A Metric Characterization of Manifolds with Boundary

by

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Introduction

This paper characterizes topological spaces admitting the structure of a smooth manifold with boundary as precisely those finite dimensional spaces admitting a (metrically) complete inner metric of bounded curvature. In proving the most interesting half of this equivalence—that a finite dimensional space of bounded curvature is a manifold with boundary—the boundary points are characterized in terms of geodesic completeness. In consequence, the Hopf-Rinow Theorem can be completely generalized to the class of toplogical manifolds (without boundary) of bounded curvature.

An *inner metric space* (X,d) is a metric space X with distance d such that for all x, y $\in$ X, d(x, y) is the infimum of the lengths of curves α joining x and y in X. Such spaces are currently of interest, due, in part, to recent work (e.g., [F], [FY], [GLP], [GP], [GPW], [GW], [P]) on the Gromov-Hausdorff convergence ([G]) of Riemannian manifolds: A limit of Riemannian spaces inherits an inner metric structure and, depending on the nature of the spaces converging to it, various other geometric properties. These properties and their topological implications are treated here abstractly, supporting the point of view that much of what is true for limits of Riemannian spaces is directly a result of the geometry they possess, not the (presumably) more special fact that they are limits.

Such an approach can be fruitful. The proof of the Convergence Theorem for Riemannian manifolds given in [P] and [GW] uses convergence of harmonic coordinates to obtain coordinates and a Riemannian metric in the limit space; however, this theorem also follows easily from earlier general results on the existence of smooth Riemannian metrics for inner metric spaces of bounded curvature ([N]). In fact, one can show without difficulty, and using only basic properties of the distance function, that a limit of Riemannian manifolds of curvature uniformly bounded above and below, volume bounded below, and diameter bounded above, has again curvature bounded above and below (in the *metric* sense recalled below), and satisfies a geodesic completeness condition. (General conditions under which bounded curvature and geodesic completeness are preserved in a limit of inner metric spaces can be found in [PD]). These three properties suffice to apply the results of [N] and reach the same conclusion as in [GW] and [P], i.e., that such a limit is a smooth manifold with $C^{1,\alpha}$ metric (with respect *abstractly constructed* harmonic coordinates!).

A natural problem to undertake now is to use these techniques of "metric geometry" to obtain information about Gromov-Hausdorff limits of spaces in the more general Grove-Petersen-Wu ([GPW]) class of n-dimensional Riemannian manifolds (i.e., with curvature and volume bounded below, and diameter bounded above). In particular, it has been conjectured that such limits are topological manifolds (in [GPW] they are shown, for $n \geq 5$, to be homology manifolds admitting a resolution). This conjecture remains unsolved, and the main difficulty, from a metric geometry perspective, is that, without the uniform upper curvature bound on the class, both the upper curvature bound *and* the geodesic completeness condition are lost in the limit. The present work treats the case of spaces having curvature bounded above and below, without geodesic completeness. Work on spaces without an upper curvature bound is in progress.

The failure of geodesic completeness is treated in this paper by introducing the notion of "geodesic terminal" to mean a point at which some geodesic "stops" (a "geodesic" is an arclength parameterized curve which is locally distance minimizing). Such points do not, of course, exist in the Riemannian case, but in limits of Riemannian spaces geodesic terminals can even occur in the *interior* of a manifold with inner metric (2.5).

The following is a summary of the main results in this paper. For definitions, see Chapter 1. For the remainder of this paper, the single word *complete* will refer to metric completeness (as distinct from geodesic completeness).

Theorem A. *Let X be a complete, locally compact inner metric space of locally bounded curvature (above and below). Then the following are equivalent:*

a) X is finite dimensional,

b) the space of directions at some point in X is precompact,

c) the set $\mathcal{T}$ of geodesic terminals is nowhere dense, and

d) X is a manifold with boundary and $\partial X = \mathcal{T}$.

Corollary B. *Let M be a topological manifold and d be an inner metric on M of locally bounded curvature. Then the following are equivalent:*

a) (M, d) is complete,

b) (M, d) is geodesically complete,

c) there exists a point $p \in M$ such that each geodesic starting at p is defined on all of $\mathbb{R}^+$, and

d) every closed, bounded subset of M is compact.

Corollary C. *A locally compact, infinite dimensional inner metric space of locally bounded curvature has a dense set of geodesic terminals.*

The "normal" coordinates used to prove Theorem A are *not* in general smooth at the boundary. However, the boundary can be "smoothed:"

Theorem D. *A topological space X admits the structure of a smooth manifold with boundary if and only if X possesses a complete metric of locally bounded curvature.*

There exist "flat" spaces (curvature bounded above and below by 0) having dense geodesic terminals (2.7), and geodesically complete spaces of curvature ≥ 0 which are also infinite dimensional (2.8).

The special case $\mathcal{T} = \emptyset$ (i.e. X is geodesically complete) was first considered in [Be], where it is proved that X is in fact smooth with a continuous Riemannian metric (smoothness of this metric is the result of [N]).

Some of the examples in Chapter 2 and the implication c) $\Rightarrow$ d) in the main theorem were included in my doctoral dissertation. I would like to thank my thesis advisor Karsten Grove for his help both before and after my thesis was written.

1. Metric Geometry

Among the several different theories of bounded curvature for metric spaces, the theory presented in [R] is most appropriate for the purposes of this paper. A few of the basics are recalled below. The only new concepts in this chapter are those of "geodesic terminal" and "comparison radius," defined at the end.

Throughout this paper, curves will always be assumed to be parameterized proportional to arclength. If α is a curve in X from x to y such that $\ell(\alpha) = d(x,y)$, where $\ell(\alpha)$ denotes the length of α, then α is called a *minimal curve.* A *geodesic* is a curve γ which is locally minimal; specifically, if γ is defined on an interval I, then for every $t \in I$ there exists an interval $J = [t-\delta, t+\delta]$, $\delta > 0$, such that $\gamma|_{J\cap I}$ is a minimal curve. In this paper, γ_{xy} will always denote a geodesic from x to y.

Under the assumptions of metric completeness and local compactness, every pair of points in X can be joined by at least one minimal curve, and every closed and bounded subset of X is compact.

In all curvature discussions in this paper, the value of $K^{-1/2}$ will be taken to be ∞ if $K \le 0$.

A *triple* $(a;b,c)$ in X is a set of three points a, b, c $\in$ X such that $a \ne b$ and $a \ne c$. For any K, let S_K denote the (complete) dimension two, simply connected Riemannian space form of constant curvature K. If $(a;b,c)$ is a triple such that $d(a,b) + d(b,c) + d(c,a) < 2\pi/\sqrt{K}$, then there is a uniquely determined (up to congruence in S_K) triangle $T_K(a;b,c)$ in S_K having sides of length $d(a,b)$, $d(b,c)$, and $d(c,a)$. Let $\alpha_K(a;b,c)$ denote the angle corresponding to a in $T_K(a;b,c)$.

Definition 1.1. *An open set U in X is said to be a* region *of* curvature $\le$ K *(resp.* $\ge$ K*) if for every triple $(a;b,c)$ in U,*

a) $(a;b,c)$ has a representative in S_K, and

b) if $a \ne b$ and $a \ne c$ and γ_{ab}, γ_{ac} are minimal curves, then the distance between any points x on γ_{ab} and y on γ_{ac} is $\le$ (resp. $\ge$) the distance between the corresponding points x' and y' in $T_K(a;b,c)$.

Theorem 1.2. *Let U be a region of curvature $\geq K$ in X. Then*

a) If γ_{ab} and γ_{ac} lie in U, then for any number $\tilde{K}$, $\lim_{s,t\to 0}\alpha_{\tilde{K}}(a;\gamma_{ab}(s),\gamma_{ac}(t))$ exists, and is independent of both $\tilde{K}$ and the parameterizations of the curves; this number is called the angle *between γ_{ab} and γ_{ac}, denoted $\alpha(\gamma_{ab},\gamma_{ac})$.*

b) The triangle inequality holds for angles.

c) If γ_{ab} and c lie in U, then for all x on γ_{ab} strictly between a and b, $\alpha(\gamma_{xa},\gamma_{xc}) + \alpha(\gamma_{xc},\gamma_{xb}) = \pi$.

d) If γ_{ab} and c lie in U, x lies on γ_{ab} strictly between a and b, and $d(c,x) = d(c,\gamma_{ab})$, then $\alpha(\gamma_{xa},\gamma_{xc}) = \alpha(\gamma_{xc},\gamma_{xb}) = \pi/2$.

In a region of curvature bounded above, conditions a) and b) still hold, but c) and d) fail in general (2.3).

A space X is said to have *curvature locally bounded below* (resp. *above*) if each $x \in X$ is contained in a region of curvature bounded below (resp. above) by some number K possibly dependent on x. X is said simply to have *locally bounded curvature* if X has curvature locally bounded above and below. If X has curvature locally bounded below, two geodesics have angle 0 if and only if they coincide on their maximal domain of definition. The angle is therefore a *bona fide* metric on the space S_p of all unit geodesics of maximal domain starting at a point $p \in X$, called the *space of directions* at p.

A point $x \in X$ is called a *branch point* if there exist distinct points a, b, c different from x and minimal curves γ_{ab}, γ_{ac} such that x lies on both γ_{ab} and γ_{ac}, the two curves coincide between a and x, and $d(a,x) = d(b,x)$. At the branch point x, the geodesic γ_{ax} "branches" to form two distinct geodesics γ_{ab} and γ_{ac}. *A region of curvature $\geq K$ contains no branch points.*

A subset A of X is called *strictly convex* if every pair of points in A is joined by a *unique* minimal curve, and that curve lies entirely in A. If $r < \pi/2\sqrt{K}$, and $\overline{B}(x,r)$ is compact and contained in a region of curvature $\le K$, then $B(x, r/2)$ is strictly convex. Hence, in a locally compact space of curvature locally bounded above, *every point is contained in a strictly convex neighborhood.*

The following lemma is well known ([CE]) and not difficult to prove. It is used frequently in this paper, both explicitly and implicitly.

Lemma 1.3. *For a triangle ABC in S_K having side lengths $< \pi \cdot K^{-1/2}$, the distance BC is a monotone increasing function of the angle at A, on $[0,\pi]$.*

In a region U of curvature $\le K$ (resp. $\ge K$), the following equivalent conditions hold, whenever the given geodesics exist in U:

A1. *If $(a;b,c)$ is a triple in U such that $b \neq c$, then $\alpha(\gamma_{ab}, \gamma_{ac}) \le$ (resp $\ge$) $\alpha_K(a;b,c)$.*

A2. *If $(a;b,c)$ is a triple such that $b \neq c$, and ABC denotes the uniquely determined triangle in S_K with $AB = d(a, b)$, $AC = d(a,c)$, angle $\alpha(\gamma_{ab}, \gamma_{ac})$ at A, and side BC of minimal length, then $d(b, c) \ge$ (resp. $\le$) BC.*

An inner metric space X is *geodesically complete* if every geodesic in X has a unit parameterization defined on all of $\mathbb{R}$.

Definition 1.4. *A point $x \in X$ is called the* terminal *of a geodesic γ_{ax} if γ_{ax} cannot be extended beyond x (as a geodesic). More generally, a point is called a* geodesic terminal *if it is the terminal of some geodesic.*

In the metrically complete case, geodesic completeness is equivalent to the absence of geodesic terminals; it is therefore both simple and convenient to have a notion of geodesic completeness for an arbitrary open set:

Definition 1.5. *If X is a complete inner metric space, an open set $U \subseteq X$ is said to be* geodesically complete *if U has no geodesic terminals.*

Definition 1.6. *If x lies in a region of curvature $\leq K$, the* upper comparison radius for K at x *is defined to be*

$$c^K(x) = \sup\{r : B(x,r) \text{ is a region of curvature } \leq K\}.$$

If x is not contained a region of curvature $\leq K$, then $c^K(x)$ is defined to be 0. A point x is called a singularity *if $c^K(x) = 0$ for all K.*

The inequality $c^K(x) \geq c^K(y) - d(x,y)$ holds for all x, $y \in X$, and shows that c^K is either everywhere infinite or a continuous map from X into the non-negative reals; if c^K is positive on X, X is said to have *curvature* $\leq$ K. Finally, $c^K(X)$ will denote $\inf_{x \in X} c^K(x)$, the *upper comparison radius* of X.

By reversing the inequalities in the above definitions, one can similarly define, for any K, the *lower* comparison radius $c_K(x)$ (with $c_K(X) = \inf_{x \in X} c_K(x)$), and curvature $\geq$ K for the whole space X.

2. Examples

Example 2.1. If M is a Riemannian manifold, then the distance induced by the Riemannian metric is *by definition* an inner metric. If the sectional curvature k on M satisfies $k \le U$, then $c^U(x) > 0$ for all x; if $k \ge L$, then by Toponogov's theorem, $c_L(M) = \infty$ ([K]). It is not known if there exist inner metric spaces having curvature $\ge L$ and $c_L(X) < \infty$. Such spaces could not be limits of Riemannian manifolds with curvature uniformly bounded below ([GP], [PD]).

Example 2.2. If X is an arbitrary metric space and $Y \subset X$ is finitely path connected (every $x, y \in Y$ can be joined by a rectifiable curve in Y), the *induced inner metric* $d_I(x, y)$ on Y is the infimum of the lengths (in the metric of X) of all curves connecting x and y in Y. This metric is topologically equivalent to the usual induced metric, but in general, $d_I(x, y) \ge d(x, y)$. In particular, every finitely path connected metric space has an inner metric. If N is a Riemannian submanifold of a Riemannian manifold M, the induced inner metric is simply the usual distance associated with the induced Riemannian metric on N.

Example 2.3. In $\mathbb{R}^3$, let X be the union of the (x,y)-plane with the z-axis, and let d be the induced inner metric from $\mathbb{R}^3$. Then X is geodesically complete, $c^0(X) = \infty$ (so X is strictly convex), and $c_0(x) > 0$ except at the origin (which is a branch point). This example illustrates that the local euclidean structure of an otherwise "nice" space is easily destroyed by a single point at which curvature is not bounded below. There is no known analogous example for the upper curvature bound. Note that the angle between the z-axis and any geodesic in the plane through the origin is π.

Example 2.4. Let X be an n-sphere ($n \geq 2$), with an open "cap" sliced off along some latitudinal "circle" above the equator. Then the boundary circle becomes a "new" geodesic in the induced inner metric, and every point on it is a branch point and a geodesic terminal. The lower curvature bound of the original sphere has been destroyed. The original upper curvature bound also no longer holds.

If the slice is made along or below the equator, the remaining closed disk is strictly convex and retains the constant curvature of the sphere, and points on the equator all become geodesic terminals.

Example 2.5. Let X_i be a flat cone with the apex smoothly rounded off at some positive distance ε_i from the end, with $\varepsilon_i \to 0$, and the wide end smoothly capped (to make the space a compact Riemannian manifold). Then the limit X of these spaces is a cone (with the wide end rounded off), which has curvature bounded above and below by 0 around, but excluding, the apex. The apex is a singularity *and* a geodesic terminal, since it is always shorter to pass around the cone than through the apex. Theorem 3.10 shows that such isolated, "interior" terminals do not occur when the curvature is locally bounded.

Example 2.6. Let $f : [a,b] \to [0,\infty)$ be a smooth function such that $f(a) = 0$ and $f|_{(a,b)} > 0$. Assume furthermore that $f(c) = 1$, where $c = (a + b) / 2$, and that f is symmetric about c, i.e., for all $0 \leq \delta \leq c - a$, $f(a + \delta) = f(b - \delta)$. For any compact Riemannian manifold X the *f-suspension* S_fX is the metric completion of the warped product ([BO], [O]) $(a,b) \times_f X$. S_fX is therefore obtained by attaching, to $(a,b) \times_f X$, a point at $a \times X$ and one at $b \times X$; S_fX is clearly homeomorphic to the topological suspension of X. Furthermore, $c \times X$ is a totally geodesic isometric embedding of X in S_fX, since $f'(c) = 0$. The induced inner metric on $(a,b) \times_f X \subset S_fX$ is simply the usual distance associated with the warped product metric.

If S^n denotes the unit n-sphere, $n \geq 2$, then S^n minus two poles is homeomorphic to $(0,\pi) \times S^{n-1}$. With the induced Riemannian metric, the fibers of this product are isometric to spheres of radius sin t, $t \in (0,\pi)$. Hence $S_{\sin}S^{n-1}$ is isometric to S^n.

Understanding the curvature of f-suspensions is a subtle problem; although the curvature of the warped product is easily computed, curvature at the end points must generally be verified directly by triangle comparison. For example, the sine-suspension of a non-simply connected space of constant curvature 1 has again constant curvature 1 on the warped product, but the curvature is not bounded at the end points (3.14).

Example 2.7. An infinite (Hilbert) cube, the product of closed intervals of lengths 1/2, 1/4, ..., can be easily verified to have curvature bounded above and below by 0. (To do this, consider it as a Gromov-Hausdorff limit of finite cubes, note that $c^0 = \infty$ for each finite cube, and apply the results of [PD] to obtain the same comparison radius for the limit; or embed the space in $\mathbb{R}^\infty$, with the flat Euclidean metric). The "faces" of the cube are all geodesic terminals, and form a dense set.

Example 2.8. An infinite torus, the product of spheres of radius 1/2, 1/4, ... (and dimension ≥ 1), is a geodesically complete space of non-negative curvature. Note that if all the spheres are 1-dimensional, the resulting infinite torus is a Gromov-Hausdorff limit of flat finite tori, but, by Theorem 3.10 cannot have curvature locally bounded above. The loss of upper curvature bound in the limit can occur in this case because the injectivity radius, and hence c^0, tends to 0 ([PD]).

3. Finite Dimensional Spaces of Locally Bounded Curvature

If X is a space of curvature locally bounded below and there exists a point $x \in X$ with at most two directions, it is easy to show that X is homeomorphic to an interval or a circle. Some of the lemmas below fail for this trivial case, and to avoid special exceptions in the statements, the direction space at each point will be assumed, when necessary, to have at least three elements.

Definition 3.1. *Let X be a space of curvature locally bounded below. The* tangent space T_p *at a point $p \in X$ is the metric space obtained from $S_p \times \mathbb{R}^+$ by identifying all points of the form $(\gamma, 0)$ (and denoting the resulting point by 0) with the following metric, where the class of (γ, t) in the identification space is denoted $t\cdot\gamma$:*

$$\delta(t\cdot\gamma,\ s\cdot\beta) = (t^2 + s^2 - 2st\cdot\cos\alpha(\gamma,\ \beta))^{1/2}.$$

For each $\gamma \in S_p$, let $T(\gamma) = \sup\{s : \gamma(s) \text{ is defined}\}$; that is, γ terminates at $\gamma(T(\gamma))$ if $T(\gamma) < \infty$, and has no terminal if

$T(\gamma) = \infty$. T will be called the *terminal map* on S_p. The *exponential map* is defined by $\exp_p(s\cdot\gamma) = \gamma(s)$, for all $s \leq T(\gamma)$. Exp_p is, by A2 (resp. A1), continuous (resp. open) on the intersection of its domain of definition with any $B(0, r)$ such that $\exp_p(B(0, r))$ is contained in a region of curvature $\geq K$ (resp. $\leq K$). Furthermore, $\exp_p$ is a radial isometry, and preserves the angle between radial geodesics (i.e., starting at p). If X is complete and has locally bounded curvature, then any sequence of geodesics whose directions are Cauchy and whose lengths have a positive lower bound has a limit which is again a geodesic; in this case $\exp_p$ is continuous on its domain of definition, and $T : S_p \to \mathbb{R} \cup \infty$ is upper semicontinuous.

The following theorem is a restatement of some of the results of [Be]; by the comments of Chapter 1 it should be clear that the assumption given in [Be], that X is a "Busemann G-space" ([B]) of bounded curvature, can be reduced to the hypothesis given below. [PD] contains a short proof of Theorem 3.2, and another is indicated at the end of the proof of Proposition 3.7.

Theorem 3.2. *Let X be a complete, locally compact inner metric space. If B(p, r) is a strictly convex, geodesically complete region of curvature bounded above and below, then T_p is isometric to some Euclidean space, and $\exp_p|_{B(0,r)}$ is a homeomorphism.*

Without geodesic completeness, the situation is somewhat less simple. In particular, finite dimensionality is not guaranteed by local compactness, nor is the direction space always compact. Suppose X is a complete, locally compact inner metric space of locally bounded curvature. Let $\overline{S}_p$ be the metric completion of S_p; then elements of the metric completion $\overline{T}_p$ of T_p can clearly be written in the form $t\overline{\gamma}$, where $\overline{\gamma} \in \overline{S}_p$, $t \in \mathbb{R}^+$, and $0\overline{\gamma} = 0$. For any $\overline{\gamma}_1$, $\overline{\gamma}_2$, $\overline{\gamma}_3 \in \overline{S}_p$, $\overline{\gamma}_2$ is said to be *between* $\overline{\gamma}_1$ and $\overline{\gamma}_3$ if $\alpha(\overline{\gamma}_1, \overline{\gamma}_3) = \alpha(\overline{\gamma}_1, \overline{\gamma}_2) + \alpha(\overline{\gamma}_2, \overline{\gamma}_3)$. For any distinct $\overline{\gamma}_1$, $\overline{\gamma}_2 \in \overline{S}_p$, the *span sp* $\{\overline{\gamma}_1, \overline{\gamma}_2\} \subseteq \overline{T}_p$ of $\overline{\gamma}_1$, $\overline{\gamma}_2$ is the set of all $t\overline{\gamma}$ such that one of $\overline{\gamma}_1$, $\overline{\gamma}_2$, $\overline{\gamma}$ is between the other two. In general, given distinct $\overline{\gamma}_1, \ldots, \overline{\gamma}_k \in \overline{S}_p$, $k > 1$, the span of $\overline{\gamma}_1, \ldots, \overline{\gamma}_k$ is the smallest subset sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_k\} \subseteq \overline{T}_p$ containing $\overline{\gamma}_1, \ldots, \overline{\gamma}_k$ such that if $\overline{\alpha}$, $\overline{\gamma} \in$ sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_k\}$, then sp $\{\overline{\alpha}, \overline{\gamma}\} \subset$ sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_k\}$. The elements $\overline{\gamma}_1, \ldots, \overline{\gamma}_k$ are said to be *independent* if $\overline{\gamma}_{j+1}$ does not lie in sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_j\}$ for any j. The notions of angle (not as a metric!) and betweeness can be generalized to the space T_p in the obvious way; e.g., for $t_1, \ldots, t_k > 0$, sp $\{t_1\overline{\gamma}_1, \ldots, t_k\overline{\gamma}_k\}$ = sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_k\}$.

Lemma 3.3. *Suppose p lies in a region of curvature* $\geq K$ *in a complete, locally compact inner metric space X. Let* $\{\gamma_i\}$ *and* $\{\eta_i\}$ *be Cauchy sequences in* S_p. *Then given any positive* $s_i \to 0$ *and* $t_i \to 0$ *such that* $s_i \leq T(\gamma_i)$ *and* $t_i \leq T(\eta_i)$, *if* $d_i = d(\gamma_i(s_i), \eta_i(t_i))$, *then*

$$\lim_{i \to \infty} \alpha(\gamma_i, \eta_i) = \lim_{i \to \infty} \cos^{-1} [(s_i^2 + t_i^2 - d_i^2) / 2 \cdot s_i \cdot t_i].$$

Proof. Given any $\varepsilon > 0$, choose a j such that for all $i > j$, $\alpha(\gamma_j, \gamma_i) < \varepsilon/2$ and $\alpha(\eta_j, \eta_i) < \varepsilon/2$. For any such i and $s \leq \min\{T(\gamma_i), T(\gamma_j)\}$ and $t \leq \min\{T(\eta_i), T(\eta_j)\}$, let $W_i(s, t)$ denote the representative in S_K of the wedge formed by $\gamma_i|_{[0,s]}$ and $\eta_i|_{[0,t]}$, and $W_j(s, t)$ denote that of the wedge formed by $\gamma_j|_{[0,s]}$ and $\eta_j|_{[0,t]}$). Curvature $\geq K$ implies $d(\gamma_i(s), \gamma_j(s))$

and $d(\eta_i(t), \eta_j(t))$ are both smaller than the distance between the endpoints of a wedge in S_K having sides of lengths s and t, with angle $\varepsilon/2$. The difference between the angles of $W_i(s, t)$ and $W_j(s, t)$ is therefore $\leq \varepsilon$, and the lemma follows.

For 3.4 - 3.7 and 3.9, let $B = B(p, r)$ be a strictly convex region of curvature $\geq k$ and $\leq K$ in a locally compact, complete inner metric space X.

Lemma 3.4. *For every* $\bar{\eta}_1, \bar{\eta}_2 \in \bar{S}_p$ *and* $a \in [0, \alpha(\bar{\eta}_1, \bar{\eta}_2)]$, *there exists some* $\bar{\zeta} \in \bar{S}_p$ *between* $\bar{\eta}_1, \bar{\eta}_2$ *such that* $\alpha(\bar{\zeta}, \bar{\eta}_1) = a$. *Furthermore, if* $\alpha(\bar{\eta}_1, \bar{\eta}_2) < \pi$, $\bar{\zeta}$ *is unique.*

Proof. Since S_p is dense in $\bar{S}_p$, for the first part of the lemma it suffices to show the following: Given minimal curves γ_{pb} and γ_{pc} in B and $a_1, a_2 > 0$ such that $a_1 + a_2 = a_3 = \alpha(\gamma_{pb}, \gamma_{pc})$, then for each $\varepsilon > 0$ there exists a minimal curve γ starting at p such that

$$|\alpha(\gamma_{pb}, \gamma) - a_1| \leq \varepsilon, \text{ and}$$
$$|\alpha(\gamma_{pc}, \gamma) - a_2| \leq \varepsilon.$$

Suppose first that $a_3 < \pi$. For all $i = 1, 2, 3, \ldots$ let b_i denote $\gamma_{pb}(2^{-i})$, c_i denote $\gamma_{pc}(2^{-i})$, and $\alpha_i : [0,1] \to B$ be minimal from b_i to c_i. Set $\beta = a_1/a_3$ and let γ_i be a unit minimal curve from p to $d_i = \alpha_i(\beta)$. Since $a_3 < \pi$, $d(b_i, c_i) < d(p, b_i) + d(p, c_i)$ for sufficiently large i, which implies $p \neq \alpha_i(\beta)$ and γ_i is nonconstant.

Let T_i be a triangle in S_K having vertices P, B_i, and D_i, with $d(P, B_i) = d(p, b_i)$, $d(P, D_i) = d(p, d_i)$, and $d(B_i, D_i) = (b_i, d_i)$. Let χ_i denote the angle of T_i at X. Then by definition of the angle (using S_K), $\lim_{i \to \infty} \chi_i = a_1$. A1 now implies that $\lim_{i\to\infty} \alpha(\gamma_{pb}, \gamma_i) \le a_1$. Similarly, $\lim_{i\to\infty} \alpha(\gamma_{pc}, \gamma_i) \le a_2$, and the above statement now follows, for $a_3 < \pi$, from $\alpha(\gamma_{pb}, \gamma_i) + \alpha(\gamma_i, \gamma_{pc}) \ge \alpha(\gamma_{pb}, \gamma_{pc}) = a_3$. If $a_3 = \pi$, choose a minimal curve γ' in a third direction. Without loss of generality, assume $a_1 \le \pi/2$ and $a_3' = \alpha(\gamma_{pb}, \gamma') \ge \pi/2$. Applying the above case to the curves γ_{pb} and γ', with $a_1' = a_1$ and $a_2' = a_3' - a_1'$ produces the required new minimal curve.

To prove the last part of the lemma, suppose again that $a_3 < \pi$, and assume that, contrary to uniqueness, there exists a $\delta > 0$ and, for $k = 1, 2$, sequences $\{\gamma_{ik}\}$ of minimal curves starting at p such that

$$|\alpha(\gamma_{pb}, \gamma_{ik}) - a_1| \le 2^{-i},$$
$$|\alpha(\gamma_{pc}, \gamma_{ik}) - a_2| \le 2^{-i},$$

with $\alpha(\gamma_{i1}, \gamma_{i2}) > \delta$ for all i. The lengths of the curves can be shortened without affecting the above assumptions; assume $\ell(\gamma_{i1}) = \ell(\gamma_{i2}) = t_i \to 0$. In the plane, choose points P, B, C, and T such that B, C, and T are collinear, $PB = d(p, b)$, $PC = d(p, c)$, $\alpha(\overline{PB}, \overline{PC}) = a_1$ and $\alpha(\overline{PC}, \overline{PT}) = a_2$. Define $r_i = t_i \cdot PB / PT$ and $s_i = t_i \cdot PC / PT$; by removing a few terms one can assume that the points $\gamma_{pc}(s_i)$ and $\gamma_{pb}(r_i)$ all lie in $B(p, r)$. Let η_i, ζ_{1i}, ζ_{2i} be the minimal curves in $B(p, r)$ from $\gamma_{pb}(r_i)$ to $\gamma_{pc}(s_i)$, $\gamma_{1i}(t_i)$, and $\gamma_{2i}(t_i)$, respectively. The assumptions on the γ_{ki} imply that $d(\gamma_{pb}(r_i), \gamma_{ki}(t_i)) + d(\gamma_{ki}(t_i), \gamma_{pc}(s_i))$ is arbitrarily close to $d(\gamma_{pb}(r_i), \gamma_{pc}(s_i))$ for large i. Choosing a representative of the triple $(\gamma_{pb}(r_i); \gamma_{ki}(t_i), \gamma_{pc}(s_i))$ in S_K and applying A1 proves that $\lim_{i\to\infty} \alpha(\eta_i, \zeta_{ki}) = 0$. From the triangle inequality it follows that $\lim_{i\to\infty} \alpha(\zeta_{1i}, \zeta_{2i}) = 0$. Let Z_{1i}, Z_{2i} be unit minimal curves in S_k, with common endpoint y and other endpoints z_{1i} and z_{2i}, respectively, such that $\ell(Z_{1i}) = \ell(\zeta_{1i})$, $\ell(Z_{2i}) = \ell(\zeta_{2i})$, and $\alpha(Z_{1i}, Z_{2i}) = \alpha(\zeta_{1i}, \zeta_{2i})$. Then $0 = \lim_{i\to\infty} d(z_{1i}, z_{2i}) / \ell(Z_{1i}) \geq \lim_{i\to\infty} d(\gamma_{1i}(t_i), \gamma_{2i}(t_i)) / \ell(\zeta_{1i}) = \lim_{i\to\infty} d(\gamma_{1i}(t_i), \gamma_{2i}(t_i))\cdot(XT/BT) / t_i$. This last limit being 0, however, implies that $\lim_{i\to\infty} \alpha(\gamma_{1i}, \gamma_{2i}) = 0$, a contradiction.

The proof of the following lemma is essentially the same as the proof of the uniqueness part of Lemma 3.4 (the case $\alpha(\bar{\eta}_1, \bar{\eta}_3) = \pi$ below follows from the absence of branch points).

Lemma 3.5. *If $\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4 \in \bar{S}_p$ and $\bar{\eta}_2$ is between $\bar{\eta}_1$ and $\bar{\eta}_3$, and between $\bar{\eta}_1$ and $\bar{\eta}_4$, with $\alpha(\bar{\eta}_1, \bar{\eta}_3) = \alpha(\bar{\eta}_1, \bar{\eta}_4)$, then $\bar{\eta}_3 = \bar{\eta}_4$.*

Lemma 3.6. *Let $\bar{\gamma}_1, \ldots, \bar{\gamma}_4 \in \bar{S}_p$ be distinct and, setting $\alpha_{ij} = \alpha(\bar{\gamma}_i, \bar{\gamma}_j)$, suppose $\alpha_{12} + \alpha_{23} = \alpha_{13} < \pi$. Then there exist unit vectors $v_i \in \mathbb{R}^3$ such that $\alpha(v_i, v_j) = \alpha_{ij}$, and a choice of v_4 any two of v_1, v_2, v_3 determines the remaining v_i.*

Proof. There exist $X_i \in \mathbb{R}^3 \setminus 0$ such that X_1, X_2, and X_3 are colinear, with $\alpha(\overline{0X}_1, \overline{0X}_2) = \alpha_{12}$, $\alpha(\overline{0X}_2, \overline{0X}_3) = \alpha_{23}$, $\alpha(\overline{0X}_1, \overline{0X}_4) = \alpha_{14}$, and $\alpha(\overline{0X}_3, \overline{0X}_4) = \alpha_{34}$. Choose $\gamma_{ij} \in S_p$ such that $\gamma_{ij} \to \gamma_i$, $i = 1, \ldots, 4$, and positive $t_j \to 0$ such that $s_{ij} = \|t_j\cdot X_i\| \leq T(\gamma_{ij})$. Let $\beta^j_{ik} : [0, 1] \to B$ be minimal from $x_{ij} = \gamma_{ij}(s_{ij})$ to $x_{kj} = \gamma_{kj}(s_{kj})$, and let α'_{2j} be the unique minimal curve from p to $x'_{2j} = \beta^j_{13}(\alpha_{12}/\alpha_{13})$. To prove the first part of the lemma, it suffices, by Lemma 3.3, to show that $\lim \alpha(\alpha'_{2j}, \alpha_{4j}) = \alpha(\overline{0X}_2, \overline{0X}_4)$; i.e., that $\lim d(x'_{2j}, x_{4j}) / t_j = X_2X_4$.

Curvature $\geq$ K and the definition of the angle (applied to representatives in S_K) imply that $\lim \alpha(\beta^j_{13}, \beta^j_{14}) \geq$

$\alpha(\overline{X_1X_3}, \overline{X_1X_4})$. In a similar way, curvature $\le k$ now implies that $\lim d(x'_{2j}, x_{4j}) / t_j \ge X_2X_4$. Now $\lim d(x'_{2j}, x_{4j}) / t_j \le X_2X_4$ can be proved in the same way, reversing the curvature bounds.

The last part of the lemma is elementary linear algebra.

Proposition 3.7. *If $\bar{\gamma}_1, \ldots, \bar{\gamma}_m \in \bar{S}_p$ are independent, then $sp\ \{\bar{\gamma}_1, \ldots, \bar{\gamma}_m\}$ is isometric to the closure of an open convex radial cone in $\mathbb{R}^m$.*

Proof. Using Lemmas 3.4 and 3.5, one can now easily map $sp\ \{\bar{\gamma}_1, \bar{\gamma}_2\}$ isometrically onto the closure C^2 of a convex open cone in $\mathbb{R}^2$, taking each $\{t\cdot\bar{\gamma} : t \in \mathbb{R}^+\}$ isometrically onto a radial ray. Note that if $\bar{\gamma}_1$ and $\bar{\gamma}_2$ lie in S_p and have extensions past p as geodesics, then the image of this map is all of $\mathbb{R}^2$.

Suppose such a map φ has been inductively constructed from $sp\ \{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$ onto the the closure C^k of a convex open cone in $\mathbb{R}^k$. In $\mathbb{R}^{k+1}$ there exists some unit vector v_{k+1} such that $\alpha(\bar{\gamma}_i, \bar{\gamma}_{k+1}) = \alpha(v_i, v_{K+1})$ for all $1 \le i \le k$. Note that if $\bar{\gamma} \in sp\ \{\bar{\alpha}, \bar{\gamma}_{k+1}\}$ for some $\bar{\alpha} \in sp\ \{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\} \cap \bar{S}_p$, then $\bar{\alpha}$ is the unique such element of $sp\ \{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\} \cap \bar{S}_p$. For if $\bar{\gamma} \in sp\ \{\bar{\alpha}', \bar{\gamma}_{k+1}\}$, with $\bar{\alpha} \ne \bar{\alpha}'$, $\bar{\gamma}_{k+1} \in sp\ \{\bar{\alpha}, \bar{\alpha}'\}$, and hence $\bar{\gamma}_{k+1} \in sp\ \{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$, a contradiction.

One can now extend φ to the union C of the spans $sp\ \{\bar{\alpha}, \bar{\gamma}_{k+1}\}$ for all $\bar{\alpha} \in sp\ \{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$ to an injective map onto the closure of a convex open radial cone in $\mathbb{R}^{k+1}$, which is an isometry on each $sp\ \{\bar{\alpha}, \bar{\gamma}_{k+1}\}$. This extension is actually an isometry: Given $\bar{\beta} \in sp\ \{\bar{\alpha}, \bar{\gamma}_{k+1}\}$, with $\bar{\alpha} \in sp\ \{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$, and $\bar{\zeta} \in sp\ \{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\} \cap \bar{S}_p$, apply Lemma 3.6 to $\bar{\zeta}, \bar{\beta}, \bar{\gamma}_{k+1}, \bar{\alpha}$, with $v_1 = \varphi(\bar{\zeta})$, $v_3 = \varphi(\bar{\gamma}_{k+1})$, and $v_4 = \varphi(\bar{\alpha})$. The unique unit vector

$v_2 \in \mathbb{R}^{k+1}$ determined by these choices is the only unit vector in $\mathbb{R}^{k+1}$ to satisfy $\alpha(v_1, v_2) = \alpha(\bar{\zeta}, \bar{\beta})$ and $\alpha(v_2, v_3) = \alpha(\bar{\beta}, \bar{\gamma}_{k+1})$, and so must coincide with $\varphi(\bar{\beta})$; i.e., $\alpha(\bar{\beta}, \bar{\alpha}) = \alpha(v_2, v_4) = \alpha(\varphi(\bar{\beta}), \varphi(\bar{\alpha}))$. Now suppose $\bar{\beta}$ and $\bar{\zeta}$ are arbitrary elements of C. Then $\bar{\zeta} \in$ sp $\{\bar{\gamma}_{k+1}\ \bar{\alpha}\}$, with $\bar{\alpha} \in$ sp $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$. The above special case shows that $\alpha(\bar{\beta}, \bar{\alpha}) = \alpha(\varphi(\bar{\beta}), \varphi(\bar{\alpha}))$, and repeating the argument shows $\alpha(\bar{\beta}, \bar{\zeta}) = \alpha(\varphi(\bar{\beta}), \varphi(\bar{\zeta}))$.

In a similar fashion, the map φ can now be extended to an isometry on the union C' of the all sp $\{\bar{\beta}, \bar{\zeta}\}$ with $\bar{\beta}, \bar{\zeta} \in C$. One need only show that if $\bar{\eta} \in$ sp $\{\bar{\beta}, \bar{\zeta}\} \cap$ sp $\{\bar{\beta}', \bar{\zeta}'\}$, then the extensions defined using the two different spans coincide; but this follows from Lemma 3.6 as in the above argument. If $\bar{\gamma} \in C'$ is *strictly* between any two elements of C', then the fact that C is the closure of a convex open cone implies that $\varphi(\bar{\gamma})$ is contained an open Euclidean subset contained in $\varphi(C')$. In addition, for any element $\bar{\mu} \in C'$ there are arbitrarily close elements strictly between $\bar{\mu}$ and some other element; in otherwords, the interior points of $\varphi(C')$ are dense in $\varphi(C')$. Since $\varphi(C')$ is a convex cone by construction, C' satisfies the requirements of the inductive step.

Finally, $C' =$ sp $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_{k+1}\}$. This follows from the fact that, since $\varphi(C)$ has non-empty interior, every element of $\mathbb{R}^{k+1}$ lies in the span of some two elements of $\varphi(C)$. Suppose $\bar{\eta} \in$ sp $\{\bar{\beta}, \bar{\zeta}\}$, with $\bar{\beta}, \bar{\zeta} \in C' \cap \bar{S}_p$. As before φ can be extended as an isometry to $C' \cup$ sp $\{\bar{\beta}, \bar{\zeta}\}$. But then $\varphi(\bar{\eta})$ lies in the span of some $\varphi(\bar{\alpha})$, $\varphi(\bar{\gamma}) \in \varphi(C)$. Since φ is an isometry, $\bar{\eta}$ lies in sp $\{\bar{\alpha}, \bar{\gamma}\}$ and so $\bar{\eta} \in C'$. This completes the inductive step.

If, in addition, $C^k = \mathbb{R}^k$, $\gamma_{k+1} \in T_p$, and γ_{k+1} has a continuation past p, then $C^{k+1} = \mathbb{R}^{k+1}$. Theorem 3.2 can now be proved. If U is geodesically complete, each geodesic starting at p is defined for at least length r, and so local compactness implies S_p is compact, and $T_p = \overline{T}_p$. Finally, compactness of S_p implies that T_p is spanned by at most finitely many elements, so $T_p = \mathbb{R}^n$ for some n.

The space sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_m\}$ will now be identified with its image in Euclidean space. For example, given any $\overline{\alpha}$, $\overline{\beta} \in \overline{S}_p$ and $s \in [0, 1]$, $s\cdot\overline{\alpha} + (1-s)\cdot\overline{\beta}$ will denote the unique element of $\overline{S}_p$ between $\overline{\alpha}$ and $\overline{\beta}$ such that $\alpha(\overline{\alpha}, (s\cdot\overline{\alpha} + (1-s)\cdot\overline{\beta})) = s\cdot\alpha(\overline{\alpha}, \overline{\beta})$.

It is not obvious at this stage that, if S_p contains m independent elements, T_p (and let alone $\exp_p^{-1}$ (B(p, r))!) contains an open subset of $\mathbb{R}^m$ (e.g., sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_m\} \setminus T_p$ could be dense in sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_m\}$).

Definition 3.8. *Let X be an inner metric space and $x \in X$. Then a subset $A \subset X$ is said to be* transverse to x *if each minimal geodesic starting at x intersects A in at most one point.*

Note that if, in the above definition, x and A both lie in a strictly convex set C, one need only consider minimal curves *in C.*

Lemma 3.9. *Suppose $\overline{\gamma}_1, \ldots, \overline{\gamma}_m \in \overline{S}_p$ are independent. Then for every $\varepsilon > o$ there exists a subset C_ε of S_p homeomorphic to an open subset of S^{m-1} which is ε-close to sp $\{\overline{\gamma}_1, \ldots, \overline{\gamma}_m\} \cap S_p$ such that the map T has a positive lower bound on C_ε.*

Proof. Recall that subsets X and Y of a metric space are ε-close (in the Hausdorff sense) if X is in an ε-neighborhood of Y, and vice versa.

The case m = 1 follows from the fact that S_p is dense in $\overline{S}_p$. For m = 2, choose α_1, $\alpha_2 \in S_p$ such that for all $s \in [0, 1]$, the angle between the line L_s from 0 through $s\cdot\alpha_1 + (1-s)\cdot\alpha_2$ and the line from 0 through $s\cdot\gamma_1 + (1-s)\cdot\gamma_2$ is $< \varepsilon / 2$. Let $\beta_t : [0, 1] \to B$ be the minimal curve from $\alpha_1(t)$ to $\alpha_2(t)$, for all t such that the endpoints are defined, and let $\zeta_{s,t}$ be the unit minimal curve from p to $\beta_t(s)$. From the proof of Lemma 3.4, for each $s \in [0, 1]$ there exists a maximal $\delta(s) \in (0, r]$ such that for all $t \le \delta(s)$, the angle between $\zeta_{s,t}$ and L_s is $\le \varepsilon / 2$. For any fixed t, as $s \to s'$, $L_s \to L_{s'}$ and $\zeta_{s,t} \to \zeta_{s',t}$, and so $e : [0, 1] \to (0, r]$ is continuous, with a positive maximum. Hence for any fixed positive $T < \delta(s)$, $\{\zeta_{s,T} : s \in (0, 1)\}$ satisfies the requirements for C_ε.

Now suppose m > 2. For any k, set

$$C_k = \{t_1\overline{\gamma}_1 + \dots + t_k\overline{\gamma}_k : t_1 + \dots + t_k = 1\},$$

and suppose there exist homeomorphisms $\varphi_i : C_{m-1} \to B$ whose images are transverse to p, with the following property: For any $a \in C_{m-1}$, let α_a^i be the unit minimal curve from p to $\varphi_i(a)$ in B and β_a be the unit vector on the radial line from 0 to a. Then $\alpha(\alpha_a^i, \beta_a)$ is uniformly small for all $a \in C_{m-1}$ and sufficiently large i. In particular, for large i, the interior of $\{\gamma \in S_p : \gamma(s) \in \varphi_i(C_{m-1}), s > 0\}$ satisfies the requirements for C_ε.

Let $\beta_i \in S_p$ such that $\alpha(\beta_i, \overline{\gamma}_m) \le 1 / i$ and $T_i = \min\{T(\beta_i), 1 / i\}$. For any a, $b \in C_{m-1}$, there exists an I > 0 such that for all $i \ge I$, $\beta_i(T_i)$ is transverse to $\{\varphi_i(a), \varphi_i(b)\}$. If otherwise, there would exist $i_k \to \infty$ and minimal curves from

$\beta_{i_k}(T_{i_k})$ through both $\varphi_{i_k}(a)$ and $\varphi_{i_k}(b)$. But then by Lemma 3.3, $\bar{\gamma}_m$ would lie in sp $\{\lim \varphi_i(a), \lim \varphi_i(b)\}$, and hence in sp $\{\bar{\gamma}_1, \ldots, \bar{\gamma}_{m-1}\}$, a contradiction. Furthermore, the choice of minimal such I is an upper semicontinuous function of $C_{m-1} \times C_{m-1}$ into the positive integers, since if $a_j \to a$ and $b_j \to b$, then for any i, the limit of minimal curves from $\beta_i(T_i)$ through both $\varphi_i(a_j)$ and $\varphi_i(b_j)$ is a minimal curve from $\beta_i(T_i)$ through both $\varphi_i(a)$ and $\varphi_i(b)$.

One can therefore choose $I > 0$ such that for all $i > I$, $\varphi_i(C_{m-1})$ is transverse to $\beta_i(T_i)$, and φ_i can be extended to a homeomorphism on C_m by letting $\varphi_i(t_1\bar{\gamma}_1 + \ldots + t_m\bar{\gamma}_m) = \gamma(t_m)$, where γ is the geodesic from $\varphi_i(t_1\bar{\gamma}_1 + \ldots + t_{m-1}\bar{\gamma}_{m-1})$ to $\beta_i(T_i)$. By Lemma 3.3, for any $a \in C_m$ and $\varepsilon > 0$, there exists a $K > 0$ such that for all $i > K$, $\alpha(\alpha_a^i, \beta_a) < \varepsilon$. As in the above argument, the choice of a minimal such K for each $a \in C_m$ is upper semicontinuous; in other words, $\alpha(\alpha_a^i, \beta_a)$ is uniformly small for large i. Finally, $\varphi_i(C_m)$ is transverse to p for large enough i, by an argument similar to the proof in the above paragraph. This completes the proof of the inductive step, and the lemma.

Theorem 3.10. *Let (X, d) be a complete, locally compact inner metric space with locally bounded curvature. Then the following are equivalent:*

a) X is finite dimensional.

b) At some point $p \in X$ the space of directions S_p is precompact.

c) The set $\mathcal{T}$ of geodesic terminals in X is nowhere dense.

d) X is a manifold with boundary, and $\partial X = \mathcal{T}$.

Proof. For any $p \in X$, let $B(p, r)$ be a strictly convex region of curvature $\le k$ and $\ge K$, and $\bar{\gamma}_1, \ldots, \bar{\gamma}_n$ be independent elements of $\bar{S}_p$.

a) $\Rightarrow$ b). Choose ε small enough that the set C_ε of Lemma 3.9 is non-empty. Then $\exp_p$ is (a homeomorphism) defined on $\{t\cdot\gamma : \gamma \in C_\varepsilon,\ t < \varepsilon\}$, which is an open radial cone in $\mathbb{R}^n$; i.e., $B(p, r)$ contains a subset of dimension n. In other words, if X is finite dimensional $\bar{S}_p$ is spanned by at most finitely elements. By Proposition 3.7, $\bar{T}_p$ is isometric to a closed, convex Euclidean cone, and $\bar{S}_p$ is a closed (and hence compact) subset of the unit sphere.

b) $\Rightarrow$ c). For all i, let C_i be from Lemma 3.9 for $\varepsilon = 2^{-i}$, and let $C = \exp_p \{t\cdot\gamma : \gamma \in C_i$ for some i, $t < \min\{T(\gamma), r\}\}$. Then C is an open dense subset of $B(p, r)$ homeomorphic to an open subset of Euclidean space. For any $z \in C$, $\exp_z^{-1}(C)$ is open in T_z and homeomorphic to an open set in $\mathbb{R}^n$; in particular, there is an open n-ball $B(0, \rho) \subset T_z$ contained in $\exp_z^{-1}(C)$. This implies that T_z is in fact isometric to $\mathbb{R}^n$, or, equivalently, that z is not a geodesic terminal. All geodesic terminals in $B(p, r)$ therefore lie in $B(p, r) \setminus C$, a nowhere dense set.

Let Y be the subset of all points $y \in X$ such that for some $\rho > 0$, the geodesic terminals in $B(y, \rho)$ are nowhere dense. Y is obviously open, and also closed: Let $w \in \bar{Y}$, and suppose $B(w, \rho)$ is a strictly convex region of curvature $\le K$ and $\ge k$. There exists in $B(w, \rho/2)$ a point $y \in Y$ and hence a point z which is contained in a geodesically complete open ball; i.e., S_z is precompact. But then by the preceding paragraph the geodesic terminals in the ball $B(z, \rho/2)$, which is a strictly convex region of curvature $\le K$ and $\ge k$, are nowhere dense. It follows that $w \in Y$. Finally, since the set $\mathcal{T}$ of geodesic terminals is nowhere dense in a region of every point, $\mathcal{T}$ is nowhere dense in all of X.

c) $\Rightarrow$ d). Let $x \in X$ and choose a strictly convex region V of curvature $\le K$ and $\ge k$ containing x, and a point $p \in V \setminus \overline{\mathcal{T}}$. Since $\mathcal{T}$ is nowhere dense, there exists a ball $B = B(p, r) \subset V$ such that B is geodesically complete (and strictly convex). By Theorem 3.2, T_p is homeomorphic to $\mathbb{R}^n$, and $\exp_p|_{B(0,r)}$ is a homeomorphism. For all $q \in V$, let γ_q denote the unique minimal curve from p to q.

The main step in the proof is showing that if x is a terminal of a geodesic in V starting at p, then x is a boundary point, and if x is not such a terminal, it is an interior point. As argued previously, the map $T : S_p \to \mathbb{R}^+$ is upper semicontinuous; at any $v \in S_p$ such that $\exp_p(tv) \in V$ for all $t \in [0, T(v)]$, T is lower semicontinuous (and hence continuous), as the following claim shows.

Claim 3.11. *If for some $v \in S_p$ and $c > 0$, $\exp_p(tv) \in V$ for all $t \in [0,c]$, then $\liminf_{w \to v} T(w) \ge c$.*

Proof. Let $y = \exp(cv)$. Note that $\overline{T}_y = \mathrm{sp}\,\{\gamma_1, \ldots, \gamma_n\}$ for some $\gamma_1, \ldots, \gamma_n$. For if $\overline{S}_y$ contained $m > n$ independent elements, Lemma 3.9 would imply that V, and hence T_p, would contain a set homeomorphic to an open subset of $\mathbb{R}^m$, a contradiction to elementary dimension theory. Likewise, since $\overline{T}_p$ is of dimension n, $\overline{T}_y$ is spanned by at least n elements.

Let $U = \exp_y^{-1}(B(p, r))$; then by Invariance of Domain U is open in $\overline{T}_y \subset \mathbb{R}^n$. Let $S(c + \varepsilon)$ denote the intersection of T_y with the $(n-1)$-sphere in $\mathbb{R}^n$ centered at 0 in of radius $c + \varepsilon$. For some small ε, $S(c+\varepsilon) \cap U$ contains an $(n-1)$-disk $\mathfrak{D}$ such that $(c+\varepsilon)\cdot v$ lies in the interior of $\mathfrak{D}$. Let S denote the $(n-1)$-sphere in T_y which is the union of $\mathfrak{D}$ with all radial lines from $\partial\,\mathfrak{D}$ to 0.

The set $Z = \exp_p^{-1}(\exp_y(S))$ is a topological $(n-1)$-sphere containing cv such that for all $t \in [0, c)$, tv lies in the n-dimensional ball bounded by Z. In particular, if $\alpha(v, w)$ is small, then the radial line through w must intersect Z near cv. In other words, $\exp_p(tw)$ is defined for t not much smaller than c. This completes the proof of the claim.

Suppose now that x is a terminal of the minimal curve γ_x. Let $D = \{w \in T_p : w = T(v)\cdot v,\ v \in S_p, \text{ and } \alpha(\exp(tw), \gamma_x)) < \varepsilon\}$, where ε is chosen, using the continuity of T, small enough that $\exp(tw) \in V$, for all $w \in D$ and $t \in [0,1]$. The continuity of T shows furthermore that D is a topological $(n-1)$-ball, and that $D' = \{tw : w \in D \text{ and } t \in (0, \|w\|]\}$ is homeomorphic to a boundary ball in n-dimensional half-space. Finally, $\exp(D')$ is an open subset containing x; for if x' is sufficiently close to x, then $\alpha(\gamma_x, \gamma_{x'}) < \varepsilon$ and $d(x',p) > r/3$. By definition, the terminal of $\gamma_{p'}$ lies in $\exp(D)$, and so $x' \in \exp(D')$.

If x is not a terminal of γ_x, then Claim 3.11 shows that nearby points are also not terminals, and so the exponential map provides a neighborhood of x homeomorphic to an open subset of Euclidean space.

X has now been shown to be a manifold with boundary, and $\partial X = \mathcal{T}'$, where $\mathcal{T}'$ is the set of terminals x with the following property: in some strictly convex region V of curvature bounded above and below containing x, there is a minimal curve γ_{px}, with $p \in V \setminus \overline{\mathcal{T}}$. Since ∂X is closed, the proof of the theorem will be complete if it is shown that each point $z \in \mathcal{T}$ is $z = \lim z_i$, with $z_i \in \mathcal{T}'$. Choose a region V containing z with curvature bounded above and below, and pick $q \in V$ so that z is a terminal of a minimal curve γ starting at q. Now choose points $q_i \to q$ such that $q_i \in V \setminus \overline{\mathcal{T}}$, and let γ_i denote the unique geodesic starting from q_i, with maximal domain of definition. Then as in the proof of the upper semi-continuity of the map T, as $i \to \infty$, the geodesics must have terminals z_i, and $z_i \to z$.

d) $\Rightarrow$ a) is a classical result. The proof of Theorem 3.10 is now complete.

Corollary 3.12. *Let M be a topological n-manifold and d be an inner metric on M of locally bounded curvature. Then the following are equivalent:*

a) (M, d) is (metrically) complete,

b) (M, d) is geodesically complete,

c) there exists a point $p \in M$ such that $\exp_p$ is defined on all of $\mathbb{R}^n$, and

d) every closed, bounded subset of M is compact.

Proof. a) $\Rightarrow$ b) If M is a manifold (without boundary) having a complete inner metric then by Theorem 3.10, $\mathcal{T} = \partial M = \emptyset$, which is equivalent to geodesic completeness in the metrically complete case (cf. Chapter 1). The proofs of the remaining implications are essentially the same as in the classical Hopf-Rinow theorem.

If the geodesic terminals $\mathcal{T}$ in a space X are nowhere dense in the neighborhood of some point p, then S_p is compact, and Theorem 3.10 implies that $\mathcal{T}$ is nowhere dense in all of X. Equivalently:

Corollary 3.13. *If (X, d) is an infinite dimensional, complete, locally compact space of locally bounded curvature, then the set $\mathcal{T}$ of geodesic terminals is dense in X.*

Corollary 3.14. *If X is a compact Riemannian manifold such that $S_f X$ has bounded curvature for some f, then X is homeomorphic to a sphere.*

Proof. The only possible terminals in $S_f X$ are the end points of the suspension, which form a nowhere dense subset of $S_f X$. On the other hand X has dimension $n \geq 1$, and two points cannot form a boundary of the (n+1)-manifold $X \times_f (a,b)$. By Theorem 3.10 the end points cannot be terminals, and so $S_f X$ is a manifold. Finally, it is a standard topological result that the suspension of a space X is a manifold if and only if X is homeomorphic to a sphere.

Any finite dimensional space can be embedded in Euclidean space, and so completeness and finite dimensionality together imply local compactness. Theorem 3.10 therefore implies that any finite dimensional space X with a complete metric of locally bounded curvature is a topological manifold with boundary. More generally, since the induced inner metric on a convex subset C of X is the same the original metric of C (as a subset of X), the following corollary holds (in the Riemannian case this was proved by Cheeger and Gromoll, cf. [CE], Chapter 8).

Corollary 3.15. *If X is a finite dimensional, complete inner metric space of locally bounded curvature, then every closed, convex subset of X is a manifold with boundary.*

In general the boundary of a space of bounded curvature need not be smooth in the "normal" coordinates of the type constructed in Theorem 3.10: A square in the plane with the induced inner metric is flat, but in no choice of normal coordinates is the boundary smooth.

Lemma 3.15. *Let X be a finite dimensional complete inner metric space of locally bounded curvature. Then ∂X is transverse to every interior point.*

Proof. Suppose $x \in \partial X$ and let γ_{ax} be minimal. There exists a point p on γ_{ax} and a strictly convex region B(p, r) of curvature bounded above and below containing x. If γ were defined beyond x, then Claim 3.11 would imply that $\exp_p^{-1}$ is a homeomorphism on an open set containing x; that is, x is contained in a Euclidean neighborhood, a contradiction. In other words, every geodesic terminates if it hits the boundary, and so cannot intersect the boundary twice.

Definition 3.16. *Suppose $B = B(p, r)$ is a strictly convex region of curvature bounded above and below, and A, $A' \subset B$ are transverse to p. Then A and A' are said to be* r-equivalent in B *if there is a (possibly not continuous) bijection $\varphi : A \to A'$ such that a and $\varphi(a)$ lie on the same radial geodesic from p. The* radial distance $\delta_r(A, A')$ *is defined to be the supremum of the distances $d(a, \varphi(a))$.*

Lemma 3.17. *Let $U = B(p, r)$ and $V = B(q, s)$ be strictly convex regions of curvature bounded above and below. Suppose A is a compact subset of $U \cap V$ such that A is transverse to both p and q. Then there exists an $\varepsilon > 0$ such that if $A' \subset U \cap V$ is r-equivalent to A in U and $\delta_r(A, A') < \varepsilon$, then A' is transverse to q.*

Proof. The set $C = \{\gamma \in S_p : \gamma(t) \in A$ for some $t\}$ is compact. If $\gamma(s)$, $\alpha(t) \in A$, there exists some $\delta > 0$ such that for all $\zeta \in (-\delta, \delta)$, $\{\gamma(s+\zeta), \alpha(t+\zeta)\}$ is transverse to q. For if otherwise, one could find $t_i \to 0$ with geodesics β_i to q starting at q passing through both $\gamma(s+t_i)$ and $\alpha(t+t_i)$. But $\lim \beta_i$ would be a minimal curve in V starting at q and passing through both $\gamma(s)$ and $\alpha(t)$, a contradiction. A similar argument shows that the function which assigns to each element of $C \times C$ the infimum of all such δ is lower semicontinuous, and therefore has a positive minimum on $C \times C$; this minimum is the desired number ε.

Theorem 3.18. *A topological space X admits the structure of a smooth manifold with boundary if and only if X possesses a complete metric of locally bounded curvature.*

Proof. Suppose X is a smooth manifold with boundary. Endow the interior of X with a Riemannian metric which is a product metric near the boundary. Extend the metric (distance) to all of X by continuity. Then X is isometrically embedded as a convex subset of the Riemannian manifold $\tilde{X}$ obtained by adding a small open collar (with the product metric) to X along the boundary. In particular, all angle comparisons in X can be carried out in $\tilde{X}$, which has locally bounded sectional curvature, and hence locally bounded curvature in the present sense.

Conversely, suppose X is finite dimensional with a complete inner metric of locally bounded curvature. For simplicity, assume X is compact, and let $\{B_i(x_i, r_i)\}$, $1 \le i \le k$, be a cover of X by balls whose *closures* lie in strictly convex regions of curvature bounded above and below, with $x_i \in \operatorname{int} X$ for all i. By the results of [Be], the sets B_i have C^1 overlap on their interiors. The terminal map $T_1 : S_{x_1} \to (0, \infty]$ was shown to be continuous on $T_1^{-1}((0, r_1])$ in the proof of Theorem 3.10. $U_1 = T_1^{-1}((0, r_1))$ is an open subset of the unit sphere S_{x_1} homeomorphic to $W_1 = B_1 \cap \partial X$ via the map $\varphi_1(v) = \exp_{x_1} \circ T_1(v)\cdot v$. One can choose a smooth map $\mathcal{T}_1 : U_1 \to (0, r)$ having a continous extension equal to T_1 on $S_{x_1} \setminus U_1$ such that, on U_1, $\mathcal{T}_1 < T_1$ and $\mathcal{T}_1$ approximates T_1 near enough that the following holds: Let $D_1 = \{\gamma(\mathcal{T}_1(\gamma)) : \gamma \in U_1\}$, that is, D_1 is obtained by "pushing" W_1 inward along radial geodesics starting at x_1 by the amount $T_1 - \mathcal{T}_1$. D_1 is r-equivalent to W_1, and so by Lemma 3.17, if $\mathcal{T}_1$ is chosen close enough to T_1, $D_1 \cap B_i$ is still transverse to r_i for all i such that $D_1 \cap B_i \neq \emptyset$. The set

$$X_1 = X \setminus \{\gamma(t) : t > \mathcal{T}_1(\gamma),\ \gamma \in U_1\}$$

is homeomorphic to X and has smooth boundary in B_1. Let B_1' be an open subset of B_1 such that $\overline{B}_1' \subset B_1$ and $\{B_1', B_2, \ldots, B_k\}$ covers X. Let T_2 be the "terminal" map for X_1, i.e., for each $\gamma \in S_{x_2}$, $T_2(\gamma) = t$ provided $\gamma(t) \in \partial X_1$, and $T_2(\gamma) = \infty$ if no such t exists. Since $\partial X_1 \cap B_2$ is transverse to r_2, T_2 is well-defined and continuous. $U_2 = T_2^{-1}((0, r_2))$ is an open subset of the unit sphere S_{x_2} homeomorphic to $W_2 = B_2 \cap \partial X_1$ via the map $\varphi_2(v) =$

$\exp_{x_2} \circ T_2(v)\cdot v$. The overlap between B_1 and B_2 is C^1 on their interiors, and $\partial X_1 \cap B_1 \cap B_2 \subset \text{int } B_1 \cap \text{int } B_2$; this implies that T_2 is smooth on $Y = \varphi_2^{-1}(\partial X_1 \cap B_1 \cap B_2)$. Setting $Y' = \varphi_2^{-1}(\partial X_1 \cap B_1' \cap B_2)$, one can now choose a smooth approximation $\mathcal{T}_2$ of T_2 on U_2 which agrees with T_2 on Y', and so that the new manifold with boundary, X_2, constructed as above, has boundary whose intersection with any B_i is transverse to r_i, and which is smooth in $B_1' \cup B_2$. This inductive procedure can be continued for a finite number of steps to obtain a manifold with boundary X_k contained in, and homeomorphic to, X, such that the restrictions of $\{B_i\}$ are C^1 coordinates for X_k.

In the noncompact case, one can use the above procedure to "smooth out" B(p, 2) for some point p. On can then cover B(p, 3) by B(p, 1.5) and a finite number of open sets which do not intersect B(p, 1). A C^1 structure can now be constructed on B(p, 3) which agrees with the previous smooth structure on B(p, 1). This process can now be continued for a countable number of steps to put a C^1 structure on all of X.

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