WIENER AMALGAM SPACES FOR THE FUNDAMENTAL IDENTITY OF GABOR ANALYSIS

HANS G. FEICHTINGER AND FRANZ LUEF

ABSTRACT. In the last decade it has become clear that one of the central themes within Gabor analysis (with respect to general time-frequency lattices) is a duality theory for Gabor frames, including the Wexler-Raz biorthogonality condition, the Ron-Shen's duality principle or Janssen's representation of a Gabor frame operator. All these results are closely connected with the so-called *Fundamental Identity of Gabor Analysis*, which we derive from an application of Poisson's summation formula for the symplectic Fourier transform. The new aspect of this presentation is the description of range of the validity of this Fundamental Identity of Gabor Analysis using Wiener amalgam spaces and Feichtinger's algebra $S_0(\mathbb{R}^d)$. Our approach is inspired by Rieffel's use of the Fundamental Identity of Gabor Analysis in the study of operator algebras generated by time-frequency shifts along a lattice, which was later independently rediscovered by Tolmieri/Orr, Janssen, and Daubechies et al., and Feichtinger/Kozek at various levels of generality, in the context of Gabor analysis.

1. Introduction

Since the work of Wexler/Raz [WR90] on the structure of the set of dual atoms for a Gabor frame many researchers have benefited from their insight that some properties of a Gabor frame have a better description with respect to the adjoint lattice. We only mention the duality principle of Ron/Shen [RS93, RS97], Janssen's representation of the Gabor frame operator [Jan95] and the investigations of Daubechies/H. Landau/Z. Landau [DLL95], which have obtained similar results on the structure of Gabor frames independently by completely different methods around the year 1995. All their methods have in common an implicit use of the adjoint lattice for separable lattices. In [FK98], Feichtinger and Kozek gained a thorough understanding of the adjoint lattice for Gabor systems with respect to a lattice Λ in $G \times \widehat{G}$, G an elementary locally compact abelian group. Furthermore, [FK98]

Key words and phrases. Modulation spaces, Fundamental Identity of Gabor Analysis, Wiener Amalgam Spaces, Poisson summation.

makes use of the symplectic Fourier transform in this context for the first time. Another ingredient in all issues concerning investigations of duality principles for Gabor systems is an identity about samples of the product of two short-time Fourier transforms for a lattice Λ and its adjoint lattice Λ^0, see Section 2. In Gabor analysis Tolimieri/Orr have pointed out the relevance of this identity for the study of Gabor systems with atoms in the Schwartz class [TO95, TO92]. In [Jan95], Janssen generalized the results of Tolimieri/Orr and called this identity the **F**undamental **I**dentity of **G**abor **A**nalysis, since he derived FIGA as a consequence of a representation of the Gabor frame operator of fundamental importance in all duality results of Gabor analysis, (which is nowadays called the Janssen representation of a Gabor frame operator). But, the results of Tolimieri/Orr and Janssen on the FIGA had been obtained by Rieffel in his construction of equivalence bimodules for the C^*-algebra $C^*(D)$ generated by time-frequency shifts of a closed subgroup D of $G \times \widehat{G}$, for a locally compact abelian group G, and the C^*-algebra $C^*(D^0)$ generated by time-frequency shifts generated of the adjoint group D^0 in 1988, [Rief88]. Furthermore, Rieffel gave a description of the adjoint lattice, which was later rediscovered by Feichtinger/Kozek, and he used the Poisson summation formula for the symplectic Fourier transform to get FIGA for functions in the Schwartz-Bruhat space $\mathcal{S}(G)$. Therefore, Tolimieri/Orr's discussion of the FIGA is just a special case of Rieffel's general result. In addition Rieffel had implicitly described Janssen's representation of a Gabor frame operator in his discussion of $C^*(D^0)$-valued inner products [Rief88].

Our discussion of FIGA follows Rieffel's discussion. Therefore, we apply the Poisson summation formula for the symplectic Fourier transform to a product $V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}$ of short-time Fourier transforms of functions resp. distributions f_1, f_2, g_1, g_2 in suitable modulation spaces, see Section 3 for the definition of modulation spaces. In our proofs we need some local properties of STFT $V_{g_1}f_1$ and $V_{g_1}f_1$ for f_1, f_2, g_1, g_2, which are naturally expressed by membership in some Wiener amalgam spaces, see Section 3. Our strategy relies heavily on the fact that Feichtinger's algebra $M^1(\mathbb{R}^{2d})$ is the biggest time-frequency homogenous Banach space, where the Poisson summation formula holds pointwise (introduced as $S_0(\mathbb{R}^d)$ in [Fei81]). We therefore look for sufficient conditions such that $V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}$ is in $M^1(\mathbb{R}^{2d})$.

In Section 2 we introduce the reader to some well-known facts of time-frequency analysis, which we will use later. In Section 3 we give a short discussion of modulation spaces and Wiener amalgam spaces and present some of their properties. In Section 4 we discuss the notion of weakly dual pairs in Gabor analysis and their connection to FIGA. Then we prove FIGA with the help of Poisson summation for the symplectic Fourier transform. Furthermore, we use Wiener amalgam spaces to describe the local behaviour of the short-time Fourier transform. In Section 5 we briefly indicate some consequences of our main result for Gabor frames.

2. Basics of Gabor Analysis

In this section we recall some well-known facts of Gabor analysis, e.g., the short-time Fourier transform and some of its properties. Our representation owes much to Gröchenig's presentation in his survey of time-frequency analysis [Gr01].

In Gabor analysis the basic objects are time-frequency shifts. More concretely, for $f \in L^2(\mathbb{R}^d)$ we define the following operators on $L^2(\mathbb{R}^d)$:

(1) the *translation* operator by

$$T_x f(t) = f(t-x), \qquad x \in \mathbb{R}^d,$$

(2) the *modulation* operator by

$$M_\omega f(t) = e^{2\pi i t \cdot \omega} f(t), \qquad \omega \in \mathbb{R}^d,$$

(3) *time-frequency shifts* by

$$\pi(x,\omega) f = M_\omega T_x f = e^{2\pi i \omega t} f(t-x), \quad (x,\omega) \in \mathbb{R}^{2d}.$$

The time-frequency shifts $(x,\omega,\tau) \mapsto \tau M_\omega T_x$ for $(x,\omega) \in \mathbb{R}^{2d}$ and $\tau \in \mathbb{C}$ with $|\tau| = 1$ define the Schrödinger representation of the Heisenberg group, consequently the time-frequency shifts $\pi(x,\omega)$ for $(x,\omega) \in \mathbb{R}^{2d}$ are a projective representation of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. More concretely, time-frequency shifts satisfy the following composition law:

$$\pi(x,\omega)\pi(y,\eta) = e^{-2\pi i x \cdot \eta} \pi(x+y, \omega+\eta), \tag{1}$$

for $(x,\omega),(y,\eta)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The noncommutativity of the time-frequency shifts leads naturally to the notion of the adjoint of a set of time-frequency shifts. Namely, let Λ be a subset of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then, the *adjoint lattice* Λ^0 of Λ is defined as the set of all time-frequency shifts in the time-frequency plane which commute with all time-frequency shifts $\{\pi(\lambda) : \lambda \in \Lambda\}$, i.e.,

$$\Lambda^0 := \{\lambda^0 \in \mathbb{R}^{2d} : \pi(\lambda)\pi(\lambda^0) = \pi(\lambda^0)\pi(\lambda) \text{ for all } \lambda \in \Lambda\}. \tag{2}$$

We include another approach to the adjoint of a lattice Λ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, because it plays a central role in the study of Gabor frames.

First we rewrite the composition law (1) of time-frequency shifts

$$\pi(x,\omega)\pi(y,\eta)=e^{-2\pi i(x\cdot\eta-\omega\cdot y)}\pi(y,\eta)\pi(x,\omega), \tag{3}$$

for $(x,\omega),(y,\eta)$ in $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$. We denote the phase-factor in (3) by $\rho(z,z')=e^{2\pi i\Omega(z,z')}$ with $z=(x,\omega)$, $z'=(u,\eta)$ and Ω denotes the standard symplectic form on $\mathbb{R}^{2d}$, i.e. $\Omega(z,z')=x\cdot\eta-\omega\cdot y$. An important fact is that ρ is a character of $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ and that every character of $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ is of the form

$$z\mapsto\rho(z,z') \text{ for some } z'\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d. \tag{4}$$

This gives an isomorphism between $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ and its dual group $\widehat{\mathbb{R}}^d\times\mathbb{R}^d$.

Let Λ be a lattice in $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ then every character of Λ extends to a character of $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ and therefore every character of Λ is of the form

$$\lambda\mapsto\rho(\lambda,z'),\ \lambda\in\Lambda, \tag{5}$$

for some $z\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d$, where z' needs not to be unique. The homomorphism from $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ to $\widehat{\Lambda}$ has as kernel: the *adjoint lattice*

$$\Lambda^0=\{z\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d \mid \rho(\lambda,z)=1 \text{ for all } \lambda\in\Lambda\}. \tag{6}$$

Therefore, the adjoint set Λ^0 of a lattice Λ has the structure of a lattice. In our discussion of FIGA we will explore further this line of reasoning.

The representation coefficients of the Schrödinger representation are up to some phase factors, equal to

$$V_gf(x,\omega):=\langle f,\pi(x,\omega)g\rangle=\int_{\mathbb{R}^d}f(t)\overline{g(t-x)}e^{-2\pi i\omega\cdot t}dt. \tag{7}$$

In Gabor analysis the representation coefficients are called the *short-time Fourier transform* (STFT) of $f\in\mathcal{S}(\mathbb{R}^d)$ with respect to a non-zero window g in Schwartz's space of testfunctions $\mathcal{S}(\mathbb{R}^d)$. For functions f with good time-frequency concentration, e.g. Schwartz functions, the STFT can be interpreted as a measure for the amplitude of the frequency band near ω at time x. The properties of STFT depend crucially on the window function g.

In our model, the time-frequency concentration of a signal is invariant under shifts in time and frequency, which is usually referred to the *covariance property* of a time-frequency representation. In harmonic analysis, a function f on $\mathbb{R}^d$ has a description in time and in frequency. A time-frequency representation of a function f encodes its properties simultaneously in time and frequency, e.g. the STFT. The following lemma expresses elementary properties of the STFT.

Lemma 2.1. *Let $f,g\in L^2(\mathbb{R}^d)$ and $(u,\eta)\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d$. Then*

(1) *Covariance Property of the STFT*

$$V_g(\pi(u,\eta)f)g(x,\omega) = e^{2\pi i u\cdot(\omega-\eta)} V_g f(x-u,\omega-\eta).$$

(2) *Basic Identity of Time-Frequency Analysis*

$$V_g f(x,\omega) = e^{-2\pi i x\cdot\omega} V_{\hat{g}}\hat{f}(\omega,-x).$$

In our proof of FIGA, we will use another basic identity of time-frequency analysis: *Moyal's formula.*

Lemma 2.2 (Moyal's Formula)**.** *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ then $V_{g_1}f_1$ and $V_{g_2}f_2$ are in $L^2(\mathbb{R}^{2d})$ and the following identity holds:*

$$\langle V_{g_1}f_1, V_{g_2}f_2\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}. \tag{8}$$

As a consequence, we get for $g \in L^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$ that

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_2,$$

for all $f \in L^2(\mathbb{R}^d)$, i.e., the STFT is an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$.

In time-frequency analysis we deal with function spaces which are invariant under time-frequency shifts. In the last years *modulation spaces* have turned out to be the correct class of Banach spaces for time-frequency analysis, [FK98, FZ98, Gr01, CG03].

3. Function Spaces for Time-Frequency Analysis

In the following we recall some well-known facts about modulation spaces and Wiener amalgam spaces. Our treatment of this notions is largely based on the excellent survey of time-frequency analysis by Gröchenig, [Gr01].

3.1. **Modulation spaces.**

In 1983 Feichtinger introduced a class of Banach spaces (see [Fei83, Fei8302]), which allow a measurement of the time-frequency concentration of a function or distribution f on $\mathbb{R}^d$, the so called *modulation spaces*. We choose the STFT $V_g f$ of f with respect to a window g with a good time-frequency concentration and as a measure we take the norm of a function space which is(isometrically) invariant under translations in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. For our investigations we restrict our study to weighted mixed-norm spaces $L^{p,q}_m$ on $\mathbb{R}^{2d}$, [Fei83]. But for the translation invariance of $L^{p,q}_m$ Feichtinger showed that the weight m has to be a moderate weight on $\mathbb{R}^{2d}$ with respect to a positive and rotational symmetric submultiplicative weight v on $\mathbb{R}^{2d}$, i.e $m(z_1+z_2) \le Cm(z_1)v(z_2)$ for $z_1, z_2 \in \mathbb{R}^{2d}$. Now for $1 \le p, q \le \infty$ we define a function or tempered distribution f to be an element of the *modulation space* $M^{p,q}_m(\mathbb{R}^d)$ if for a fixed g in Schwartz space $\mathcal{S}(\mathbb{R}^d)$ the norm

$$\|f\|_{M^{p,q}_m} := \|V_g f\|_{L^{p,q}_m} = \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|V_g f(x,\omega)|^p m(x,\omega)^p dx\Big)^{q/p} d\omega\Big)^{1/q}$$

is finite. Then $M^{p,q}_m(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g. We always measure the $M^{p,q}_m$-norm with a fixed non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and that for any non-zero $g_1 \in M^1_v(\mathbb{R}^d)$ the norm equivalence of $\|f\|_{M^{p,q}_m}$ with $\|V_{g_1} f\|_{L^{p,q}_m}$ holds.

One reason for the usefulness of modulation spaces is that many well-known function spaces can be identified with modulation spaces for certain weights:

(1) $M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.
(2) $M^1(\mathbb{R}^d)$ is *Feichtinger's algebra*, which is sometimes denoted by $S_0(\mathbb{R}^d)$.
(3) If $m(x,\omega) = (1+x^2)^{s/2}$ then $M^{2,2}_m = L^2_s = \{f \in \mathcal{S}'(\mathbb{R}^d) : (\int_{\mathbb{R}^d}|f(x)|^2(1+x^2)^s dx)^{1/2} < \infty\}$ is a weighted L^2-space.
(4) If $m(x,\omega) = (1+\omega^2)^{s/2}$ then $M^{2,2}_m = H_s = \{f \in \mathcal{S}'(\mathbb{R}^d) : (\int_{\mathbb{R}^d}|\hat{f}(\omega)|^2(1+\omega^2)^s d\omega)^{1/2} < \infty\}$ is a *Sobolev space.*
(5) If $m(x,\omega) = (1+x^2+\omega^2)^{s/2}$ then $M^{2,2}_m = Q_s = L^2_s \cap H^s$, where Q_s is the *Shubin class*, see [Shu01].

Modulation spaces inherit many properties from the mixed norm spaces, e.g. duality. In the following theorem we state some of their properties, that are of interest in the later discussion.

Theorem 3.1. *Let $1 \le p,q < \infty$ and m a v-moderate weight on $\mathbb{R}^{2d}$.*

(1) *The dual space of $M^{p,q}_m(\mathbb{R}^d)$ is $M^{p',q'}_{1/m}(\mathbb{R}^d)$ with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$ and the duality is given by*
$$\langle f, h\rangle = \iint_{\mathbb{R}^{2d}} V_g f(x,\omega)\overline{V_g h(x,\omega)}dx d\omega,$$
for $f \in M^{p,q}_m(\mathbb{R}^d)$ and $h \in M^{p',q'}_{1/m}(\mathbb{R}^d)$.
(2) *$M^{p,q}_m(\mathbb{R}^d)$ is invariant under time-frequency shifts:*
$$\|\pi(u,\eta)f\|_{M^{p,q}_m} \le Cv(u,\eta)\|f\|_{M^{p,q}_m} \quad \text{for} \quad (u,\eta) \in \mathbb{R}^{2d}.$$
(3) *If $p = q$ and $m(\omega,-x) \le Cm(x,\omega)$ then $M^{p,p}_m(\mathbb{R}^d)$ is invariant under Fourier transform.*

Proof. All these statements are well-known and the interested reader may find a proof of statement (1) in Chapter 11 of [Gr01]. We only give the arguments for statements (2) and (3), because they provide the reader with some insight about our choice of weights.

(2) The time-frequency invariance of $M^{p,q}_m(R^d)$ is a direct consequence of the definition of moderate weights and the Covariance Property of the STFT , see Lemma 2.1. Let $z = (u, \eta)$ be a point of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the following holds:

$$\begin{aligned}
\|\pi(u,\eta)f\|_{M^{p,q}_m} &= \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|V_gf(x-u,\omega-\eta)|^p m(x,\omega)^p dx\Big)^{q/p} d\omega\Big)^{1/q} \\
&= \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|V_gf(x,\omega)|^p m(x+u,\omega+\eta)^p dx\Big)^{q/p} d\omega\Big)^{1/q} \\
&\leq C\Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|V_gf(x,\omega)|^p v(u,\eta)^p m(x,\omega)^p dx\Big)^{q/p} d\omega\Big)^{1/q} \\
&= Cv(z)\|f\|_{M^{p,q}_m}.
\end{aligned}$$

(3) The key of the argument is an application of the basic identity of Gabor analysis, see Lemma 2.1, to a Fourier invariant window g and the independence of the definition of $M^{p,q}_m$ for $g \in \mathcal{S}(\mathbb{R}^d)$. For simplicity we choose g to be the standard Gaussian $g_0(x) = 2^{-d/4}e^{-\pi x^2}$.

$$\begin{aligned}
\|\hat{f}\|_{M^{p,p}_m} &= \Big(\iint_{\mathbb{R}^{2d}}|V_{g_0}f(x,\omega)|^p m(x,\omega)^p dx d\omega\Big)^{1/p} \\
&\leq \Big(\iint_{\mathbb{R}^{2d}}|V_{\widehat{g_0}}\hat{f}(x,\omega)|^p m(x,\omega)^p dx d\omega\Big)^{1/p} \\
&= \Big(\iint_{\mathbb{R}^{2d}}|V_{g_0}f(-\omega,x)|^p m(x,\omega)^p dx d\omega\Big)^{1/p} \\
&= \Big(\iint_{\mathbb{R}^{2d}}|V_{g_0}f(x,\omega)|^p m(\omega,-x)^p dx d\omega\Big)^{1/p} \\
&\leq C\|f\|_{M^{p,p}_m}.
\end{aligned}$$

□

In the following Corollary, we state some of the properties of the modulation space $M^{1,1}(\mathbb{R}^d)$. In harmonic analysis $M^{1,1}(\mathbb{R}^d)$ is the so-called *Feichtinger algebra* and some authors use the notation $S_0(\mathbb{R}^d)$ to indicate that Feichtinger's algebra is a Segal algebra, too. There is another reason for this notation, because $S_0(\mathbb{R}^d)$ shares many properties with the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of test functions, e.g., Feichtinger's algebra is invariant under Fourier transform. In the rest of our paper we will denote Feichtinger's algebra by $M^1(\mathbb{R}^d)$.

Corollary 3.2. *Feichtinger's algebra $M^1(\mathbb{R}^d)$ has the following properties:*

(1) $M^1(\mathbb{R}^d)$ *is a Banach algebra under pointwise multiplication.*
(2) $M^1(\mathbb{R}^d)$ *is a Banach algebra under convolution.*
(3) $M^1(\mathbb{R}^d)$ *is invariant under time-frequency shifts.*
(4) $M^1(\mathbb{R}^d)$ *is invariant under Fourier transform.*

Before we present the proof we recall that the STFT can be written as a convolution. Namely, let $g^*(x) = \overline{g(-x)}$ be the involution of $g \in L^2(\mathbb{R}^d)$. Then, STFT of $f \in L^2(\mathbb{R}^d)$ has the following form

$$V_g f(x,\omega) = e^{-2\pi i x\cdot\omega}(f * M_\omega g^*)(x). \tag{9}$$

For other formulations of the STFT and its relation to other time-frequency representations, such as the Wigner distribution or the ambiguity function we refer the reader to Gröchening's book [Gr01].

Proof. (1) By (9) the M^1-norm of f is given by

$$\|f\|_{M^1} = \int_{\mathbb{R}^d} \|f * M_\omega g^*\|_{L^1} d\omega.$$

Therefore, we get the following estimate for $h * f$, where $f \in M^1(\mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^d)$:

$$\begin{aligned} \|h * f\|_{M^1} &= \int_{\mathbb{R}^d} \|h * f * M_\omega g^*\|_{L^1} d\omega \\ &\leq \int_{\mathbb{R}^d} \|h\|_{L^1} \|f * M_\omega g^*\|_{L^1} d\omega \\ &= \|h\|_{L^1}\|f\|_{M^1}. \end{aligned}$$

(2) The statement follows from (1) by applying Fourier transforms.
(3) The statement is a special case of our general result for modulation spaces, Theorem 3.1.
(4) The statement is again a special case of our general result for modulation spaces, Theorem 3.1.

□

Despite the above stated properties, Feichtinger observed that $M^1(\mathbb{R}^d)$ is the *minimal* time-frequency homogenous Banach space, [Fei81]. Another pleasant property of Feichtinger's algebra $M^1(\mathbb{R}^d)$ is that it is the largest Banach space which allows an application of Poisson's summation formula, [Fei81]. Our main results about FIGA rely heavily on this fact. We will discuss this topic further after the introduction of the symplectic Fourier transform.

3.2. **Wiener Amalgam Spaces.** Around 1980 Feichtinger introduced a class of Banach spaces, which allow measurement of local and of global properties of functions, see [Fei81a]. Feichtinger's work was motivated by some spaces Wiener had used in his study of the Fourier transform, see [Wie]. Nowadays those spaces are called *Wiener amalgam spaces* and they are a generalization of Fournier and Stewart's amalgam spaces, [FS85]. Wiener amalgam spaces have turned out to be very useful in harmonic analysis and time-frequency analysis, e.g. [Fei90, FZ98, Gr01].

More concretely, let g be an element of the space of test functions $\mathcal{D}(\mathbb{R}^d)$, whose translates generate a partition of unity over $\mathbb{R}^{2d}$, i.e., $\sum_{m\in\mathbb{Z}^{2d}} T_m g \equiv 1$. Let $V(\mathbb{R}^{2d})$ be a translation invariant Banach space of functions (or distributions) over $\mathbb{R}^{2d}$ with the property that $\mathcal{D}\cdot V \subset V$. Then the *Wiener amalgam space* $W(X, L^{p,q}_m)$ with local component X and global component $L^{p,q}_m$ is defined as the space of all functions resp. distributions for which the norm

$$\|f\|_{W(X,L^{p,q}_m)} = \Big(\int_{\mathbb{R}^d}\big(\int_{\mathbb{R}^d}\|f\cdot T_{(z_1,z_2)}\overline{g}\|^p_X m(z_1,z_2)^p dz_1\big)^{q/p} dz_2\Big)^{1/q}$$

is finite. We note that different choices of $g\in\mathcal{D}$ give the same space and yield equivalent norms [Fei81a].

In [Fei81a] also an extension of Hölder's inequality to Wiener amalgam spaces is given: If X be a Banach algebra with respect to pointwise multiplication, then for $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$ ones has

$$\|f\|_{W(X,L^1)} \le \|f\|_{W(X,L^{p,q}_m)}\|f\|_{W(X,L^{p',q'}_{1/m})} \quad . \tag{10}$$

There are many characterizations of Feichtinger's algebra $M^1(\mathbb{R}^d)$. Later we shall need a result of Feichtinger that $M^1(\mathbb{R}^d) = W(\mathcal{F}L^1, L^1)$ with equivalent norms [Fei81]. One of these norms has a formulation by means of the STFT:

$$\begin{aligned}
\|f\|_{W(\mathcal{F}L^1,L^1)} &= \iint_{\mathbb{R}^{2d}} \|f\cdot T_{(z_1,z_2)}g\|_{\mathcal{F}L^1} dz_1 dz_2 \\
&= \iint_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d}|(f\cdot T_{(z_1,z_2)}\overline{g})\widehat{\ }(\omega)|d\omega\Big)dz_1dz_2 \\
&= \iint_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d}|V_g f(z_1,z_2,\omega)|d\omega\Big)dz_1dz_2.
\end{aligned}$$

Analogous expressions for the norm of Wiener amalgam spaces $W(\mathcal{F}L^1, L^{p,q}_m)$ with local component $\mathcal{F}L^1$ and global component $L^{p,q}_m$ can be derived in terms of the STFT.

4. The Fundamental Identity of Gabor Analysis

In this section we first recall some elementary facts about Gabor frame operators and the symplectic Fourier transform. These basic facts and a result [CG03] of Cordero/Gröchenig on local properties for $V_g f$ with window $g \in M^1_v(\mathbb{R}^d)$ and $f \in M^{p,q}_m(\mathbb{R}^d)$ will allow us to derive our main result about the Fundamental Identity of Gabor Analysis.

Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and let $g \in L^2(\mathbb{R}^d)$ be a Gabor atom then $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a *Gabor system.* Since the Balian-Low principle tells us that it is not possible to construct (this is in contrast to the situation with wavelets) an orthonormal basis for $L^2(\mathbb{R}^d)$ of this form, starting e.g. from a Schwartz function g, interest in Gabor frames arose. A milestone was the paper by Daubechies, Grossmann and Meyer, [DGM86], where the "painless use" of (tight) Gabor frames was suggested.

In our case the Gabor frame operator has the following form:

$$S_{g,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g, \quad f \in L^2(\mathbb{R}^d).$$

A Gabor system $\mathcal{G}(g,\Lambda)$ is called a *Gabor frame* if the Gabor frame operator $S_{g,\Lambda}$ is invertible, i.e., if there exist some finite, positive real numbers A, B such that

$$A \cdot \mathrm{I} \le S_{g,\Lambda} \le B \cdot \mathrm{I}$$

or equivalently,

$$A\|f\|^2 \le \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le B\|f\|^2,$$

for all f in $L^2(\mathbb{R}^d)$. Gabor frames $\mathcal{G}(g,\Lambda)$ allow the following reconstruction formulas

$$f = (S_{g,\Lambda})^{-1} S_{g,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)(S_{g,\Lambda})^{-1} g \tag{11}$$

$$f = S_{g,\Lambda}(S_{g,\Lambda})^{-1} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)(S_{g,\Lambda})^{-1} g\rangle \pi(\lambda)g. \tag{12}$$

Due to its appearance in the reconstruction formulas $\gamma_0 := (S_{g,\Lambda})^{-1} g$ is called the (canonical) *dual Gabor atom.* Note that the non-orthogonality of the time-frequency shifts yields that the coefficients in the reconstruction formula (11) are not unique and therefore there are other *dual atoms* $\gamma \in L^2(\mathbb{R}^d)$ with $S_{g,\gamma,\Lambda} := \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)g = \mathrm{I}$. Some authors call (g,γ) a *dual pair* of Gabor atoms if $S_{g,\gamma,\Lambda} = \mathrm{I}$.

If one considers Gabor systems $\mathcal{G}(\Lambda, g)$ beyond L^2-setting, then the operator identity $S_{g,\gamma,\Lambda} = \mathrm{I}$ has to be interpreted in weak sense. This approach was suggested by Feichtinger and Zimmermann in [FZ98]. They called two elements $g, \gamma \in L^2(\mathbb{R}^d)$ a *weakly dual pair* with respect to Λ, if

$$\langle f, h\rangle = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle\langle\pi(\lambda)g, h\rangle \tag{13}$$

holds. One can show that absolute convergence of the series on the right, for all $f, h \in M^1(\mathbb{R}^d)$. Although the absolute convergence of the series seems very restrictive at first sight also for the case that $g \in M^1(\mathbb{R}^d)$ and $\gamma \in M^\infty(\mathbb{R}^d)$. By the symmetry of the definition in g and γ we get the same result for $\gamma \in M^1(\mathbb{R}^d)$ and $g \in M^\infty(\mathbb{R}^d)$. Therefore, in the sequel we will only state our results for one setting. We also mention without proof that under the above assumption on the pair (g, γ) the operator mapping the pair (f, h) to $\sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)\gamma\rangle\langle\pi(\lambda)g, h\rangle$ is continuous on $M^1(\mathbb{R}^d) \times M^1(\mathbb{R}^d)$ and that the corresponding Gabor frame operator $S_{g,\gamma,\Lambda}$ maps $M^1(\mathbb{R}^d)$ into $M^\infty(\mathbb{R}^d)$, [FZ98]. Another direct consequence of the definiton is that a pair (g, γ) is weakly dual if and only if $S_{g,\gamma,\Lambda} = \mathrm{I}$. For further properties we refer the interested reader to [FZ98].

The preceding discussion leads in a natural way to the study of the product of two STFT's $V_\gamma f \cdot \overline{V_g h}$ restricted to a lattice Λ. In Gabor analysis Tolmieri and Orr realized (in the one-dimensional case and for a product lattice $\alpha\mathbb{Z}\times\beta\mathbb{Z}$, $\alpha, \beta > 0$) that such sums should be evaluated with the help of Poisson's summation formula. But in his research on Morita equivalence of noncommuative tori [Rief88], Rieffel had used this identity for functions f, g, h, γ in Schwartz-Bruhat space $\mathcal{S}(G)$ for an elementary locally compact abelian group G and restrictions to a closed subgroup D of $G \times \widehat{G}$ in 1988! Only recently one of us has realized the connection between Rieffel's results and Gabor analysis [Lu05]. Therefore the research in Gabor anlaysis has been undertaken independently of Rieffel's work, despite its great relevance for Gabor anlaysis. We will explore this further in a subsequent paper.

In the following we define the symplectic Fourier transform and some of its basic properties, which was implicitely used by Rieffel in his derivation of the Fundamental Identity of Gabor Analysis. Due to the work of Feichtinger and Kozek [FK98] we also have gained the insight that the symplectic Fourier transform might be of some relevance in this context.

We continue our investigations of the character ρ of the commutation relation (3). The antisymmetry of Ω implies that ρ is a skew-bicharacter of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Nevertheless, ρ gives a Fourier transform $\widehat{F}^s$

on the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$

$$\begin{aligned} \widehat{F}^s(z) &= \iint_{\mathbb{R}^{2d}} \rho(z, z')F(z')dz' \\ &= \iint_{\mathbb{R}^{2d}} e^{2\pi i \Omega(z,z')}F(z')dz' \\ &= \iint_{\mathbb{R}^{2d}} e^{2\pi i(y\cdot\omega - x\cdot\eta)}F(y, \eta)dyd\eta, \end{aligned}$$

for $z = (x, \omega)$ and $z' = (y, \eta)$ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. We call $\widehat{F}^s$ the *symplectic Fourier transform* of a function F in $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, because it is induced by the symplectic form Ω of $\mathbb{R}^{2d}$.

The Poisson summation formula is one of the most powerful tools in harmonic analysis. In our derivation of FIGA we need a Poisson summation formula for the symplectic Fourier transform, which relates values of a function F on a lattice Λ in the time-frequency plane with the samples of its symplectic Fourier transform on the adjoint lattice Λ^0.

In the following theorem we give some properties of the symplectic Fourier transform.

Theorem 4.1. *Let $M^1(\mathbb{R}^{2d})$ be Feichtinger's algebra over the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.*

(1) *The symplectic Fourier transform is selfinverse on $L^2(\mathbb{R}^{2d})$.*
(2) *$M^1(\mathbb{R}^{2d})$ is invariant under the symplectic Fourier transform.*

Proof.

(1) The statement is a consequence of the anti-symmetry of the symplectic form Ω.
(2) We make the observation that ρ arises from $e^{2\pi i(x\cdot\omega+y\cdot\eta)}$ by a rotation of $\pi/2$, i.e. the symplectic Fourier transform is a rotated version of the Fourier transform on $\mathbb{R}^{2d}$. Therefore the result follows from the main properties of Feichtinger's algebra, see Theorem 3.1.

□

Traditionally a harmless use of the Poisson summation formula is only known for Schwartz functions. That for Feichtinger's algebra Poisson summation holds pointwise and with absolute convergence is quite unexpected. We only remind you on the work of Katznelson and of Kahane et al., where they give striking examples for the failure of Poisson's summation formula, [KL94, Kat67]. In [Gr96] Gröchenig pointed out the relevance of Poisson's summation formula for $M^1(\mathbb{R}^d)$ in his study of uncertainty principles and embeddings of various function spaces into $M^1(\mathbb{R}^d)$.

In the following theorem we state Poisson's summation formula for the symplectic Fourier transform.

Theorem 4.2. *[Poisson Summation] Let $F \in M^1(\mathbb{R}^{2d})$ then*

$$\sum_{\lambda\in\Lambda} F(\lambda) = |\Lambda|^{-1} \sum_{\lambda^0\in\Lambda^0} \widehat{F}^s(\lambda^0) \tag{14}$$

holds pointwise and with absolute convergence of both sums.

In [BP04] Benedetto/Pfander constructed a function $g \in L^2(\mathbb{R})$ such that $|V_g g|^2 \notin S_0(\mathbb{R}^2)$ and therefore the symplectic Fourier transform is not valid for $|V_g g|^2$.

Before we present our results on the validity of the FIGA, we compute the symplectic Fourier transform of $V_g f$ for $f, g \in M^1(\mathbb{R}^d)$.

Lemma 4.3. *Let $f, g \in M^1(\mathbb{R}^d)$ then the following holds:*

(1) $V_g f \in M^1(\mathbb{R}^{2d})$.
(2) $\widehat{V_g f}^s(z) = f(x)\overline{g(\omega)}e^{-2\pi i x\cdot\omega}$ *for* $z = (x, \omega) \in \mathbb{R}^{2d}$.

Proof. (1) Follows from the functorial properties and the minimality of $M^1(\mathbb{R}^d)$, see [Fei81]. Feichtinger and Kozek give a different proof in [FK98].
(2) Straightforward computation.

□

We recall that *Rihaczek's distribution* for $f, g \in L^2(\mathbb{R}^d)$ is defined as $R(f, g)(x, \omega) = f(x)\overline{g(\omega)}e^{-2\pi i x\cdot\omega}$ for $z = (x, \omega) \in \mathbb{R}^{2d}$. It is a very popular time-frequency representation in engineering, [HM01, Gr04]. An application of Theorem 4.2 yields a generalization of a formula of Kaiblinger [Ka05] which also established a connection to the Rihaczek distribution.

Proposition 4.4. *Let f, g be in $M^1(\mathbb{R}^d)$ and Λ a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the following relation holds:*

$$\sum_{\lambda\in\Lambda} V_g f(\lambda) = |\Lambda|^{-1} \sum_{\lambda^0\in\Lambda^0} R(f, g)(\lambda^0),$$

An application of Theorem (4.2) to a product of two STFT's combined with Lemma (4.3)(2) yields the FIGA.

Theorem 4.5 (Basic FIGA). *Assume that $f_1, f_2, g_1, g_2 \in M^1(\mathbb{R}^d)$. Then*

$$\sum_{\lambda\in\Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0\in\Lambda^0} V_{g_1} g_2(\lambda^0) \cdot \overline{V_{f_1} f_2(\lambda^0)} \tag{15}$$

Proof. Our argument is just the computation of the symplectic Fourier transform of $F = V_{g_1}f_1(\lambda)\cdot\overline{V_{g_2}f_2(\lambda)}$ and a use of Theorem (4.2).

$$\begin{aligned}
\widehat{F}^s(Y) &= \iint_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_{g_1}f_1(X)\overline{V_{g_2}f_2(X)}\rho(Y,X)dY\\
&= \iint_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \langle\pi(Y)f_1,\pi(Y)\pi(X)g_1\rangle\overline{\langle f_2,\pi(X)g_2\rangle}\rho(X,Y)dY\\
&= \iint_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \langle\pi(Y)f_1,\pi(X)\pi(Y)g_1\rangle\overline{\langle f_2,\pi(X)g_2\rangle}\rho(X,Y)dY\\
&= \langle f_1,\pi(Y)f_2\rangle\overline{\langle g_1,\pi(Y)g_2\rangle},
\end{aligned}$$

where in the last step we used Moyal's formula (8). $\square$

The Fourier invariance of $M^1(\mathbb{R}^d)$ and the basic identity of Gabor analysis (2) yield the following reformulation of (15):

$$\sum_{\lambda\in\mathcal{J}\Lambda} V_{\hat{g}_1}\hat{f}_1(\lambda)\cdot\overline{V_{\hat{g}_2}\hat{f}_2(\lambda)} = |\Lambda|^{-1}\sum_{\lambda^0\in\mathcal{J}\Lambda^\perp} V_{g_1}g_2(\lambda^0)\cdot,\overline{V_{f_1}f_2(\lambda^0)}$$

where $\mathcal{J} = \begin{pmatrix} 0 & \mathrm{I}_d\\ -\mathrm{I}_d & \end{pmatrix}$ denotes a rotation by $\pi/2$ of the time-frequency plane $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$. We have stated this reformulation of (15) for two reasons: (1) It is another manifestation of the fact that V_gf encodes information about f and $\hat{f}$,(2) It shows that an application of the Fourier transform on the level of signals and windows corresponds to a rotation of the lattice Λ in $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ and that the application of the symplectic Fourier transform to a time-frequency representation of f yields a rotation of the dual lattice $\Lambda^\perp$ in $\widehat{\mathbb{R}}^d\times\mathbb{R}^d$.

What properties of $f,g\in S_0(\mathbb{R}^d)$ imply that $V_gf\in S_0(\mathbb{R}^{2d})$? Recall that $S_0(\mathbb{R}^d) = W(\mathcal{F}L^1,L^1)$. This fact suggests that for $f\in M^{p,q}_m(\mathbb{R}^d)$ and $g\in M^1(\mathbb{R}^d)$ the STFT $V_gf\in W(\mathcal{F}L^1,L^{p,q}_m)$. In [CG03] Cordero and Gröchenig have proved this result in their investigations of localization operators. Before we state their result on local properties of the STFT, we introduce a weighted version of Feichtinger's algebra $M^1(\mathbb{R}^d)$.

By assumption our weight m is v-moderate for v a submultiplicative weight on $\mathbb{R}^{2d}$. The space $M^1_v(\mathbb{R}^d)$ is now the correct weighted version of Feichtinger's algebra $M^1(\mathbb{R}^d)$.

Proposition 4.6 (Cordero/Gröchenig)**.** *Let* $1\le p,q\le\infty$. *If* $f\in M^{p,q}_m(\mathbb{R}^d)$ *and* $g\in M^1_v(\mathbb{R}^d)$, *then* $V_gf\in W(\mathcal{F}L^1,L^{p,q}_m)$ *with*

$$\|V_gf\|_{W(\mathcal{F}L^1,L^{p,q}_m)}\le C\|f\|_{M^{p,q}_m}\|g\|_{M^1_v}. \tag{16}$$

In other words, the norms of $L^{p,q}_m$ and of $W(\mathcal{F}L^1, L^{p,q}_m)$ are equivalent on the range of V_g.

We refer the reader to [CG03] for a proof of Proposition 4.6.

The remaining part of this section we look for conditions on the quadruple f_1, f_2, g_1, g_2, which imply that $V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}$, belongs to Feichtinger's algebra $M^1(\mathbb{R}^d)$, because then a careless application of Poisson formula summation is allowed.

The following theorem is our main result, which is an extension of the validity of range of FIGA.

Theorem 4.7. *[Main Result] There exists $C > 0$ independent of p, q, v, m such that for $f_1 \in M^{p,q}_m$, $f_2 \in M^{p',q'}_{1/m}$ and $g_1, g_2 \in M^1_v$, then the following holds:*

$$\sum_{\lambda \in \Lambda} V_{g_1}f_1(\lambda) \cdot \overline{V_{g_2}f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1}g_2(\lambda^0) \cdot \overline{V_{f_1}f_2(\lambda^0)}.$$

Proof. By Proposition 4.6 we have that $V_{g_1}f_1 \in W(\mathcal{F}L^1, L^{p,q}_m)$ and $V_{g_2}f_2 \in W(\mathcal{F}L^1, L^{p',q'}_{1/m})$. Therefore an application of Hölder's inequality (10) for Wiener amalgam spaces yields that $V_{g_1}f_1 \cdot \overline{V_{g_2}f_2} \in W(\mathcal{F}L^1, L^1)$. The inequalities (10) and (16) imply the desired norm estimate:

$$\begin{aligned}
\|V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}\|_{M^1} &\leq \|V_{g_1}f_1 \cdot \overline{V_{g_2}f_2}\|_{W(\mathcal{F}L^1,L^1)} \\
&\leq C\|V_{g_1}f_1\|_{W(\mathcal{F}L^1,L^{p,q}_m)}\|V_{g_2}f_2\|_{W(\mathcal{F}L^1,L^{p',q'}_{1/m})} \\
&\leq C\|g_1\|_{M^1_v}\|g_2\|_{M^1_v}\|f_1\|_{M^{p,q}_m}\|f_2\|_{M^{p',q'}_{1/m}}.
\end{aligned}$$

Therefore our object of interest is in $M^1(\mathbb{R}^d)$ and an application of Poisson summation yields the desired result. □

As an application of Theorem 4.7 we derive the known results about the validity of the FIGA. The first result was obtained by Feichtinger/Zimmermann in their discussion of weakly dual pairs [FZ98].

Corollary 4.8 (Feichtinger-Zimmermann)**.** *Let g_1, g_2 be in $M^1(\mathbb{R}^d)$. If $f_1 \in M^1(\mathbb{R}^d)$ and $f_2 \in M^\infty(\mathbb{R}^d)$ or $f_1, f_2 \in L^2(\mathbb{R}^d)$ then*

$$\sum_{\lambda \in \Lambda} V_{g_1}f_1(\lambda) \cdot \overline{V_{g_2}f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1}g_2(\lambda^0) \cdot \overline{V_{f_1}f_2(\lambda^0)}.$$

holds.

Proof. The corollary covers the cases $f_1 \in M^1(\mathbb{R}^d)$ and $f_2 \in \big(M^1(\mathbb{R}^d)\big)' = M^\infty(\mathbb{R}^d)$ and $f_1, f_2 \in M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Therefore the proof is a direct consequence of Theorem 4.7 □

The second result covers the case of Tolimieri/Orr of the validity of the FIGA for Schwartz funtions [TO95]. The proof consists of the well-known fact [Gr01] that the modulation spaces $M^1_{v_s}(\mathbb{R}^d)$ for $v_s(x,\omega) = (1+x^2+\omega^2)^{s/2}$ are the building blocks of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, namely

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{s\geq 0} M^1_{v_s}(\mathbb{R}^d).$$

By duality we get a description of tempered distributions

$$\mathcal{S}'(\mathbb{R}^d) = \bigcup_{s\geq 0} M^\infty_{1/v_s}(\mathbb{R}^d).$$

Corollary 4.9 (Tolimieri-Orr)**.** *Let f_1, g_1, g_2 be in $\mathcal{S}(\mathbb{R}^d)$ and $f_2 \in \mathcal{S}'(\mathbb{R}^d)$ then we have the following identity:*

$$\sum_{\lambda\in\Lambda} V_{g_1}f_1(\lambda)\cdot\overline{V_{g_2}f_2(\lambda)} = |\Lambda|^{-1}\sum_{\lambda^0\in\Lambda^0} V_{g_1}g_2(\lambda^0)\cdot\overline{V_{f_1}f_2(\lambda^0)}.$$

Proof. The statement is true for every building block of $\mathcal{S}(\mathbb{R}^d)$ and of $\mathcal{S}'(\mathbb{R}^d)$, respectively. Therefore our statement is a direct consequence of our main result Theorem 4.7. □

We stop here our list of examples and leave it to the reader to choose a pairing of his interest.

5. Biorthogonality condition of Wexler-Raz

In this section we present some consequences of our results on the FIGA, especially its relation to Janssen's representation of Gabor frame operators and the biorthogonality condition of Wexler-Raz.

Many researchers have drawn deep consequences from Janssen's representation, e.g. Gröchenig/Leinert in their proof of the "irrational case"-conjecture [GL04], Feichtinger/Kaiblinger in their work on the continuous dependence of the dual atom for a Gabor atom in $M^1(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$ [FK04].

In [Jan95] Janssen obtained his representation for Gabor frame operators $S_{g,\Lambda}$ with $g \in \mathcal{S}(\mathbb{R}^d)$ and $\Lambda = \alpha\mathbb{Z}\times\beta\mathbb{Z}^d$. The general form for arbitrary lattices was obtained by Feichtinger in collaboration with Kozek and Zimmermann in [FK98, FZ98].

Let Λ be a lattice in $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ and g a Gabor atom then the frame operator

$$S_{g,\Lambda}f = \sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)g.$$

Janssen's insight consists on a formal level of the following observation

$$\begin{aligned}\langle S_{g,\Lambda}f,h\rangle &= \Big\langle \sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)g,h\Big\rangle \\ &\overset{(FIGA)}{=} \Big\langle |\Lambda|^{-1}\sum_{\lambda^0\in\Lambda^0}\langle g,\pi(\lambda^0)g\rangle\pi(\lambda^0)f,h\Big\rangle \\ &= \langle |\Lambda|^{-1}S_{f,\Lambda^0}g,h\Big\rangle.\end{aligned}$$

But $S_{f,\Lambda^0}g$ is a series of time-frequency shifts operators acting on f. More concretely, the following representation of the Gabor frame operator was obtained by Janssen in [Jan95]

$$S_{g,\Lambda} = |\Lambda|^{-1}\sum_{\lambda^0\in\Lambda^0}\langle g,\pi(\lambda^0)g\rangle\pi(\lambda^0). \tag{17}$$

But the series on the right side of (17) only defines a bounded operator with the additional assumption that

$$\text{(A)} \qquad \sum_{\lambda^0\in\Lambda^0}|\langle g,\pi(\lambda^0)g\rangle| < \infty. \tag{18}$$

The last condition was introduced by Tolimieri/Orr in their discussion of Gabor frames [TO95].

The preceding observations lead Janssen to consider operators of the form

$$A = \sum_{\lambda^0\in\Lambda^0} a(\lambda^0)\pi(\lambda^0)$$

for $\big(a(\lambda^0)\big)\in\ell^1(\Lambda^0)$. But in [Rief88] Rieffel used such operators to introduce on $\mathcal{S}(\mathbb{R}^d)$ a Hilbert C^*-module structure for the C^*-algebra of all time-frequency shifts generated by Λ^0. This fact is the reason for the relation between Rieffel's work on Morita equivalence of noncommutative tori and Gabor analysis [Lu05].

We now extend the main results of Feichtinger/Zimmermann about weakly dual pairs [FZ98] to our setting. The notion of weakly dual pairs is the proper concept for the interpretation of a Gabor frame operator $S_{g,\gamma,\Lambda}$ in a weak sense.

First we recall the biorthogonality condition of Wexler-Raz. In Section 4 we have shown that the Gabor frame operator $S_{g,\Lambda}$ of a Gabor system $\mathcal{G}(g,\Lambda)$ for a lattice $\Lambda\in\mathbb{R}^{2d}$ gives rise to a reconstruction formula

$$f = (S_{g,\Lambda})^{-1}S_{g,\Lambda}f = \sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)\gamma_0 \tag{19}$$

for signals $f \in L^2(\mathbb{R}^d)$. We also mentioned the existence of other dual functions γ, which give rise for reconstruction formulas. In [WR90] Wexler/Raz gave a characterization of all dual functions γ for periodic discrete Gabor systems, which was the motivation for the work of Janssen, Tolimieri/Orr and Daubechies et al. [Jan95, DLL95, TO95]. The main result of Wexler/Raz consists in our setting of the following condition:

Let Λ^0 be a lattice in $\mathbb{R}^{2d}$. A pair $(g, \gamma) \in M^{p,q}_m \times M^{p',q'}_{1/m}(\mathbb{R}^d)$ satisfies the *Wexler-Raz condition* with respect to Λ^0, if

$$|\Lambda|^{-1}\langle \gamma, \pi(\lambda^0) g\rangle = \delta_{0,\lambda^0}, \tag{20}$$

where δ_{0,λ^0} denotes the Kronecker delta for the set Λ^0. In terms of Gabor systems the Wexler-Raz condition expresses the *biorthogonality* of the two sets $\mathcal{G}(g, \Lambda^0)$ and $\mathcal{G}(\gamma, \Lambda^0)$ to each other on $L^2(\mathbb{R}^d)$.

The importance of the Wexler-Raz condition arises from the fact, that under certain assumptions it characterizes all dual atoms of a given Gabor frame $\mathcal{G}(g, \Lambda)$.

The following theorem is the proposed extension of Feichtinger and Zimmermann's result [FZ98].

Theorem 5.1. *Let Λ be a lattice in $\mathbb{R}^{2d}$ and let (g, γ) be a dual pair in $M^{p,q}_m \times M^{p',q'}_{1/m}(\mathbb{R}^d)$. Then the following holds:*

(1) *(Wexler-Raz Identity)*

$$S_{g,\gamma,\Lambda} f = |\Lambda|^{-1} S_{f,\gamma,\Lambda^0} g \text{ in } M^\infty(\mathbb{R}^d) \tag{21}$$

for all $f \in M^1(\mathbb{R}^d)$.

(2) *(Janssen Representation)*

$$S_{g,\gamma,\Lambda} f = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_\gamma g(\lambda^0)\pi(\lambda^0) \tag{22}$$

is a bounded operator from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$ and the series converges unconditionally in the strong sense.

The proof is just a reformulation of the FIGA and the arguments of [FZ98] are also valid in our situation.

The usefulness of weakly dual pairs relies on the fact, that it is equivalent to the Wexler-Raz condition.

Theorem 5.2 (Feichtinger-Zimmermann)**.** *Let Λ be a lattice in $\mathbb{R}^{2d}$. Then a pair (g, γ) in $M^{p,q}_m \times M^{p',q'}_{1/m}(\mathbb{R}^d)$ is weakly dual with respect to Λ if and only if (g, γ) satisfies the Wexler-Raz condition with respect to Λ^0.*

The proof of Feichtinger/Zimmermann can again be adapted to our situation [FZ98].

Acknowledgement: The authors want to thank W. Czaja for many useful suggestions. The second named author wants to thank the Max Planck Institute of Mathematics at Bonn for their hospitality and Prof. Y. Manin for his kind invitation, because large part of the manuscript was written during this stay.

References

[BP04] J.J. Benedetto, G. Pfander *Frame expansions for Gabor multiliers*, Preprint, submitted.

[CL02] P.G. Casazza and M.C. Lammers, *Analyzing the Weyl-Heisenberg frame identity*, Appl. Comput. Harmon. Anal., 12(2), 171-178, 2002.

[CG03] E. Cordero, K. Gröchenig *Time-frequency analysis of localization operators*, J. Funct. Anal., 205(1):107 - 131, 2003.

[DGM86] I. Daubechies, A. Grossmann, Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., 27(5):1271-1283, 1986.

[DLL95] I. Daubechies, H.J. Landau, Z. Landau, *Gabor time-frequency lattices and the Wexler-Raz idenity*, J. Fourier Anal. Appl., 1(4):437-478, 1995.

[Fei81] H.G. Feichtinger, *On a new Segal algebra*, Mh. Math, 92:269-289, 1981.

[Fei81a] H.G. Feichtinger, *Banach convolution algebras of Wiener type*, In **Function Series, Operators**, Vol.I, II (Budapest 1980), pages 509-524. North-Holland, Amsterdam 1983.

[Fei83] H.G. Feichtinger, *Modulation Spaces on locally compact abelian groups*, Technical Report, University of Vienna, 1983.

[Fei90] H.G. Feichtinger, *Generalized Amalgams, with applications to Fourier transform*, Canad. J. Math., 42(3):395-409, 1990.

[Fei03] H. G. Feichtinger, Modulation spaces of locally compact Abelian groups, in: "Wavelets and their Applications" (Chennai, January 2002), M. Krishna, R. Radha and S. Thangavelu, eds., Allied Publishers, New Delhi (2003), 1–56.

[Fei8302] H. G. Feichtinger. Modulation spaces of locally compact Abelian groups. In R. Radha M. Krishna and S. Thangavelu, editors, *Proc. Internat. Conf. on Wavelets and Applications*, pages 1–56, Chennai, January 2002, 2003. Allied Publishers, New Delhi.

[FK98] H.G. Feichtinger and W. Kozek, *Quantization of TF–lattice invariant operators on elementary LCA groups*, In H.G. Feichtinger and T. Strohmer, editors, **Gabor Analysis and Algorithms: Theory and Applications**, pages 233–266. Birkhäuser, Boston, 1998.

[FK04] H.G. Feichtinger and N. Kaiblinger, *Varying the time-frequency lattice of Gabor frames*, Trans. Amer. Math. Soc., 356(5), 2001-2023, 2004.

[FS85] J.J.F. Fournier and J. Stewart, *Amalgams of L^p and ℓ^q*, Bull. Amer. Math. Soc. (N.S.), 13(1):1-21, 1985.

[FZ98] H.G. Feichtinger and G. Zimmermann, *A Banach space of test functions for Gabor analysis*, In H.G. Feichtinger and T. Strohmer, editors, **Gabor Analysis and Algorithms: Theory and Applications**, pages 123–170. Birkhäuser, Boston, 1998.

[Gr96] K. Gröchenig, *An uncertainty principle related to the Poisson summation formula*, Studia Math., 121(1):87-104, 1996.

[Gr01] K. Gröchenig, **Foundations of Time-Frequency Analysis**, Birkhäuser, Boston, 2001.

[GL04] K. Gröchenig and M. Leinert, *Wiener's lemma for twisted convolution and Gabor frames*, *J. Amer. Math.*, 1:1-17, 2004.

[Gr04] K. Gröchenig, *A pedestrian approach to pseudodifferential operators*, Preprint, submitted, 2004.

[HM01] F. Hlawatsch and G. Matz, **Quadratic time-frequency analysis of linear time-varying systems**, In L. Debnath, editor, Wavelet transforms and time-frequency signal analysis, Birkhäuser, Boston, pages 235-287, 2001.

[Jan95] A.J.E.M. Janssen, *Duality and biorthogonality for Weyl-Heisenberg frames*, J. Four. Anal. Appl., 1(4):403-436, 1995.

[Jan01] A.J.E.M. Janssen, *Representations of Gabor frame operators*, In J.S. Byrnes, editor, Twentieth century harmonic analysis–a ¿ celebration. Proceedings of the NATO Advanced Study Institute, Il ¿ Ciocco, Italy, July 2-15, 2000, Kluwer Academic,pages 73-101, 2001.

[KL94] J.P. Kahane and P.G. Lemarie-Rieusset. Remarques sur la formule sommatoire de Poisson. *Studia Mathematica*, 109/3:303–316, 1994.

[Ka05] N. Kaiblinger, *Approximation of the Fourier transform and the dual Gabor window*, J. Four. Anal. Appl., Vol. 11/2.

[Kat67] Y. Katznelson. Une remarque concernant la formule de Poisson. *Studia Math.*, 19:107–108, 1967.

[Lu05] F. Luef, *Gabor analysis, noncommutative tori and Feichtinger's algebra*, Preprint.

[Rief88] M.A. Rieffel, *Projective modules over higher-dimensional noncommutative tori*, *Can. J. Math.*, 40:257-388, 1988.

[RS93] A. Ron and Z. Shen, *Weyl-Heisenberg frames and stable bases*, Talk at Oberwolfach conference on Approx. Theory,1993.

[RS97] A. Ron and Z. Shen, *Weyl-Heisenberg frames and Riesz bases in* $L^2(\mathbb{R}^d)$, Duke Math.J., 89(2):237-282, 1997.

[Shu01] M.A. Shubin, **Pseudodifferential Operators and Spectral Theory**, Springer-Verlag, Berlin, 2^{nd} edition, 2001.

[TO92] R. Tolimieri and R.S. Orr, *OCharacterization of Weyl-Heisenberg frames via Poisson summation relationships*, Trans. Inform. Theory., 4:277-280, 1992.

[TO95] R. Tolimieri and R.S. Orr, *On Poisson summation, the ambiguity Function, and the theory of Weyl-Heisenberg frames*, J. Four. Anal. Appl., 1(3):233-247, 1995.

[TA98] Richard Tolimieri and Myoung An. **Time-frequency representations.** Applied and Numerical Harmonic Analysis. Boston: Birkhäuser, 1998.

[WR90] J. Wexler and S. Raz, *Discrete Gabor expansions*, Signal Processing, 21(3):207-221, 1990.

[Wie] N. Wiener, **The Fourier Integral and certain of its Applications**, Cambridge Univ. Press, 1933.

Fakultät für Mathematik, Nordbergstrasse 15, 1090 Wien, Austria
E-mail address: Hans.Feichtinger@univie.ac.at

Fakultät für Mathematik, Nordbergstrasse 15, 1090 Wien, Austria,

Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: Franz.Luef@univie.ac.at