cases when the double-canonical map of an irreducible pointed nodal curve with at least one marked point are the cases 1) and 2) of the Theorem 1.1. For compact irreducible nodal curves the only exceptions are of genus 2: cases 3a), 3b), 3c), of the Theorem 1.1. The problem of distinguishes two points, located on distinct components of the curve X is reduced to the case of rational irreducible components in the same way as in Lemma 6.1. The surjectivity of 6.3 allows us to choose any desired values of biresidues at the singular points of our two components. It is easy to check that the only exceptional case is 3d) of Theorem 1.1.

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